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FORMATION OF RADIAL PATTERNS VIA MIXED ATTRACTIVE AND REPULSIVE INTERACTIONS FOR SCHRÖDINGER SYSTEMS*

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Dedicated to Paul H. Rabinowitz on the occasion of his 80th birthday

Abstract. This paper is concerned with the asymptotic behavior of least energy vector solutions for nonlinear Schrödinger systems with mixed couplings of attractive and repulsive forces. We focus here on the radially symmetric case while the general studies were already conducted in our earlier work [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Math. Pures Appl.* (9), 106 (2016), pp. 477–511], [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Fixed Point Theory Appl.*, 19 (2017), pp. 559–583]. Though there is still the general phenomenon of component-wise pattern formation with co-existence of partial synchronization and segregation for positive least energy vector solutions as in [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Math. Pures Appl.* (9), 106 (2016), pp. 477–511], [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Fixed Point Theory Appl.*, 19 (2017), pp. 559–583], in our case of radially symmetric domains, it turns out that the energy of synchronization part may be concentrated either on the center of the domain or on the boundary of the domain depending on the spatial dimension of the domain. This is a distinct new feature from [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Math. Pures Appl.* (9), 106 (2016), pp. 477–511], [J. Byeon, Y. Sato, and Z.-Q. Wang, *J. Fixed Point Theory Appl.*, 19 (2017), pp. 559–583] due to the radially symmetric property. Our approach develops techniques of multiscale asymptotic estimates.

Key words. formation of patterns, mixed couplings, coupled Schrödinger equations

AMS subject classifications. 35J57, 35J50, 35J60, 35B40

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1. Introduction.

In this paper, we are concerned with the existence and asymptotic behavior of radially symmetric positive vector solutions for the following coupled nonlinear Schrödinger system:

(1.1)
$$-\Delta u_i + \lambda_i u_i = \mu_i u_i^3 + \sum_{j\neq i}^{3} \beta_{ij} u_i u_j^2, \quad u_i > 0 \quad \text{in} \quad \Omega, \quad u_i \in H_0^1(\Omega) \quad (i = 1, 2, 3).$$

Here Ω is the ball centered at the origin with radius R_0 in $\mathbb{R}^n$ for $n \leq 3$.

We continue our study on the qualitative effects of large mixed coupling on the system (1.1). More precisely, we examine the case of large $\beta_{12} > 0, \beta_{13} < 0, \beta_{23} < 0$ when the domain is a ball. We are interested in the positive least energy vector

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solutions of (1.1) in the class of radially symmetric functions. The existence of a least energy vector solution can be derived from the arguments in [7, 8]. Our concern in this paper is the asymptotic behavior of the positive least energy solution in the class of radial functions when $|\beta_{i,j}|$ is large for $i \neq j$ and $\beta_{12} > 0, \beta_{13} < 0, \beta_{23} < 0$. We will show that there is an aggregation phenomenon for the first and second components and segregation between the first two components and the third component. As $|\beta_{i,j}|$ is getting larger, the first two components are getting smaller. Even if the first two components are small, they concentrate on a sphere with a radius $r_0 \geq 0$. We will see that $r_0 = R_0$ for $n = 1, 2$ and $r_0 = 0$ for $n = 3$, that is, the concentration behavior strongly depends on the spatial dimension n.

The systems considered in this paper arise when we study standing wave solutions of the time-dependent m-coupled Schrödinger systems

$$(1.2)\qquad \begin{cases} -i\frac{\partial}{\partial t}\Phi_j = \Delta\Phi_j - V_j(y)\Phi_j + \mu_j|\Phi_j|^2\Phi_j + \Phi_j \sum\limits_{i=1, i\neq j}^{m} \beta_{ij}|\Phi_i|^2 \quad \text{in } \mathbb{R}^n, \\ \Phi_j = \Phi_j(y,t) \in \mathbb{C},\ t > 0,\ j = 1, \ldots, m. \end{cases}$$

These systems of equations, also known as coupled Gross–Pitaevskii equations, have applications in Bose–Einstein condensates theory for multispecies Bose–Einstein condensates (see [9, 28]). Physically, β_{jj} and β_{ij} $(i \neq j)$ are the intraspecies and interspecies scattering lengths, respectively. The sign of the scattering length determines whether the interactions of states are repulsive or attractive. In the attractive case ($\beta_{ij} > 0$ for $i \neq j$) the components tend to be synchronized while in the repulsive case ($\beta_{ij} < 0$ for $i \neq j$) the components segregate componentwisely, leading to much more complicated behaviors of solutions. Various aspects of mathematical analysis on coupled nonlinear Schödinger equations have progressed extensively in recent years, e.g., [1, 2, 3, 4, 6, 7, 8, 9, 10, 11, 14, 16, 17, 19, 20, 21, 22, 24, 26, 27, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44] and references therein. Most of these works have been done for the purely attractive case (i.e., all coupling constants are positive) or for the purely repulsive case (i.e., all coupling constants are negative). In the attractive case, solutions tend to go synchronization componentwise, leading to simpler structure of positive solutions (e.g., [1, 3, 4, 16, 17, 20, 21, 22, 24, 27, 41, 44] for more details). For the repulsive case, phase separation has been proved in several works causing more complicated solution structure of multiple positive solutions (e.g., [2, 9, 10, 11, 19, 35, 36, 37, 38, 39, 40, 42, 43]).

In terms of these previous studies, it is quite natural to consider the case of mixed couplings, i.e., there are both attractive and repulsive couplings simultaneously. Starting with an interesting case study of mixed coupling in [16], there have been results which appeared in [7, 8, 16, 32, 33, 35, 36]. In [16], Lin and Wei showed the existence of a bound state solution for a system of three equations with mixed couplings (two positive coupling and one negative coupling constants), and they demonstrated that the solution possesses component-segregation asymptotically when the couplings are small of different scales. For a general m system with mixed couplings, Soave in [35] established the existence of solutions with at least k positive components for every $k \leq m$. In [36], Soave and Tavares gave general conditions for the existence and symmetry result for least energy solutions with simultaneous mixed couplings on bounded domains or the whole space. In [32, 33], Sato and Wang proved the existence and the asymptotic behavior of least energy solution for the three coupled systems on the bounded domain with large attractive coupling constant. Furthermore, Byeon, Sato, and Wang showed the new interesting componentwise asymptotic patter formations including co-existence of partial synchronization and segregation, when not only the attractive constant but also repulsive constant are large, under Dirichlet condition in [7] and Neumann condition in [8]. The current work continues the studies in this direction. As we show for the radially symmetric solutions, the asymptotic behavior is more delicate with spatial dimensions. For $n = 1, 2$, we see that for large mixed couplings, the third component concentrates at the origin while the other two components develop into a small synchronized boundary layer concentration. For $n = 3$, the first two components also concentrate at the origin with synchronized small peaks. We expect that the energy estimation for the synchronization part in this paper would provide a motivation to study concentration behavior on higher dimensional manifolds, for example, interior spheres.

From now on, we consider a domain of ball with radius R_0,

$$\Omega \equiv B_{R_0}(0) = \{x \in \mathbb{R}^n \mid |x| < R_0\}.$$

Since we are interested in the case that two coupling constants are repulsive and one coupling constant is attractive and that $|\beta_{ij}|$ is large for $i \neq j$, we rewrite μ_j by β_{jj}, β_{12} by $\alpha\beta_{12}$, β_{13} by $-\beta\beta_{13}$, and β_{23} by $-\beta\beta_{23}$, and consider the following system:

$$(1.3)\quad \begin{cases} \frac{d^2u_1}{dr^2} + \frac{(n-1)}{r}\frac{du_1}{dr} - \lambda_1 u_1 + \beta_{11}(u_1)^3 + \alpha\beta_{12}u_1(u_2)^2 - \beta\beta_{13}u_1(u_3)^2 = 0, \\ \frac{d^2u_2}{dr^2} + \frac{(n-1)}{r}\frac{du_2}{dr} - \lambda_2 u_2 + \alpha\beta_{21}(u_1)^2u_2 + \beta_{22}(u_2)^3 - \beta\beta_{23}u_2(u_3)^2 = 0, \\ \frac{d^2u_3}{dr^2} + \frac{(n-1)}{r}\frac{du_3}{dr} - \lambda_3 u_3 - \beta\beta_{31}(u_1)^2u_3 - \beta\beta_{32}(u_2)^2u_3 + \beta_{33}(u_3)^3 = 0, \\ u_i(r) > 0 \ \text{ for } r < R_0, \\ u_1(R_0) = u_2(R_0) = u_3(R_0) = 0, \end{cases}$$

where $\beta_{ij} = \beta_{ji} > 0$ for $1 \le i, j \le 3$, $\alpha, \beta > 0$. In this paper we need a control of multiscale convergence rates for α, $\beta \to \infty$. We will use the following notation: for $a, b \in \mathbb{R}$,

$$\alpha^a \# \beta^b \to \infty \text{ if and only if } \alpha, \beta \to \infty \text{ with } \lim_{\alpha,\beta\to\infty} \frac{\alpha^a}{\beta^b} = \infty.$$

When we say $\alpha^a \# \beta^b$ is large, it means that all three quantities $\alpha, \beta, \frac{\alpha^a}{\beta^b}$ are large. To state our results we introduce some notation and limiting problems. Let us define

$$\mathbf{H}_r(\Omega) \equiv (H^1_{0,r}(\Omega))^3, \quad \text{where } H^1_{0,r}(\Omega) = \{u \in H^1_0(\Omega) \mid u(x) = u(|x|) \ \text{ for all } x \in \Omega\}.$$

For $p \ge 1$ and $\lambda > 0$, we use the following notation:

$$|u|^p_{p,\Omega} = \int_\Omega |u|^p\, dx, \ \ ||u||^2_{\lambda,\Omega} = |\nabla u|^2_{2,\Omega} + \lambda |u|^2_{2,\Omega}, \ \ \|\vec{u}\| = \left(\sum_{i=1}^3 ||u_i||^2_{\lambda_i,\Omega}\right)^{\frac{1}{2}}.$$

We define the energy functional for the system (1.3),

$$I(\vec{u}) = \sum_{i=1}^3 \left(\frac{1}{2}||u_i||^2_{\lambda_i,\Omega} - \frac{1}{4}\beta_{ii}|u_i|^4_{4,\Omega}\right) - \frac{1}{2}(\alpha\beta_{12}|u_1u_2|^2_{2,\Omega} - \beta\beta_{13}|u_1u_3|^2_{2,\Omega} - \beta\beta_{23}|u_2u_3|^2_{2,\Omega})$$

for $\vec{u} = (u_1, u_2, u_3) \in \mathbf{H}_r(\Omega)$. Then critical points of I correspond to solutions of the system (1.3). A solution $\vec{u} = (u_1, u_2, u_3)$ is called a vector solution if $u_i \neq 0$ for each

$i = 1, 2, 3$. A positive least energy vector solution of (1.3) is a vector solution whose components are all positive and whose energy is the least among all vector solutions.

It is well known that the following equation has a positive least energy solution:

$$\Delta u - \lambda_3 u + \beta_{33}(u)^3 = 0, \quad u > 0 \text{ in } \Omega, \quad u = 0 \text{ on } \partial\Omega. \tag{1.4}$$

We denote the least energy level for (1.4) by L. It is well known that for a ball $\Omega = B_R(0)$, the positive solution of (1.4) is unique [15]. We will show that the third component of the least energy solution of (1.3) converges to the positive solution of a limit problem (1.4). On the other hand, the first and second components have a different type of convergence and the corresponding limit problem depends on the space dimension n as follows (here, we will use t as the one dimensional variable for the limiting problem).

(1) $n = 1, 2$ (concentration on the boundary). Given the positive least energy solution U_3 of (1.4), we consider the following problem:

$$\begin{cases} \frac{d^2 v_1}{dt^2} - \beta_{13} \left(\frac{\partial U_3}{\partial r}(R_0)\right)^2 t^2 v_1 + \beta_{12} v_1 (v_2)^2 = 0, \ v_1 > 0 & \text{in } \mathbb{R}^1_+, \\ \frac{d^2 v_2}{dt^2} - \beta_{23} \left(\frac{\partial U_3}{\partial r}(R_0)\right)^2 t^2 v_2 + \beta_{21} (v_1)^2 v_2 = 0, \ v_2 > 0 & \text{in } \mathbb{R}^1_+, \\ v_i(0) = 0, \ \lim_{|x|\to\infty} v_i(x) = 0, \ i = 1, 2. \end{cases} \tag{1.5}$$

Here we note that the term $\frac{(n-1)}{t}\frac{d}{dt}$ does not appear in the equation above since (1.5) would be a limiting equation for (3.29), and the first derivative term for the solution in (3.29) would vanish as $\beta \to \infty$.

