A VARIANT OF CLARK'S THEOREM AND ITS APPLICATIONS FOR NONSMOOTH FUNCTIONALS WITHOUT THE PALAIS–SMALE CONDITION*

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Abstract. By introducing a new notion of the genus with respect to the weak topology in Banach spaces, we prove a variant of Clark's theorem for nonsmooth functionals without the Palais–Smale condition. In this new theorem, the Palais–Smale condition is replaced by a weaker assumption, and a sequence of critical points converging weakly to zero with nonpositive energy is obtained. As applications, we obtain infinitely many solutions for a quasi-linear elliptic equation which is very degenerate and lacks strict convexity, and we also prove the existence of infinitely many homoclinic orbits for a second-order Hamiltonian system for which the functional is not in C^1 and does not satisfy the Palais–Smale condition. These solutions cannot be obtained via existing abstract theory.

Key words. the Clark theorem, weak genus, very degenerate quasi-linear equation, homoclinic orbit of Hamiltonian system

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1. Introduction and main result. The Clark theorem (see [6]) is a classical theorem in critical point theory that has numerous applications in the existence theory of differential equations and dynamic systems (see, for instance, [5, 16, 20, 21, 23, 26, 27]). In [16], Heinz stated the Clark theorem in the following form.

THEOREM A (see [16]). *Let X be a Banach space, $J \in C^1(X, \mathbb{R})$. Assume J satisfies the Palais–Smale condition, is even and bounded from below, and $J(0) = 0$. If for any $k \in \mathbb{N}$, there exists a k-dimensional subspace X_k of X and $\rho_k > 0$ such that $\sup_{X_k \cap S_{\rho_k}} J < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$, then J has a sequence of critical values $c_k < 0$ satisfying $c_k \to 0$ as $k \to \infty$.*

This theorem was improved by Kajikiya in [18], where it is proved that under the same assumptions as in Theorem A, there exists a sequence $\{u_k\}$ of critical points of J such that $J(u_k) \leq 0$, $u_k \neq 0$, and $\lim_{k \to \infty} \|u_k\| = 0$. The additional information $\lim_{k \to \infty} \|u_k\| = 0$ of the critical point sequence has been proved to be very important in many applications (see [18] and [21]). In [21], Liu and Wang extended the Clark theorem in various situations. As well as giving a new proof for the Kajikiya result, they proved a new version of the Clark theorem for nonsmooth functionals which can

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be applied to various quasi-linear elliptic equations for the existence of infinitely many solutions. They also gave a surprising variant of the Clark theorem for strongly indefinite functionals and used it to obtain infinitely many periodic solutions for sublinear first-order Hamiltonian systems.

However, all those variants of the Clark theorem mentioned above rely on the Palais–Smale condition, which makes it impossible to apply those abstract results to differential equations in many important situations. The present paper is devoted to establishing a new variant of the Clark theorem for nonsmooth functionals which do not satisfy the Palais–Smale condition. The motivations are from applications to very degenerate quasi-linear equations and from applications to homoclinic orbits of Hamiltonian systems. In these situations, we will see that the usual Palais–Smale condition cannot hold and that the sequence of solutions weakly converging to zero cannot converge to zero in the strong norm, let alone in the L^∞-norm. The quasi-linear elliptic equations considered here are very degenerate and lack strict convexity, and the second-order Hamiltonian systems treated here do not have a C^1 variational formulation and do not satisfy the Palais–Smale condition.

We recall some notations and definitions before stating our main result. Let X be a Banach space and X' be its dual space. The weak convergence in X is denoted by "$\rightharpoonup$". Let J be a functional defined on X. J is said to be weakly sequentially lower semicontinuous if $\liminf_{n\to\infty} J(u_n) \geq J(u)$ for any $u \in X$ and $\{u_n\} \subset X$ satisfying $u_n \rightharpoonup u$.

Let J be a continuous functional defined on X and E a dense subspace of X. As in [19], we say that J is E-differentiable if

(a) for all $u \in X$ and all $h \in E$, the derivative of J at u in the direction h, denoted by $\langle J'(u), h \rangle$, exists, that is,

$$\langle J'(u), h \rangle = \lim_{t\to 0+} \frac{J(u+th) - J(u)}{t};$$

(b) the mapping $(u, h) \mapsto \langle J'(u), h \rangle$ satisfies
   (i) $\langle J'(u), h \rangle$ is linear in h,
   (ii) $\langle J'(u), h \rangle$ is continuous in u, that is, $\langle J'(u_n), h \rangle \to \langle J'(u), h \rangle$ as $u_n \to u$ in X.

The slope of an E-differentiable functional J at u denoted by $|J'(u)|$ is an extended number in $[0, +\infty]$:

$$|J'(u)| = \sup\{\langle J'(u), h \rangle \mid h \in E, \ \|h\| = 1\},$$

where $\|\cdot\|$ denotes the norm in X. A point $u \in X$ is said to be a critical point of J if $|J'(u)| = 0$.

The main result of this paper is the following theorem.

THEOREM 1.1. *Let X be a separable and reflexive Banach space with norm $\|\cdot\|$ and let E be a dense subspace of X. Assume that J is a continuous functional defined on X which is E-differentiable. Suppose that J satisfies the following conditions.*

($\mathbf{A_1}$) *J is an even functional, i.e., $J(-u) = J(u)$ for every $u \in X$, and it is bounded from below.*

($\mathbf{A_2}$) *If $u \in X$, $\{u_n\} \subset X$, $|J'(u_n)| \to 0$, and $u_n \rightharpoonup u$ as $n \to \infty$, then*

$$|J'(u)| = 0.$$

($\mathbf{A_3}$) *J is weakly sequentially lower semicontinuous.*

($\mathbf{A_4}$) *The set $\{u \in X \mid J(u) \leq J(0)\}$ is bounded in X.*
($\mathbf{A_5}$) *For any positive integer k, there exists a k-dimensional subspace X_k of X*
and $\rho_k > 0$ such that $\sup_{X_k \cap S_{\rho_k}} J < J(0)$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$.
Then, J has infinitely many pairs of critical points $\{\pm u_k \mid k = 1, 2, \ldots\}$ satisfying

$$J(\pm u_k) \leq J(0), \quad u_k \neq 0 \text{ for } k = 1, 2, \ldots, \quad \text{and } u_k \rightharpoonup 0 \text{ as } k \to \infty.$$

Remark 1.2. We will assume, without loss of generality, $J(0) = 0$.

Remark 1.3. The assumption ($\mathbf{A_2}$) can be deduced from the following assumption.
($\mathbf{A_2'}$) If $\{u_n\} \subset X$, $u \in X$, and $u_n \rightharpoonup u$ in X, then

$$\langle J'(u_n), h \rangle \to \langle J'(u), h \rangle \quad \forall h \in E.$$

Therefore, the result of Theorem 1.1 is true if the assumption ($\mathbf{A_2}$) is replaced by
($\mathbf{A_2'}$).

Obviously, either condition ($\mathbf{A_2}$) or ($\mathbf{A_2'}$) is weaker than the Palais–Smale condition for nonsmooth functionals (see [21, Definition 3.8]). Concerning various differential equations, there are many cases in which the functionals satisfy ($\mathbf{A_1}$)−($\mathbf{A_5}$), but do not satisfy the Palais–Smale condition. Using a new genus with respect to the weak topology of X and the method of descending flow, we will give the proof of Theorem 1.1 in the following three sections.

In the literature, intensive work was carried out in the 1990s on critical point theory for nonsmooth functionals. We discuss and compare some related work here. Corvellec, Degiovanni, and Marzocchi in [9] and Degiovanni and Marzocchi in [13] gave the following definition of the weak slope for a continuous functional defined in a metric space.

DEFINITION 1.4 (Definition 2.1 of [9] or Definition 2.1 of [13]). *Let Y be a metric space endowed with the metric d. Let $f : Y \to \mathbb{R}$ be a continuous function and $u \in Y$. We denote by $|df|(u)$ the supremum of the σ's in $[0, +\infty]$ such that there exist $\delta > 0$ and $\mathcal{H} : B(u, \delta) \times [0, \delta] \to Y$ continuous with*

$$d(\mathcal{H}(v, t), v) \leq t \quad \text{and} \quad f(\mathcal{H}(v, t)) \leq f(v) - \sigma t,$$

where $B(u, \delta) = \{v \in Y \mid d(u, v) < \delta\}$. The extended real number $|df|(u)$ is called the weak slope of f at u, and u is called a critical point of f if $|df|(u) = 0$.

Using this, the authors of [9] also gave the definition of the weak slope for a lower semicontinuous functional f defined in the metric space Y. More precisely, let Y be a metric space endowed with the metric d and $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function. Let $\mathcal{D}(f) = \{u \in Y \mid f(u) < +\infty\}$, epi$(f) = \{(u, \xi) \in Y \times \mathbb{R} \mid f(u) \leq \xi\}$, and

$$\mathcal{G}_f : \text{epi}(f) \to \mathbb{R}, \quad \mathcal{G}_f(u, \xi) = \xi.$$

The set $Y \times \mathbb{R}$ is endowed with the metric $\varrho((u, \xi), (v, \mu)) = (d^2(u, v) + (\xi - \mu)^2)^{1/2}$ and epi(f) with the induced metric.

DEFINITION 1.5 (Definition 2.6 of [9] or Definition 2.4 of [13]). *Let Y be a metric space endowed with the metric d. Let $f : Y \to \mathbb{R} \cup \{+\infty\}$ be a lower semicontinuous function and $u \in \mathcal{D}(f)$. The weak slope of f at u is defined by*

$$|df|(u) = \begin{cases} \frac{|d\mathcal{G}_f|(u, f(u))}{\sqrt{1 - (|d\mathcal{G}_f|(u, f(u)))^2}} & \text{if } |d\mathcal{G}_f|(u, f(u)) < 1, \\ +\infty & \text{if } |d\mathcal{G}_f|(u, f(u)) = 1. \end{cases}$$

$u \in \mathcal{D}(f)$ *is called a critical point of f if $|df|(u) = 0$.*

The idea of Definition 1.5 is to reduce the study of the lower semicontinuous function f to that of the continuous function $\mathcal{G}_f$. In view of critical point theory, it is necessary to introduce a condition which implies a bijective correspondence between the critical points of f and those of $\mathcal{G}_f$.

DEFINITION 1.6 (Definition 2.2 of [15] or (4.1) of [9]). *We say that f satisfies the condition* epi *on Y if, for any $b > 0$,*

$$\inf\{|d\mathcal{G}_f|(u,\xi) \mid u \in \mathcal{D}(f),\ f(u) < \xi,\ |\xi| \le b\} > 0.$$

The epi condition guarantees that a critical point of $\mathcal{G}_f$ corresponds to a critical point of the original functional f. Therefore, using the weak slope to obtain critical points of a lower semicontinuous functional, the epi condition must be verified.

