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STABILITY OF TRAVELING-WAVE SOLUTIONS FOR A SCHRÖDINGER SYSTEM WITH POWER-TYPE NONLINEARITIES

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ABSTRACT. In this article, we consider the Schrödinger system with power-type nonlinearities,

$$i\frac{\partial}{\partial t}u_j + \Delta u_j + a|u_j|^{2p-2}u_j + \sum_{k=1,k\neq j}^{m} b|u_k|^p|u_j|^{p-2}u_j = 0; \quad x \in \mathbb{R}^N,$$

where $j = 1, \dots, m$, u_j are complex-valued functions of $(x,t) \in \mathbb{R}^{N+1}$, a, b are real numbers. It is shown that when $b > 0$, and $a + (m-1)b > 0$, for a certain range of p, traveling-wave solutions of this system exist, and are orbitally stable.

1. Introduction

It is well-understood that the nonlinear Schrödinger (NLS) equation

$$iu_t + \Delta u \pm |u|^2 u = 0 \tag{1.1}$$

where u is a complex-valued function of $(x,t) \in \mathbb{R}^{N+1}$, arises in a generic situation. The equation describes evolution of small amplitude, slowly varying wave packets in a nonlinear media [4]. Indeed, it has been derived in such diverse fields as deep water waves [34], plasma physics [35], nonlinear optical fibers [11, 12], magneto-static spin waves [36], to name a few. The m-coupled nonlinear Schrödinger (CNLS) system

$$i\frac{\partial}{\partial t}u_j + \Delta u_j + a_j|u_j|^{2p-2}u_j + \sum_{k=1,k\neq j}^{m} b_{kj}|u_k|^p|u_j|^{p-2}u_j = 0; \quad x \in \mathbb{R}^N, \tag{1.2}$$

for $j = 1, \dots, m$, where u_j are complex-valued functions of $(x,t) \in \mathbb{R}^{N+1}$, a_j and $b_{jk} = b_{kj}$ are real numbers, arise physically under conditions similar to those described by (1.1). The CNLS system also models physical systems in which the field has more than one components; for example, in optical fibers and waveguides, the propagating electric field has two components that are transverse to the direction of propagation. When $m = 2$, the CNLS system also arises in the Hartree-Fock theory for a double condensate; i.e., a binary mixture of Bose-Einstein condensates

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in two different hyperfine states. Readers are referred to the works [4, 11, 12, 34, 35] for the derivation as well as applications of this system.

The system admits the conserved quantities

$$\begin{aligned} &E(u_1, u_2, \dots, u_m) \\ &= \int_{\mathbb{R}^N} \Big[\sum_{j=1}^{m} \big(|\nabla u_j(x)|^2 - \frac{a_j}{p} |u_j(x)|^{2p} \big) - \sum_{j,k=1; j\neq k}^{m} \frac{b_{jk}}{p} |u_k(x)|^p |u_j(x)|^p \Big] dx, \end{aligned}$$

$$Q(u_j) = \int_{\mathbb{R}^N} |u_j(x,t)|^2 dx,$$

for $j = 1, 2, \dots, m$.

We are interested in traveling-wave solutions for (1.2) of the form $u(x,t) = (u_1, u_2, \dots, u_m)$, where for $j = 1, 2, \dots, m$,

$$u_j(x,t) = e^{i[(\omega_j - \frac{1}{4}|\theta|^2)t + \frac{1}{2}\theta x + m_j]} \varphi_{j,\omega_j}(x - \theta t) \tag{1.3}$$

for m_j, ω_j real constants, $\theta \in \mathbb{R}^N$ with $\omega_j - \frac{1}{4}|\theta|^2 > 0$ and $\varphi_{j,\omega_j} : \mathbb{R}^N \to \mathbb{R}$ are functions of one variable whose values are small when $|\xi| = |x - \theta t|$ is large. An important special case arises when $m_j = 0$, $\theta = \vec{0}$ and $\omega_j = \Omega_j > 0$. These special solutions (where, to emphasize the dependence on the parameters, we write φ_{j,ω_j} as ϕ_{j,Ω_j})

$$u_j(x,t) = e^{i\Omega_j t} \phi_{j,\Omega_j}(x) \tag{1.4}$$

are often referred to as *standing waves*. It is easy to see that, for example, standing waves are solutions of (1.2) if and only if $(u_1, u_2, \dots, u_m)$ is a critical point for the functional $E(u_1, u_2, \dots, u_m)$, when the functions $u_j(x)$ are varied subject to the m constraints that $Q(u_j)$ be held constant. If $(u_1, u_2, \dots, u_m)$ is not only a critical point but in fact a global minimizer then the standing wave is called a *ground-state solution*. In some cases, namely when $p = 2$, $N = 1$ and certain conditions on a_j, b_{jk} (see, for example, [24, 25, 21]), it is possible to show further that the ground-state solutions are solitary waves with the usual sech-profile.

One question unique to such type of nonlinear systems as (1.2) is to study the existence and stability of *nontrivial solutions* $(u_1, \dots, u_m)$; that is, all components of the solutions are non-zero, and they may be refereed to as co-existing solutions or vector solutions. For the system (1.2), there are many semi-trivial solutions (or collapsing solutions) which are solutions with at least one component being zero. In those cases, the system collapses into system of lower orders. For example, our result in [24] shows that for the 2-coupled system, (that is, when $m = p = 2$ and $N = 1$) there are obstructions to the existence and stability of nontrivial solutions with all components being positive. Roughly speaking, our result says that in order to have positive non-trivial solutions, the nonlinear couplings have to be either small or large. Thus this is a situation where multiple solutions exist and classifying and distinguishing the solutions becomes an important and difficult issue. Intensive work has been done in the last few years, see [1, 2, 3, 9, 14, 15, 18, 19, 27, 29, 31, 32]. All these works have been mainly on 2-systems or with small couplings. Despite the partial progress made so far, many difficult questions remain open and little is known for $m-$systems for $m \geq 3$.

2. Statement of results

In this work, we concentrate on the case when $a_j = a$ and $b_{kj} = b_{jk} = b$. However, we also discuss how the method can be extended to include the general case. In particular, we will employ the techniques used in [24, 25, 26] to show the existence and stability of ground state solutions to the system

$$\begin{gathered} i\frac{\partial}{\partial t}u_j + \Delta u_j + a|u_j|^{2p-2}u_j + \sum_{k=1,k\neq j}^{m} b|u_k|^p|u_j|^{p-2}u_j = 0; \quad x \in \mathbb{R}^N, \\ \big(u_1(\cdot,0),\dots,u_m(\cdot,0)\big) = (u_{10},\dots,u_{m0}) \end{gathered} \tag{2.1}$$

for $j = 1,\dots,m$ and for a certain range of p.

Logically, prior to a discussion of stability in terms of perturbations of the initial data should be a theory for the initial-valued problem itself. This issue has been studied in a previous work of ours ([23]). In that work, the contraction mapping technique based on Strichartz estimates was used to first establish local well-posedness in $H^1_{\mathbb{C}}(\mathbb{R}^N) \times \dots \times H^1_{\mathbb{C}}(\mathbb{R}^N) :\equiv X^{(m)}$ for $2 \le p < N/(N-2)$. To show the Lipschitz continuity for the nonlinear terms, the approach necessitates $2 \le p$. This condition puts a restriction on the applicable range of p for dimension $1 \le N \le 3$ for the proof of local existence. It is worth pointing out that there are cases when $1 \le p$ is allowed. For example, if $u_j = A_j u$ for some real constants A_j, then the system (1.2) is uncoupled and the result follows directly from [7], provided the initial data are related accordingly. One technical point deserves some comments here. For the single NLS equation of the type (1.1), the nonlinear term $g(u) = |u|^\alpha u$ with $\alpha \ge 0$ satisfies the Lipschitz continuity for some exponents $r_j, \rho_j \in [2, 2N/(N-2))$, ($r_j, \rho_j \in [2,\infty]$ if $N = 1$)

$$\|g(u) - g(v)\|_{L^{\rho_j'}} \le C(M)\|u - v\|_{L^{r_j}}$$

where $\frac{1}{\rho} + \frac{1}{\rho'} = 1$, for all $u, v \in H^1(\mathbb{R}^N)$ such that $\|u\|_{H^1}, \|v\|_{H^1} \le M$. Using this fact, it was shown (see for example, [7]) that the Cauchy problem for NLS equation of the type (1.1) is well-posed. It was claimed by Fanelli and Montefusco [10] and Song [30] that the local well-posedness result for (1.2) for $m = 2$ follows from the contraction mapping argument for $1 \le p < N/(N-2)$ (the power has been re-scaled here for comparison). The system in this case takes the form

$$\begin{gathered} iu_{1t} + u_{1xx} + (a|u_1|^{2p-2} + b|u_2|^p|u_1|^{p-2})u_1 = 0, \\ iu_{2t} + u_{2xx} + (b|u_1|^p|u_2|^{p-2} + c|u_2|^{2p-2})u_2 = 0. \end{gathered}$$