The corresponding energy functional B_{U_3} is defined by

$$\begin{aligned} &B_{U_3}(v_1, v_2) \\ &\equiv \frac{1}{2} \int_{\mathbb{R}^1_+} \left|\frac{dv_1}{dt}\right|^2 + \left|\frac{dv_2}{dt}\right|^2 + t^2 \left(\frac{\partial U_3}{\partial r}(R_0)\right)^2 (\beta_{13} v_1^2 + \beta_{23} v_2^2) - \beta_{12} v_1^2 v_2^2 \, dt \end{aligned} \tag{1.6}$$

for $(v_1, v_2) \in H^1_b$, where H^1_b is defined by the completion of $C_0^\infty(\mathbb{R}^1_+)$ with respect to the norm

$$\|(v_1, v_2)\|_b^2 \equiv \int_{\mathbb{R}^1_+} \left|\frac{dv_1}{dt}\right|^2 + \left|\frac{dv_2}{dt}\right|^2 + t^2 (\beta_{13} v_1^2 + \beta_{23} v_2^2) \, dt.$$

We note that for $\breve{v} = (v_1, v_2)$

$$B'_{U_3}(\breve{v})(\breve{v}) = \int_{\mathbb{R}^1_+} \left|\frac{dv_1}{dt}\right|^2 + \left|\frac{dv_2}{dt}\right|^2 + t^2 \left(\frac{\partial U_3}{\partial r}(R_0)\right)^2 (\beta_{13} v_1^2 + \beta_{23} v_2^2) - 2\beta_{12} v_1^2 v_2^2 \, dt.$$

Then, we consider the following minimization problem:

$$M_b = M_b(U_3) \equiv \inf\{B_{U_3}(\breve{v}) \mid B'_{U_3}(\breve{v})(\breve{v}) = 0, \breve{v} \in H^1_b \setminus \{(0, 0)\}\}. \tag{1.7}$$

In view of [7, Proposition 4], we see that there exists a minimizer $(v_{1,b}, v_{2,b})$ of (1.7) which is a least energy solution of (1.5). Moreover, there exist constants $c, C > 0$ satisfying

$$0 < v_{i,b}(t) \le C e^{-ct^2} \quad \text{for any} \quad t > 0, \ i = 1, 2. \tag{1.8}$$

By the scaling $v_{i,b}(t) = \sqrt{\left|\frac{\partial U_3}{\partial r}(R_0)\right|}v_i^0(\sqrt{\left|\frac{\partial U_3}{\partial r}(R_0)\right|}t)$, $i = 1, 2$, we see that

$$M_b = M_b(U_3) = \left|\frac{\partial U_3}{\partial r}(R_0)\right|^{\frac{3}{2}} M_b^0, \tag{1.9}$$

where M_b^0 is the energy of a least energy solution $(v_{1,b}^0, v_{2,b}^0) \in H_b^1(\mathbb{R}_+^1)$ of the following normalized problem:

$$\begin{cases} \frac{d^2 v_{1,b}^0}{dt^2} - \beta_{13}t^2 v_{1,b}^0 + \beta_{12}v_{1,b}^0(v_{2,b}^0)^2 = 0,\ v_{1,b}^0 > 0 & \text{in } \mathbb{R}_+^1, \\ \frac{d^2 v_{2,b}^0}{dt^2} - \beta_{23}t^2 v_{2,b}^0 + \beta_{21}(v_{1,b}^0)^2 v_{2,b}^0 = 0,\ v_{2,b}^0 > 0 & \text{in } \mathbb{R}_+^1, \\ v_{i,b}^0(0) = 0,\ \lim_{|x|\to\infty} v_{i,b}^0(x) = 0,\ i = 1, 2. \end{cases} \tag{1.10}$$

(2) $n = 3$ (concentration on the center). For the positive least energy solution U_3 of (1.4), we consider the following problem:

$$\begin{cases} \Delta v_1 - \beta_{13}\left(U_3(0)\right)^2 v_1 + \beta_{12}v_1(v_2)^2 = 0,\ v_1 > 0, \\ \Delta v_2 - \beta_{23}\left(U_3(0)\right)^2 v_2 + \beta_{21}(v_1)^2 v_2 = 0,\ v_2 > 0, \\ \qquad v_1, v_2 \in H_{0,r}^1(\mathbb{R}^n). \end{cases} \tag{1.11}$$

The corresponding energy functional C_{U_3} is defined by

$$C_{U_3}(v_1, v_2) \equiv \frac{1}{2}\int_{\mathbb{R}^n} \left(|\nabla v_1|^2 + |\nabla v_2|^2 + \left(U_3(0)\right)^2 (\beta_{13}v_1^2 + \beta_{23}v_2^2) - \beta_{12}v_1^2 v_2^2\right) dy.$$

We also note that for $\breve{v} = (v_1, v_2) \in H_c^1 \equiv (H_{0,r}^1(\mathbb{R}^n))^2$,

$$C'_{U_3}(\breve{v})(\breve{v}) \equiv \int_{\mathbb{R}^n} \left(|\nabla v_1|^2 + |\nabla v_2|^2 + \left(U_3(0)\right)^2 (\beta_{13}v_1^2 + \beta_{23}v_2^2) - 2\beta_{12}v_1^2 v_2^2\right) dy.$$

We consider the following minimization problem:

$$M_c = M_c(U_3) \equiv \inf\{C_{U_3}(\breve{v}) \mid C'_{U_3}(\breve{v})(\breve{v}) = 0,\quad \breve{v} \in H_c^1 \setminus \{(0, 0)\}\}. \tag{1.12}$$

In view of [8, Proposition 4], there exists a minimizer $(v_{1,c}, v_{2,c})$ of (1.12) which is a least energy solution of (1.11) and radially symmetric up to a translation. Moreover, there are constants $c, C > 0$ satisfying

$$0 < v_{i,c}(y) \le Ce^{-c|y|} \quad \text{for any} \quad y \in \mathbb{R}^n,\ i = 1, 2. \tag{1.13}$$

By the scaling $v_{i,c}(y) = U_3(0)v_{i,c}^0(U_3(0)y)$, $i = 1, 2$, we note that

$$M_c(U_3) = (U_3(0))^{4-n} M_c^0, \tag{1.14}$$

where M_c^0 is the energy of a least energy solution of the following normalized problem:

$$\begin{cases} \Delta v_{1,c}^0 - \beta_{13}v_{1,c}^0 + \beta_{12}v_{1,c}^0(v_{2,c}^0)^2 = 0,\ v_{1,c}^0 > 0, \\ \Delta v_{2,c}^0 - \beta_{23}v_{2,c}^0 + \beta_{21}(v_{1,c}^0)^2 v_{2,c}^0 = 0,\ v_{2,c}^0 > 0, \\ \qquad v_{1,c}^0, v_{2,c}^0 \in H_{0,r}^1(\mathbb{R}^n). \end{cases} \tag{1.15}$$

From now on, for any domain $O \subset \mathbb{R}^n$, any function $u \in H_0^1(O)$ will also be regarded as an element in $H^1(\mathbb{R}^n)$ by the zero extension u on $\mathbb{R}^n \setminus O$.

Throughout this paper, we often use t as the one dimensional variable for the limiting problem, $x = r\theta$, where $r = |x|$ and $\theta = \frac{x}{|x|}$, as the original variable in Ω, and y, $s = |y|$ as the scaled variables such that

$$\begin{cases} y &= \beta^{\frac{1}{4}}(R_0 - r)\theta, \ s = |y| \ \text{ if } \ n = 1, 2, \\ y &= \sqrt{\beta}x, \ s = |y| \qquad \text{ if } \ n = 3. \end{cases} \tag{1.16}$$

Our main result is the following theorem.

THEOREM 1. *We assume that $\Omega = B_{R_0}(0)$. We take any $\delta_1, \delta_2 \geq \frac{1}{4}$ for $n = 1, 2$ and any $\delta_3 > 0$ for $n = 3$. Then the following holds.*

(a) *There exists a constant $\alpha_0 > 0$, independent of $\beta > 0$ such that for any $\alpha \geq \alpha_0$, $\beta > 0$, the system (1.3) has a least energy vector solution $\vec{u}^{\alpha,\beta} = (u_1^{\alpha,\beta}, u_2^{\alpha,\beta}, u_3^{\alpha,\beta})$ satisfying*

$$\lim_{\alpha\to\infty} I(\vec{u}^{\alpha,\beta}) = L \ \textit{uniformly for } \beta > 0 \textit{ and}$$

$$\lim_{\alpha\to\infty} \|\vec{u}^{\alpha,\beta} - (0, 0, U_3)\| = 0 \quad \textit{uniformly for } \beta > 0,$$

where L is the least energy level of (1.4), and U_3 is the unique least energy solution of (1.4).

(b)

$$I(\vec{u}^{\alpha,\beta}) = \begin{cases} L + \frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1} + o(1)) \quad \textit{as } \alpha\#\beta^{1+\delta_n} \to \infty \quad \textit{if} \quad n = 1, 2, \\ L + \frac{\beta^{\frac{1}{2}}}{\alpha}(M_c + o(1)) \quad \textit{as } \alpha\#\beta^{1+\delta_n} \to \infty \quad \textit{if} \quad n = 3, \end{cases} \tag{1.17}$$

where $|S^{n-1}|$ is the area of the unit sphere.

(c) *For $n = 1, 2$ and $u_i^{\alpha,\beta}(x) = u_i^{\alpha,\beta}(|x|)$, we define*

$$v_{i,b}^{\alpha,\beta}(|y|) \equiv \frac{\sqrt{\alpha}}{\beta^{\frac{1}{4}}} u_i^{\alpha,\beta}\Big(R_0 - \frac{|y|}{\beta^{\frac{1}{4}}}\Big), \ i = 1, 2, \ \textit{for } |y| \in \Omega_{\beta,b} \equiv \Big\{|y| \in \mathbb{R} \ | \ 0 \leq |y| \leq \beta^{\frac{1}{4}} R_0\Big\}. \tag{1.18}$$

Then there is a positive least energy vector solution $(v_{1,b}, v_{2,b})$ of (1.5) such that $(v_{1,b}^{\alpha,\beta}, v_{2,b}^{\alpha,\beta}) \to (v_{1,b}, v_{2,b})$ in $[H^1(\mathbb{R})]^2$, up to a subsequence, as $\alpha\#\beta^{1+\delta_n} \to \infty$. Moreover, there are constants $c, C, D > 0$, independent of large $\alpha\#\beta^{1+\delta_n}$ such that for $i = 1, 2$,

$$D\frac{\beta^{\frac{1}{4}}}{\sqrt{\alpha}} \leq \max_{\Omega} u_i^{\alpha,\beta}, \quad u_i^{\alpha,\beta}(x) \leq C\frac{\beta^{\frac{1}{4}}}{\sqrt{\alpha}} \exp(-c(\beta^{\frac{1}{4}}(R_0 - |x|))), \quad 0 \leq |x| \leq R_0. \tag{1.19}$$

(d) *For $n = 3$, we define*

$$v_{i,c}^{\alpha,\beta}(y) = \sqrt{\frac{\alpha}{\beta}} u_i^{\alpha,\beta}\Big(\frac{y}{\sqrt{\beta}}\Big), \quad i = 1, 2, \quad \textit{for} \ \ y \in \Omega_{\beta,c} \equiv \Big\{y \in \mathbb{R}^n \ \Big| \ |y| \leq \sqrt{\beta}R_0\Big\}. \tag{1.20}$$

Then there is a positive least energy vector solution $(v_{1,c}, v_{2,c})$ of (1.11) such that

$$(v_{1,c}^{\alpha,\beta}, v_{2,c}^{\alpha,\beta}) \to (v_{1,c}, v_{2,c}) \quad \textit{in} \quad [H^1(\mathbb{R}^n)]^2, \textit{ up to a subsequence, as} \quad \alpha\#\beta^{1+\delta_n} \to \infty.$$

Moreover, there are constants $c, C, D > 0$, independent of large $\alpha\#\beta^{1+\delta_n}$ such that for $i = 1, 2$,

$$D\sqrt{\frac{\beta}{\alpha}} \leq \max_{\Omega} u_i^{\alpha,\beta}, \quad u_i^{\alpha,\beta}(x) \leq C\sqrt{\frac{\beta}{\alpha}} \exp(-c\sqrt{\beta}|x|), \quad 0 \leq |x| < R_0. \tag{1.21}$$

Theorem 1 indicates that the concentration region of the least energy vector solution for the synchronization part depends on the dimension of the domain. As we saw in [7], without the radial constraint, a concentration point of the first and second components moves to the boundary even though the Dirichlet boundary condition is imposed. In the radially symmetric case, if $n = 3$, the expense of the concentration of the first and second components on the sphere is much higher than the concentration at the center of a ball. In order to obtain the asymptotic behavior of the synchronization part, we divide the domain into two parts and use the different norms on each regions.

The paper is organized as follows. In section 2, we review some preliminaries to establish the existence results for (1.3). In section 3, we introduce a combined norm to obtain energy estimates depending on the dimension of the domain, and complete the proof of Theorem 1.