From [4, Theorem 1.1.2], it is easy to see that if J is E-differentiable, then $|J'(u)| \le |dJ|(u)$ for any $u \in X$.

Based on the weak slope, the authors of [9] and [13] established some deformation theorems and then developed a nonsmooth critical point theory for lower semicontinuous functionals. This theory has many important applications; see, for example, [4, 14, 15]. In particular, in [15] Degiovanni and Schuricht obtained a Clark-type theorem for lower semicontinuous functionals defined in metric spaces (see [15, Theorem 2.5]). The following theorem is a direct consequence of [15, Theorem 2.5].

THEOREM 1.7. *Let Y be a complete metric space which is a subset of some linear space X in the set-theoretic sense and for which $0 \in Y$ and $-u \in Y$ whenever $u \in Y$. Assume that $J : Y \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous functional defined on Y. Suppose that J satisfies the following conditions.*

(i) *J is an even functional, i.e., $J(-u) = J(u)$ for every $u \in Y$, and it is bounded from below.*

(ii) *J satisfies the* epi *condition.*

(iii) *For any $c < J(0)$, the function J satisfies the condition $(PS)_c$.*

(iv) *For any positive integer k, there exists a continuous odd map ψ from the k-dimensional sphere S^k to Y such that*

$$\sup_{u \in S^k} J(\psi(u)) < J(0).$$

Then, J has infinitely many pairs of critical points $\{\pm u_k \mid k = 1, 2, \ldots\}$ satisfying

$$J(\pm u_k) < J(0) \quad for\ k = 1, 2, \ldots.$$

We point out that Theorem 1.1 cannot be derived from the above result. The reason for this is that, under the weak topology of a separable and reflexive Banach space, it is very difficult to verify the epi condition for a lower semicontinuous functional. More precisely, let X be a separable and reflexive Banach space and J the functional satisfying the conditions $(\mathbf{A_1})$–$(\mathbf{A_5})$ in Theorem 1.1. Because X is a separable and reflexive Banach space and J satisfies the conditions $(\mathbf{A_3})$ and $(\mathbf{A_4})$, if we choose $Y = \{u \in X \mid J(u) \le J(0)\}$ endowed with the weak topology of X, then Y is a compact metric space (see Lemma 3.1 in section 3). Now the functional J in Theorem 1.1 is merely a lower semicontinuous functional in Y and not a continuous functional in Y in general. Because J satisfies the conditions $(\mathbf{A_1})$–$(\mathbf{A_5})$ in Theorem 1.1, it is easy to verify that the functional J in Theorem 1.1 satisfies the conditions (i), (iii), and (iv) in Theorem 1.7. In particular, we must point out that J satisfies

the $(PS)_c$ condition, i.e., the condition (**iii**) in Theorem 1.7, since Y is a compact metric space. However, the conditions ($\mathbf{A_1}$)–($\mathbf{A_5}$) in Theorem 1.1 cannot imply that the functional J satisfies the condition (**ii**), i.e., the epi condition. To the best of our knowledge, under the weak topology of X, the epi condition is only verified for the functional $f : X \to \mathbb{R}$ with a very special form: $f = f_0 + f_1$, where $f_0 : X \to \mathbb{R}$ is a C^1, convex, and weakly lower semicontinuous functional, $f_1 : X \to \mathbb{R}$ is of class C^1, and $f_1' : X \to X'$ is completely continuous (see [12, Proposition 5.2]). It would be a very interesting question to pursue in the future whether a condition like ($\mathbf{A_2}$) could replace the PS condition so that it is incorporated into the critical point theory developed by Degiovanni and his collaborators.

As an application of Theorem 1.1, in section 5 we consider the following quasi-linear elliptic equation:

$$(1.1) \quad -\mathrm{div}(D_\xi \Psi(x, u, \nabla u)) + D_s \Psi(x, u, \nabla u) = K(x)|u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$). The conditions on the function $\Psi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$, $(x, s, \xi) \mapsto \Psi(x, s, \xi)$ will allow the quasi-linear elliptic operator

$$-\mathrm{div}(D_\xi \Psi(x, u, \nabla u)) + D_s \Psi(x, u, \nabla u)$$

to be very degenerate. In fact, under our assumptions on Ψ, $\Psi(x, s, \cdot)$ may vanish in an open set containing $\xi = 0$ in $\mathbb{R}^N$ for any (x, s) (see the example (5.7) in section 5), and Ψ may be not strictly convex with respect to ξ. Therefore, the Euler–Lagrange functional corresponding to (1.1) does not satisfy the Palais–Smale condition. The very degenerate-type elliptic operators involved in (1.1) are related to some traffic congestion dynamic models [3, 8]. In section 5, using Theorem 1.1, we prove that this equation has infinitely many solutions. In a sense, our result improves the previous results on quasi-linear elliptic equations in [11, 22].

As another application of Theorem 1.1, in section 6 we consider the existence of homoclinic orbits for the following second-order Hamiltonian system:

$$(1.2) \qquad\qquad\qquad \ddot{u} + \nabla_u V(t, u) = 0,$$

where V is even with respect to u, $\nabla_u V$ exists, $\nabla_u V \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$, $V(t, u) = -K(t, u) + W(t, u)$, W is sublinear with respect to u, and K only satisfies the "pinching" condition [17]

$$\tau_1 |u|^2 \leq K(t, u) \leq \tau_2 |u|^2$$

for some constants $\tau_2 > \tau_1 > 0$. Under these assumptions, the Euler–Lagrange functional corresponding to (1.2) is not C^1 in the Sobolev space $H^1(\mathbb{R})$ and does not satisfy the Palais–Smale condition in general. Using Theorem 1.1, we prove that this equation has infinitely many homoclinic orbits. Our result improves the previous results in [17, 24, 27].

2. The genus with respect to the weak topology and its properties. Let X be a separable and reflexive Banach space with norm $\| \cdot \|$ and let X' be its dual space. Let $\{\phi_k\}$ be a sequence dense in the unit ball of X' and define as in [12], for every $u, v \in X$,

$$(2.1) \qquad\qquad (u, v)_w = \sum_{k=1}^{\infty} \frac{1}{2^k} \phi_k(u)\phi_k(v).$$

Then $(\cdot, \cdot)_w$ is a scalar product on X whose associated norm $\| \cdot \|_w$ satisfies

$$\|u\|_w \leq \|u\| \quad \forall u \in X$$

and induces the weak topology of X on every $\| \cdot \|$-bounded subset of X. We denote the normed space $(X; \| \cdot \|_w)$ by X_w. It is easy to see that if $\{u_n\} \subset X$ and $\{\|u_n\|\}$ is bounded, then

$$(2.2) \qquad\qquad u_n \rightharpoonup u \Leftrightarrow \|u_n - u\|_w \to 0, \quad n \to \infty.$$

The topology generated by $\| \cdot \|_w$ is denoted by w, and all topological notations related to $\| \cdot \|_w$ will include this symbol.

Let $S(X)$ be the class of subsets of $X \setminus \{0\}$ symmetric with respect to the origin and closed with respect to the $\| \cdot \|$-topology. Recall that the genus of $A \in S(X)$, denoted by $\gamma(A)$, is the least integer n such that there exists an odd and continuous map $\phi : A \to \mathbb{R}^n \setminus \{0\}$.

DEFINITION 2.1. Let $S(X_w)$ be the class of subsets of $X \setminus \{0\}$ symmetric with respect to the origin and closed with respect to the w-topology. If $A \in S(X_w)$, the weak genus of A, denoted by $\gamma_w(A)$, is the least integer m such that there exists an odd and w-continuous map $\psi : A \to \mathbb{R}^m \setminus \{0\}$.

Remark 2.2. Since a closed set in X_w is also a closed set in the Banach space X, we deduce that $S(X_w) \subset S(X)$ and, for any $A \in S(X_w)$,

$$(2.3) \qquad\qquad \gamma(A) \leq \gamma_w(A).$$

Remark 2.3. Assume $x_0 \in X$ and $\|x_0\| > 1$. Let $A = \{x \mid \|x - x_0\| = 1\} \cup \{x \mid \|x + x_0\| = 1\}$. Then $A \in S(X)$, but $A \notin S(X_w)$.

Remark 2.4. Let X be an infinite-dimensional and separable Hilbert space and $\{e_n\}$ an orthonormal base of X. Define

$$L = \bigcup_{0 \leq t \leq 1} \left(\{(1 - 2t)e_1 + te_2\} \bigcup \left(\bigcup_{n=2}^{\infty} \{(1 - t)e_n + te_{n+1} - e_1\} \right) \right).$$

Then L is closed with respect to the $\| \cdot \|$-topology, but is not closed with respect to the w-topology. Indeed, $cl_w(L) = L \cup \{-e_1\}$, where $cl_w(L)$ denotes the closure of L in X_w. Let $A = L \cup (-L)$, where $-L = \{u \mid -u \in L\}$. Then $A \in S(X_w)$. From the construction of A we see that

$$\gamma(A) = 1, \quad \gamma_w(A) = 2.$$

Therefore, γ and γ_w do not coincide on $S(X_w)$.

For the properties of the genus γ, one may consult [7] and [6]. Our following proposition shows that the weak genus γ_w and the genus γ share similar properties. The proof of the properties of γ_w is similar to the proof of the properties of γ, and one may consult p. 17 of [7].

PROPOSITION 2.5. *The weak genus γ_w has the following properties.*
 (**i**) *If $A_1, A_2 \in S(X_w)$ and if there exists an odd and w-continuous map $f : A_1 \to A_2$, then $\gamma_w(A_1) \leq \gamma_w(A_2)$. In particular, if $A_1 \subset A_2$, then $\gamma_w(A_1) \leq \gamma_w(A_2)$.*
 (**ii**) *For $A_1, A_2 \in S(X_w)$, $\gamma_w(A_1 \cup A_2) \leq \gamma_w(A_1) + \gamma_w(A_2)$.*

(**iii**) *If $A \in S(X_w)$ and A is w-compact, then $\gamma_w(A) < +\infty$ and A has a neighborhood U open with respect to the w-topology such that $cl_w(U) \in S(X_w)$ and $\gamma_w(cl_w(U)) = \gamma_w(A)$.*

(**iv**) *If $G \in S(X_w)$ and if there exists an odd homeomorphism of the n-sphere onto G, then $\gamma_w(G) = n + 1$.*

(**v**) *If $\{G_n\}$ is a decreasing sequence of w-compact sets belonging to $S(X_w)$ and if $G = \cap_{n=1}^\infty G_n$, then $G \in S(X_w)$ and*

$$\gamma_w(G) = \lim_{n \to \infty} \gamma_w(G_n).$$

Remark 2.6. There is no ambiguity regarding homeomorphism in Proposition 2.5(**iv**), since $\psi : G \to S^n$ is a homeomorphism with respect to the $\| \cdot \|$-topology if and only if it is a homeomorphism with respect to the w-topology. This is easily seen via the compactness of S^n.