While it is true that there are instances when $1 \le p$ is acceptable as mentioned above, it appears the range for p cannot be extended to include $p < 2$ in general without loss of Lipschitz continuity and thus the claim is doubtful. It may be possible that other methods allow for the local well-posedness when $1 \le p < N/(N-2)$ in which case the result for local existence holds for all dimensions N. To extend the local existence result to a global one, all that is needed is $p < 1 + 2/N$. The condition $p < 1 + 2/N$ when coupled with $2 \le p < N/(N-2)$ for local existence implies that $N = 1$.

In light of the above mentioned well-posedness results, the assumption that $2 \le p < 3$ is needed (which implies that $N = 1$). (See also Remark 1 below.) The precise statements of our main results are as follows. Let $\phi(x)$ be the unique positive, spherically symmetric and decreasing solution in $H^1_{\mathbb{C}}(\mathbb{R}^N)$ of

$$-\Delta f + f = |f|^{2p-2} f,$$

and for any $\Omega > 0$, let

$$\phi_{\Omega,a+(m-1)b}(x) = \Big(\frac{\Omega}{a+(m-1)b}\Big)^{\frac{1}{2(p-1)}} \phi(\sqrt{\Omega}x).$$

In Section 3, we establish the stability result for ground-state solutions of (2.1).

Proposition 2.1. *For $2 \le p < 3$ and $N = 1$, let $a,b \in \mathbb{R}$ such that $b > 0$ and $a + (m-1)b > 0$. Then for any $\Omega > 0$, the ground-state solutions*

$$\big(e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x), \dots, e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x)\big)$$

of (2.1) are orbitally stable in the following sense: for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u_{10}, \dots, u_{m0}) \in X^{(m)}$ with

$$\inf_{\gamma_i, y \in \mathbb{R}} \Big\{ \sum_{j=1}^m \big\| u_{j0} - e^{i\gamma_j}\phi_{\Omega,a+(m-1)b}(\cdot + y)\big\|_{H^1} \Big\} < \delta.$$

The solution $\big(u_1(x,t), \dots, u_m(x,t)\big)$ with $\big(u_1(\cdot,0), \dots, u_m(\cdot,0)\big) = (u_{10}, \dots, u_{m0})$ satisfies

$$\inf_{\theta_i, y \in \mathbb{R}} \Big\{ \sum_{j=1}^m \big\| u_j - e^{i\theta_j}\phi_{\Omega,a+(m-1)b}(\cdot + y)\big\|_{H^1} \Big\} < \epsilon$$

uniformly for all $t \ge 0$.

Remark 2.2. (1) As mentioned above, when $1 < p < 1 + 2/N$ there still exist solutions for the initial-valued problem, provided that $\big(u_1(x,0), \dots, u_m(x,0)\big) \in X^{(m)}$ and satisfies

$$u_j = \Big(\frac{1}{a+(m-1)b}\Big)^{\frac{1}{2(p-1)}} u \quad \text{for } j = 1, 2, \dots, m \tag{2.2}$$

for then the system reduces to one equation which is the nonlinear cubic Schrödinger equation. As we must require that the initial data satisfy (2.2), the uniqueness for the Cauchy problem is preserved only for the subspace

$$\begin{aligned} Y^{(m)} :\equiv \big\{ &\mathbf{u}(x,t) \in X^{(m)} : \|u_j(x,t)\|_{L^2} = \big(\frac{1}{a+(m-1)b}\big)^{\frac{1}{2(p-1)}} \|u\|_{L^2},\ \forall t \big\} \\ &\subset X^{(m)}. \end{aligned}$$

Hence, instead of establishing the same stability theory as stated in Theorem 2.1, using our methods we can still obtain stability for a much more restricted subspace $Y^{(m)}$ in the case $1 < p < 1 + 2/N$ which is valid for any space dimension. We omit details here.

(2) Item (1) sheds some lights on why Proposition 2.1 is to be expected for $2 \le p < 3$ and $N = 1$. Because the solution to the Cauchy problem for (2.1) is unique in this case (see [23]), it follows that $\{\mathbf{u} \in Y^{(m)},\ \mathbf{u} \text{ solves } (2.1)\} = \{\mathbf{u} \in X^{(m)},\ \mathbf{u} \text{ solves } (2.1)\}$.

Next, we show that instead of allowing the ground-state solutions to wander around at random, one can pick unique trajectory and phase shifts that the ground-state solutions must follow. Precisely, we have

Theorem 2.3. *For $2 \le p < 3$ and $N = 1$, let a and $b \in \mathbb{R}$ such that $b > 0$ and $a + (m-1)b > 0$. Then for any $\Omega > 0$, the ground-state solutions*

$$\big(e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x), \dots, e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x)\big)$$

of (2.1) are orbitally stable in the sense that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $(u_{10}, \dots, u_{m0}) \in X^{(m)}$ satisfies

$$\inf_{\theta_j,\eta\in\mathbb{R}} \Big\{ \sum_{j=1}^{m} \|u_{j0} - e^{i\theta_j}\phi_{\Omega,a+(m-1)b}(\cdot + \eta)\|_{H^1}. \Big\} < \delta \tag{2.3}$$

There exist C^1 mappings $\theta_j, \eta : \mathbb{R} \to \mathbb{R}$ for which the solution $(u_1(x,t), \dots, u_m(x,t))$ with initial data $(u_1(\cdot,0), \dots, u_m(\cdot,0)) = (u_{10}, \dots, u_{m0})$ satisfies

$$\sum_{j=1}^{m} \|u_j(\cdot,t) - e^{i\theta_j(t)}\phi_{\Omega,a+(m-1)b}(\cdot + \eta(t))\|_{H^1} < \epsilon \tag{2.4}$$

for all $t \ge 0$. Moreover,

$$\eta'(t) = O(\epsilon), \quad \theta_j'(t) = \Omega + O(\epsilon), \tag{2.5}$$

for $j = 1, \dots, m$ as $\epsilon \to 0$, uniformly in t.

This result is then extended in Section 4 to include traveling-wave solutions. For $\theta \in \mathbb{R}$, define the operator $T_\theta : H^1_{\mathbb{C}}(\mathbb{R}) \to H^1_{\mathbb{C}}(\mathbb{R})$ by

$$(T_\theta u)(x) = \exp\Big(\frac{i\theta x}{2}\Big)u(x).$$

For any pair $(\omega, \theta) \in \mathbb{R} \times \mathbb{R}$ such that $\Omega = \omega - \frac{1}{4}\theta^2 > 0$, let $\varphi_\omega = T_\theta\phi_\Omega$. It is straightforward to see that if $(e^{i\Omega t}\phi_\Omega, \dots, e^{i\Omega t}\phi_\Omega)$ is a ground-state solution of (2.1), then $(e^{i\omega t}\varphi_\omega, \dots, e^{i\omega t}\varphi_\omega)$ is a traveling-wave solution of (2.1).