2. Preliminary. For $\vec{u} = (u_1, u_2, u_3) \in \mathbf{H}_r(\Omega)$, we define

$$
\begin{aligned}
(2.1)\qquad l(u_1,u_2,u_3) \equiv \frac{1}{2}\sum_{i=1}^{3}\left(|\nabla u_i|^2 + \lambda_i (u_i)^2 - \frac{1}{2}\beta_{ii}|u_i|^4\right)\\
-\frac{1}{2}\left(\alpha\beta_{12}|u_1u_2|^2 - \beta\beta_{13}|u_1u_3|^2 - \beta\beta_{23}|u_2u_3|^2\right).
\end{aligned}
$$

Recall the energy functional $I(\vec{u}) = \int_\Omega l(u_1, u_2, u_2)dx$ for $\vec{u} = (u_1, u_2, u_3) \in \mathbf{H}_r(\Omega)$, and define

$$
\begin{aligned}
\mathcal{D}_{\alpha,\beta} &\equiv \{\vec{u} \in \mathbf{H}_r(\Omega)\,|\, I'(\vec{u})(u_1,u_2,0) = 0, I'(\vec{u})(0,0,u_3) = 0, (u_1,u_2) \neq (0,0), u_3 \neq 0\},\\
\widetilde{\mathcal{D}}_{\alpha,\beta} &\equiv \{\vec{u} \in \mathbf{H}_r(\Omega)\,|\; I'(\vec{u})(u_1,u_2,0) \le 0, I'(\vec{u})(0,0,u_3) \le 0, (u_1,u_2) \neq (0,0), u_3 \neq 0\},\\
\widehat{\mathcal{D}}_{\alpha,\beta} &\equiv \{\vec{u} \in \mathbf{H}_r(\Omega)\,|\; \det A_{\alpha,\beta}(\vec{u}) > 0\},
\end{aligned}
$$

where the matrix $A_{\alpha,\beta}(\vec{u})$ is given by

(2.2)

$$
\begin{bmatrix}
\beta_{11}|u_1|^4_{4,\Omega} + \beta_{22}|u_2|^4_{4,\Omega} + 2\alpha\beta_{12}|u_1u_2|^2_{2,\Omega} & -(\beta\beta_{13}|u_1u_3|^2_{2,\Omega} + \beta\beta_{23}|u_2u_3|^2_{2,\Omega})\\
-(\beta\beta_{13}|u_1u_3|^2_{2,\Omega} + \beta\beta_{23}|u_2u_3|^2_{2,\Omega}) & \beta_{33}|u_3|^4_{4,\Omega}
\end{bmatrix}.
$$

LEMMA 2 (see [7, Lemma 5 and Remark 6]). (i) *For any $\vec{u} \in \widetilde{\mathcal{D}}_{\alpha,\beta} \cup \widehat{\mathcal{D}}_{\alpha,\beta}$, there exists a unique $(s_{\alpha,\beta}(\vec{u}), t_{\alpha,\beta}(\vec{u})) \in (0,1] \times (0,1]$ such that*

$$
\begin{aligned}
&(s_{\alpha,\beta}(\vec{u})u_1, s_{\alpha,\beta}(\vec{u})u_2, t_{\alpha,\beta}(\vec{u})u_3) \in \mathcal{D}_{\alpha,\beta},\\
&I(s_{\alpha,\beta}(\vec{u})u_1, s_{\alpha,\beta}(\vec{u})u_2, t_{\alpha,\beta}(\vec{u})u_3) = \max_{s,t>0} I(su_1, su_2, tu_3).
\end{aligned}
$$

(ii) $I(s_{\alpha,\beta}(\vec{u})u_1, s_{\alpha,\beta}(\vec{u})u_2, t_{\alpha,\beta}(\vec{u})u_3) = \frac{H(\vec{u})}{4G(\vec{u})}$, *where*

$$
\begin{aligned}
G(\vec{u}) &= (\beta_{11}|u_1|^4_{4,\Omega} + \beta_{22}|u_2|^4_{4,\Omega} + 2\alpha\beta_{12}|u_1u_2|^2_{2,\Omega})\beta_{33}|u_3|^4_{4,\Omega}\\
&\quad - (\beta\beta_{13}|u_1u_3|^2_{2,\Omega} + \beta\beta_{23}|u_2u_3|^2_{2,\Omega})^2,\\
H(\vec{u}) &= \beta_{33}|u_3|^4_{4,\Omega}(\|u_1\|^2_{\lambda_1,\Omega} + \|u_2\|^2_{\lambda_2,\Omega})^2\\
&\quad + 2(\beta\beta_{13}|u_1u_3|^2_{2,\Omega} + \beta\beta_{23}|u_2u_3|^2_{2,\Omega})(\|u_1\|^2_{\lambda_1,\Omega} + \|u_2\|^2_{\lambda_2,\Omega})\|u_3\|^2_{\lambda_3,\Omega}\\
&\quad + (\beta_{11}|u_1|^4_{4,\Omega} + \beta_{22}|u_2|^4_{4,\Omega} + 2\alpha\beta_{12}|u_1u_2|^2_{2,\Omega})\|u_3\|^4_{\lambda_3,\Omega}.
\end{aligned}
$$

We consider a minimization problem

$$
c_{\alpha,\beta} = \inf_{\vec{u}\in\mathcal{D}_{\alpha,\beta}} I(\vec{u}) = \inf_{\vec{u}\in\mathcal{D}_{\alpha,\beta}} \frac{1}{4}\left(||u_1||^2_{\lambda_1,\Omega} + ||u_2||^2_{\lambda_2,\Omega} + ||u_3||^2_{\lambda_3,\Omega}\right).
$$

With the aid of [29, 5], we argue as in [7] to have the following proposition.

PROPOSITION 3 (see [7, Propositions 8–12]). *The minimum $c_{\alpha,\beta}$ is achieved by an element $\vec{u}^{\alpha,\beta} = (u_1^{\alpha,\beta}, u_2^{\alpha,\beta}, u_3^{\alpha,\beta}) \in \mathcal{D}_{\alpha,\beta}$ which satisfies* (1.3). *Moreover, the following hold uniformly for $\beta > 0$:*
(i) $u_1^{\alpha,\beta}, u_2^{\alpha,\beta}, u_3^{\alpha,\beta} > 0$ *in Ω for large $\alpha > 0$.*
(ii) *the set $\{|u_3^{\alpha,\beta}|_{\infty,\Omega}\}_{\alpha\geq 1,\beta>0}$ is bounded.*
(iii) $\lim_{\alpha\to\infty} \|\vec{u}^{\alpha,\beta} - (0,0,U_3)\| = 0$, *where U_3 is the least energy solution of* (1.4).
(iv) $\lim_{\alpha\to\infty} c_{\alpha,\beta} = L$, *where L is the least energy level of* (1.4).
(v) $\lim_{\alpha\to\infty} \|u_1^{\alpha,\beta}\|^2_{\lambda_1,\Omega} = \lim_{\alpha\to\infty} \|u_2^{\alpha,\beta}\|^2_{\lambda_2,\Omega} = 0$.
(vi) $\lim_{\alpha\to\infty} \beta \int_\Omega (u_1^{\alpha,\beta})^2(u_3^{\alpha,\beta})^2 + (u_2^{\alpha,\beta})^2(u_3^{\alpha,\beta})^2 dx = \lim_{\alpha\to\infty} \alpha \int_\Omega (u_1^{\alpha,\beta})^2(u_2^{\alpha,\beta})^2 dx = 0$.

Proof. The proof follows from the arguments in [7], and we omit the details. Indeed, the minimum $c_{\alpha,\beta}$ is achieved by an element $\vec{u}^{\alpha,\beta} = (u_1^{\alpha,\beta}, u_2^{\alpha,\beta}, u_3^{\alpha,\beta}) \in \mathcal{D}_{\alpha,\beta}$ which satisfies (1.3) by the same arguments in [7, Proposition 8]. Moreover, (iv)–(vi) are obtained from the arguments in [7, Proposition 9]. By using the arguments in [7, Proposition 10], (i) can be proved. Finally, (ii) and (iii) can be obtained from the arguments in [7, Proposition 11] and [7, Proposition 12], respectively. ∎

3. A refined convergence by a renormalization.

We recall $\vec{u}^{\alpha,\beta} = (u_1^{\alpha,\beta}, u_2^{\alpha,\beta}, u_3^{\alpha,\beta})$ as a minimizer of $c_{\alpha,\beta}$. In this section, we renormalize $(u_1^{\alpha,\beta}, u_2^{\alpha,\beta})$ so that the renormalized solution converges to a least energy solution of the elliptic system discussed in section 1. After analyzing the refined convergence, we will prove Theorem 1 at the end.

3.1. Basic energy estimates.

PROPOSITION 4. *$c_{\alpha,\beta}$ has the following upper estimate:*

$$\begin{aligned} c_{\alpha,\beta} &\leq \min\left\{L + \frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1} + o(1)), L + \frac{\beta^{2-\frac{n}{2}}}{\alpha}(M_c + o(1))\right\} \\ &= \begin{cases} L + \frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1} + o(1)) & \text{as } \alpha\#\beta^{\frac{3}{4}} \to \infty \quad \text{if} \quad n = 1,2, \\ L + \frac{\beta^{\frac{1}{2}}}{\alpha}(M_c + o(1)) & \text{as } \alpha\#\beta^{\frac{1}{2}} \to \infty \quad \text{if} \quad n = 3, \end{cases} \end{aligned} \tag{3.1}$$

where L is the least energy level of (1.4), *and M_b and M_c are defined in* (1.7) *and* (1.12), *respectively.*

Proof. We choose a cut-off function $\chi \in C^2(\mathbb{R}^1, [0,1])$ such that $0 \leq \chi \leq 1$ and

$$\chi(t) = \begin{cases} 1 & \text{if } |t| \leq \frac{1}{4}, \\ 0 & \text{if } |t| \geq \frac{1}{2}. \end{cases} \tag{3.2}$$

We consider the following cases depending on the possible location of the concentration part for the first and second components:

Case 1: Concentration on the boundary. We take the least energy solution U_3 for (1.4) and a least energy solution $(v_{1,b}, v_{2,b})$ for (1.5). We note that

$$4L = \|U_3\|^2_{\lambda_3,\Omega} = \beta_{33}|U_3|^4_{4,\Omega}, \tag{3.3}$$

(3.4)
$$4M_b(U_3) = |\nabla v_{1,b}|^2_{2,\mathbb{R}^1_+} + |\nabla v_{2,b}|^2_{2,\mathbb{R}^1_+} + \int_{\mathbb{R}^1_+} \left| \frac{\partial U_3}{\partial r}(R_0) \right|^2 t^2 \left(\beta_{13}(v_{1,b})^2 + \beta_{23}(v_{2,b})^2 \right) dt$$
$$= 2\beta_{12}|v_{1,b}v_{2,b}|^2_{2,\mathbb{R}^1_+}.$$

Now, for $x \in \Omega$, we define

$$\begin{aligned}
&(w^{\alpha,\beta}_{1,b}(x), w^{\alpha,\beta}_{2,b}(x), w_3(x))\\
&\equiv \Big(\frac{\beta^{\frac14}}{\sqrt{\alpha}} v_{1,b}(\beta^{\frac14}(R_0 - |x|))\chi\Big(\beta^{\frac{1}{12}}\Big(1 - \frac{|x|}{R_0}\Big)\Big),\\
&\frac{\beta^{\frac14}}{\sqrt{\alpha}} v_{2,b}(\beta^{\frac14}(R_0 - |x|))\chi\Big(\beta^{\frac{1}{12}}\Big(1 - \frac{|x|}{R_0}\Big)\Big), U_3(x)\Big).
\end{aligned}$$

Then, $(w^{\alpha,\beta}_{1,b}, w^{\alpha,\beta}_{2,b}, w_3) \in \mathbf{H}_r(\Omega)$. We denote $x = r\theta \in \Omega$, where $r = |x|$ and $\theta = \frac{x}{|x|}$. By (1.8) and a change of variables $y = \beta^{\frac14}(R_0 - r)\theta$ and $s = |y|$, it holds that for each $i = 1, 2$,

$$\|w^{\alpha,\beta}_{i,b}\|^2_{\lambda_i,\Omega} = \frac{\beta^{\frac34}}{\alpha} \left\{ |\nabla v_{i,b}|^2_{2,\mathbb{R}^1_+} + o_\beta(1) \right\} |S^{n-1}| R_0^{n-1}, \tag{3.5}$$

$$|w^{\alpha,\beta}_{i,b}|^4_{4,\Omega} = \frac{\beta^{\frac34}}{\alpha^2} \left\{ |v_{i,b}|^4_{4,\mathbb{R}^1_+} + o_\beta(1) \right\} |S^{n-1}| R_0^{n-1}, \tag{3.6}$$

(3.7)
$$\alpha\beta_{12}|w^{\alpha,\beta}_{1,b} w^{\alpha,\beta}_{2,b}|^2_{2,\Omega} = \frac{\beta^{\frac34}}{\alpha} \left\{ \beta_{12}|v_{1,b}v_{2,b}|^2_{2,\mathbb{R}^1_+} + o_\beta(1) \right\} |S^{n-1}| R_0^{n-1},$$

$$\beta\beta_{i3}|w^{\alpha,\beta}_{i,b} w_3|^2_{2,\Omega} = \frac{\beta^{\frac34}}{\alpha} \left\{ \beta_{i3} \textstyle\int_{\mathbb{R}^1_+} \left| \frac{\partial U_3}{\partial r}(R_0) \right|^2 s^2 (v_{i,b})^2\, ds + o_\beta(1) \right\} |S^{n-1}| R_0^{n-1}, \tag{3.8}$$

where $o_\beta(1) \to 0$ as $\beta \to \infty$. We prove only (3.8) since (3.5)–(3.7) can be proved in a similar way. Setting $y = \beta^{\frac14}(R_0 - r)\theta$ and $s = |y|$, we see that $ds = -\beta^{\frac14} dr$ and $r^{n-1} = (R_0 - \beta^{-\frac14} s)^{n-1}$. Thus it follows that
(3.9)
$$\begin{aligned}
&\beta\beta_{i3}|w^{\alpha,\beta}_{i,b} w_3|^2_{2,\Omega}\\
&= \frac{\beta^{\frac32}}{\alpha}\beta_{i3} \int_0^{R_0} \left\{ v_{i,b}(\beta^{\frac14}(R_0 - r))\chi\left(\beta^{\frac{1}{12}}\left(1 - \frac{r}{R_0}\right)\right) U_3(r) \right\}^2 r^{n-1} dr |S^{n-1}|\\
&= \frac{\beta^{\frac32 - \frac14}}{\alpha}\beta_{i3} \int_0^{\beta^{\frac16}\frac{R_0}{2}} \left\{ v_{i,b}(s)\chi\left(\frac{s}{\beta^{\frac16} R_0}\right) U_3(R_0 - \beta^{-\frac14} s) \right\}^2 (R_0 - \beta^{-\frac14} s)^{n-1} ds |S^{n-1}|.
\end{aligned}$$

We see from the fact $U_3(R_0) = 0$ and the mean value theorem that for $s \in [0, \frac{\beta^{\frac16} R_0}{2}]$,

$$U_3(R_0 - \beta^{-\frac14} s) = U_3(R_0 - \beta^{-\frac14} s) - U_3(R_0) = -\frac{dU_3}{dr}(R_0)\beta^{-\frac14} s + O(\beta^{-\frac12} s^2). \tag{3.10}$$

From the exponential decay of $v_{1,b}, v_{2,b}$ in (1.8), we see that (3.9) and (3.10) imply (3.8).