3. A gradient field and the related descending flow. Throughout this section, let X be a separable and reflexive Banach space with norm $\| \cdot \|$, E a dense subspace of X, and $J : X \to \mathbb{R}$ an E-differentiable functional. For any $c \in \mathbb{R}$, set

$$(3.1) \qquad J^c = \{u \in X \mid J(u) \le c\}.$$

LEMMA 3.1. *If J satisfies the conditions $(\mathbf{A_3})$ and $(\mathbf{A_4})$ and $c \le 0$, then J^c is w-compact.*

Proof. We may assume $J^c \neq \emptyset$. Since $c \le 0$, J^c is bounded in X by $(\mathbf{A_4})$. Let $\{u_n\} \subset J^c$. Then $\{\|u_n\|\}$ is bounded. Since X is a reflexive Banach space, there exists $u \in X$ such that, up to a subsequence, $u_n \rightharpoonup u$. Since J is weakly sequentially lower semicontinuous, we have $\liminf_{n \to \infty} J(u_n) \ge J(u)$. It follows that $u \in J^c$. Therefore, J^c is w-compact. $\qquad \square$

Let

$$(3.2) \qquad \mathcal{K} = \{u \in X \mid J(u) \le 0, \ |J'(u)| = 0\}.$$

If J is an even functional, then $0 \in \mathcal{K}$.

For $c \le 0$, let

$$(3.3) \qquad \mathcal{K}^c = \mathcal{K} \cap J^c = \{u \in X \mid J(u) \le c, \ |J'(u)| = 0\}.$$

LEMMA 3.2. *If J satisfies the conditions $(\mathbf{A_2})$–$(\mathbf{A_4})$ and $c \le 0$, then $\mathcal{K}^c$ is w-compact.*

Proof. We may assume $\mathcal{K}^c \neq \emptyset$. By Lemma 3.1, we only need to prove that $\mathcal{K}^c$ is w-closed in J^c. Let $\{u_n\} \subset \mathcal{K}^c$. The proof of Lemma 3.1 shows that $u_n \rightharpoonup u$ for some $u \in J^c$. Since $|J'(u_n)| = 0$, the condition $(\mathbf{A_2})$ implies that $|J'(u)| = 0$. It follows that $u \in \mathcal{K}^c$. Therefore, $\mathcal{K}^c$ is a w-compact set. $\qquad \square$

LEMMA 3.3. *If J satisfies the conditions $(\mathbf{A_2})$–$(\mathbf{A_4})$, V is a w-open neighborhood of $\mathcal{K}$, and $J^0 \setminus V \neq \emptyset$, then*

$$(3.4) \qquad \inf\{|J'(u)| \mid u \in J^0 \setminus V\} > 0.$$

Proof. If (3.4) were false, then there would be $u_n \in J^0 \setminus V$ such that $|J'(u_n)| \to 0$. By Lemma 3.1, there exists $u \in J^0 \setminus V$ such that, up to a subsequence, $u_n \rightharpoonup u$. By $(\mathbf{A_2})$, we have $|J'(u)| = 0$, which is a contradiction. $\qquad \square$

To prove Theorem 1.1, we will use an argument of contradiction. For this, we assume that there exists $\iota > 0$ such that

$$(3.5) \qquad\qquad \mathcal{K} \cap B_{X_w}(0, \iota) = \{0\},$$

where

$$B_{X_w}(u, r) = \{v \in X_w \mid \|v - u\|_w < r\}.$$

Let

$$(3.6) \qquad\qquad \mathcal{K}' = \mathcal{K} \setminus \{0\}.$$

Then, by (3.5) and Lemma 3.2, $\mathcal{K}'$ is a w-compact set and

$$(3.7) \qquad\qquad d_w(0, \mathcal{K}') \geq \iota,$$

where

$$d_w(u, A) = \inf_{v \in A} \|u - v\|_w.$$

For $r > 0$, let

$$(\mathcal{K}')_r = \{u \in X \mid d_w(u, \mathcal{K}') < r\}.$$

LEMMA 3.4. *Assume that J satisfies the conditions $(\mathbf{A_1})$–$(\mathbf{A_4})$ and suppose that (3.7) holds. Then there exists*

$$\zeta : \tilde{X} \to E \setminus \{0\},$$

where $\tilde{X} := \{u \in X \mid |J'(u)| \neq 0, \ J(u) < 0\}$, satisfying the following.
  (**i**) *ζ is an equivariant locally Lipschitz mapping, i.e., for any $u \in \tilde{X}$, $\zeta(-u) = -\zeta(u)$ and there exist $L_u > 0$ and $r_u > 0$ such that for any $v_1, v_2 \in B_X(u, r_u)$,*

$$\|\zeta(v_1) - \zeta(v_2)\| \leq L_u \|v_1 - v_2\|,$$

  where $B_X(u, r) = \{v \in X \mid \|v - u\| < r\}$.
  (**ii**) *For any $u \in \tilde{X}$, $\|\zeta(u)\| \leq 1$ and $\langle J'(u), \zeta(u) \rangle > 0$.*
  (**iii**) *If $c < 0$ and $J^c \setminus (\mathcal{K}')_{\iota/20} \neq \emptyset$, then*

$$(3.8) \qquad\qquad \inf\{\langle J'(u), \zeta(u) \rangle \mid u \in J^c \setminus (\mathcal{K}')_{\iota/20}\} > 0.$$

  (**iv**) *If $\{u \in X \mid J(u) < 0, \ u \in cl_w((\mathcal{K}')_{\iota/3} \setminus (\mathcal{K}')_{\iota/4})\} \neq \emptyset$, then*

$$(3.9) \qquad \inf\{\langle J'(u), \zeta(u) \rangle \mid J(u) < 0, \ u \in cl_w((\mathcal{K}')_{\iota/3} \setminus (\mathcal{K}')_{\iota/4})\} > 0.$$

Proof. For any $u \in \tilde{X}$, by the definition of $|J'(u)|$, there exists $h_u \in E$ satisfying

$$(3.10) \qquad\qquad \|h_u\| = 1$$

and

$$\langle J'(u), h_u \rangle \geq \frac{3}{4} \min\{|J'(u)|, \ 1\}.$$

From the definition of E-differentiability and the fact that J is continuous in X, there exists δ_u such that

$$(3.11) \qquad\qquad 0 < \delta_u < \iota/60,$$

$$0 > \sup_{v \in B_X(u, \delta_u)} J(v) \geq \inf_{v \in B_X(u, \delta_u)} J(v) > \frac{3}{2} J(u), \tag{3.12}$$

and

$$\langle J'(v), h_u \rangle \geq \frac{1}{2} \min\{|J'(u)|, \ 1\} > 0 \quad \text{if } v \in B_X(u, \delta_u). \tag{3.13}$$

Then

$$\mathcal{A} = \{B_X(u, \delta_u) \mid |J'(u)| \neq 0, \ J(u) < 0\}$$

is an open covering of $\tilde{X}$.

Since $\tilde{X}$ is a metric space and hence a paracompact space, there exists a locally finite open covering $\mathcal{M} = \{M_i \mid i \in \Lambda\}$ of $\tilde{X}$ finer than $\mathcal{A}$. Define

$$\lambda_i(u) = \frac{d(u, X \setminus M_i)}{\sum_{j \in \Lambda} d(u, X \setminus M_j)}, \quad u \in \tilde{X},$$

where

$$d(u, A) = \inf_{v \in A} \|u - v\|.$$

Since each $u \in \tilde{X}$ has a neighborhood intersecting only finitely many M_j, $\sum_{j \in \Lambda} d(u, X \setminus M_j)$ is only a finite sum, and each λ_i is a Lipschitz continuous functional on $\tilde{X}$ with respect to the $\|\cdot\|$ norm.

For any $i \in \Lambda$, choose one $u_i \in \tilde{X}$ such that $M_i \subset B_X(u_i, \delta_{u_i})$. Define

$$\zeta^*(u) := \sum_{i \in \Lambda} \lambda_i(u) h_{u_i} \in E, \quad u \in \tilde{X}, \tag{3.14}$$

and

$$\zeta(u) := \frac{1}{2}(\zeta^*(u) - \zeta^*(-u)), \quad u \in \tilde{X}. \tag{3.15}$$

For every $u \in \tilde{X}$, the sum in (3.14) is only a finite sum. Clearly, ζ satisfies (**i**). Since J is an even functional, we have

$$\langle J'(-u), h \rangle = \langle -J'(u), h \rangle \quad \forall u \in X \ \forall h \in E. \tag{3.16}$$

Then by (3.10)–(3.16), we deduce that ζ satisfies (**ii**).

Now we prove (**iii**). For $u \in J^c \setminus (\mathcal{K}')_{\iota/20}$, there exist only a finite number of elements from $\mathcal{M}$, denoted by $M_1, M_2, \ldots, M_k$, such that for all $1 \leq i \leq k$, either $u \in M_i$ or $-u \in M_i$. Then

$$\zeta(u) = \frac{1}{2} \sum_{i=1}^{k} \lambda_i(u) h_{u_i} - \frac{1}{2} \sum_{i=1}^{k} \lambda_i(-u) h_{u_i}. \tag{3.17}$$

Clearly, $\pm u \in M_i$ if and only if $\lambda_i(\pm u) \neq 0$, in which case $\pm u \in B_X(u_i, \delta_{u_i})$ and

$$\langle J'(\pm u), h_{u_i} \rangle \geq \frac{1}{2} \min\{|J'(u_i)|, \ 1\}. \tag{3.18}$$

We first consider the case $u \in M_i$. Since $J(u) \leq c$, by (3.12), we have

$$J(u_i) < \frac{2}{3} c. \tag{3.19}$$

Since $u \in B_X(u_i, \delta_{u_i})$ and $u \notin (\mathcal{K}')_{\iota/20}$, by (3.11), we see that

$$(3.20) \qquad\qquad u_i \notin (\mathcal{K}')_{\iota/40}.$$

Combining (3.19) and (3.20) yields

$$(3.21) \qquad\qquad u_i \in J^{\frac{2}{3}c} \setminus (\mathcal{K}')_{\iota/40}.$$

In the case $-u \in M_i$, the same argument from (3.19) to (3.21), with $-u$ instead of u, holds. Since $\frac{2}{3}c < 0$, by Lemma 3.3,

$$(3.22) \qquad |J'(u_i)| \geq m_c := \inf\{|J'(u)| \mid u \in J^{\frac{2}{3}c} \setminus (\mathcal{K}')_{\iota/40}\} > 0.$$

By (3.17), (3.18), and (3.22), we have

$$\begin{aligned}
\langle J'(u), \zeta(u)\rangle &= \frac{1}{2}\sum_{i=1}^{k}\lambda_i(u)\langle J'(u), h_{u_i}\rangle - \frac{1}{2}\sum_{i=1}^{k}\lambda_i(-u)\langle J'(u), h_{u_i}\rangle \\
&= \frac{1}{2}\sum_{i=1}^{k}\lambda_i(u)\langle J'(u), h_{u_i}\rangle + \frac{1}{2}\sum_{i=1}^{k}\lambda_i(-u)\langle J'(-u), h_{u_i}\rangle \\
&\geq \frac{1}{4}\min\{1,\ |J'(u_i)|,\ i=1,2,\ldots,k\}\left(\sum_{i=1}^{k}\lambda_i(u) + \sum_{i=1}^{k}\lambda_i(-u)\right) \\
(3.23) \qquad &= \frac{1}{2}\min\{1,\ |J'(u_i)|,\ i=1,2,\ldots,k\} \geq \min\{1,\ m_c\}/2 > 0.
\end{aligned}$$

This implies (**iii**).