Corollary 2.4. *For $2 \le p < 3$ and $N = 1$, let a and $b \in \mathbb{R}$ such that $b > 0$ and $a + (m-1)b > 0$. The traveling-wave solutions $(e^{i\omega t}\varphi_\omega, \dots, e^{i\omega t}\varphi_\omega)$ are orbitally stable in the sense that for any $\epsilon > 0$ given, there exists $\delta = \delta(\varphi) > 0$ such that if*

$$\inf_{\vec\gamma,y} \Big\{ \sum_{j=1}^{m} \|u_{0j} - e^{i\gamma_j}\phi(\cdot + y)\|_{H^1} \Big\} < \delta$$

then there are C^1 mappings $p_j, q : \mathbb{R} \to \mathbb{R}$ for which the solution $\vec u = (u_1, \dots, u_m)$ with initial data $\vec u_0 = (u_{01}, \dots, u_{0m})$ satisfies, for all $j = 1, 2, \dots, m$

$$\sum_{j=1}^{m} \|u_j(\cdot,t) - e^{ip_j(t)}\varphi_\omega(\cdot + q(t))\|_{H^1} \le \epsilon$$

for all $t \ge 0$. Moreover, p_j and q are close to ω and θ in the sense that

$$p_j'(t) = \omega + O(\epsilon) \qquad q'(t) = \theta + O(\epsilon)$$

as $\epsilon \to 0$, uniformly in t.

Remark 2.5. It follows immediately from Remark 1 that Theorem 2.3 and Corollary 2.4 hold in $Y^{(m)}$ for the case $1 < p < 1 + 2/N$.

This article concludes with some comments and a discussion about how the method could be extended to include the system (1.2).

3. Stability results for the ground-state solutions

3.1. **Variational problem.** Let $u_1, \dots, u_m \in H^1_{\mathbb{C}}(\mathbb{R}^N)$ and consider the following functional associated with conserved quantity of (2.1):

$$
\begin{aligned}
&E^{(m)}(u_1, \dots, u_m)\\
&= \int_{\mathbb{R}^N} \Big[\sum_{i=1}^m \big(|\nabla u_i(x)|^2 - \frac{a}{p}|u_i(x)|^{2p}\big) - \sum_{i,j=1; i\neq j}^m \frac{b}{p}|u_i(x)|^p|u_j(x)|^p\Big]dx.
\end{aligned}
\tag{3.1}
$$

In the remainder of this article, it is assumed that $2 \le p < 3$ (which implies that $N = 1$) and that $b > 0$ and $a + (m-1)b > 0$.

Remark 3.1. As mentioned previously, to establish stability results as well as to extend the local existence to a global one, all that is needed is $p < 1 + 2/N$. This condition when coupled with $2 \le p < N/(N-2)$ for local existence implies that $N = 1$. However, there are instances when $p < 2$ is permissible (see, for example Remark 1). Thus, to allow for the adaptability of the proofs obtained when $2 \le p < 3$ to those instances, we refrain from taking $N = 1$ directly, with the understanding that when $2 \le p < 3$ then $N = 1$.

For $u \in H^1_{\mathbb{C}}(\mathbb{R}^N)$, define

$$
E_1^{(m)}(u) = \int_{\mathbb{R}^N} \big(|\nabla u(x)|^2 - \frac{a+(m-1)b}{p}|u(x)|^{2p}\big)dx. \tag{3.2}
$$

It is clear that for any $\Omega > 0$,

$$
\phi_{\Omega, a+(m-1)b}(x) = \Big(\frac{\Omega}{a+(m-1)b}\Big)^{\frac{1}{2(p-1)}}\phi(\sqrt{\Omega}x)
$$

is the unique positive, spherically symmetric and decreasing solution in $H^1_{\mathbb{C}}(\mathbb{R}^N)$ of

$$
-\Delta f + \Omega f = \big(a + (m-1)b\big)|f|^{2p-2}f,
$$

and

$$
\|\phi_{\Omega, a+(m-1)b}\|_{L^2} = \big(a+(m-1)b\big)^{-\frac{1}{2(p-1)}}\Omega^{\frac{1}{2(p-1)}-\frac{N}{4}}\|\phi\|_{L^2}.
$$

Fix an $\Omega > 0$ and let

$$
\lambda = \big(a+(m-1)b\big)^{-\frac{1}{(p-1)}}\Omega^{\frac{1}{(p-1)}-\frac{N}{2}}\|\phi\|^2_{L^2}. \tag{3.3}
$$

For fixed $\Omega > 0$ (hence $\lambda > 0$ is also fixed) and any $\mu_1, \dots, \mu_{m-1} > 0$, define the real numbers $I^{(m)}, I_1^{(m)}$ as follows:

$$
\begin{aligned}
&I^{(m)}(\lambda, \mu_1, \dots, \mu_{m-1})\\
&= \inf\Big\{E^{(m)} : u_1, \dots, u_m \in H^1_{\mathbb{C}}(\mathbb{R}^N), \|u_1\|^2_{L^2} = \lambda, \|u_j\|^2_{L^2} = \mu_{j-1},\ j = 2, \dots, m\Big\},
\end{aligned}
$$

and

$$
I_1^{(m)}(\lambda) = \inf\{E_1^{(m)}(u) : u \in H^1_{\mathbb{C}}(\mathbb{R}^N), \|u\|^2_{L^2} = \lambda\}.
$$

The sets of minimizers for $I^{(m)}(\lambda, \mu_1, \dots, \mu_{j-1})$ and $I_1^{(m)}(\lambda)$ for $j = 2, \dots, m$ are, respectively,

$$
\begin{aligned}
&G^{(m)}(\lambda, \mu_1, \dots, \mu_{j-1})\\
&= \Big\{(u_1, \dots, u_m) \in X^{(m)} : I^{(m)}(\lambda, \mu_1, \dots, \mu_{j-1}) = E^{(m)}(u_1, \dots, u_m),\\
&\quad \|u\|^2_{L^2} = \lambda, \|u_j\|^2_{L^2} = \mu_{j-1}\Big\},
\end{aligned}
$$

$$G_1^{(m)}(\lambda) = \{u \in H^1_{\mathbb{C}}(\mathbb{R}^N) : I_1^{(m)}(\lambda) = E_1^{(m)}(u), \|u\|^2_{L^2} = \lambda\}.$$

The following 2 Lemmas are clear. (For example, the proofs can be easily modified from those presented in [24].)

Lemma 3.2. *For all $\lambda > 0$, one has $-\infty < I^{(m)}(\lambda, \dots, \lambda) < 0$.*

Lemma 3.3. *If $\{(u_{1n}, \dots, u_{mn})\}$ is a minimizing sequence for $I^{(m)}(\lambda, \dots, \lambda)$, then there exist constants $B > 0$ and $\delta > 0$ such that*

- (i) *$\sum_{j=1}^m \|u_{jn}\|_{H^1} \le B$ for all n, and*
- (ii) *$\sum_{j=1}^m \|u_{jn}\|^{2p}_{L^{2p}} \ge \delta$ for all sufficiently large n.*

Let $\{(u_{1n}, \dots, u_{mn})\} \in X^{(m)}$ be a minimizing sequence for $E^{(m)}$ and consider a sequence of nondecreasing functions $M_n : [0, \infty) \to [0, m\lambda]$ as follows

$$M_n(s) = \sup_{y \in \mathbb{R}^N} \int_{|x-y|<s} \sum_{j=1}^m |u_{jn}(x)|^2 dx.$$

As $M_n(s)$ is a uniformly bounded sequence of nondecreasing functions in s, one can show using, for example, Helly's selection theorem (see [13]) that it has a subsequence, which is still denoted as M_n, that converges point-wisely to a nondecreasing limit function $M(s) : [0, \infty) \to [0, m\lambda]$. Let

$$\rho = \lim_{s\to\infty} M(s) :\equiv \lim_{s\to\infty} \lim_{n\to\infty} M_n(s) = \lim_{s\to\infty} \lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y|<s} \sum_{j=1}^m |u_{jn}(x)|^2 dx.$$

Then $0 \le \rho \le m\lambda$.