From (3.4), (3.5), (3.7), and (3.8), it follows that

$$\sum_{i=1,2}(\|w_{i,b}^{\alpha,\beta}\|_{\lambda_i,\Omega}^2+\beta\beta_{i3}|w_{i,b}^{\alpha,\beta}w_3|_{2,\Omega}^2)=\frac{\beta^{\frac{3}{4}}}{\alpha}(4M_b+o_\beta(1))|S^{n-1}|R_0^{n-1},\tag{3.11}$$

$$2\alpha\beta_{12}|w_{1,b}^{\alpha,\beta}w_{2,b}^{\alpha,\beta}|_{2,\Omega}^2=\frac{\beta^{\frac{3}{4}}}{\alpha}(4M_b+o_\beta(1))|S^{n-1}|R_0^{n-1}.\tag{3.12}$$

From (3.3), (3.11), (3.12), $\vec{w}^{\alpha,\beta}\equiv(w_{1,b}^{\alpha,\beta},w_{2,b}^{\alpha,\beta},w_3)\in\widehat{\mathcal{D}}_{\alpha,\beta}$ as $\alpha\#\beta^{\frac{3}{4}}\to\infty$. Thus from Lemma 2(i), there exists a unique $(s,t)\in\mathbb{R}^2$ such that $(sw_{1,b}^{\alpha,\beta},sw_{2,b}^{\alpha,\beta},tw_3)\in\mathcal{D}_{\alpha,\beta}$. By the expression in Lemma 2(ii) for $I(sw_{1,b}^{\alpha,\beta},sw_{2,b}^{\alpha,\beta},tw_3)$, we deduce that

$$I(sw_{1,b}^{\alpha,\beta},sw_{2,b}^{\alpha,\beta},tw_3)=\frac{\|w_3\|_{\lambda_3,\Omega}^4}{4\beta_{33}|w_3|_{4,\Omega}^4}+\frac{K(\vec{w}^{\alpha,\beta})}{4\beta_{33}|w_3|_{4,\Omega}^4G(\vec{w}^{\alpha,\beta})},\tag{3.13}$$

where

$$\begin{aligned}K(\vec{w})&=\beta_{33}^2|w_3|_{4,\Omega}^8(\|w_1\|_{\lambda_1,\Omega}^2+\|w_2\|_{\lambda_2,\Omega}^2)^2\\&\quad+2\beta_{33}|w_3|_{4,\Omega}^4(\beta\beta_{13}|w_1w_3|_{2,\Omega}^2+\beta\beta_{23}|w_2w_3|_{2,\Omega}^2)(\|w_1\|_{\lambda_1,\Omega}^2+\|w_2\|_{\lambda_2,\Omega}^2)\|w_3\|_{\lambda_3,\Omega}^2\\&\quad+(\beta\beta_{13}|w_1w_3|_{2,\Omega}^2+\beta\beta_{23}|w_2w_3|_{2,\Omega}^2)^2\|w_3\|_{\lambda_3,\Omega}^4\end{aligned}\tag{3.14}$$

and

$$\begin{aligned}G(\vec{w})=G(w_1,w_2,w_3)=&(\beta_{11}|w_1|_{4,\Omega}^4+\beta_{22}|w_2|_{4,\Omega}^4+2\alpha\beta_{12}|w_1w_2|_{2,\Omega}^2)\beta_{33}|w_3|_{4,\Omega}^4\\&-(\beta\beta_{13}|w_1w_3|_{2,\Omega}^2+\beta\beta_{23}|w_2w_3|_{2,\Omega}^2)^2.\end{aligned}\tag{3.15}$$

Now, by using (3.3), (3.13) is simplified to

$$\begin{aligned}&I(sw_{1,b}^{\alpha,\beta},sw_{2,b}^{\alpha,\beta},tw_3)\\&=L+\frac{L(\|w_{1,b}^{\alpha,\beta}\|_{\lambda_1,\Omega}^2+\|w_{2,b}^{\alpha,\beta}\|_{\lambda_2,\Omega}^2+\beta\beta_{13}|w_{1,b}^{\alpha,\beta}w_3|_{2,\Omega}^2+\beta\beta_{23}|w_{2,b}^{\alpha,\beta}w_3|_{2,\Omega}^2)^2}{G(\vec{w}^{\alpha,\beta})}.\end{aligned}\tag{3.16}$$

From (3.6), (3.11), and (3.12),

$$G(\vec{w}^{\alpha,\beta})=\frac{\beta^{\frac{3}{4}}}{\alpha}(16M_bL|S^{n-1}|R_0^{n-1}+o(1))\quad\text{as}\quad\alpha\#\beta^{\frac{3}{4}}\to\infty.\tag{3.17}$$

Thus, from (3.11) and (3.17), it follows that as $\alpha\#\beta^{\frac{3}{4}}\to\infty$,

$$c_{\alpha,\beta}\le I(sw_{1,b}^{\alpha,\beta},sw_{2,b}^{\alpha,\beta},tw_3)=L+\frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1}+o(1)).\tag{3.18}$$

Since a least energy solution U_3 of (1.4) is unique, we obtain one of the upper estimates for (3.1).

Case 2: Concentration on the center. We take the least energy solution U_3 for (1.4) and a least energy solution $(v_{1,c},v_{2,c})$ for (1.11). We note that

$$4L=\|U_3\|_{\lambda_3,\Omega}^2=\beta_{33}|U_3|_{4,\Omega}^4,\tag{3.19}$$

$$\begin{aligned}4M_c(U_3)&=|\nabla v_{1,c}|_{2,\mathbb{R}^n}^2+|\nabla v_{2,c}|_{2,\mathbb{R}^n}^2+(U_3(0))^2\left\{\beta_{13}|v_{1,c}|_{2,\mathbb{R}^n}^2+\beta_{23}|v_{2,c}|_{2,\mathbb{R}^n}^2\right\}\\&=2\beta_{12}|v_{1,c}v_{2,c}|_{2,\mathbb{R}^n}^2.\end{aligned}\tag{3.20}$$

Now, for $x \in \Omega$, we define

$$
\begin{aligned}
&(w_{1,c}^{\alpha,\beta}(x), w_{2,c}^{\alpha,\beta}(x), w_3(x)) \\
&\equiv \left(\sqrt{\frac{\beta}{\alpha}} v_{1,c}(\sqrt{\beta}x)\chi(\beta^{\frac{1}{4}} R_0^{-1}|x|), \sqrt{\frac{\beta}{\alpha}} v_{2,c}(\sqrt{\beta}x)\chi(\beta^{\frac{1}{4}} R_0^{-1}|x|),\ U_3(x) \right).
\end{aligned}
$$

We have $(w_{1,c}^{\alpha,\beta}, w_{2,c}^{\alpha,\beta}, w_3) \in \mathbf{H}_r(\Omega)$. By (1.13) and a change of variables $x = \beta^{-\frac{1}{2}}y$, it holds that for each $i = 1, 2$,

$$
\|w_{i,c}^{\alpha,\beta}\|_{\lambda_i,\Omega}^2 = \frac{\beta^{2-\frac{n}{2}}}{\alpha} \left\{ |\nabla v_{i,c}|_{2,\mathbb{R}^n}^2 + o_\beta(1) \right\}, \tag{3.21}
$$

$$
|w_{i,c}^{\alpha,\beta}|_{4,\Omega}^4 = \frac{\beta^{2-\frac{n}{2}}}{\alpha^2} \left\{ |v_{i,c}|_{4,\mathbb{R}^n}^4 + o_\beta(1) \right\}, \tag{3.22}
$$

$$
\alpha\beta_{12}|w_{1,c}^{\alpha,\beta} w_{2,c}^{\alpha,\beta}|_{2,\Omega}^2 = \frac{\beta^{2-\frac{n}{2}}}{\alpha} \left\{ \beta_{12}|v_{1,c}v_{2,c}|_{2,\mathbb{R}^n}^2 + o_\beta(1) \right\}, \tag{3.23}
$$

$$
\beta\beta_{i3}|w_{i,c}^{\alpha,\beta} w_3|_{2,\Omega}^2 = \frac{\beta^{2-\frac{n}{2}}}{\alpha} \left\{ \beta_{i3}(U_3(0))^2|v_{i,c}|_{2,\mathbb{R}^n}^2 + o_\beta(1) \right\}, \tag{3.24}
$$

where $o_\beta(1) \to 0$ as $\beta \to \infty$. From (3.20), (3.21), (3.23), and (3.24), it follows that

$$
\sum_{i=1,2} (\|w_{i,c}^{\alpha,\beta}\|_{\lambda_i,\Omega}^2 + \beta\beta_{i3}|w_{i,c}^{\alpha,\beta} w_3|_{2,\Omega}^2) = \frac{\beta^{2-\frac{n}{2}}}{\alpha}(4M_c(U_3) + o_\beta(1)), \tag{3.25}
$$

$$
2\alpha\beta_{12}|w_{1,c}^{\alpha,\beta} w_{2,c}^{\alpha,\beta}|_{2,\Omega}^2 = \frac{\beta^{2-\frac{n}{2}}}{\alpha}(4M_c(U_3) + o_\beta(1)). \tag{3.26}
$$

By the similar arguments with Case 1 above, it follows that as $\alpha\#\beta^{2-\frac{n}{2}} \to \infty$,

$$
c_{\alpha,\beta} \le L + \frac{\beta^{2-\frac{n}{2}}}{\alpha}(M_c + o(1)). \tag{3.27}
$$

By comparing (3.18) and (3.27), we complete the proof of Proposition 4. $\square$

3.2. A renormalization and basic estimates. As we will use a multiscale renormalization we introduce some notation here. We will consider two different scalings according to the scaled regions: the neighborhood of the boundary or the neighborhood of the center. In order to distinguish these different scalings and scaled regions, we will use the subscript "b" for the notation related to the scalings on the neighborhood of the boundary, and the subscript "c" for the notation related to the scalings on the neighborhood of the center.

The scaling on the neighborhood of the boundary. Let

$$
u_1^{\alpha,\beta}(x) = \frac{\beta^{\frac{1}{4}}}{\sqrt{\alpha}} v_{1,b}^{\alpha,\beta}(\beta^{\frac{1}{4}}(R_0 - |x|)),\ u_2^{\alpha,\beta}(x) = \frac{\beta^{\frac{1}{4}}}{\sqrt{\alpha}} v_{2,b}^{\alpha,\beta}(\beta^{\frac{1}{4}}(R_0 - |x|)),\ u_3^{\alpha,\beta}(x) = v_3^{\alpha,\beta}(x). \tag{3.28}
$$

We denote $x = r\theta \in \Omega$, where $r = |x|$ and $\theta = \frac{x}{|x|}$. By letting $y = \beta^{\frac{1}{4}}(R_0 - r)\theta$, $s = |y|$, and $\Omega_b^\beta = \{s \in \mathbb{R} \mid 0 \le s \le \frac{\beta^{\frac{1}{4}} R_0}{2}\}$, we see that $\vec{v}^{\alpha,\beta} \equiv (v_{1,b}^{\alpha,\beta}, v_{2,b}^{\alpha,\beta}, v_3^{\alpha,\beta})$

satisfies

(3.29)

$$\begin{cases} \frac{d^2 v_{1,b}^{\alpha,\beta}}{ds^2} - \frac{(n-1)\beta^{-\frac{1}{4}}}{R_0-\beta^{-\frac{1}{4}}s}\frac{dv_{1,b}^{\alpha,\beta}}{ds} - \frac{\lambda_1}{\sqrt{\beta}}v_{1,b}^{\alpha,\beta} + \frac{\beta_{11}}{\alpha}(v_{1,b}^{\alpha,\beta})^3 + \beta_{12}v_{1,b}^{\alpha,\beta}(v_{2,b}^{\alpha,\beta})^2 \\ = \sqrt{\beta}\beta_{13}v_{1,b}^{\alpha,\beta}(v_3^{\alpha,\beta}(R_0-\beta^{-\frac{1}{4}}s))^2 \quad \text{in } \Omega_b^\beta, \\ \frac{d^2 v_{2,b}^{\alpha,\beta}}{ds^2} - \frac{(n-1)\beta^{-\frac{1}{4}}}{R_0-\beta^{-\frac{1}{4}}s}\frac{dv_{2,b}^{\alpha,\beta}}{ds} - \frac{\lambda_2}{\sqrt{\beta}}v_{2,b}^{\alpha,\beta} + \beta_{21}(v_{1,b}^{\alpha,\beta})^2 v_{2,b}^{\alpha,\beta} + \frac{\beta_{22}}{\alpha}(v_{2,b}^{\alpha,\beta})^3 \\ = \sqrt{\beta}\beta_{23}v_{2,b}^{\alpha,\beta}(v_3^{\alpha,\beta}(R_0-\beta^{-\frac{1}{4}}s))^2 \quad \text{in } \Omega_b^\beta, \\ \Delta v_3^{\alpha,\beta} - \lambda_3 v_3^{\alpha,\beta} - \left(\sum_{i=1}^2 \beta_{3i}(v_{i,b}^{\alpha,\beta}(\beta^{\frac{1}{4}}(R_0-|x|)))^2\right)\frac{\beta^{\frac{3}{2}}}{\alpha}v_3^{\alpha,\beta} + \beta_{33}(v_3^{\alpha,\beta})^3 = 0 \quad \text{in } \Omega. \end{cases}$$