It suffices to prove (**iv**). Assume $J(u) < 0$ and $u \in cl_w((\mathcal{K}')_{\iota/3} \setminus (\mathcal{K}')_{\iota/4})$. As in the proof of (**iii**), there exist finite $u_i \in \tilde{X}$, $1 \leq i \leq k$, such that (3.17) and (3.18) hold. For any i, since either $u \in B_X(u_i, \delta_{u_i})$ or $-u \in B_X(u_i, \delta_{u_i})$ and since $u \in cl_w((\mathcal{K}')_{\iota/3} \setminus (\mathcal{K}')_{\iota/4})$, (3.11) and (3.7) can be used to deduce that

$$u_i \in J^0 \setminus (B_{X_w}(0, \iota/5) \cup (\mathcal{K}')_{\iota/5}).$$

This, together with Lemma 3.3, implies

$$|J'(u_i)| \geq m_* := \inf\{|J'(u)| \mid J(u) \leq 0,\ u \notin B_{X_w}(0, \iota/5) \cup (\mathcal{K}')_{\iota/5}\} > 0.$$

We then conclude as (3.23). $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Set $A = (\mathcal{K}')_{\iota/40}$ and $B = X \setminus (\mathcal{K}')_{\iota/20}$, and define

$$\varrho(u) = \frac{d_w(u, A)}{d_w(u, A) + d_w(u, B)}, \quad u \in X.$$

Then ϱ is an even functional which is Lipschitz continuous with respect to the $\|\cdot\|_w$-norm, $\varrho|_A = 0$, $\varrho|_B = 1$, and $0 \leq \varrho \leq 1$. The fact that $\|v\|_w \leq \|v\|$ for $v \in X$ implies that ϱ is also Lipschitz continuous with respect to the $\|\cdot\|$-norm. Let

$$(3.24) \qquad\qquad \chi(u) = -\varrho(u)\zeta(u), \quad u \in \tilde{X}.$$

The conclusion (**i**) in Lemma 3.4 implies that χ is Lipschitz continuous with respect to the $\|\cdot\|$-norm. Therefore, the initial value problem

$$\begin{cases} \frac{d\eta}{dt} = \chi(\eta), \\ \eta(0, u) = u \in \hat{X} := \{v \in X \mid J(v) < 0\} \end{cases}$$

has a unique solution in $\hat{X}$, denoted by $\eta(t, u)$, with right maximal interval of existence $[0, T(u))$. The conclusion (**ii**) in Lemma 3.4, together with the construction of χ, implies that $\langle J'(u), \chi(u) \rangle \leq 0$ for all $u \in \hat{X}$. This fact, together with the equation

$$\frac{d}{dt} J(\eta(t, u)) = \langle J'(\eta(t, u)), \chi(\eta(t, u)) \rangle,$$

implies that J is nonincreasing along trajectories of η. Therefore, $\hat{X}$ is an invariant set of the flow η. The conclusion (**ii**) in Lemma 3.4, together with the construction of χ, also implies $\|\chi(u)\| \leq 1$ for all $u \in \hat{X}$. From this and the fact that $\hat{X}$ is an invariant set of the flow η, we see that $T(u) = +\infty$ for any $u \in \hat{X}$. From the conclusion (**i**) in Lemma 3.4 and the definition of χ, we have $\chi(-u) = -\chi(u)$ for all $u \in \hat{X}$. It follows that, for any $(t, u) \in [0, +\infty) \times \hat{X}$,

$$(3.25) \qquad\qquad\qquad \eta(t, -u) = -\eta(t, u).$$

LEMMA 3.5. *Assume J satisfies the conditions* $(\mathbf{A_1})$–$(\mathbf{A_4})$ *and* $\inf_X J < 0$. *Suppose that* (3.7) *holds. Then there exists $r_0 > 0$ satisfying that, for any $c < 0$, there exists $T_c > 0$ such that for any $u \in J^c$,*

$$\|\eta(T_c, u)\|_w \geq r_0.$$

Proof. Arguing indirectly, assume that, for any $0 < r < \iota/6$, there exist $c_r < 0$ and a sequence $\{u_n\} \subset J^{c_r}$ and $T_n \to +\infty$ such that

$$(3.26) \qquad\qquad\qquad \|\eta(T_n, u_n)\|_w < r.$$

We shall prove that, if n is large enough, then there exists $0 < t_n < T_n$ such that

$$(3.27) \qquad\qquad\qquad \eta(t_n, u_n) \in (\mathcal{K}')_{\iota/4}.$$

If this were not true, then, up to a subsequence,

$$\eta([0, T_n], u_n) \subset J^{c_r} \setminus (\mathcal{K}')_{\iota/4}, \ n = 1, 2, \ldots.$$

By the conclusion (**iii**) in Lemma 3.4, we have

$$(3.28) \qquad a_r := \inf\{\langle J'(u), \zeta(u) \rangle \mid u \in J^{c_r} \setminus (\mathcal{K}')_{\iota/4}\} > 0.$$

It follows that as $n \to \infty$,

$$J(\eta(T_n, u_n)) - J(u_n) = -\int_0^{T_n} \langle J'(\eta(t, u_n)), \varrho(\eta(t, u_n))\zeta(\eta(t, u_n)) \rangle dt$$

$$(3.29) \qquad\qquad = -\int_0^{T_n} \langle J'(\eta(t, u_n)), \zeta(\eta(t, u_n)) \rangle dt \leq -a_r T_n \to -\infty.$$

This contradicts the assumption that J is bounded from below.

By (3.27) and (3.26), we see that, when n is large enough, there exist $t_n < t'_n < t''_n \leq T_n$ such that

$$\eta(t'_n, u_n) \in cl_w((\mathcal{K}')_{\iota/4}), \quad \eta(t''_n, u_n) \in J^{c_r} \setminus (\mathcal{K}')_{\iota/3},$$

and

$$(3.30) \qquad \eta([t'_n, t''_n], u_n) \subset J^{c_r} \cap (cl_w((\mathcal{K}')_{\iota/3}) \setminus (\mathcal{K}')_{\iota/4}).$$

It follows that

$$\frac{\iota}{12} = \frac{\iota}{3} - \frac{\iota}{4} \leq \|\eta(t_n'', u_n) - \eta(t_n', u_n)\|_w$$

$$= \left\| -\int_{t_n'}^{t_n''} \varrho(\eta(t, u_n))\zeta(\eta(t, u_n))dt \right\|_w$$

$$= \left\| \int_{t_n'}^{t_n''} \zeta(\eta(t, u_n))dt \right\|_w \quad \text{(by (3.30))}$$

$$(3.31) \qquad \leq \int_{t_n'}^{t_n''} \|\zeta(\eta(t, u_n))\|_w dt \leq \int_{t_n'}^{t_n''} \|\zeta(\eta(t, u_n))\|dt.$$

By $\|\zeta\| \leq 1$ (see Lemma 3.4), we have, as a consequence of (3.31),

$$(3.32) \qquad t_n'' - t_n' \geq \frac{\iota}{12}.$$

By the conclusion (**iv**) in Lemma 3.4, we have

$$(3.33) \qquad b := \inf\{\langle J'(u), \zeta(u)\rangle \mid J(u) < 0, \ u \in cl_w((\mathcal{K}')_{\iota/3} \setminus (\mathcal{K}')_{\iota/4})\} > 0.$$

Clearly, b is independent of r. Note that

$$J(\eta(t_n'', u_n)) = J(\eta(t_n', u_n)) - \int_{t_n'}^{t_n''} \langle J'(\eta(t, u_n)), \varrho(\eta(t, u_n))\zeta(\eta(t, u_n))\rangle dt$$

$$(3.34) \qquad = J(\eta(t_n', u_n)) - \int_{t_n'}^{t_n''} \langle J'(\eta(t, u_n)), \zeta(\eta(t, u_n))\rangle dt.$$

By (3.30) and (3.32)–(3.34), we see that

$$(3.35) \qquad J(\eta(T_n, u_n)) \leq J(\eta(t_n'', u_n)) \leq J(\eta(t_n', u_n)) - \frac{b\iota}{12} \leq -\frac{b\iota}{12} := c_*,$$

where c_* is a constant independent of r. Since J^{c_*} is w-compact (see Lemma 3.1) and $0 \notin J^{c_*}$, there exists a constant $r_* > 0$ which is independent of r such that

$$(3.36) \qquad J^{c_*} \cap B_{X_w}(0, r_*) = \emptyset.$$

Choosing $0 < r < r_*$ and using (3.26), (3.35), and (3.36), we arrive at a contradiction. This finishes the proof. □

4. Proof of Theorem 1.1.

Proof of Theorem 1.1. Arguing indirectly, assume that there exists $\iota > 0$ such that (3.5) holds. Let $r_0 > 0$ be the constant given in Lemma 3.5. By Lemma 3.1,

$$B := J^0 \setminus B_{X_w}(0, r_0)$$

is a w-compact set, and by the condition that J is an even functional (see ($\mathbf{A_1}$)), we have $B \in S(X_w)$. Then, by (2.3) and the property (**iii**) in Proposition 2.5, we have

$$\gamma(B) \leq \gamma_w(B) < +\infty.$$

Therefore, we can choose an integer $m > \gamma(B)$. And by the condition $(\mathbf{A_5})$, we can choose $A \in S(X)$ such that $\gamma(A) \geq m$ and

$$c := \sup_{u \in A} J(u) < 0.$$

Then, by Lemma 3.5,

(4.1) $$\eta(T_c, A) \subset B.$$

Since η is equivariant (see (3.25)) and continuous, by the properties of the genus, we get that

$$m \leq \gamma(A) \leq \gamma(\eta(T_c, A)) \leq \gamma(B) < m.$$

This contradiction finishes the proof. $\qquad\square$

5. Application 1: Degenerate quasi-linear elliptic equations. In this section, we consider the following quasi-linear elliptic equation:

(5.1) $$-\mathrm{div}(D_\xi \Psi(x, u, \nabla u)) + D_s \Psi(x, u, \nabla u) = K(x)|u|^{q-2}u \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega,$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$ ($N \geq 2$).