Lions' Concentration Compactness Lemma [16, 17] shows that there are three possibilities for the value of ρ:

Case 1: (Vanishing) $\rho = 0$. Since $M(s)$ is non-negative and non-decreasing, this is equivalent to saying

$$M(s) = \lim_{n\to\infty} M_n(s) = \lim_{n\to\infty} \sup_{y \in \mathbb{R}^N} \int_{|x-y|<s} \sum_{j=1}^m |u_{jn}(x)|^2 dx = 0$$

for all $s < \infty$, or

Case 2: (Dichotomy) $\rho \in (0, m\lambda)$, or

Case 3: (Compactness) $\rho = m\lambda$, which implies that there exists $\{y_n\}_{n=1} \in \mathbb{R}^N$ such that for any $\epsilon > 0$, there exists $s < \infty$ such that

$$\int_{|x-y_n|<s} \sum_{j=1}^m |u_{jn}(x)|^2 dx \ge m\lambda - \epsilon.$$

The next Lemma will be useful in ruling out the vanishing of minimizing sequences.

Lemma 3.4. *There exists a constant C such that for all $u_j \in H^1_{\mathbb{C}}(\mathbb{R}^N)$, $j = 1, \dots, m$*

$$\int_{\mathbb{R}^N} \sum_{j=1}^m |u_j|^{\frac{2N+4}{N}} dx \le C\Big(\sup_{y\in\mathbb{R}^N} \int_{|x-y|<s} \sum_{j=1}^m |u_j|^2 dx\Big)^{2/N} \sum_{j=1}^m \|u_j\|^2_{H^1}.$$

Proof. Let $(Q_j)_{j\geq 0}$ be a sequence of open, unit cubes of $\mathbb{R}^N$ such that $Q_j \bigcap Q_k = \emptyset$ if $j \neq k$ and $\overline{\bigcup_{j\geq 0} Q_j} = \mathbb{R}^N$. It is well-known (see, for example, [7, Lemma 1.7.7]) that there exists a constant K independent of j such that for all $f \in H^1_{\mathbb{C}}(Q_j)$

$$\int_{Q_j} |f(x)|^{\frac{2N+4}{N}} dx \leq K\Big(\int_{Q_j} |f(x)|^2 dx\Big)^{2/N} \|f\|^2_{H^1(Q_j)}.$$

Consequently, if $u_1, u_2, \dots, u_m \in H^1_{\mathbb{C}}(Q_j)$,

$$\int_{Q_j} |u_j(x)|^{\frac{2N+4}{N}} dx \leq C\Big(\int_{Q_j} |u_j(x)|^2 dx\Big)^{2/N} \|u_j\|^2_{H^1(Q_j)}.$$

The Lemma follows immediately from summing over j. $\square$

The following identities are well-known. (See, for example, [7, Lemma 8.1.2].)

Lemma 3.5. *Let $a, \Omega > 0$. If $-\Delta f + \Omega f = a|f|^{2p-2} f$, then*

$$\int_{\mathbb{R}^N} (|\nabla f|^2 + \Omega |f|^2) dx = a \int_{\mathbb{R}^N} |f|^{2p} dx,$$
$$(N-2)\int_{\mathbb{R}^N} |\nabla f|^2 dx + N\Omega \int_{\mathbb{R}^N} |f|^2 dx = \frac{Na}{p} \int_{\mathbb{R}^N} |f|^{2p} dx.$$

Using the above identities, the next Lemma can be derived easily.

Lemma 3.6. *The following statements hold :*

(1) *for any $\lambda, \mu_{j-1} \geq 0$ and $j = 2, \dots, m$,*

$$I^{(m)}(\lambda, \mu_1, \dots, \mu_{m-1}) \geq I_1^{(m)}(\lambda) + \sum_{j=2}^{m} I_1^{(m)}(\mu_{j-1});$$

(2)

$$I_1^{(m)}(\lambda) = E_1^{(m)}(\phi_{\Omega, a+(m-1)b}) = -\frac{N+2-Np}{N+2p-Np} \lambda \Big(\frac{\lambda\big(a+(m-1)b\big)^{\frac{1}{p-1}}}{\|\phi\|^2_{L^2}}\Big)^{\frac{2(p-1)}{2-N(p-1)}};$$

(3) *$I^{(m)}(\lambda, \dots, \lambda) = m I_1^{(m)}(\lambda)$ for $\lambda > 0$, and $(\phi_{\Omega, a+(m-1)b}, \dots, \phi_{\Omega, a+(m-1)b}) \in G^{(m)}(\lambda, \dots, \lambda)$;*

(4) *$I_1^{(m-k)}(\lambda) > I_1^{(m)}(\lambda)$, for all $k \in (0, m)$ and $\lambda > 0$.*

Corollary 3.7. *For any $\Omega > 0$ fixed,*

$$\Big\{\Big(e^{i\alpha_1}\phi_{\Omega, a+(m-1)b}(\cdot + y), \dots, e^{i\alpha_m}\phi_{\Omega, a+(m-1)b}(\cdot + y)\Big)\Big\} \subset G^{(m)}\big(\lambda(\Omega), \dots, \lambda(\Omega)\big)$$

where $\alpha_j \in \mathbb{R}$, $j = 1, 2, \dots, m$; $y \in \mathbb{R}^N$.

The following Lemma provides *strict sub-additivity* of the function $I^{(m)}$ needed to rule out the dichotomy of minimizing sequences.

Lemma 3.8. *For any $\beta_j \in [0, \lambda]$, $j = 1, \dots, m$ satisfying $0 < \sum_{j=1}^m \beta_i < m\lambda$, we have*

$$I^{(m)}(\lambda, \dots, \lambda) < I^{(m)}(\beta_1, \dots, \beta_m) + I^{(m)}(\lambda - \beta_1, \lambda - \beta_2, \dots, \lambda - \beta_m).$$

Proof. We consider separately the following cases.

Case 1: $\beta_j \in (0, \lambda)$ for $j = 1, \dots, m$. From (3) in Lemma 3.6, we have

$$I^{(m)}(\lambda, \dots, \lambda) = mI_1^{(m)}(\lambda);$$

and from [8, Theorem II.1] we have, for any $\beta_j \in (0, \lambda)$

$$I_1^{(m)}(\lambda) < I_1^{(m)}(\beta_j) + I_1^{(m)}(\lambda - \beta_j).$$

Consequently, we obtain

$$\begin{aligned} I^{(m)}(\lambda, \dots, \lambda) = mI_1^{(m)}(\lambda) &< \sum_{j=1}^{m} I_1^{(m)}(\beta_j) + \sum_{j=1}^{m} I_1^{(m)}(\lambda - \beta_j) \\ &\le I^{(m)}(\beta_1, \dots, \beta_m) + I^{(m)}(\lambda - \beta_1, \dots, \lambda - \beta_m) \end{aligned}$$

where item (1) in Lemma 3.6 has been used. Thus case 1 is proved.