The scaling on the neighborhood of the center. Let

(3.30)

$$u_1^{\alpha,\beta}(x) = \sqrt{\frac{\beta}{\alpha}}v_{1,c}^{\alpha,\beta}(\sqrt{\beta}x), \quad u_2^{\alpha,\beta}(x) = \sqrt{\frac{\beta}{\alpha}}v_{2,c}^{\alpha,\beta}(\sqrt{\beta}x), \quad u_3^{\alpha,\beta}(x) = v_3^{\alpha,\beta}(x).$$

By letting $y = \beta^{\frac{1}{2}}x$ and $\Omega_c^\beta = \{y \in \mathbb{R}^n \mid |y| \le \frac{\sqrt{\beta}R_0}{2}\}$, we see that $\vec{v}^{\alpha,\beta} \equiv (v_{1,c}^{\alpha,\beta}, v_{2,c}^{\alpha,\beta}, v_3^{\alpha,\beta})$ satisfies

(3.31)

$$\begin{cases} \Delta v_{1,c}^{\alpha,\beta} - \frac{\lambda_1}{\beta}v_{1,c}^{\alpha,\beta} + \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^3 + \beta_{12}v_{1,c}^{\alpha,\beta}(v_{2,c}^{\alpha,\beta})^2 = \beta_{13}v_{1,c}^{\alpha,\beta}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 \text{ in } \Omega_c^\beta, \\ \Delta v_{2,c}^{\alpha,\beta} - \frac{\lambda_2}{\beta}v_{2,c}^{\alpha,\beta} + \beta_{21}(v_{1,c}^{\alpha,\beta})^2 v_{2,c}^{\alpha,\beta} + \frac{\beta_{22}}{\alpha}(v_{2,c}^{\alpha,\beta})^3 = \beta_{23}v_{2,c}^{\alpha,\beta}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 \text{ in } \Omega_c^\beta, \\ \Delta v_3^{\alpha,\beta} - \lambda_3 v_3^{\alpha,\beta} - \left(\sum_{i=1}^2 \beta_{3i}(v_{i,c}^{\alpha,\beta}(\sqrt{\beta}x))^2\right)\frac{\beta^2}{\alpha}v_3^{\alpha,\beta} + \beta_{33}(v_3^{\alpha,\beta})^3 = 0 \qquad \text{in } \Omega. \end{cases}$$

Here we denote

$$s = |y|, \quad \Delta = \frac{d^2}{ds^2} + \frac{(n-1)}{s}\frac{d}{ds}, \quad \text{and} \quad |\nabla\phi|^2_{2,\Omega_c^\beta} = |S^{n-1}|\int_0^{\frac{\sqrt{\beta}R_0}{2}} \left|\frac{d\phi}{ds}\right|^2 s^{n-1}ds.$$

Now we have the following result.

PROPOSITION 5. *It holds that*

(3.32)

$$
\begin{aligned}
&\frac{\beta^{2-\frac{n}{2}}}{\alpha}\left\{\frac{\beta_{11}}{\alpha}|v_{1,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4+\frac{\beta_{22}}{\alpha}|v_{2,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4+2\beta_{12}|v_{1,c}^{\alpha,\beta}v_{2,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2\right\}\\
&+\frac{\beta^{\frac{3}{4}}|S^{n-1}|}{\alpha}\left\{\frac{\beta_{11}}{\alpha}|v_{1,b}^{\alpha,\beta}(R_0-\beta^{-\frac14}|y|)^{\frac{n-1}{4}}|_{4,\Omega_b^\beta}^4+\frac{\beta_{22}}{\alpha}|v_{2,b}^{\alpha,\beta}(R_0-\beta^{-\frac14}|y|)^{\frac{n-1}{4}}|_{4,\Omega_b^\beta}^4\right.\\
&\left.+2\beta_{12}|v_{1,b}^{\alpha,\beta}v_{2,b}^{\alpha,\beta}(R_0-\beta^{-\frac14}|y|)^{\frac{n-1}{2}}|_{2,\Omega_b^\beta}^2\right\}\\
&=\frac{\beta^{2-\frac{n}{2}}}{\alpha}\left\{\sum_{i=1}^2\left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2+\frac{\lambda_i}{\beta}|v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2+\beta_{i3}\int_{\Omega_c^\beta}(v_3^{\alpha,\beta}(\beta^{-\frac12}y))^2(v_{i,c}^{\alpha,\beta})^2\,dy\right)\right\}\\
&+\frac{\beta^{\frac34}|S^{n-1}|}{\alpha}\left\{\sum_{i=1}^2\left(\left|\nabla v_{i,b}^{\alpha,\beta}(R_0-\beta^{-\frac14}|y|)^{\frac{n-1}{2}}\right|_{2,\Omega_b^\beta}^2+\frac{\lambda_i}{\sqrt{\beta}}v_{i,b}^{\alpha,\beta}(R_0-\beta^{-\frac14}y)^{\frac{n-1}{2}}|_{2,\Omega_b^\beta}^2\right)\right.\\
&\left.+\sum_{i=1}^2\beta_{i3}\int_{\Omega_b^\beta}\sqrt{\beta}(v_3^{\alpha,\beta}(R_0-\beta^{-\frac14}|y|))^2(v_{i,b}^{\alpha,\beta})^2(R_0-\beta^{-\frac14}|y|)^{n-1}dy\right\}\\
&\le\min\left\{\frac{\beta^{\frac34}}{\alpha}(4M_b|S^{n-1}|R_0^{n-1}+o(1)),\frac{\beta^{2-\frac{n}{2}}}{\alpha}(4M_c+o(1))\right\}\\
&=\begin{cases}\frac{\beta^{\frac34}}{\alpha}(4M_b|S^{n-1}|R_0^{n-1}+o(1)) & \text{as } \alpha\#\beta^{\frac34}\to\infty \ \text{ if } \ n=1,2,\\ \frac{\beta^{\frac12}}{\alpha}(4M_c+o(1)) & \text{as } \alpha\#\beta^{\frac12}\to\infty \ \text{ if } \ n=3.\end{cases}
\end{aligned}
$$

Proof. In view of [7, Proposition 14], we have the following estimation:

$$
\begin{aligned}
&\beta_{11}|u_1^{\alpha,\beta}|_{4,\Omega}^4+\beta_{22}|u_2^{\alpha,\beta}|_{4,\Omega}^4+2\alpha\beta_{12}|u_1^{\alpha,\beta}u_2^{\alpha,\beta}|_{2,\Omega}^2\\
&=\|u_1^{\alpha,\beta}\|_{\lambda_1,\Omega}^2+\|u_2^{\alpha,\beta}\|_{\lambda_2,\Omega}^2+\beta\beta_{13}|u_1^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2+\beta\beta_{23}|u_2^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2\\
&\le\min\left\{\frac{\beta^{\frac34}}{\alpha}(4M_b|S^{n-1}|R_0^{n-1}+o(1)),\frac{\beta^{2-\frac{n}{2}}}{\alpha}(4M_c+o(1))\right\}\\
&=\begin{cases}\frac{\beta^{\frac34}}{\alpha}(4M_b|S^{n-1}|R_0^{n-1}+o(1)) & \text{as } \alpha\#\beta^{\frac34}\to\infty \ \text{ if } \ n=1,2,\\ \frac{\beta^{\frac12}}{\alpha}(4M_c+o(1)) & \text{as } \alpha\#\beta^{\frac12}\to\infty \ \text{ if } \ n=3.\end{cases}
\end{aligned}
\tag{3.33}
$$

The proof of Proposition 5 comes from (3.33) by taking a change of variables. ∎

PROPOSITION 6. *For any $R\in(0,R_0)$, there exists a constant $C>0$ such that*

$$
\begin{aligned}
|u_1^{\alpha,\beta}|_{\infty,\Omega}+|u_2^{\alpha,\beta}|_{\infty,\Omega}&\le C\frac{\beta^{\frac38}}{\sqrt{\alpha}} && \text{as } \ \alpha\#\beta^{\frac34}\to\infty \quad \text{if} \quad n=1;\\
|u_1^{\alpha,\beta}|_{\infty,\{x\in\Omega|\ |x|\ge R\}}+|u_2^{\alpha,\beta}|_{\infty,\{x\in\Omega|\ |x|\ge R\}}&\le C\frac{\beta^{\frac38}}{\sqrt{\alpha}} && \text{as } \alpha\#\beta^{\frac34}\to\infty \quad \text{if} \quad n=2;\\
|u_1^{\alpha,\beta}|_{\infty,\Omega}+|u_2^{\alpha,\beta}|_{\infty,\Omega}&\le C\sqrt{\frac{\beta}{\alpha}} && \text{as } \ \alpha\#\beta^{\frac12}\to\infty \quad \text{if} \quad n=3.
\end{aligned}
$$

Proof. For $n=1$, there exists $C_1>0$, independent of $\beta>0$, such that for $i=1,2$, $|u_i^{\alpha,\beta}|_{\infty,\Omega}\le C_1\|u_i^{\alpha,\beta}\|_{\lambda_i,\Omega}$. Then, (3.33) implies that there exists some $C_2>0$, independent of $\beta>0$, such that for large $\alpha\#\beta^{\frac34}$,

$$|u_i^{\alpha,\beta}|_{\infty,\Omega}\le C_2\left(\frac{\beta^{\frac34}}{\alpha}\right)^{\frac12}=C_2\frac{\beta^{\frac38}}{\sqrt{\alpha}},\quad i=1,2.$$

For $n = 2$, since $u_1^{\alpha,\beta}$ and $u_2^{\alpha,\beta}$ are radially symmetric, there exists $C_3 > 0$, independent of $\beta > 0$, that for $i = 1, 2$, $|u_i^{\alpha,\beta}|_{\infty,\{x\in\Omega|\ |x|\geq R\}} \leq C_3\|u_i^{\alpha,\beta}\|_{\lambda_i,\{x\in\Omega|\ |x|\geq R\}}$. Then, by the same argument with the case $n = 1$, there exists a constant $C_4 > 0$ such that for large $\alpha\#\beta^{\frac{3}{4}}$,

$$|u_i^{\alpha,\beta}|_{\infty,\{x\in\Omega|\ |x|\geq R\}} \leq C_4\left(\frac{\beta^{\frac{3}{4}}}{\alpha}\right)^{\frac{1}{2}} = C_4\frac{\beta^{\frac{3}{8}}}{\sqrt{\alpha}}, \quad i = 1, 2.$$

For $n = 3$, it suffices to prove that $|v_{1,c}^{\alpha,\beta}|_{\infty,\Omega}$ and $|v_{2,c}^{\alpha,\beta}|_{\infty,\Omega}$ is bounded for large $\alpha\#\beta^{\frac{1}{2}}$. Since $(v_{1,c}^{\alpha,\beta}, v_{2,c}^{\alpha,\beta})$ satisfies

$$\begin{aligned}
\Delta v_{1,c}^{\alpha,\beta} &- \frac{\lambda_1}{\beta}v_{1,c}^{\alpha,\beta} + \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^3 + \beta_{12}v_{1,c}^{\alpha,\beta}(v_{2,c}^{\alpha,\beta})^2 \\
&- \beta_{13}v_{1,c}^{\alpha,\beta}(u_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y)^2 = 0 \text{ in } B_{\sqrt{\beta}R_0}(0) \\
\Delta v_{2,c}^{\alpha,\beta} &- \frac{\lambda_2}{\beta}v_{2,c}^{\alpha,\beta} + \beta_{21}v_{2,c}^{\alpha,\beta}(v_{1,c}^{\alpha,\beta})^2 + \frac{\beta_{22}}{\alpha}(v_{2,c}^{\alpha,\beta})^3 \\
&- \beta_{23}v_{2,c}^{\alpha,\beta}(u_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 = 0 \text{ in } B_{\sqrt{\beta}R_0}(0),
\end{aligned}$$

$$-\Delta v_{1,c}^{\alpha,\beta} \leq \left(\frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 + \beta_{12}(v_{2,c}^{\alpha,\beta})^2\right)v_{1,c}^{\alpha,\beta} \quad \text{in } B_{\sqrt{\beta}R_0}(0), \tag{3.34}$$

$$-\Delta v_{2,c}^{\alpha,\beta} \leq \left(\beta_{21}(v_{1,c}^{\alpha,\beta})^2 + \frac{\beta_{22}}{\alpha}(v_{2,c}^{\alpha,\beta})^2\right)v_{2,c}^{\alpha,\beta} \quad \text{in } B_{\sqrt{\beta}R_0}(0). \tag{3.35}$$

Then, for each $l \geq 0$, we multiply both sides of (3.34) by $(v_{1,c}^{\alpha,\beta})^{2l+1}$, (3.35) by $(v_{2,c}^{\alpha,\beta})^{2l+1}$ and integrate parts and add two inequalities. Then for some $C > 0$, independent of $\alpha, \beta > 1$, $l \geq 0$,

$$\begin{aligned}
&\int_{B_{\sqrt{\beta}R_0}(0)} |\nabla(v_{1,c}^{\alpha,\beta})^{l+1}|^2 + |\nabla(v_{2,c}^{\alpha,\beta})^{l+1}|^2 dy \\
&\leq C(l+1)\int_{B_{\sqrt{\beta}R_0}(0)} (v_{1,c}^{\alpha,\beta})^{2l+4} + (v_{1,c}^{\alpha,\beta})^{2l+2}(v_{2,c}^{\alpha,\beta})^2 + (v_{1,c}^{\alpha,\beta})^2(v_{2,c}^{\alpha,\beta})^{2l+2} + (v_{2,c}^{\alpha,\beta})^{2l+4}dy.
\end{aligned}$$