The function

$$\Psi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}, \quad (x, s, \xi) \mapsto \Psi(x, s, \xi)$$

satisfies the following conditions.

$(\mathbf{\Psi_1})$ $\Psi : \Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ is measurable in x for all $(s, \xi) \in \mathbb{R} \times \mathbb{R}^N$, of class C^1 in (s, ξ) for almost every $x \in \Omega$. For almost every $x \in \Omega$ and $s \in \mathbb{R}$, the function $\Psi(x, s, \cdot)$ is convex.

$(\mathbf{\Psi_2})$ For any $(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N$, $\Psi(x, s, \xi) \geq 0$, $sD_s\Psi(x, s, \xi) \geq 0$, and $\Psi(x, -s, -\xi) = \Psi(x, s, \xi)$. For almost every (x, s), $\Psi(x, s, 0) = 0$. And for almost every (x, ξ), $D_s\Psi(x, 0, \xi) = 0$. Moreover, for $s \neq 0$, the function

(5.2) $$sD_s\Psi(x, s, \xi) \text{ is convex in } \xi.$$

$(\mathbf{\Psi_3})$ There exist $1 < p < N$, $\alpha > 0$, and $\lambda > 0$ such that

(5.3) $$\Psi(x, s, \xi) \geq \lambda|\xi|^p$$

for $(x, s) \in \Omega \times \mathbb{R}$ and $|\xi| \geq \alpha$.

$(\mathbf{\Psi_4})$ There exist $L > 0$, $C > 0$, and $M > 0$ such that

(5.4) $$|D_\xi\Psi(x, s, \xi)| \leq L|\xi|^{p-1} \quad \forall(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

(5.5) $$MD_\xi\Psi(x, s, \xi) \cdot \xi \pm D_s\Psi(x, s, \xi) \geq -C \quad \forall(x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N,$$

and for $x \in \Omega$ a.e. and all $s \in \mathbb{R}$, the functions

(5.6) $$MD_\xi\Psi(x, s, \xi) \cdot \xi \pm D_s\Psi(x, s, \xi) \text{ are convex in } \xi.$$

Remark 5.1.
(**i**) Under the assumptions $(\mathbf{\Psi_1})$–$(\mathbf{\Psi_4})$, the function $\Psi(x, s, \xi)$ may not be strictly convex in ξ. This case has been considered by several authors in some special situations (see, for example, [11] and [22]).

(**ii**) Under the assumptions $(\boldsymbol{\Psi_1})$–$(\boldsymbol{\Psi_4})$, the quasi-linear elliptic operator

$$-\operatorname{div}(D_\xi \Psi(x, u, \nabla u)) + D_s \Psi(x, u, \nabla u)$$

may be very degenerate. An example of such a Ψ is the following:

$$(5.7) \qquad \Psi(x, s, \xi) = (1 + (\arctan s)^2)g(|\xi|),$$

where

$$g(r) = \begin{cases} 0, & 0 \leq r \leq 2, \\ 2 - \sqrt{4 - (r-2)^2}, & 2 \leq r \leq 3, \\ ar^p + br + c, & r \geq 3, \end{cases}$$

and

$$a = \frac{4 \times 3^{1/2-p}}{p(p-1)}, \quad b = \frac{p-5}{\sqrt{3}(p-1)}, \quad c = 2 - \frac{2\sqrt{3}(p-2)}{p}.$$

Elementary calculations show that this Ψ satisfies the assumptions $(\boldsymbol{\Psi_1})$–$(\boldsymbol{\Psi_4})$. Equation (5.1) with $\Psi(x, s, \cdot)$ very degenerate and not strictly convex has never been studied before, as far as we know.

(**iii**) Equations (5.4) and (5.5) imply that

$$(5.8) \qquad |D_s \Psi(x, s, \xi)| \leq ML|\xi|^p + C \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

For the nonlinear term on the right side of (5.1), we assume

$$(5.9) \qquad\qquad\qquad\qquad 1 < q < p$$

and the following assumptions.

($\mathbf{K_1}$) $K \in L^\infty(\Omega)$.
($\mathbf{K_2}$) The following holds:

$$(5.10) \qquad\qquad\qquad \operatorname{essinf}_{x \in \Omega} K(x) > 0.$$

From (5.4) and $\Psi(x, s, 0) \equiv 0$ (see $(\boldsymbol{\Psi_2})$), we deduce that

$$(5.11) \qquad\qquad \Psi(x, s, \xi) \leq \frac{L}{p}|\xi|^p \quad \forall (x, s, \xi) \in \Omega \times \mathbb{R} \times \mathbb{R}^N.$$

Under the assumptions $(\boldsymbol{\Psi_1})$–$(\boldsymbol{\Psi_4})$, (5.9), and $(\mathbf{K_1})$–$(\mathbf{K_2})$, since we have (5.11), the following functional is well defined:

$$(5.12) \qquad J(u) = \int_\Omega \Psi(x, u, \nabla u)dx - \frac{1}{q}\int_\Omega K(x)|u|^q dx, \quad u \in X := W_0^{1,p}(\Omega),$$

where $W_0^{1,p}(\Omega)$ is the Sobolev space with the standard norm

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{1/p}.$$

By (5.4), (5.8), (5.9), and $(\mathbf{K_1})$–$(\mathbf{K_2})$, it is easy to see that for

$$E := W_0^{1,p}(\Omega) \cap L^\infty(\Omega),$$

J is E-differentiable, and for any $u \in X$ and $h \in E$,

(5.13)
$$\langle J'(u), h \rangle = \int_\Omega D_\xi \Psi(x, u, \nabla u) \cdot \nabla h dx + \int_\Omega D_s \Psi(x, u, \nabla u) h dx - \int_\Omega K(x)|u|^{q-2} u h dx.$$

It follows that the critical points of J are weak solutions of (5.1).

In general, this functional does not satisfy the Palais–Smale condition. For example, in the case of (5.7), we can choose a sequence $\{u_n\} \subset X$ satisfying that $\sup_{x \in \Omega} |\nabla u_n| \leq 1$ for all n, $u_n \rightharpoonup 0$ in X and $\inf_n \|u_n\| > 0$. Then $\{u_n\}$ is a Palais–Smale sequence of J, but it does not contain a convergent subsequence.

Such a sequence can be constructed as follows. Denote

$$I = \{x \in \mathbb{R}^N \mid x = (x_1, x_2, \ldots, x_N),\ 0 \leq x_j \leq 1\}$$

and

$$s_{ji} = \{x \in I \mid x_j = i\}, \quad I_{ji} = \text{conv}(s_{ji} \cup \{(1/2, 1/2, \ldots, 1/2)\})$$

for $j = 1, 2, \ldots, N$ and $i = 0, 1$, where $\text{conv}(A)$ means the closed convex hull of a set A. Then $I = \cup_{0 \leq j \leq 1,\ i=0,1} I_{ji}$. Define, for $x \in I$,

$$v(x) = \begin{cases} x_j & \text{if } x \in I_{j0}, \\ 1 - x_j & \text{if } x \in I_{j1}. \end{cases}$$

Then v is well defined, $v \in C_0(I)$, and $|\nabla v| = 1$ a.e. in I. Now suppose, without loss of generality, that we can translate I into a new cube I^* which is contained in Ω. For any n, divide I^* equally into n^N small cubes. Define, for $x \in \frac{1}{n}I := \{x \mid nx \in I\}$,

$$v_n(x) = \frac{1}{n} v(nx).$$

Define u_n on I^* in such a way that its constraint on each small cube is a translation of v_n. Outside of I^*, u_n is defined to be 0. For such a sequence, we have $\{u_n\} \subset X$, $|\nabla u_n| = 1$ a.e. in I^*, and $|\nabla u_n| = 0$ in $\Omega \setminus I^*$ for all n, and therefore $\|u_n\| = 1$ for all n. Moreover, $\|u_n\|_{L^\infty(\Omega)} \to 0$ and thus $u_n \rightharpoonup 0$ in X as $n \to \infty$.

The main result of this section is the following theorem.

THEOREM 5.2. *Under the assumptions* $(\mathbf{\Psi_1})$–$(\mathbf{\Psi_4})$, (5.9), *and* $(\mathbf{K_1})$–$(\mathbf{K_2})$, (5.1) *has infinitely many solutions.*

Proof. We shall verify that the functional J defined in (5.12) satisfies the assumptions $(\mathbf{A_1})$–$(\mathbf{A_5})$ in Theorem 1.1.

Since $1 < q < p$, by the Sobolev inequality and $(\mathbf{K_1})$, there exists $C > 0$ such that

(5.14)
$$\int_\Omega K(x)|u|^q dx \leq C\|u\|^q \quad \forall u \in X.$$

By $(\mathbf{\Psi_3})$, (5.11), and (5.14), we have

$$J(u) = \int_{|\nabla u| \geq \alpha} \Psi(x, u, \nabla u) dx + \int_{|\nabla u| < \alpha} \Psi(x, u, \nabla u) dx - \frac{1}{q} \int_\Omega K(x)|u|^q dx$$

$$\geq \lambda \int_{|\nabla u| \geq \alpha} |\nabla u|^p dx - \frac{L}{p} \int_{|\nabla u| < \alpha} |\nabla u|^p dx - \frac{C}{q} \|u\|^q$$

(5.15)
$$\geq \lambda \|u\|^p - \frac{C}{q} \|u\|^q - \left(\frac{L}{p} + \lambda\right) \alpha^p \text{meas}(\Omega),$$

where $\mathrm{meas}(A)$ denotes the Lebesgue measure of a set A. Therefore, J is coercive and bounded from below. It follows that $\{u \in X \mid J(u) \leq 0\}$ is bounded in X.

By $(\boldsymbol{\Psi_1})$ and $(\boldsymbol{\Psi_2})$, we see that $\int_\Omega \Psi(x, u, \nabla u) dx$ is weakly sequentially lower semicontinuous (see [25, Theorem 1.6]). Since the embedding $X \hookrightarrow L^q(\Omega)$ is compact (see, for example, [1]), the functional $\int_\Omega K(x)|u|^q dx$ is weakly sequentially continuous. Therefore, J is weakly sequentially lower semicontinuous. We have verified that J satisfies the assumptions $(\mathbf{A_1})$, $(\mathbf{A_3})$, and $(\mathbf{A_4})$ in Theorem 1.1.