Case 2: Exactly k of $\{\beta_1, \beta_2, \dots, \beta_m\}$ vanish, $k = 2, \dots, m-1$, and without loss of generality, we may assume that

$$\beta_{m-k+1} = \dots = \beta_m = 0; \quad \beta_j \in (0, \lambda], \text{ for } j = 1, 2, \dots, m-k.$$

The variational problem then becomes

$$\begin{aligned} &\inf \Big\{ \int_{\mathbb{R}^N} \Big[\sum_{j=1}^{m-k} \big(|\nabla u_j|^2 - \frac{a}{p} |u_j|^{2p} \big) - \sum_{i,j=1; i \ne j}^{m-k} \frac{b}{p} |u_i|^p |u_j|^p \Big] dx : \|u_j\|_{L^2}^2 = \beta_j \Big\} \\ &= \inf \Big\{ E^{(m-k)}(u_1, \dots, u_{m-k}) : \|u_j\|_{L^2}^2 = \beta_j, \ j = 1, 2, \dots, m-k \Big\} \end{aligned}$$

which is the $(m-k)$-case. Thus, item (1) in Lemma 3.6 implies that

$$I^{(m)}(\beta_1, \dots, \beta_{m-k}, 0, \dots, 0) = I^{(m-k)}(\beta_1, \dots, \beta_{m-k}) \ge \sum_{j=1}^{m-k} I_1^{(m-k)}(\beta_j). \tag{3.4}$$

On the other hand, part (4) in Lemma 3.6 says that for all $k \in (0, m)$

$$I_1^{(m-k)}(\beta_j) > I_1^{(m)}(\beta_j),$$

we obtain that $I^{(m)}(\beta_1, \dots, \beta_{m-k}, 0, \dots, 0) > \sum_{j=1}^{m-k} I_1^{(m)}(\beta_j)$. Thus,

$$\begin{aligned} &I^{(m)}(\lambda, \dots, \lambda) \\ &= mI_1^{(m)}(\lambda) \le kI_1^{(m)}(\lambda) + \sum_{j=1}^{m-k} \Big(I_1^{(m)}(\beta_j) + I_1^{(m)}(\lambda - \beta_j) \Big) \\ &\le I^{(m)}(\lambda - \beta_1, \dots, \lambda - \beta_{m-k}, \lambda, \dots, \lambda) + \sum_{j=1}^{m-k} I_1^{(m)}(\beta_j) \\ &< I^{(m)}(\beta_1, \dots, \beta_{m-k}, 0, \dots, 0) + I^{(m)}(\lambda - \beta_1, \dots, \lambda - \beta_{m-k}, \lambda, \dots, \lambda) \end{aligned}$$

proving case 2. Thus the Lemma is proved. □

With all the calculations in hand, one can proceed straightforwardly (see, for example, [24]) to show that minimizing sequences are compact and that the set of minimizers $G^{(m)}(\lambda, \dots, \lambda)$ is stable. Precisely, we have the following.

Lemma 3.9. *For every $\epsilon > 0$ given, there exists $\delta > 0$ such that if*

$$\inf_{(\Phi_1,\dots,\Phi_m)\in G^{(m)}} \|(u_{10},\dots,u_{m0}) - (\Phi_1,\dots,\Phi_m)\|_{X^{(m)}} < \delta,$$

then the solution $\big(u_1(x,t),\dots,u_m(x,t)\big)$ of (2.1) with $\big(u_1(x,0),\dots,u_m(x,0)\big) = (u_{10},\dots,u_{m0})$ satisfies

$$\inf_{(\Phi_1,\dots,\Phi_m)\in G^{(m)}} \|\big(u_1(\cdot,t),\dots,u_m(\cdot,t)\big) - (\Phi_1,\dots,\Phi_m)\|_{X^{(m)}} < \epsilon$$

for all $t \in \mathbb{R}$.

3.2. **Stability of ground-state solutions.** In this subsection, we will show that the set of minimizers $G^{(m)}(\lambda,\dots,\lambda)$ contains just a single $m-$tuple of functions (modulo translations and phase shifts), and that this $m-$tuple of functions is indeed a ground-state solution of (2.1) given by

$$\big(\Phi_1(x,t),\dots,\Phi_m(x,t)\big) = \Big(e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x),\dots,e^{i\Omega t}\phi_{\Omega,a+(m-1)b}(x)\Big).$$

Proposition 2.1 then follows directly from this fact and Lemma 3.9.

We start first with the following Lemma that relates the functions $\Phi_1,\dots,\Phi_m$ whenever $(\Phi_1,\dots,\Phi_m) \in G^{(m)}(\lambda,\dots,\lambda)$.

Lemma 3.10. *Let $(\Phi_1,\dots,\Phi_m) \in G^{(m)}(\lambda,\dots,\lambda)$. Then for any $x \in \mathbb{R}^N$,*

$$|\Phi_1(x)| = |\Phi_2(x)| = \dots = |\Phi_m(x)|.$$

Proof. It follows from Lemma 3.6 that for any $(\Phi_1,\dots,\Phi_m) \in G^{(m)}(\lambda,\dots,\lambda)$

$$I^{(m)}(\lambda,\dots,\lambda) = E^{(m)}(\Phi_1,\dots,\Phi_m) \ge \sum_{j=1}^m E_1^{(m)}(\Phi_j) \ge m I_1^{(m)}(\lambda) = I^{(m)}(\lambda,\dots,\lambda).$$

Thus,

$$\frac{a}{p}\sum_{j=1}^m \|\Phi_j\|_{L^{2p}}^{2p} + \frac{b}{p}\int_{\mathbb{R}^N} \sum_{i,j=1;i\neq j}^m |\Phi_i|^p|\Phi_j|^p dx = \frac{a+(m-1)b}{p}\sum_{j=1}^m \|\Phi_j\|_{L^{2p}}^{2p}$$

which implies that

$$\int_{\mathbb{R}^N} \sum_{i,j=1;i\neq j}^m |\Phi_i|^p|\Phi_j|^p dx = (m-1)\sum_{j=1}^m \|\Phi_j\|_{L^{2p}}^{2p}. \tag{3.5}$$

We can rewrite (3.5) as

$$\int_{\mathbb{R}^N} \sum_{i,j=1;i\neq j}^m \Big||\Phi_i(x)|^p - |\Phi_j(x)|^p\Big|^2 dx = 0,$$

from which the statement of the Lemma immediately follows. $\square$

Next, we show the following.

Lemma 3.11. *For any $\Omega > 0$ fixed,*

$$\Big\{\Big(e^{i\alpha_1}\phi_{\Omega,a+(m-1)b}(\cdot+y),\dots,e^{i\alpha_m}\phi_{\Omega,a+(m-1)b}(\cdot+y)\Big)\Big\} = G^{(m)}\big(\lambda(\Omega),\dots,\lambda(\Omega)\big)$$

where $\alpha_j \in \mathbb{R}$, $j = 1,2,\dots,m$; $y \in \mathbb{R}^N$.

Proof. It has been established (Corollary 3.7) that for any $\Omega > 0$ fixed and for $\alpha_j \in \mathbb{R}$, $j = 1, 2, \dots, m$ and $y \in \mathbb{R}^N$,

$$\Big(e^{i\alpha_1}\phi_{\Omega,a+(m-1)b}(\cdot + y), \dots, e^{i\alpha_m}\phi_{\Omega,a+(m-1)b}(\cdot + y)\Big) \in G^{(m)}\big(\lambda(\Omega), \dots, \lambda(\Omega)\big).$$

Hence, the Lemma is proved if we show that any minimizer in $G^{(m)}(\lambda(\Omega), \dots, \lambda(\Omega))$ must be of the form given above. Now, since the constrained minimizer for the variational problem exists, there are Lagrange multipliers $\Omega_1, \dots, \Omega_m \in \mathbb{R}$ such that for $j = 1, 2, \dots, m$

$$-\Delta\Phi_j + \Omega_j\Phi_j = a|\Phi_j|^{2p-2}\Phi_j + b\sum_{i=1;i\neq j}^{m}|\Phi_i|^p|\Phi_j|^{p-2}\Phi_j. \tag{3.6}$$

Using Lemma 3.8, we can rewrite this system as m-uncoupled equations

$$-\Delta\Phi_j + \Omega_j\Phi_j = \big(a + (m-1)b\big)|\Phi_j|^{2p-2}\Phi_j. \tag{3.7}$$

A bootstrap argument shows that any m-tuple L^2-distribution solution of (3.7) must indeed be smooth and given by (see, for example, [7])

$$\Phi_j(x) = e^{i\alpha_j}\phi_{\Omega_j,a+(m-1)b}(x + y_j),$$

where $\alpha_j \in \mathbb{R}$, $y_j \in \mathbb{R}^N$ and $\Omega_j > 0$ for $j = 1, 2, \dots, m$. Now recall that for any $x \in \mathbb{R}^N$, we must have

$$|\Phi_1(x)| = |\Phi_2(x)| = \dots = |\Phi_m(x)|,$$

and

$$\|\Phi_1\|_{L^2}^2 = \|\Phi_2\|_{L^2}^2 = \dots = \|\Phi_m\|_{L^2}^2 = \lambda = \big(a + (m-1)b\big)^{-\frac{1}{(p-1)}}\Omega^{\frac{1}{(p-1)}-\frac{N}{2}}\|\phi\|_{L^2}^2.$$

It is easy to see then that $y_1 = y_2 = \dots = y_m$, and

$$\Omega = \Omega_1 = \Omega_2 = \dots = \Omega_m > 0.$$

The Lemma is thus established. □

The above proposition follows from Lemmas 3.9 and 3.11.