Then, using the Sobolev inequality and Hölder's inequality, we see that for some $C' > 0$, independent of $l \geq 0$,

$$\begin{aligned}
\left(\int_{B_{\sqrt{\beta}R_0}(0)} (v_{1,c}^{\alpha,\beta})^{6(l+1)} + (v_{2,c}^{\alpha,\beta})^{6(l+1)}dy\right)^{\frac{1}{3}} \leq C'(l+1)\int_{B_{\sqrt{\beta}R_0}(0)} &(v_{1,c}^{\alpha,\beta})^{2l+4} \\
&+ (v_{2,c}^{\alpha,\beta})^{2l+4}dy.
\end{aligned}$$

In view of (3.32), we get that $\sum_{i=1}^2 |\nabla v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta}$ is bounded for large $\alpha\#\beta^{\frac{1}{2}}$. Then we see that for any $q \geq 6$, $\int_{B_{\sqrt{\beta}R_0}(0)} (v_{1,c}^{\alpha,\beta})^q + (v_{2,c}^{\alpha,\beta})^q dy$ is bounded for large $\alpha\#\beta^{\frac{1}{2}}$. Then, applying Theorems 9.20 and 9.26 in [13], we get the boundedness of $|v_{1,c}^{\alpha,\beta}|_{\infty,B_{\sqrt{\beta}R_0}(0)}$ and $|v_{2,c}^{\alpha,\beta}|_{\infty,B_{\sqrt{\beta}R_0}(0)}$ for large $\alpha\#\beta^{\frac{1}{2}}$. These complete the proof of Proposition 6. □

3.3. Finer estimates for the renormalized equations. From now on, we assume that $\alpha\#\beta^{1+\delta_n} \to \infty$ for $\delta_1, \delta_2 \geq \frac{1}{4}$ if $n = 1, 2$, and $\delta_3 > 0$ if $n = 3$. We note that

$$\alpha\#\beta^{1+\delta_n} \to \infty \Longrightarrow \begin{cases} \alpha\#\beta^{\frac{3}{4}} \to \infty & \text{if} \quad n = 1, 2, \\ \alpha\#\beta^{\frac{1}{2}} \to \infty & \text{if} \quad n = 3. \end{cases}$$

Now, from Proposition 3, we see that for the least energy solution U_3 of (1.4), $u_3^{\alpha,\beta} \to U_3$ in $H_0^1(\Omega)$ as $\alpha\#\beta^{1+\delta_n} \to \infty$. In order to prove Theorem 1, it is required to get $u_3^{\alpha,\beta} \to U_3$ in C^1 near the boundary for $n = 1, 2$, but $u_3^{\alpha,\beta} \to U_3$ in C^0 near the center for $n = 3$.

PROPOSITION 7. *We take any $R \in (0, R_0)$. For any $\delta_1 \geq \frac{1}{4}$, $\delta_2 \geq \frac{1}{4}$, and $\delta_3 > 0$,*
(i) $u_3^{\alpha,\beta} \to U_3$ *in* $C^1(\overline{\Omega})$ *as* $\alpha\#\beta^{1+\delta_n} \to \infty$ *if* $n = 1$;
(ii) $u_3^{\alpha,\beta} \to U_3$ *in* $C^1(\overline{\{x \in \Omega \mid |x| \geq R\}})$ *as* $\alpha\#\beta^{1+\delta_n} \to \infty$ *if* $n = 2$;
(iii) $u_3^{\alpha,\beta} \to U_3$ *in* $C(\overline{\Omega})$ *as* $\alpha\#\beta^{1+\delta_n} \to \infty$ *if* $n = 3$.

Proof. Note that $u_3^{\alpha,\beta}$ and U_3 satisfy

$$\Delta u_3^{\alpha,\beta} - \lambda_3 u_3^{\alpha,\beta} + \beta_{33}(u_3^{\alpha,\beta})^3 = \Big(\beta\beta_{13}(u_1^{\alpha,\beta})^2 + \beta\beta_{23}(u_2^{\alpha,\beta})^2\Big)u_3^{\alpha,\beta} \quad \text{in} \quad \Omega,$$
$$\Delta U_3 - \lambda_3 U_3 + \beta_{33}(U_3)^3 = 0 \quad \text{in} \quad \Omega.$$

Then, defining $w^{\alpha,\beta} \equiv u_3^{\alpha,\beta} - U_3$, we see

$$-\Delta w^{\alpha,\beta} + \lambda_3 w^{\alpha,\beta} = \beta_{33}(u_3^{\alpha,\beta})^3 - \beta_{33}(U_3)^3 - \Big(\beta\beta_{13}(u_1^{\alpha,\beta})^2 + \beta\beta_{23}(u_2^{\alpha,\beta})^2\Big)\, u_3^{\alpha,\beta}.$$

We define $f \equiv \beta_{33}(u_3^{\alpha,\beta})^3 - \beta_{33}(U_3)^3$, $\quad g \equiv \big(\beta\beta_{13}(u_1^{\alpha,\beta})^2 + \beta\beta_{23}(u_2^{\alpha,\beta})^2\big)u_3^{\alpha,\beta}$. By Proposition 3, $\{|u_3^{\alpha,\beta}|_{\infty,\Omega}\}_{\alpha\geq 1,\beta>0}$ is bounded. Thus, for any $p > 1$, there exists a constant $C > 0$, independent of large $\alpha, \beta > 0$ such that

$$|f|_{p,\Omega} \leq C|w^{\alpha,\beta}|_{p,\Omega}, \quad |g|_{p,\Omega} \leq C\beta|(u_1^{\alpha,\beta})^2 u_3^{\alpha,\beta}|_{p,\Omega} + C\beta|(u_2^{\alpha,\beta})^2 u_3^{\alpha,\beta}|_{p,\Omega}.$$

For $n = 1$, we take $p = 2$. Then, we see from (3.33) and Proposition 6 that for some $C' > 0$, independent of large $\alpha, \beta > 0$,

$$|g|_{2,\Omega} \leq C\beta \sum_{i=1}^{2} |u_i^{\alpha,\beta}|_{\infty,\Omega} |u_i^{\alpha,\beta} u_3^{\alpha,\beta}|_{2,\Omega} \leq C'\beta \frac{\beta^{\frac{3}{8}}}{\alpha^{\frac{1}{2}}} \frac{1}{\alpha^{\frac{1}{2}}\beta^{\frac{1}{8}}} = C'\frac{\beta^{1+\frac{1}{4}}}{\alpha}.$$

Together with Proposition 3, we see that $|f|_{2,\Omega}$ and $|g|_{2,\Omega}$ converge to 0 as $\alpha\#\beta^{1+\frac{1}{4}} \to \infty$. Then it follows from the $W^{2,2}$-estimate [13, Theorem 8.12] that $w^{\alpha,\beta} \to 0$ in $W^{2,2}(\Omega)$ as $\alpha\#\beta^{1+\frac{1}{4}} \to \infty$. This implies that $w^{\alpha,\beta} \to 0$ in $C^1(\overline{\Omega})$ as $\alpha\#\beta^{1+\frac{1}{4}} \to \infty$.

For $n = 2$, we take $p = 2$. Then, we see from Proposition 3, (3.33), and Proposition 6 that for some $C' > 0$, independent of large $\alpha, \beta > 0$,

$$|g|_{2,\{x\in\Omega \mid |x|\geq R\}} \leq C\beta \sum_{i=1}^{2} |u_i^{\alpha,\beta}|_{\infty,\{x\in\Omega \mid |x|\geq R\}} |u_i^{\alpha,\beta} u_3^{\alpha,\beta}|_{2,\Omega} \leq C'\frac{\beta^{1+\frac{1}{4}}}{\alpha}.$$

By the same arguments for $n = 1$, we see that $w^{\alpha,\beta} \to 0$ in $W^{2,2}(\{x \in \Omega \mid |x| \geq 2R\})$ as $\alpha\#\beta^{1+\frac{1}{4}} \to \infty$. Since $w^{\alpha,\beta}$ is radially symmetric, we get that

$$w^{\alpha,\beta} \to 0 \quad \text{in} \quad C^1(\overline{\{x \in \Omega \mid |x| \geq 2R\}}) \quad \text{as} \quad \alpha\#\beta^{1+\frac{1}{4}} \to \infty.$$

For $n = 3$, we take any $p > \frac{3}{2}$. Then, we see from Propositions 3, 5, and 6 that for some $C' > 0$, independent of large $\alpha, \beta > 0$,

$$|g|_{p,\Omega} \le C\beta \sum_{i=1}^{2} (|u_i^{\alpha,\beta}|_{\infty,\Omega})^{\frac{2p-2}{p}} (|u_i^{\alpha,\beta} u_3^{\alpha,\beta}|_{2,\Omega})^{\frac{2}{p}} \le C'\beta \frac{\beta^{\frac{2p-2}{2p}}}{\alpha^{\frac{2p-2}{2p}}} \frac{1}{\alpha^{\frac{1}{p}} \beta^{\frac{1}{2p}}} = C' \frac{\beta^{1+\frac{2p-3}{2p}}}{\alpha}.$$

Thus, it follows from Lemma 9.17 and Theorem 9.19 in [13] that

$$w^{\alpha,\beta} \to 0 \quad \text{in} \quad W^{2,p}(\Omega) \quad \text{as} \quad \alpha \# \beta^{1+\frac{2p-3}{2p}} \to \infty.$$

Since $W^{2,p}(\Omega) \hookrightarrow C(\overline{\Omega})$ for $p > \frac{3}{2}$, it follows that $w^{\alpha,\beta} \to 0$ in $C(\overline{\Omega})$ as $\alpha \# \beta^{1+\frac{2p-3}{2p}} \to \infty$. Since $1 + \frac{2p-3}{2p} = 1$ for $p = \frac{3}{2}$, we conclude that for any $\delta_3 > 0$, $w^{\alpha,\beta} \to 0$ in $C(\overline{\Omega})$ as $\alpha \# \beta^{1+\delta_3} \to \infty$. This completes the proof. ∎

Proposition 8. (i) *There is a constant $C > 0$ such that for large $\alpha \# \beta^{1+\delta_n}$,*

$$\begin{cases} \sum_{i=1}^2 \left(|\nabla v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta} + |v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta} \right) \le C & \text{if} \quad n = 1, 2, \\ \sum_{i=1}^2 \left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta} + |v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta} \right) \le C & \text{if} \quad n = 3. \end{cases}$$

(ii) *There exists a constant $\delta > 0$ such that*

$$\begin{cases} \sum_{i=1}^2 |v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta} \ge \delta \quad \text{and} \\ \lim_{\alpha\#\beta^{1+\delta_n} \to \infty} \beta^{\frac{5}{8}-\frac{n}{4}} (\sum_{i=1}^2 |v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta})^2 = 0 \quad \text{if} \quad n = 1, 2, \\ \sum_{i=1}^2 |v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta} \ge \delta \quad \text{and} \\ \lim_{\alpha\#\beta^{1+\delta_n} \to \infty} \beta^{-\frac{5}{8}+\frac{n}{4}} (\sum_{i=1}^2 |v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta})^2 = 0 \quad \text{if} \quad n = 3. \end{cases}$$

Proof. (i) Let $\Omega_{B,R_0} = \{x \in \Omega \mid \mathrm{dist}(x, \partial\Omega) \le \frac{R_0}{2}\} = \{x \mid \frac{R_0}{2} \le |x| \le R_0\}$. Since we have the convergence in Proposition 7, as in the proof of [7, Proposition 17], we deduce that for some $C > 0$, independent of large $\alpha \# \beta^{1+\delta_n}$,

$$|u_i^{\alpha,\beta}|^2_{2,\Omega_{B,R_0}} \le C \left(\beta^{-\frac{1}{2}} |\nabla u_i^{\alpha,\beta}|^2_{2,\Omega_{B,R_0}} + \sqrt{\beta} |u_3^{\alpha,\beta} u_i^{\alpha,\beta}|^2_{2,\Omega_{B,R_0}} \right) \quad (i = 1, 2). \tag{3.36}$$

Then we have

$$\sum_{i=1}^2 |v_{i,b}^{\alpha,\beta}|^2_{2,\Omega_b^\beta} \le C \sum_{i=1}^2 \left[|\nabla v_{i,b}^{\alpha,\beta}|^2_{2,\Omega_b^\beta} + \sqrt{\beta} \int_{\Omega_b^\beta} (v_3^{\alpha,\beta}(R_0 - \beta^{-\frac{1}{4}}|y|))^2 (v_{i,b}^{\alpha,\beta})^2 dy \right]. \tag{3.37}$$

If $n = 1, 2$, then Proposition 5 implies the first claim of (i).

If $n = 3$, then Proposition 5 implies that there is a constant $c > 0$ satisfying

$$\sum_{i=1}^2 \left(|\nabla v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + \int_{\Omega_c^\beta} (v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}} y))^2 (v_{i,c}^{\alpha,\beta})^2 \, dy \right) \le c. \tag{3.38}$$

In view of Proposition 7 and $\inf_{B_{\frac{R_0}{2}}(0)} U_3 > 0$, we get that $\inf_{\Omega_c^\beta} v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}} y) \ge C$ for some constant $C > 0$, and thus

$$\begin{aligned} &\sum_{i=1}^2 \left(|\nabla v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + C^2 |v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} \right) \\ &\le \sum_{i=1}^2 \left(|\nabla v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + \int_{\Omega_c^\beta} (v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}} y))^2 (v_{i,c}^{\alpha,\beta})^2 \, dy \right) \le c. \end{aligned} \tag{3.39}$$

Now we complete the proof of Proposition 8(i).