Now, we prove that J satisfies the assumption $(\mathbf{A_2})$ in Theorem 1.1. Let $u \in X$ and $\{u_n\} \subset X$ satisfy that $|J'(u_n)| \to 0$ and $u_n \rightharpoonup u$ as $n \to \infty$. Choose $H \in C^1(\mathbb{R})$ such that $H(s) = 1$ for $|s| \leq 1/2$, $H(s) = 0$ for $|s| \geq 1$, $0 \leq H(s) \leq 1$, and $|H'(s)| \leq 4$ for all s. Let $\psi \in E$ with $\psi \geq 0$ and let $T \in \mathbb{R}$ with $T \geq 1$. As in [4], take $\varphi_n = \psi H(u_n/T) \exp(-M u_n^+)$ as a test function in $\langle J'(u_n), \varphi_n \rangle = o(1)$, where $M > 0$ is the constant in (5.5) and $u_n^+ = \max\{u_n, 0\}$. By (5.13), we have

(5.16)
$$o(1) = -\int_\Omega (M D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla u_n^+ - D_s \Psi(x, u_n, \nabla u_n)) \psi H(u_n/T) \exp(-M u_n^+) dx$$
$$+ \int_\Omega \exp(-M u_n^+) D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla(\psi H(u_n/T)) dx$$
$$- \int_\Omega K(x)|u_n|^{q-2} u_n \psi H(u_n/T) \exp(-M u_n^+) dx.$$

Letting $\Omega_n^\pm := \{x \in \Omega \mid \pm u_n(x) \geq 0\}$, we rewrite (5.16) as

(5.17)
$$o(1) = -\int_{\Omega_n^+} (M D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla u_n - D_s \Psi(x, u_n, \nabla u_n)) \psi H(u_n/T) \exp(-M u_n^+) dx$$
$$+ \int_{\Omega_n^-} D_s \Psi(x, u_n, \nabla u_n) \psi H(u_n/T) \exp(-M u_n^+) dx$$
$$+ \int_\Omega \exp(-M u_n^+) D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla(\psi H(u_n/T)) dx$$
$$- \int_\Omega K(x)|u_n|^{q-2} u_n \psi H(u_n/T) \exp(-M u_n^+) dx.$$

Since $u_n \rightharpoonup u$ in X, we have, up to a subsequence,

(5.18) $u_n \to u$ a.e. in Ω and $u_n \to u$ in $L^t(\Omega)$ for all $t \in [1, Np/(N-p))$.

By (5.5) and (5.6), we may apply [25, Theorem 1.6] to the function

$$(M D_\xi \Psi(x, s, \xi) \cdot \xi - D_s \Psi(x, s, \xi)) \psi H(s/T) \exp(-Ms\chi_+(s)) \chi_+(s),$$

where $\chi_+(s) = 1$ if $s > 0$ and $\chi_+(s) = 0$ if $s \leq 0$, to obtain that

(5.19)
$$\liminf_{n \to \infty} \int_{\Omega_n^+} (M D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla u_n - D_s \Psi(x, u_n, \nabla u_n)) \psi H(u_n/T) \exp(-M u_n^+) dx$$
$$\geq \int_{\Omega^+} (M D_\xi \Psi(x, u, \nabla u) \cdot \nabla u - D_s \Psi(x, u, \nabla u)) \psi H(u/T) \exp(-M u^+) dx,$$

where $\Omega^+ = \{x \in \Omega \mid u(x) \geq 0\}$ and $u^+ = \max\{u, 0\}$. According to $(\mathbf{\Psi_2})$, we have $D_s\Psi(x, s, \xi) \leq 0$, and $-D_s\Psi(x, s, \xi)$ is convex in ξ if $s \leq 0$. Using [25, Theorem 1.6] again, we have

$$\limsup_{n \to \infty} \int_{\Omega_n^-} D_s\Psi(x, u_n, \nabla u_n)\psi H(u_n/T)\exp(-Mu_n^+)dx$$

$$(5.20) \qquad \leq \int_{\Omega^-} D_s\Psi(x, u, \nabla u)\psi H(u/T)\exp(-Mu^+)dx,$$

where $\Omega^- = \{x \in \Omega \mid u(x) \leq 0\}$.

In order to study the third term on the right side of (5.17), we use an argument from the proof of [2, Theorem 2.1] (see also [10]) to verify that, up to a subsequence,

$$(5.21) \qquad e_n := (D_\xi\Psi(x, u_n, \nabla u_n) - D_\xi\Psi(x, u, \nabla u)) \cdot (\nabla u_n - \nabla u) \to 0 \quad \text{a.e. in } \Omega.$$

The convexity of Ψ in ξ implies that $e_n \geq 0$ in Ω, while (5.4), together with the fact that $\{u_n\}$ is bounded in X, implies that $\{e_n\}$ is bounded in $L^1(\Omega)$. For $\eta > 0$, define $T_\eta(s) = s$ for $|s| \leq \eta$ and $T_\eta(s) = (s/|s|)\eta$ for $|s| > \eta$. Using $\varphi_n = T_\eta(u_n - u)$ as a test function in $\langle J'(u_n), \varphi_n \rangle = o(1)$, we have

$$o(1) = \int_\Omega (D_\xi\Psi(x, u_n, \nabla u_n) - D_\xi\Psi(x, u, \nabla u)) \cdot \nabla(T_\eta(u_n - u))dx$$

$$+ \int_\Omega D_s\Psi(x, u_n, \nabla u_n)T_\eta(u_n - u)dx$$

$$+ \int_\Omega D_\xi\Psi(x, u, \nabla u) \cdot \nabla(T_\eta(u_n - u))dx$$

$$(5.22) \qquad - \int_\Omega K(x)|u_n|^{q-2}u_n T_\eta(u_n - u)dx.$$

For any $\eta > 0$, by (5.4), $(\mathbf{K_1})$, and (5.18), since $\nabla(T_\eta(u_n - u)) \rightharpoonup 0$ weakly in $L^p(\Omega)$ and $T_\eta(u_n - u) \to 0$ strongly in $L^t(\Omega)$ for all $t \in [1, Np/(N-p))$, the last two terms in (5.22) converge to 0. As a consequence of (5.8), the second term on the right side of (5.22) is estimated as

$$\int_\Omega |D_s\Psi(x, u_n, \nabla u_n)T_\eta(u_n - u)|dx \leq C\eta.$$

Therefore, (5.22) implies

$$(5.23) \qquad \limsup_{n \to \infty} \int_{\{|u_n - u| \leq \eta\}} e_n dx \leq C\eta.$$

Fix a number $\theta \in (0, 1)$. Using the Hölder inequality, we see that

$$\int_\Omega e_n^\theta dx \leq \left(\int_{\{|u_n - u| \leq \eta\}} e_n dx\right)^\theta (\text{meas}(\Omega))^{1-\theta}$$

$$(5.24) \qquad + \left(\int_\Omega e_n dx\right)^\theta (\text{meas}(\{|u_n - u| \geq \eta\}))^{1-\theta}.$$

Since $\{e_n\}$ is bounded in $L^1(\Omega)$ and u_n converges to u in measure, combining (5.23) and (5.24) yields

$$\limsup_{n \to \infty} \int_\Omega e_n^\theta dx \leq C\eta^\theta$$

for any $\eta > 0$. Since $\eta > 0$ is arbitrary, one has

$$\lim_{n \to \infty} \int_\Omega e_n^\theta dx = 0,$$

which implies (5.21) up to a subsequence. From (5.18) and (5.21), we use Lemma 7.1 to deduce that

(5.25) $$D_\xi \Psi(x, u_n, \nabla u_n) \to D_\xi \Psi(x, u, \nabla u) \quad \text{a.e. in } \Omega.$$

By (5.4), we have

$$\sup_n \int_\Omega |\exp(-Mu_n^+) H(u_n/T) D_\xi \Psi(x, u_n, \nabla u_n)|^{\frac{p}{p-1}} dx < +\infty,$$

which combined with (5.25) implies

(5.26) $$\begin{aligned} &\exp(-Mu_n^+) H(u_n/T) D_\xi \Psi(x, u_n, \nabla u_n) \rightharpoonup \\ &\exp(-Mu^+) H(u/T) D_\xi \Psi(x, u, \nabla u) \quad \text{in } L^{\frac{p}{p-1}}(\Omega). \end{aligned}$$

Therefore,

(5.27) $$\begin{aligned} &\lim_{n \to \infty} \int_\Omega \exp(-Mu_n^+) H(u_n/T) D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla \psi dx \\ &= \int_\Omega \exp(-Mu^+) H(u/T) D_\xi \Psi(x, u, \nabla u) \cdot \nabla \psi dx. \end{aligned}$$

Denoting

$$D_0 = \sup_n \|u_n\|^p, \quad \mu = \|\psi\|_{L^\infty(\Omega)},$$

by (5.4) and the fact that $|H'(s)| \leq 4$ for all s, we have

(5.28) $$\limsup_{n \to \infty} \frac{1}{T} \int_\Omega |\psi \exp(-Mu_n^+) H'(u_n/T) D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla u_n| dx \leq \frac{4\mu L D_0}{T}$$

and

(5.29) $$\frac{1}{T} \int_\Omega |\psi \exp(-Mu^+) H'(u/T) D_\xi \Psi(x, u, \nabla u) \cdot \nabla u| dx \leq \frac{4\mu L D_0}{T}.$$

Combining (5.27), (5.28), and (5.29) leads to

(5.30) $$\begin{aligned} &\limsup_{n \to \infty} \Big| \int_\Omega \exp(-Mu_n^+) D_\xi \Psi(x, u_n, \nabla u_n) \cdot \nabla(\psi H(u_n/T)) dx \\ &- \int_\Omega \exp(-Mu^+) D_\xi \Psi(x, u, \nabla u) \cdot \nabla(\psi H(u/T)) dx \Big| \leq \frac{8\mu L D_0}{T}. \end{aligned}$$

By (5.18), we have

(5.31) $$\begin{aligned} &\lim_{n \to \infty} \int_\Omega K(x)|u_n|^{q-2} u_n \psi H(u_n/T) \exp(-Mu_n^+) dx \\ &= \int_\Omega K(x)|u|^{q-2} u \psi H(u/T) \exp(-Mu^+) dx. \end{aligned}$$