Next, we will show that instead of allowing the ground-state solutions to wander around at random, one can pick unique trajectory and phase shifts that the ground-state solutions must follow. Denote $\vec{\theta} = (\theta_1, \dots, \theta_m)$. Following the idea used in [6, 33], the functions θ_j $(j = 1, 2, \dots, m)$ and η are found through minimizing the function $R = R(\vec{\theta}, \eta) : \mathbb{R}^{m+1} \to \mathbb{R}$,

$$\begin{aligned} R(\vec{\theta}, \eta) = \sum_{j=1}^{m} \Big[&\Omega\|u_j(x) - e^{i\theta_j}\phi_{\Omega,a+(m-1)b}(x + \eta)\|_{L^2}^2 \\ &+ \|u_j'(x) - e^{i\theta_j}\phi_{\Omega,a+(m-1)b}'(x + \eta)\|_{L^2}^2\Big]. \end{aligned} \tag{3.8}$$

From now on, denote $\phi(x) = \phi_{\Omega,a+(m+1)b}(x)$ for simplicity. Due to the symmetry, we only need to consider one component

$$\begin{aligned} R_j(\theta_j, \eta) &= \Omega\|u_j(x) - e^{i\theta_j}\phi(x + \eta)\|_{L^2}^2 + \|u_j'(x) - e^{i\theta_j}\phi'(x + \eta)\|_{L^2}^2 \\ &= \int_{\mathbb{R}^N}\Big(\Omega|u_j(x) - e^{i\theta_j}\phi(x + \eta)|^2 + |(u_j)_x - e^{i\theta_j}\phi'(x + \eta)|^2\Big)dx. \end{aligned} \tag{3.9}$$

Then

$$\begin{aligned}\frac{\partial R_j}{\partial \eta} &= \int_{\mathbb{R}^N} \big[-2\Omega \operatorname{Re}(u_j(x)e^{-i\theta_j})\phi'(x+\eta) - 2\operatorname{Re}((u_j)_x e^{-\theta_j}\phi''(x+\eta)\big]dx \\ &= 2\operatorname{Re}\int_{\mathbb{R}^N} u_j(x)e^{-i\theta_j}(\phi''(x+\eta)-\Omega\phi(x+\eta))'dx \\ &= -2[a+(m-1)b]\operatorname{Re}\int_{\mathbb{R}^N} u_j(x)e^{-i\theta_j}(\phi^{2p-1}(x+\eta))'dx \\ &= -2(2p-1)[a+(m-1)b]\int_{\mathbb{R}^N}\operatorname{Re}\big(u_j(x)e^{-i\theta_j}\big)\phi^{2p-2}(x+\eta)\phi'(x+\eta)dx,\end{aligned} \tag{3.10}$$

and

$$\frac{\partial R_j}{\partial \theta_j} = i[a+(m-1)b]\int_{\mathbb{R}^N}\operatorname{Im}\big(u_j(x)e^{-i\theta_j}\big)\phi^{2p-1}(x+\eta)dx. \tag{3.11}$$

Define vector-valued function $Q : X \times \mathbb{R}^{m+1} \to \mathbb{R}^{m+1}$,

$$Q(\vec{\psi},\vec{\theta},\eta) = (F(\vec{\psi},\vec{\theta},\eta), \vec{G}(\vec{\psi},\vec{\theta},\eta))$$

where

$$\begin{gathered} F(\vec{\psi},\vec{\theta},\eta) = \sum_{j=1}^m \int_{\mathbb{R}^N}\operatorname{Re}\big(\psi_j e^{-i\theta_j}\big)\phi^{2p-2}(x+\eta)\phi'(x+\eta)dx; \\ G_j(\vec{\psi},\vec{\theta},\eta) = \int_{\mathbb{R}^N}\operatorname{Im}\big(\psi_j e^{-i\theta_j}\big)\phi^{2p-1}(x+\eta)dx.\end{gathered} \tag{3.12}$$

Next, we verify the conditions needed for using the Implicit Function Theorem.

Lemma 3.12. *Denote $\vec{\phi} = (\phi,\dots,\phi)$. Then:*

(i) $Q(\vec{\phi},\vec{0},0) = (0,\vec{0})$.
(ii) $|\nabla Q| < 0$.

Proof. Statement (i) follows from the facts that $-\Delta\phi + \Omega\phi = (a+(m-1)b)\phi^{2p-1}$ and

$$\begin{gathered} F(\vec{\phi},\vec{0},0) = \sum_{j=1}^m \int_{\mathbb{R}^N}\phi^{2p-2}\phi' dx = 0; \\ G_j(\vec{\phi},\vec{0},0) = \int_{\mathbb{R}^N}\operatorname{Im}(\phi\cdot 1)\phi^{2p-1}dx = 0.\end{gathered}$$

To prove (ii), notice that

$$\begin{aligned}\frac{\partial F}{\partial \theta_j}\Big|_{(\vec{\phi},\vec{0},0)} &= \operatorname{Re}\Big(-i\int_{\mathbb{R}^N}\operatorname{Im}\big(\psi_j(x)e^{-i\theta_j}\big)\phi^{2p-2}(x+\eta)\phi'(x+\eta)dx\Big)\Big|_{(\vec{\phi},\vec{0},0)} = 0, \\ \frac{\partial F}{\partial \eta}\Big|_{(\vec{\phi},\vec{0},0)} &= \frac{\partial}{\partial\eta}\Big[\sum_{j=1}^m\int_{\mathbb{R}^N}\operatorname{Re}\big(\psi_j(x)e^{-\theta_j}\big)\phi^{2p-2}(x+\eta)\phi'(x+\eta)dx\Big]\Big|_{(\vec{\phi},\vec{0},0)} \\ &= \frac{\partial}{\partial\eta}\Big[\sum_{j=1}^m\int_{\mathbb{R}^N}\operatorname{Re}\big(\psi_j(x-\eta)e^{-\theta_j}\big)\phi^{2p-2}(x)\phi'(x)dx\Big]\Big|_{(\vec{\phi},\vec{0},0)} \\ &= \sum_{j=1}^m\int_{\mathbb{R}^N}-\phi'(x)\phi^{2p-2}(x)\phi'(x)dx \\ &= -m\int_{\mathbb{R}^N}\phi^{2p-2}(x)\big[\phi'(x)\big]^2dx,\end{aligned}$$

$$\frac{\partial G_j}{\partial \eta}\Big|_{(\vec{\phi},\vec{0},0)} = (2p-1)\int_{\mathbb{R}^N} \operatorname{Im}\big(\psi_j(x)e^{-i\theta_j}\big)\phi^{2p-2}(x+\eta)\phi'(x+\eta)dx\Big|_{(\vec{\phi},\vec{0},0)} = 0,$$
$$\frac{\partial G_j}{\partial \theta_j}\Big|_{(\vec{\phi},\vec{0},0)} = \int_{\mathbb{R}^N} i\big(\psi_j(x)e^{-\theta_j} - \overline{\psi}_j(x)e^{i\theta_j}\big)\phi^{2p-1}(x+\eta)dx\Big|_{(\vec{\phi},\vec{0},0)} = \int_{\mathbb{R}^N} \phi^{2p}(x)dx.$$