(ii) In view of (3.32), (3.37), and (3.39), there is a constant $c > 1$ satisfying

$$
\begin{aligned}
& c \min\left\{\frac{\beta^{2-\frac{n}{2}}}{\alpha}\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4 + \frac{\beta^{\frac{3}{4}}}{\alpha}\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}^4, \frac{\beta^{\frac{3}{4}}}{\alpha}, \frac{\beta^{2-\frac{n}{2}}}{\alpha}\right\} \\
& \geq \frac{\beta^{2-\frac{n}{2}}}{\alpha}\sum_{i=1}^{2}\left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2 + \int_{\Omega_c^\beta}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2(v_{i,c}^{\alpha,\beta})^2\,dy\right) \\
& + \frac{\beta^{\frac{3}{4}}}{\alpha}\sum_{i=1}^{2}\left(|\nabla v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2 + \int_{\Omega_b^\beta}\sqrt{\beta}(v_3^{\alpha,\beta}(R_0 - \beta^{-\frac{1}{4}}|y|))^2(v_{i,b}^{\alpha,\beta})^2dy\right) \\
& \geq \frac{1}{c}\left\{\frac{\beta^{2-\frac{n}{2}}}{\alpha}\sum_{i=1}^{2}\left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2 + |v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2\right)\right. \\
& \left. + \frac{\beta^{\frac{3}{4}}}{\alpha}\sum_{i=1}^{2}\left(|\nabla v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2 + |v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2\right)\right\}.
\end{aligned} \tag{3.40}
$$

Together with the Sobolev inequality, we get that for some constant $C > 0$,

$$
\begin{aligned}
& \min\left\{\left[\frac{\beta^{1-\frac{n}{4}}}{\alpha^{\frac{1}{2}}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \frac{\beta^{\frac{1}{2}-\frac{1}{8}}}{\alpha^{\frac{1}{2}}}\left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right]^2, \frac{\beta^{\frac{3}{4}}}{\alpha}, \frac{\beta^{2-\frac{n}{2}}}{\alpha}\right\} \\
& \geq C\left\{\frac{\beta^{2-\frac{n}{2}}}{\alpha}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \frac{\beta^{\frac{3}{4}}}{\alpha}\left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right\}.
\end{aligned} \tag{3.41}
$$

If $n = 1, 2$, then $\frac{5}{4} - \frac{n}{2} > 0$. Dividing (3.41) by $\frac{\beta^{\frac{3}{4}}}{\alpha}$, we get that

$$
\begin{aligned}
& \min\left\{\left[\beta^{\frac{5}{8}-\frac{n}{4}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right]^2, 1\right\} \\
& \geq C\left\{\beta^{\frac{5}{4}-\frac{n}{2}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right\} \\
& \geq C\left\{\beta^{\frac{5}{8}-\frac{n}{4}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right\}.
\end{aligned} \tag{3.42}
$$

Then we have

$$
\frac{1}{C} \geq \beta^{\frac{5}{8}-\frac{n}{4}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 + \left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2 \geq C, \tag{3.43}
$$

and thus

$$
\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2 = 0. \tag{3.44}
$$

The first and second line of (3.42) also implies that

$$2\left(\beta^{\frac{5}{4}-\frac{n}{2}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^4+\left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^4\right)$$
$$\geq C\left(\beta^{\frac{5}{4}-\frac{n}{2}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2+\left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2\right).$$

Together with (3.43), we see that $\beta^{\frac{5}{4}-\frac{n}{2}}(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta})^2=O(1)$. By (3.44), we have

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\beta^{\frac{5}{8}-\frac{n}{4}}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)^2=\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}O\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}\right)=0.$$

In view of (3.43), we get that $\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta})^2\geq C$ if $n=1,2$. The similar arguments also hold for the case $n=3$. Now we complete the proof of Proposition 8(ii). ◻

PROPOSITION 9. *There is a constant $C\geq 1$ satisfying for large $\alpha\#\beta^{1+\delta_n}$,*

$$\begin{cases}\frac{1}{C}\leq\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\Omega_b^\beta)}\leq C & i=1,2\quad \textit{if}\quad n=1,2,\\ \frac{1}{C}\leq\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)}\leq C & i=1,2\quad \textit{if}\quad n=3.\end{cases}\tag{3.45}$$

Moreover,

$$\begin{cases}\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)}\right)=\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}O(\beta^{\frac{n}{4}-\frac{5}{8}})=0\quad \textit{if}\quad n=1,2,\\ \lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\Omega_b^\beta)}\right)=\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}O(\beta^{-\frac{n}{4}+\frac{5}{8}})=0\quad \textit{if}\quad n=3.\end{cases}\tag{3.46}$$

Proof. First, we consider the case $n=3$. In view of Propositions 5 and 8 and (3.40), we have for some $c\geq 1$,

$$\begin{aligned}c&\geq\frac{|v_{1,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4}{\alpha}+\frac{|v_{2,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4}{\alpha}+|v_{1,c}^{\alpha,\beta}v_{2,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2+\beta^{-\frac{5}{4}+\frac{n}{2}}\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}^4\\&\geq\frac{1}{c}\sum_{i=1}^{2}\left\{\left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2+|v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2\right)+\beta^{-\frac{5}{4}+\frac{n}{2}}\left(|\nabla v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2+|v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2\right)\right\}.\end{aligned}\tag{3.47}$$

We recall from Proposition 8 that

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\beta^{-\frac{5}{8}+\frac{n}{4}}\left(\sum_{i=1}^{2}|v_{i,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}\right)^2=0.\tag{3.48}$$

Suppose that $\liminf_{\alpha\#\beta^{1+\delta_n}\to\infty}|v_{1,c}^{\alpha,\beta}|_{\infty,\Omega_c^\beta}=0$ or $\liminf_{\alpha\#\beta^{1+\delta_n}\to\infty}|v_{2,c}^{\alpha,\beta}|_{\infty,\Omega_c^\beta}=0$, we see from (3.47) and (3.48) that

$$\liminf_{\alpha\#\beta^{1+\delta_n}\to\infty}\sum_{i=1}^{2}\left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2+|v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2\right)=0,$$

which implies that $\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}|v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4\right)=0$ by the Sobolev inequality. This contradicts Proposition 8, and thus there is a constant $C>0$ such that for large

$\alpha\#\beta^{1+\delta_n}$, $\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)} \geq C$, $i = 1,2$. We also note that Proposition 6 implies that $v_{i,c}^{\alpha,\beta}$ is uniformly bounded from above in $L^\infty(\Omega_b^\beta)$. Therefore, the proof of (3.45) for $n = 3$ is obtained. Note that $-\frac{5}{4} + \frac{n}{2} > 0$ if $n = 3$. Then (3.47) and the Sobolev imbedding theorem imply the estimation (3.46) for $n = 3$.

Second, we consider cases $n = 1,2$. In view of Propositions 8 and 5 and (3.40), we have for some $c \geq 1$,

$$\begin{aligned} c &\geq \beta^{\frac{5}{4}-\frac{n}{2}} \sum_{i=1}^{2} |v_{i,c}^{\alpha,\beta}|_{4,\Omega_c^\beta}^4 + \frac{|v_{1,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}^4}{\alpha} + \frac{|v_{2,b}^{\alpha,\beta}|_{4,\Omega_b^\beta}^4}{\alpha} + |v_{1,b}^{\alpha,\beta} v_{2,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2 \\ &\geq \frac{1}{c} \sum_{i=1}^{2} \left\{ \beta^{\frac{5}{4}-\frac{n}{2}} \left(|\nabla v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2 + |v_{i,c}^{\alpha,\beta}|_{2,\Omega_c^\beta}^2 \right) + \left(|\nabla v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2 + |v_{i,b}^{\alpha,\beta}|_{2,\Omega_b^\beta}^2 \right) \right\}. \end{aligned} \tag{3.49}$$

By a similar argument with the case $n = 3$ above, there is a constant $C > 0$ such that $\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\Omega_b^\beta)} \geq C$ for $i = 1,2$. In view of (3.49) and the Sobolev imbedding theorem, we also note that $v_{i,b}^{\alpha,\beta}$ is uniformly bounded from above in $L^\infty(\Omega_b^\beta)$.

If $n = 1$, then the Sobolev imbedding theorem and (3.49) imply the estimation (3.46). If $n = 2$, then the Sobolev imbedding theorem and (3.49) imply $\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} |v_{i,c}^{\alpha,\beta}|_{p,\Omega_c^\beta} = 0$ for any $p > 1$. By applying $W^{2,p}$ estimates to (3.31), we see that the estimation (3.46) holds. Now we complete the proof of Proposition 9. ∎

PROPOSITION 10. *There exists a constant $R > 0$ such that*

$$\begin{cases} \liminf_{\alpha\#\beta^{1+\delta_n}\to\infty} \left(\sum_{i=1}^2 \int_0^R |v_{i,b}^{\alpha,\beta}|^4 dy \right) > 0 & \text{if } n = 1,2, \\ \liminf_{\alpha\#\beta^{1+\delta_n}\to\infty} \left(\sum_{i=1}^2 \int_{B_R(0)} |v_{i,c}^{\alpha,\beta}|^4 dy \right) > 0 & \text{if } n = 3. \end{cases} \tag{3.50}$$

Proof. First, we consider cases $n = 1,2$. Suppose that for any fixed $R > 0$,

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} \left(\sum_{i=1}^{2} \int_0^R |v_{i,b}^{\alpha,\beta}|^4 dy \right) = 0. \tag{3.51}$$

Since $v_3^{\alpha,\beta}(R_0) = 0$ and $\frac{dv_3^{\alpha,\beta}}{dr}\Big|_{r=R_0} < 0$, there exists a constant $c > 0$ such that

$$\sqrt{\beta}(v_3^{\alpha,\beta}(R_0 - \beta^{-\frac{1}{4}}|y|))^2 \geq c|y|^2 \quad \text{for} \quad 0 \leq |y| \leq \frac{\beta^{\frac{1}{4}} R_0}{2}.$$

In view of Proposition 5, we have

$$\int_{R \leq s \leq \frac{\beta^{\frac{1}{4}} R_0}{2}} \sum_{i=1}^{2} |v_{i,b}^{\alpha,\beta}|^2 ds = O(R^{-2}). \tag{3.52}$$

By (3.51), (3.52), and Proposition 9, we get that $\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} \left(\int_{\Omega_b^\beta} \sum_{i=1}^2 |v_{i,b}^{\alpha,\beta}|^4 dy \right) = 0$, which contradicts Proposition 8. This implies that there is a constant $R > 0$ satisfying

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} \left(\sum_{i=1}^{2} \int_0^R |v_{i,b}^{\alpha,\beta}|^4 dy \right) > 0 \quad \text{if } n = 1,2.$$

Second, we consider a case $n = 3$. From (3.31), we have

$$\begin{cases} \Delta v_{1,c}^{\alpha,\beta} = O(1)(|v_{1,c}^{\alpha,\beta}| + |v_{2,c}^{\alpha,\beta}|), \\ \Delta v_{2,c}^{\alpha,\beta} = O(1)(|v_{1,c}^{\alpha,\beta}| + |v_{2,c}^{\alpha,\beta}|). \end{cases} \tag{3.53}$$

To the contrary, suppose that for any fixed $R > 0$,

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\int_{B_R(0)}|v_{i,c}^{\alpha,\beta}|^4 dy\right) = \lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\int_0^R |v_{i,c}^{\alpha,\beta}|^4 s^{n-1} ds |S^{n-1}|\right) = 0.$$

Then $W^{2,p}$-estimates and (3.53) imply that

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(B_{\frac{R}{2}}(0))}\right) = 0 \quad \text{for any fixed} \quad R > 0. \tag{3.54}$$

In view of Proposition 8, we see that $\int_{\frac{R}{2}\le s\le \frac{\sqrt{\beta}R_0}{2}} \sum_{i=1}^{2} \left(\left|\frac{dv_{i,c}^{\alpha,\beta}}{ds}\right|^2 + |v_{i,c}^{\alpha,\beta}|^2\right) s^{n-1} ds \le C$ for some constant $C > 0$; thus

$$\int_{\frac{R}{2}\le s\le \frac{\sqrt{\beta}R_0}{2}} \sum_{i=1}^{2} \left(\left|\frac{dv_{i,c}^{\alpha,\beta}}{ds}\right|^2 + |v_{i,c}^{\alpha,\beta}|^2\right) ds = O(R^{-(n-1)}). \tag{3.55}$$

By Sobolev inequality and (3.54), we get that $\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} \sum_{i=1}^{2} \|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)} = 0$, which contradicts Proposition 9. Thus there is a constant $R > 0$ such that for $n = 3$,

$$\lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\left(\sum_{i=1}^{2}\int_{B_R(0)}|v_{i,c}^{\alpha,\beta}|^4 dy\right) > 0.$$

We are going to improve the estimations (3.52) and (3.55). $\square$

PROPOSITION 11. *There are constants $c, C > 0$, independent of large $\alpha\#\beta^{1+\delta_n}$, satisfying*

$$\begin{cases} v_{i,b}^{\alpha,\beta}(y) \le C\exp(-c|y|) \quad \textit{for } 0 \le |y| \le \beta^{\frac{1}{4}}R_0 \quad \textit{if} \quad n = 1, 2, \\ v_{i,c}^{\alpha,\beta}(y) \le C\exp(-c|y|) \quad \textit{for } 0 \le |y| \le \sqrt{\beta}R_0 \quad \textit{if} \quad n = 3. \end{cases} \tag{3.56}$$