Combining (5.17), (5.19), (5.20), (5.30), and (5.31) yields

$$\int_\Omega D_\xi \Psi(x,u,\nabla u)\nabla(\psi H(u/T)\exp(-Mu^+))dx$$

$$+ \int_\Omega D_s \Psi(x,u,\nabla u)\psi H(u/T)\exp(-Mu^+)dx$$

$$(5.32) \qquad - \int_\Omega K(x)|u|^{q-2}u\psi H(u/T)\exp(-Mu^+)dx \geq -8\mu L D_0/T.$$

Letting $T \to +\infty$ in (5.32), we obtain

$$\int_\Omega D_\xi \Psi(x,u,\nabla u)\nabla(\psi\exp(-Mu^+))dx + \int_\Omega D_s\Psi(x,u,\nabla u)\psi\exp(-Mu^+)dx$$

$$(5.33) \quad - \int_\Omega K(x)|u|^{q-2}u\psi\exp(-Mu^+)dx \geq 0.$$

Let $\varphi \in E$ with $\varphi \geq 0$. For $n \in \mathbb{N}$, define

$$u^{+n}(x) = \left\{ \begin{array}{ll} 0, & u(x) \leq 0 \\ u(x), & 0 < u(x) \leq n, \\ n, & u(x) > n, \end{array} \right.$$

and take $\psi = \varphi\exp(Mu^{+n})$ in (5.33). It follows that

$$\int_\Omega D_\xi\Psi(x,u,\nabla u)\nabla(\varphi\exp(-M(u^+ - u^{+n})))dx$$

$$+ \int_\Omega D_s\Psi(x,u,\nabla u)\varphi\exp(-M(u^+ - u^{+n}))dx$$

$$(5.34) \qquad - \int_\Omega K(x)|u|^{q-2}u\varphi\exp(-M(u^+ - u^{+n}))dx \geq 0.$$

Letting $n \to +\infty$ in (5.34), we deduce that, for any $\varphi \in E$ with $\varphi \geq 0$,

$$(5.35) \quad \int_\Omega D_\xi\Psi(x,u,\nabla u)\nabla\varphi dx + \int_\Omega D_s\Psi(x,u,\nabla u)\varphi dx - \int_\Omega K(x)|u|^{q-2}u\varphi dx \geq 0.$$

Repeating the argument above with test function $\varphi_n = \psi H(u_n/T)\exp(-Mu_n^-)$, where $u_n^- = u_n^+ - u_n$, we obtain, for any $\varphi \in E$ with $\varphi \geq 0$,

$$(5.36) \quad \int_\Omega D_\xi\Psi(x,u,\nabla u)\nabla\varphi dx + \int_\Omega D_s\Psi(x,u,\nabla u)\varphi dx - \int_\Omega K(x)|u|^{q-2}u\varphi dx \leq 0.$$

By (5.35) and (5.36), we have $|J'(u)| = 0$. Therefore, $(\mathbf{A_2})$ holds for J.

Finally, by (5.11), we have

$$J(u) \leq \frac{L}{p}\|u\|^p - \frac{1}{q}\int_\Omega K(x)|u|^q dx, \quad u \in X.$$

Since $1 < q < p$, by (5.10), for any $k \in \mathbb{N}$, if X_k is a k-dimensional subspace of $C_0^\infty(\Omega)$ and $\rho_k > 0$ is sufficiently small, then $\sup_{X_k \cap S_\rho} J < 0$, where $S_\rho = \{u \in X \mid \|u\| = \rho\}$. Therefore, J satisfies $(\mathbf{A_1})$–$(\mathbf{A_5})$ in Theorem 1.1. By Theorem 1.1, J has infinitely many critical points $\{u_k\}$ satisfying $u_k \neq 0$ for all k and $u_k \rightharpoonup 0$ as $k \to \infty$. $\qquad\qquad \Box$

Remark 5.3. Let us remark that the assumptions on K in Theorem 5.2 are not the weakest assumptions. For the example of (5.7), the assumption ($\mathbf{K_2}$) can be weakened as $\operatorname{essinf}_{x\in\Omega}K(x) \geq 0$. In fact, if $K = 0$ a.e. on a subset $\tilde{\Omega} \subset \Omega$ having positive measure, then all u with $u = 0$ on $\Omega\setminus\tilde{\Omega}$ and $|Du| \leq 2$ a.e. is a solution of (5.1), and in this case we have trivially a continuum of solutions, while if $\operatorname{essinf}_{x\in\Omega}K(x) > 0$, then the existence of infinitely many solutions is not a trivial problem but can be derived from Theorem 5.2.

6. Application 2: Homoclinic orbits of second-order Hamiltonian systems. In this section, we consider the second-order Hamiltonian system

$$(6.1) \qquad \ddot{u} + \nabla_u V(t, u) = 0,$$

where $t \in \mathbb{R}$ and the function $V : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ satisfies the following assumptions.

($\mathbf{H_1}$) $V(t, u) = -K(t, u) + W(t, u)$, $\nabla_u V$ exists, and $\nabla_u V \in C(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}^n)$.

($\mathbf{H_2}$) There are constants $\tau_1, \tau_2 > 0$ such that for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$,

$$(6.2) \qquad \tau_1|u|^2 \leq K(t, u) \leq \tau_2|u|^2.$$

($\mathbf{H_3}$) There exists $t_0 \in \mathbb{R}$ and $\varsigma > 0$ such that uniformly in $t \in (t_0 - \varsigma, t_0 + \varsigma)$,

$$(6.3) \qquad \lim_{|u|\to 0} \frac{W(t, u)}{|u|^2} = +\infty.$$

($\mathbf{H_4}$) There exists $\vartheta \in (1, 2)$ such that for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$,

$$(6.4) \qquad |W(t, u)| \leq b(t)|u|^\vartheta,$$

where $b : \mathbb{R} \to \mathbb{R}^+$ is a function such that $b \in L^\xi(\mathbb{R})$ for some $1 \leq \xi \leq 2/(2 - \vartheta)$.

($\mathbf{H_5}$) For all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$, $V(t, -u) = V(t, u)$.

We say that a classical solution $u \in C^2(\mathbb{R}, \mathbb{R}^n)$ of (6.1) is a homoclinic (to 0) solution if $u(t) \to 0$ and $u'(t) \to 0$ as $t \to \infty$. Moreover, if $u(t) \not\equiv 0$, $u(t)$ is called a nontrivial homoclinic solution.

THEOREM 6.1. *Under the assumptions* ($\mathbf{H_1}$)$-$($\mathbf{H_5}$)*, (6.1) possesses infinitely many nontrivial homoclinic solutions.*

The functional corresponding to (6.1) is

$$(6.5) \qquad J(u) = \frac{1}{2}\int_{\mathbb{R}}|u'|^2 dt + \int_{\mathbb{R}} K(t, u)dt - \int_{\mathbb{R}} W(t, u)dt, \quad u \in X := H^1(\mathbb{R}),$$

where

$$H^1(\mathbb{R}) = \left\{ u : \mathbb{R} \to \mathbb{R}^n \ \Big| \ \int_{\mathbb{R}}|u'|^2 dt + \int_{\mathbb{R}}|u|^2 dt < +\infty \right\}$$

is the standard Sobolev space with the inner product and the norm

$$(u, v) = \int_{\mathbb{R}} u' \cdot v' dt + \int_{\mathbb{R}} u \cdot v dt, \quad \|u\| = (u, u)^{1/2},$$

respectively.

Under the assumptions ($\mathbf{H_1}$)–($\mathbf{H_5}$), J is in $C(X)$ but is not in $C^1(X)$. A typical example is the following:

$$K(t, u) = \frac{1}{2}\left(1 + \frac{1}{2}\sin\left(\frac{1}{|u|^\alpha}\right)\right)|u|^2, \quad W(t, u) = b(t)|u|^\vartheta,$$

where $0 < \alpha < 1$, $1 < \vartheta < 2$, $b \in C(\mathbb{R}, \mathbb{R}) \cap L^\xi(\mathbb{R})$ for some $1 \leq \xi \leq 2/(2 - \vartheta)$, $b \geq 0$, and $b \not\equiv 0$. For $u \in X$ and $v \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$, the derivative of J at u in the direction of v is

$$\langle J'(u), v \rangle = \int_{\mathbb{R}} u' \cdot v' dt + \int_{\mathbb{R}} u \cdot v dt + \frac{1}{2} \int_{\mathbb{R}} \sin\left(\frac{1}{|u|^\alpha}\right) u \cdot v dt$$

(6.6)
$$- \frac{\alpha}{4} \int_{\mathbb{R}} |u|^{-\alpha} \cos\left(\frac{1}{|u|^\alpha}\right) u \cdot v dt - \vartheta \int_{\mathbb{R}} b(t) |u|^{\vartheta - 2} u \cdot v dt.$$

There exist $u, v \in X$ such that the integral $\int_{\mathbb{R}} |u|^{-\alpha} \cos(|u|^{-\alpha}) u \cdot v dt$ is divergent, and therefore J is not a C^1 functional in X. Indeed, define, for $t \in \mathbb{R}$,

$$\kappa(t) = \frac{1}{1 + |t|^{\frac{1}{2-\alpha}}},$$

$$u(t) = (\kappa(t), 0, \ldots, 0), \quad v(t) = (\kappa(t) \cos(|\kappa(t)|^{-\alpha}), 0, \ldots, 0).$$

Then calculations show that $u, v \in X$ and $\int_{\mathbb{R}} |u|^{-\alpha} \cos(|u|^{-\alpha}) u \cdot v dt = \infty$.

To appeal to the classical Clark theorem, which is in the framework of C^1 functionals, the following assumption on K is imposed by many authors in their works on (6.1) (see, for example, [27]).

$(\mathbf{H_2'})$ $K(t, u) = (L(t)u) \cdot u$, where $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a symmetric and positive definite matrix for all $t \in \mathbb{R}$ and there are constants $0 < \tau_1 < \tau_2$ such that for all $(t, u) \in \mathbb{R} \times \mathbb{R}^n$,
$$\tau_1 |u|^2 \leq (L(t)u) \cdot u \leq \tau_2 |u|^2.$$

It is easy to verify that if V satisfies $(\mathbf{H_1})$, $(\mathbf{H_2'})$, and $(\mathbf{H_3})$–$(\mathbf{H_5})$, then J is a C^1 functional in X and satisfies the Palais–Smale condition, and the existence of infinitely many homoclinic orbits can be derived from the classical Clark theorem. However, since no growth constraints are imposed on $\nabla_u V$ in the setting of $(\mathbf{H_1})$–$(\mathbf{H_5})$, the functional J defined in (6.5) is neither in C^1 nor does it satisfy the Palais–Smale condition.