Thus,

$$\det(\nabla G) = -m\Big(\int_{\mathbb{R}^N} \phi^{2p-2}(x)\big[\phi'(x)\big]^2 dx\Big)\Big(\int_{\mathbb{R}^N} \phi^{2p}(x)dx\Big)^{m-1} < 0.$$

□

Define an equivalent norm in $X^{(m)}$ by

$$\|\vec{u}\|^2_{X^{(m)}} = \sum_{j=1}^{m} \big[\Omega\|u_j(x)\|^2_{L^2} + \|u_j'(x)\|^2_{L^2}\big]; \quad \vec{u} = (u_1, \dots, u_m) \in X^{(m)}. \tag{3.13}$$

Let the $X^{(m)}$-neighborhood of the trajectory of $(e^{i\theta_1}\phi, \dots, e^{i\theta_m}\phi)$ be defined by

$$\mathcal{U}_\beta = \Big\{\vec{\phi} \in X^{(m)} : \inf_{\vec{\theta},\eta} \big\{\|\vec{\psi}(x) - e^{i\vec{\theta}_j}\phi(x+\eta)\|_{X^{(m)}} < \beta\big\}\Big\}.$$

Lemma 3.13. *There exist a $\beta > 0$ and C^1 maps $\vec{\theta}, \eta : \mathcal{U}_\beta \to \mathbb{R}$ such that*

$$F(\vec{\psi}, \vec{\theta}(\vec{\psi}), \eta(\vec{\psi})) \equiv 0, \quad G_j(\vec{\psi}, \vec{\theta}(\vec{\psi}), \eta(\vec{\psi})) \equiv 0,$$

for all $\vec{\psi} \in \mathcal{U}_\beta$.

Proof. It is easy to see that the function Q is C^1 on its' domain. Lemma 3.12 verifies all the conditions needed to apply the Implicit Function Theorem. Thus Lemma 3.13 follows provided $\beta > 0$ is small enough. □

Because of the stability result stated in Proposition 2.1, $\vec{u}(\cdot, t) \in \mathcal{U}_\beta$ and hence the corresponding functions $\vec{\theta}$ and η are defined on $\vec{u}(\cdot, t)$. One can therefore consider the functions $\vec{\theta}$ and η from $\mathbb{R} \to \mathbb{R}$ as

$$\eta(t) = \eta\big(\vec{u}(\cdot, t)\big),$$

and for $i = 1, 2, \dots, m$

$$\theta_i(t) = \theta_i\big(\vec{u}(\cdot, t)\big).$$

The next Lemma is clear. (See, for example, [22].)

Lemma 3.14. *The function Q is continuously differentiable with respect to t.*

We are now ready for the following proof.

Proof of Theorem 2.3. The first part of the Theorem is an immediate consequence of Lemma 3.13 and Proposition 2.1. Indeed, for fixed $\epsilon > 0$, one can first apply Lemma 3.13 to find a proper $\beta > 0$ such that the continuous maps exist. Then, the result of [23] implies the existence of some $\delta > 0$ such that when the initial data satisfies assumption (2.3), the resulting perturbations from η and θ_j's for all $t \geq 0$ will remain in the ball $\mathcal{U}_\beta$. Thus the estimate (2.4) holds.

It is left to show (2.5). Define the m functions

$$h_j(x, t) = e^{-i\theta_j(t)}u(x, t) - \phi(x + \eta(t)) = h_{j1} + ih_{j2}$$

for $j = 1, \dots, m$. According to (2.4), $\sum_{j=1}^m \|h_{j1}\|_{H^1} + \sum_{j=1}^m \|h_{j2}\|_{H^1} = O(\epsilon)$ for all $t \ge 0$. Differentiating F with respect to t, we obtain

$$
\begin{aligned}
F_t &= \sum_{j=1}^m \int_{\mathbb{R}^N} \operatorname{Re}\Big([(u_j)_t - (u_j)_x \eta'(t)] e^{-i\theta_j(t)} - i\theta_j' u_j e^{-i\theta_j(t)}\Big) \phi^{2p-2}\phi' dx \\
&= \operatorname{Re} \sum_{j=1}^m \int_{\mathbb{R}^N} \Big[(u_j)_t e^{-i\theta_j(t)} \phi^{2p-2}\phi' - (u_j)_x \eta'(t) e^{-i\theta_j(t)} \phi^{2p-2}\phi' \\
&\quad - i\theta_j'(t) u_j e^{-i\theta_j(t)} \phi^{2p-2}\phi'\Big] dx \\
&= \sum_{j=1}^m (I_{j1} - I_{j2} - I_{j3}) = 0.
\end{aligned}
$$

Notice that

$$
\begin{gathered}
\begin{aligned}
I_{j1} &= \operatorname{Re} \int_{\mathbb{R}^N} i\Big[(u_j)_{xx} + \big(a|u_j|^p + b\sum_{k\neq j} |u_k|^p\big)|u_j|^{p-2}u_j\Big] e^{-i\theta_j(t)} \phi^{2p-2}\phi' dx \\
&= -\int_{\mathbb{R}^N} \Big[(h_2)_{xx} + \big(a|u_j|^p + b\sum_{k\neq j} |u_k|^p\big)|u_j|^{p-2}h_2\Big] \phi^{2p-2}\phi' dx = O(\epsilon), \\
I_{j2} &= \int_{\mathbb{R}^N} \Big(\phi^{2p-2}(\phi')^2 \eta'(t) + (h_1)_x \eta'(t) \phi^{2p-2}\phi'\Big) dx = c\eta'(t) + O(\epsilon),
\end{aligned} \\
I_{j3} = \int_{\mathbb{R}^N} \theta_j'(t) h_2 \phi^{2p-2}\phi' dx = O(\epsilon)\theta_j'(t).
\end{gathered}
$$

Thus we have

$$
\eta'(t) = O(\epsilon) + O(\epsilon) \sum_{j=1}^m \theta_j'(t).
$$

Similarly, the other m equations for $\vec{\theta}$ give

$$
\theta_j'(t) = \Omega + O(\epsilon) + O(\epsilon)\eta'(t), \quad j = 1, \dots, m.
$$

The statement (2.5) follows immediately. Thus, the Theorem is proved. $\square$

4. Stability of traveling-wave solutions

The result obtained in Section 3 is now broadened to include traveling-wave solutions and improved by providing a more detailed view of the connection between the functions η and θ_i. For $\theta \in \mathbb{R}$, define the operator $T_\theta : H^1_{\mathbb{C}}(\mathbb{R}) \to H^1_{\mathbb{C}}(\mathbb{R})$ by

$$
(T_\theta u)(x) = \exp\Big(\frac{i\theta x}{2}\Big) u(x).
$$

For any pair $(\omega, \theta) \in \mathbb{R} \times \mathbb{R}$ such that $\Omega = \omega - \frac{1}{4}\theta^2 > 0$, let $\varphi_\omega = T_\theta \phi_\Omega$. The following Lemma is straightforward.

Lemma 4.1. *If* $(e^{i\Omega t}\phi_\Omega, \dots, e^{i\Omega t}\phi_\Omega)$ *is a ground-state solution of* (2.1)*, then* $(e^{i\omega t}\varphi_\omega, \dots, e^{i\omega t}\varphi_\omega)$ *is a traveling-wave solution of* (2.1).