Although a variant of Clark's theorem for nonsmooth functionals was recently proved in [21], the E-differentiable functional in [21, Theorem 3.9] still needs to satisfy the Palais–Smale condition, and therefore [21, Theorem 3.9] is not applicable here either. When K satisfies only the "pinching" condition (6.2), as far as we know, there are no known results on the existence of multiple homoclinic orbits for (6.1).

Proof of Theorem 6.1. Let $E = H^1(\mathbb{R}) \cap C_0^\infty(\mathbb{R})$. Since $H^1(\mathbb{R})$ can be embedded continuously into $L^\infty(\mathbb{R})$, we deduce that under the assumption $(\mathbf{H_1})$, J is E-differentiable and, for any $u \in X$ and $h \in E$,

(6.7)
$$\langle J'(u), h \rangle = \int_{\mathbb{R}} u' \cdot h' dx + \int_{\mathbb{R}} \nabla_u V(t, u) \cdot h dx.$$

J is even as a consequence of $(\mathbf{H_5})$. By (6.2) and (6.4), if $\xi = 1$, then

$$J(u) \geq \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dt + \tau_1 \int_{\mathbb{R}} |u|^2 dt - \|u\|_{L^\infty(\mathbb{R})}^\vartheta \int_{\mathbb{R}} |b(t)| dt$$

$$\geq \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dt + \tau_1 \int_{\mathbb{R}} |u|^2 dt - \|u\|^\vartheta \int_{\mathbb{R}} |b(t)| dt$$

(6.8)
$$\geq \min\left\{\frac{1}{2}, \tau_1\right\} \|u\|^2 - \|u\|^\vartheta \int_{\mathbb{R}} |b(t)| dt,$$

where the inequality $\|u\|_{L^\infty(\mathbb{R})} \leq \|u\|$ has been used. If $1 < \xi \leq 2/(2 - \vartheta)$, then $\vartheta\xi/(\xi - 1) \geq 2$ and we have

$$J(u) \geq \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dt + \tau_1 \int_{\mathbb{R}} |u|^2 dt - \left(\int_{\mathbb{R}} |b(t)|^\xi dt \right)^{1/\xi} \left(\int_{\mathbb{R}} |u|^{\frac{\vartheta\xi}{\xi-1}} dt \right)^{\frac{\xi-1}{\xi}}$$

$$\geq \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dt + \tau_1 \int_{\mathbb{R}} |u|^2 dt - \|u\|^\vartheta \left(\int_{\mathbb{R}} |b(t)|^\xi dt \right)^{1/\xi}$$

$$(6.9) \qquad \geq \min\left\{ \frac{1}{2}, \tau_1 \right\} \|u\|^2 - \|u\|^\vartheta \left(\int_{\mathbb{R}} |b(t)|^\xi dt \right)^{1/\xi},$$

where the inequality $\|u\|_{L^{\vartheta\xi/(\xi-1)}(\mathbb{R})} \leq \|u\|$ has been used. It follows from (6.8) and (6.9) that J is coercive and bounded from below. Therefore, J satisfies the conditions $(\mathbf{A_1})$ and $(\mathbf{A_4})$.

Let $u_n \rightharpoonup u$ in X and $h \in E$. Then

$$(6.10) \qquad \int_{\mathbb{R}} u'_n \cdot h' dt + \int_{\mathbb{R}} u_n \cdot h dt \to \int_{\mathbb{R}} u' \cdot h' dt + \int_{\mathbb{R}} u \cdot h dt.$$

Since $h \in C_0^\infty(\mathbb{R})$, by the Lebesgue convergence theorem, we have

$$(6.11) \qquad -\int_{\mathbb{R}} u_n \cdot h dt + \int_{\mathbb{R}} \nabla_u V(t, u_n) \cdot h dt \to -\int_{\mathbb{R}} u \cdot h dt + \int_{\mathbb{R}} \nabla_u V(t, u) \cdot h dt.$$

Combining (6.10) and (6.11) yields

$$(6.12) \qquad \langle J'(u_n), h_n \rangle \to \langle J'(u), h \rangle.$$

Therefore, the condition $(\mathbf{A'_2})$ in Remark 1.3 holds for J. Moreover, if $u_n \rightharpoonup u$ in X, then by [25, Theorem 1.6] we have

$$(6.13) \qquad \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}} |u'_n|^2 dt \geq \frac{1}{2} \int_{\mathbb{R}} |u'|^2 dt.$$

Applying Fatou's lemma and using (6.2) leads to

$$(6.14) \qquad \liminf_{n \to \infty} \frac{1}{2} \int_{\mathbb{R}} K(t, u_n) dt \geq \frac{1}{2} \int_{\mathbb{R}} K(t, u) dt,$$

while (6.4) implies

$$(6.15) \qquad \lim_{n \to \infty} \int_{\mathbb{R}} W(t, u_n) dt = \int_{\mathbb{R}} W(t, u) dt.$$

By (6.13)–(6.15), we see that J satisfies the condition $(\mathbf{A_3})$.

Finally, as in the proof of [21, Theorem 3.1], by (6.3), we deduce that the functional (6.5) satisfies the condition $(\mathbf{A_5})$. Then by Theorem 1.1 and Remark 1.3, (6.1) has infinitely many homoclinic solutions. $\square$

7. Appendix. In this appendix, we prove the following lemma, which is an adaptation of [22, Lemma 2.1] and which is used in the proof of Theorem 5.2.

LEMMA 7.1. *Let $\Phi, \Phi_k \in C^1(\mathbb{R}^N, \mathbb{R})$ be convex functions satisfying that $\Phi_k(\zeta) \to \Phi(\zeta)$ and $\nabla\Phi_k(\zeta) \to \nabla\Phi(\zeta)$ for any $\zeta \in \mathbb{R}^N$ as $k \to \infty$, and that there exists $\gamma > 1$ such that*

$$(7.1) \qquad \liminf_{|\zeta| \to \infty} \frac{\Phi_k(\zeta)}{|\zeta|^\gamma} > 0$$

holds uniformly for k. Let $\xi, \xi_k \in \mathbb{R}$ be such that

$$(7.2) \qquad \lim_{k \to \infty} (\nabla\Phi_k(\xi_k) - \nabla\Phi(\xi)) \cdot (\xi_k - \xi) = 0.$$

Then $\{\nabla\Phi_k(\xi_k)\}$ converges to $\nabla\Phi(\xi)$.

Proof. First, we prove that $\{\xi_k\}$ is bounded in $\mathbb{R}^N$. Note that the convexity of Φ_k implies

$$\Phi_k(\xi_k) \le \Phi_k(\xi) + \nabla\Phi_k(\xi_k) \cdot (\xi_k - \xi)$$
$$= \Phi_k(\xi) + \nabla\Phi(\xi) \cdot (\xi_k - \xi) + (\nabla\Phi_k(\xi_k) - \nabla\Phi(\xi)) \cdot (\xi_k - \xi).$$

If $\{\xi_k\}$ were unbounded, using (7.2) and the fact that $\lim_{k \to \infty} \Phi_k(\xi) = \Phi(\xi)$, we would have

$$\limsup_{k \to \infty} \frac{\Phi_k(\xi_k)}{|\xi_k|^\gamma} \le 0,$$

which contradicts (7.1).

Now we use an argument borrowed from the proof of [22, Lemma 2.1]. Define $\hat{\Phi}_k(\zeta) = \Phi_k(\zeta) - \nabla\Phi_k(\xi) \cdot \zeta$ and $\hat{\Phi}(\zeta) = \Phi(\zeta) - \nabla\Phi(\xi) \cdot \zeta$ for $\zeta \in \mathbb{R}$. Then, since $\{\xi_k\}$ is bounded, $\hat{\Phi}_k$ and $\hat{\Phi}$ satisfy all the assumptions of the present lemma. Moreover, ξ is a minimizer of both $\hat{\Phi}_k$ and $\hat{\Phi}$. Therefore, we may assume, without loss of generality, that ξ is a minimizer of both Φ_k and Φ, and we need to prove that $\lim_{k \to \infty} \nabla\Phi_k(\xi_k) = 0$.

Assume by contradiction that there exists $\delta > 0$ such that, up to a subsequence, $|\nabla\Phi_k(\xi_k)| > \delta$ for every $k \in \mathbb{N}$. Let $t_k \in (0,1)$ be such that $|\nabla\Phi_k((1-t_k)\xi + t_k\xi_k)| = \delta$ and let $\zeta_k = (1-t_k)\xi + t_k\xi_k$. Up to a subsequence, $\nabla\Phi_k(\zeta_k)$ is convergent to some $\alpha \in \mathbb{R}^N$ with $|\alpha| = \delta$. Since

$$0 \le (\nabla\Phi_k(\xi_k) - \nabla\Phi_k(\zeta_k)) \cdot (\xi_k - \zeta_k) = (1-t_k)(\nabla\Phi_k(\xi_k) - \nabla\Phi_k(\zeta_k)) \cdot (\xi_k - \xi),$$

one has

$$0 \le \nabla\Phi_k(\zeta_k) \cdot (\zeta_k - \xi) = t_k \nabla\Phi_k(\zeta_k) \cdot (\xi_k - \xi) \le t_k \nabla\Phi_k(\xi_k) \cdot (\xi_k - \xi).$$

This, together with (7.2), implies

$$(7.3) \qquad \lim_{k \to \infty} \nabla\Phi_k(\zeta_k) \cdot (\zeta_k - \xi) = 0.$$

Taking the limit in the inequality

$$\Phi_k(\xi) \ge \Phi_k(\zeta_k) + \nabla\Phi_k(\zeta_k) \cdot (\xi - \zeta_k)$$

and using (7.3), since $\lim_{k \to \infty} \Phi_k(\xi) = \Phi(\xi)$ and since ξ is a minimizer of Φ_k, we see that

$$(7.4) \qquad \lim_{k \to \infty} \Phi_k(\zeta_k) = \Phi(\xi).$$

For every $\eta \in \mathbb{R}^N$, passing to the limit as $k \to \infty$ in the inequality

$$\Phi_k(\eta) \geq \Phi_k(\zeta_k) + \nabla\Phi_k(\zeta_k) \cdot (\eta - \zeta_k) = \Phi_k(\zeta_k) + \nabla\Phi_k(\zeta_k) \cdot (\eta - \xi) + \nabla\Phi_k(\zeta_k) \cdot (\xi - \zeta_k)$$

and using (7.3), (7.4), and the fact that $\lim_{k\to\infty} \nabla\Phi_k(\zeta_k) = \alpha$, we arrive at

$$\Phi(\eta) \geq \Phi(\xi) + \alpha \cdot (\eta - \xi) \quad \forall \eta \in \mathbb{R}^N.$$

Since $\alpha \neq 0$, this contradicts the fact that ξ is a minimizer of Φ and Φ is of class C^1. □

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