Proof of Corollary 2.4. Similar arguments as used in [6, 21, 33] allow us to extend the stability result obtained above to include traveling-wave solutions as well. Readers are referred to, for example, [21] for the proof of this. $\square$

5. Conclusion

The traveling-wave solutions of (2.1) have been shown to be orbitally stable in $X^{(m)}$ when $2 \le p < 3$ and $N = 1$ and orbitally stable in $Y^{(m)}$ when $1 < p < 1 + 2/N$. Notice that when $N = 1$ and $p = 2$, the system (2.1) reduces to the 2-coupled system considered in [24] (when $m = 2$) and to the 3−coupled system considered in [25] (when $m = 3$ and $a_j = a$ and $b_{kj} = b_{jk} = b$). Thus, when $a_j = a$ and $b_{kj} = b_{jk} = b$, the results in this manuscript generalize the ones obtained in [24, 25] to include the case of $m-$coupled nonlinear Schrödinger system. The assumptions $2 \le p < 3$ and $N = 1$ are necessary for the global existence to hold. In particular, the concentration compactness used in establishing the stability theory here only requires that $1 < p \le 1 + 2/N$ and nothing more. It may be possible that other methods allow for the well-posedness of the Cauchy problem when $1 \le p < N/(N - 2)$ in which case the stability results in this paper hold in $X^{(m)}$ for $1 < p \le 1 + 2/N$.

Another interesting question arises naturally. How about the existence and stability theories for the general case (1.2)? As explained earlier, the crucial idea beside keeping the constraints on the L^2-norms of components related and having the coefficients give rise to positive numbers A_m such that the Euler-Lagrange equations can be rewritten as uncoupled equations, is that the strict sub-additivity of the function $I^{(m)}$ must be established. This means that one needs to analyze all the collapsing cases that may occur. A good starting point for this had been suggested in the conclusion of our previous work [25].

References

[1] A. Ambrosetti, E. Colorado; *Bound and ground states of coupled nonlinear Schrödinger equations*, C. R. Math. Acad. Sci. Paris, 342 (2006), 453-458.

[2] T. Bartsch, Z.-Q. Wang; *Note on ground states of nonlinear Schrödinger systems*, J. Partial Differential Equations, 19 (2006), 200-207.

[3] T. Bartsch, Z.-Q. Wang, J. Wei; *Bound states for a coupled Schrödinger system*, J. Fixed Point Theory Appl., 2 (2007), 353-367.

[4] D. J. Benney, A. C. Newell; *The propagation of nonlinear wave envelopes*, Jour. Math. Phys., 46 (1967), 133-139.

[5] J. L. Bona, Y. Liu, N. V. Nguyen; *Stability of solitary waves in higher-order Sobolev spaces*, Comm. Math Sci., 2 (2004), 35-52.

[6] J. L. Bona, A. Soyeur; *On the stability of solitary-wave solutions of model equations for long waves*, Jour. Nonlin. Sci., 4 (1994), 449-470.

[7] T. Cazenave; *Semilinear Schrödinger equations*, AMS-Courant Lecture Notes, 10, (2003).

[8] T. Cazenave, P.-L. Lions; *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys., 85 (1982), 549-561.

[9] E. N. Dancer, J. Wei, T. Weth; *A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 27 (2010), 953-969.

[10] L. Fanelli, E. Montefusco; *On the blow-up threshold for weakly coupled nonlinear Schrödinger equations*, Jour. Phys. A: Math. and Theor., 40 (2007), 14139-14150.

[11] A. Hasegawa, F. Tappert; *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers I. Anomalous dispersion*, Appl. Phys. Lett., 23 (1973), 142.

[12] A. Hasegawa, F. Tappert; *Transmission of stationary nonlinear optical pulses in dispersive dielectric fibers II. Normal dispersion*, Appl. Phys. Lett., 23 (1973), 171.

[13] F. Hirsch, G. Lacombe; *Elements of functional analysis*, translated by S. Levy, Grad. Texts Math., Springer (1999).

[14] T.-C. Lin, J. Wei; *Ground state of N coupled nonlinear Schrödinger equations in R^n, $n \le 3$*, Comm. Math. Phys., 255 (2005), 629-653.

[15] T.-C. Lin, J. Wei; *Spikes in two coupled nonlinear Schrödinger equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 22 (2005), 403-439.
[16] P.-L. Lions; *The concentration-compactness principle in the calculus of variations*. The locally compact case. I, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), 109-145.
[17] P.-L. Lions; *The concentration-compactness principle in the calculus of variations. The locally compact case. II*, Ann. Inst. H. Poincaré Anal. Non Linéaire, 1 (1984), 223-283.
[18] Z. Liu and Z.-Q. Wang; *Multiple bound states of nonlinear Schrödinger systems*, Comm. Math. Phys., 282 (2008), 721-731.
[19] L. A. Maia, E. Montefusco, B. Pellacci; *Positive solutions for a weakly coupled nonlinear Schrödinger system*, Jour. Diff. Eqns, 229 (2006), 743-767.
[20] F. Merle, L. Vega; *L^2 stability of solitons for KdV equations*, Int. Math. Res. Notes, 13 (2003), 735-753.
[21] N. V. Nguyen; *On the orbital stability of solitary waves for the 2-coupled nonlinear Schr"odinger system*, Comm. Math. Sci., Vol 9, (2011), 997-1012.
[22] N. V. Nguyen; *Stability of solitary waves for the vector nonlinear Schrodinger equation in higher-order Sobolev spaces*, accepted, in press, Jour. Math. Anal. Appl.
[23] N. V. Nguyen, R.-S. Tian, B. Deconinck, N. Sheils; *Global existence for a system of Schrödinger equations with power-type nonlinearities*, Jour. Math. Phys., 54, 011503 (2013), doi: 10.1063/1.4774149.
[24] N. V. Nguyen, Z.-Q. Wang; *Orbital stability of solitary waves for a nonlinear Schrödinger system*, Adv. Diff. Eqns., Vol 16, 9-10 (2011), 977-1000.
[25] N. V. Nguyen, Z.-Q. Wang; *Orbital stability of solitary waves of a 3-coupled nonlinear Schrödinger system*, Jour. Nonlin. Anal. Series A: Theory, Meth. & Appl., Vol 90, pp 1-26 (2013); doi: 10.1016/j.na.2013.05.027.
[26] M. Ohta; *Stability of solitary waves for coupled nonlinear Schrödinger equations*, Nonlinear Anal.: Theory, Methods & Appl., 26 (1996), 933-939.
[27] A. Pomponio; *Coupled nonlinear Schrödinger systems with potentials*, Jour. Diff. Eqn, 227 (2006), 258-281.
[28] G. J. Roskes; *Some nonlinear multiphase interactions*, Stud. Appl. Math., 55 (1976), 231.
[29] B. Sirakov; *Least energy solitary waves for a system of nonlinear Schrödinger equations in $\mathbb{R}^n$*, Comm. Math. Phys., 271 (2007), 199-221.
[30] X. Song; *Sharp thresholds of global existence and blowup for a system of Schrödinger equations with combined power-type nonlinearities*, Jour. Math. Phys., 51, 033509 (2010).
[31] J. C. Wei, T. Weth; *Nonradial symmetric bound states for a system of two coupled Schrödinger equations*, Rend. Lincei Mat. Appl., 18 (2007), 279-294.
[32] J. C. Wei, T. Weth; *Radial solutions and phase separation in a system of two coupled Schrödinger equations*, Arch. Rational Mech. Anal., 190 (2008), 83-106.
[33] M. Weinstein; *Lyapunov stability of ground states of nonlinear dispersive evolution equations*, Comm. Pure Appl. Math., 39 (1986), 51-68.
[34] V. E. Zakharov; *Stability of periodic waves of finite amplitude on the surface of a deep fluid*, Sov. Phys. Jour. Appl. Mech. Tech. Phys., 4 (1968), 190-194.
[35] V. E. Zakharov; *Collapse of Langmuir waves*, Sov. Phys. JETP, 35 (1972), 908-914.
[36] A. K. Zvezdin, A. F. Popkov; *Contribution to the nonlinear theory of magnetostatic spin waves*, Sov. Phys. JETP, 2 (1983), 350.

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