Proof. We note from (3.29) that

$$\frac{d^2 v_{i,b}^{\alpha,\beta}}{ds^2} - \frac{(n-1)\beta^{-\frac{1}{4}}}{R_0 - \beta^{-\frac{1}{4}}s}\frac{dv_{i,b}^{\alpha,\beta}}{ds} \ge -C(v_{1,b}^{\alpha,\beta} + v_{2,b}^{\alpha,\beta}) \quad \text{in } \Omega_b^\beta, \quad i = 1, 2. \tag{3.57}$$

Together with (3.52) and [13, Theorem 9.20], we have

$$\lim_{R\to+\infty}\sum_{i=1}^{2}\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(R\le|y|\le\beta^{\frac{1}{4}}\frac{R_0}{2})} = 0 \quad \text{if} \quad n = 1, 2. \tag{3.58}$$

Moreover, (3.55) and the Sobolev inequality imply that

$$\lim_{R\to+\infty}\sum_{i=1}^{2}\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(R\le|y|\le\sqrt{\beta}\frac{R_0}{2})} = 0 \quad \text{if} \quad n = 3. \tag{3.59}$$

In view of (3.28) and (3.30), we have

$$v_{i,b}^{\alpha,\beta}(\beta^{\frac14}(R_0-|x|)) = \beta^{\frac14} v_{i,c}^{\alpha,\beta}(\sqrt{\beta}x). \tag{3.60}$$

By (3.46), we see that for $i=1,2$,

$$\begin{cases} \|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\beta^{\frac14}\frac{R_0}{2}\le|y|\le\beta^{\frac14}R_0)} = \beta^{\frac14}\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)} = O(\beta^{\frac{2n-3}{8}}) = o(1) \text{ if } \ n=1,\\ \|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\beta^{\frac14}\frac{R_0}{2}\le|y|\le\beta^{\frac14}R_0)} = \beta^{\frac14}\|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\Omega_c^\beta)} = O(\beta^{\frac{2n-3}{8}}) = O(\beta^{\frac18}) \text{ if } \ n=2,\\ \|v_{i,c}^{\alpha,\beta}\|_{L^\infty(\sqrt{\beta}\frac{R_0}{2}\le|y|\le\sqrt{\beta}R_0)} = \beta^{-\frac14}\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\Omega_b^\beta)} = O(\beta^{\frac{3-2n}{8}}) = o(1) \ \text{ if } \ \ n=3. \end{cases} \tag{3.61}$$

Now we shall prove (3.56).

First, we consider cases $n=1,2$. Although our proof is based on [7], we need more careful analysis in detail since $\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(|y|\le\beta^{\frac14}R_0)}$ might not be sufficiently small for $n=2$ due to (3.61). Let $w_i^{\alpha,\beta}(x) = v_{i,b}^{\alpha,\beta}(\beta^{\frac14}R_0-|x|)$ for $i=1,2$ and $w_3^{\alpha,\beta}(x) = v_3^{\alpha,\beta}(x)$. For the exponential decay property of $w_1^{\alpha,\beta}$ and $w_2^{\alpha,\beta}$, we take the first eigenfunction $\varphi>0$ in Ω of $-\Delta$ with $\varphi=0$ on $\partial\Omega$ and $\max_{x\in\Omega}\varphi(x)=1$. Let $\mu_1>0$ be the corresponding first eigenvalue. For $a>0$, which will be determined later, we define

$$\Phi(y)\equiv\exp(-a(\beta^{\frac14}\varphi(\beta^{-\frac14}y))^2), \quad y=(\beta)^{\frac14}x\in\Omega_\beta^*=\{y=(\beta)^{\frac14}x \mid x\in\Omega\}.$$

We note that

$$-\Delta w_1^{\alpha,\beta}+\left(\frac{\lambda_1}{\sqrt{\beta}}+\beta^{\frac12}\beta_{13}(w_3^{\alpha,\beta}(\beta^{-\frac14}y))^2-\frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2-\beta_{12}(w_2^{\alpha,\beta})^2\right)w_1^{\alpha,\beta}=0 \quad\text{in}\quad \Omega_\beta^*,$$

and from Proposition 7 that for some $e_1>0$, independent of $j\ge1$,

$$\beta^{\frac12}\beta_{13}(w_3^{\alpha,\beta}(\beta^{-\frac14}y))^2\ge e_1(d(y,\Omega_\beta^*))^2, \quad y\in\Omega_\beta^*.$$

We also note that

$$-\Delta\Phi(y)=(-2a\mu_1(\varphi(x))^2+2a|\nabla_x\varphi(x)|^2-4a^2\beta^{\frac12}(\varphi(x))^2|\nabla_x\varphi(x)|^2)\Phi(y),$$

and that for some $e_2>0$, independent of $\beta\ge1$,

$$\beta^{\frac12}(\varphi(x))^2=\beta^{\frac12}(\varphi(\beta^{-\frac14}y))^2\le e_2(d(y,\Omega_\beta^*))^2, \quad y\in\Omega_\beta^*.$$

Then, for $e_3\equiv\max_{x\in\Omega}|\nabla\varphi(x)|^2$,

$$\begin{aligned} &-\Delta\Phi+\left(\frac{\lambda_1}{\sqrt{\beta}}+\beta^{\frac12}\beta_{13}(w_3^{\alpha,\beta}(\beta^{-\frac14}y))^2-\frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2-\beta_{12}(w_2^{\alpha,\beta})^2\right)\Phi\\ &=\left(\frac{\lambda_1}{\sqrt{\beta}}+\beta^{\frac12}\beta_{13}(w_3^{\alpha,\beta}(\beta^{-\frac14}y))^2-\frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2-\beta_{12}(w_2^{\alpha,\beta})^2\right)\Phi\\ &\quad-\Big(2a\mu_1(\varphi(\beta^{-\frac14}y))^2-2a|\nabla_x\varphi(\beta^{-\frac14}y)|^2+4a^2(\beta)^{\frac12}(\varphi(\beta^{-\frac14}y))^2|\nabla_x\varphi(\beta^{-\frac14}y)|^2\Big)\Phi\\ &\ge\Big(e_1(d(y,\Omega_\beta^*))^2-4a^2e_2e_3(d(y,\Omega_\beta^*))^2\Big)\Phi\\ &\quad-\left(\frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2+\beta_{12}(w_2^{\alpha,\beta})^2+2a\mu_1(\varphi(\beta^{-\frac14}y))^2\right)\Phi\\ &=\left(\frac{e_1}{2}(d(y,\Omega_\beta^*))^2-4a^2e_2e_3(d(y,\Omega_\beta^*))^2\right)\Phi\\ &\quad+\left(\frac{e_1}{2}(d(y,\Omega_\beta^*))^2-\frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2-\beta_{12}(w_2^{\alpha,\beta})^2-2a\mu_1(\varphi(\beta^{-\frac14}y))^2\right)\Phi. \end{aligned}$$

In view of (3.58) and (3.61), we see that if $d(y, \partial\Omega^*_\beta) \geq m$ for large $m > 0$,

$$\frac{e_1}{2}(d(y, \Omega^*_\beta))^2 - \frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2 - \beta_{12}(w_2^{\alpha,\beta})^2 - 2a\mu_1(\varphi(\beta^{-\frac{1}{4}}y))^2 > 0.$$

We take small $a > 0$ so that $\frac{e_1}{2} - 4a^2 e_2 e_3 \geq \frac{e_1}{4}$. Then for $d(y, \partial\Omega^*_\beta) \geq m$,

$$-\Delta\Phi + \left(\frac{\lambda_1}{\sqrt{\beta}} + \beta^{\frac{1}{2}}\beta_{13}(w_3^{\alpha,\beta}(\beta^{-\frac{1}{4}}y))^2 - \frac{\beta_{11}}{\alpha}(w_1^{\alpha,\beta})^2 - \beta_{12}(w_2^{\alpha,\beta})^2\right)\Phi > 0.$$

By (3.58), we also have $\lim_{R\to+\infty}\|v_{i,b}^{\alpha,\beta}\|_{L^\infty(\partial B_R(0))} = 0$ for $i = 1, 2$. Thus, it follows from the comparison principle that for some large $m > 0$, independent of large $\alpha\#\beta^{1+\delta_n}$, there exists a constant $D > 0$ such that

$$w_1^{\alpha,\beta}(y) \leq D\Phi(y), \quad d(y, \partial\Omega^*_\beta) \geq m.$$

Note that $e_4 d(x, \partial\Omega) \leq \varphi(x)$, $x \in \Omega$ for some $e_4 > 0$ and $\{w_1^{\alpha,\beta}, w_2^{\alpha,\beta}\}_j$ is bounded in $L^\infty(d(y, \partial\Omega^*_\beta) \leq m)$ by Proposition 9. Therefore, there exist constant $C, c > 0$ such that

$$w_1^{\alpha,\beta}(y) \leq C\exp(-c(d(y, \partial\Omega^*_\beta)^2)), \quad x \in \Omega^*_\beta.$$

By the same argument with $w_1^{\alpha,\beta}$, we get the same estimate for $w_2^{\alpha,\beta}$. This implies the proof of (3.56) for $n = 1, 2$.

Second, we consider a case $n = 3$. We recall from (3.31) that

$$\Delta v_{1,c}^{\alpha,\beta} - \left(\frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2\right)v_{1,c}^{\alpha,\beta} = 0. \tag{3.62}$$

Let

$$\Psi(y) = \exp(-\sigma_1|y|) + \exp\left(-\frac{\sigma_2}{\sqrt{\beta}}\left(|y| - \frac{3\sqrt{\beta}}{4}R_0\right)^2\right), \tag{3.63}$$

where $\sigma_1, \sigma_2 > 0$ will be determined later. We have

$$\begin{aligned} &\Delta\Psi - \left(\frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2\right)\Psi \\ &= -\exp(-\sigma_1|y|)A(y) - \exp\left(-\frac{\sigma_2}{\sqrt{\beta}}\left(|y| - \frac{3\sqrt{\beta}}{4}R_0\right)^2\right)B(y), \end{aligned} \tag{3.64}$$

where

$$A(y) = \frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 - \sigma_1^2 + \frac{2\sigma_1}{|y|} \tag{3.65}$$

and

$$\begin{aligned} B(y) = &\frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 \\ &- \frac{4\sigma_2^2}{\beta}(|y| - \frac{3\sqrt{\beta}}{4}R_0)^2 + \frac{2\sigma_2}{\sqrt{\beta}} + \frac{4\sigma_2}{\sqrt{\beta}}\frac{(|y| - \frac{3\sqrt{\beta}}{4}R_0)}{|y|}. \end{aligned} \tag{3.66}$$

Fix a constant large $R > 0$. We claim that

$$A(y) > 0 \quad \text{and} \quad B(y) > 0 \quad \text{if} \quad R < |y| \le \frac{3R_0\sqrt{\beta}}{4}. \tag{3.67}$$

Proposition 7 yields

$$c_0(R_0 - \beta^{-\frac{1}{2}}|y|) \le v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y) \le c_0^{-1}(R_0 - \beta^{-\frac{1}{2}}|y|) \quad \text{for some constant} \quad 0 < c_0 < 1. \tag{3.68}$$

By (3.68), (3.59), and (3.61), if $\sigma_1 > 0$ is sufficiently small and $R > 0$ is large enough, then there is a constant $c_1, c_2 > 0$ satisfying

$$\begin{aligned} A(y) &\ge \frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}c_0^2(R_0 - \beta^{-\frac{1}{2}}|y|)^2 - \sigma_1^2 + \frac{2\sigma_1}{|y|} \\ &\ge c_1R_0^2 - \sigma_1^2 + \frac{2\sigma_1}{|y|} > c_2 > 0 \quad \text{if} \quad R < |y| \le \frac{3R_0\sqrt{\beta}}{4}. \end{aligned} \tag{3.69}$$

Moreover, we also see from (3.68), (3.59), and (3.61) that if $\sigma_2 > 0$ is sufficiently small and $R > 0$ is large enough, then there is a constant $c_3 > 0$ such that

$$\begin{aligned} B(y) \ge{}& \frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}c_0^2(R_0 - \beta^{-\frac{1}{2}}|y|)^2 \\ &- 4\sigma_2^2\left(\frac{3R_0}{4} - \beta^{-\frac{1}{2}}|y|\right)^2 - 2\sigma_2\left(\left(\frac{3R_0}{4} - \beta^{-\frac{1}{2}}|y|\right)^2 + |y|^{-2}\right) \\ &> c_3 > 0 \quad \text{if} \quad R < |y| \le \frac{3R_0\sqrt{\beta}}{4}. \end{aligned} \tag{3.70}$$

Now we see that the claim (3.67) holds. By the comparison principle, (3.59), and (3.61), there is a constant $c > 0$ satisfying

$$v_{1,c}^{\alpha,\beta}(y) \le C\Psi(y) \quad \text{for} \quad R \le |y| \le \frac{3\sqrt{\beta}R_0}{4}. \tag{3.71}$$

Let $\Upsilon(y) = \exp\big(-\frac{\sigma_3}{\sqrt{\beta}}\big(|y| - \sqrt{\beta}R_0\big)^2 - \frac{\sigma_3}{\beta^{\frac{1}{4}}}\big(|y| - \frac{\sqrt{\beta}R_0}{2}\big)\big)$, where $\sigma_3 > 0$ will be determined later. Now we are going to show that there is a constant $C > 0$ satisfying

$$v_{1,c}^{\alpha,\beta}(y) \le C\Upsilon(y) \quad \text{for} \quad \frac{\sqrt{\beta}R_0}{2} \le |y| \le \sqrt{\beta}R_0. \tag{3.72}$$

We see that

$$\Delta\Upsilon - \left(\frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2\right)\Upsilon(y) = -C(y)\Upsilon(y),$$

where

$$\begin{aligned} C(y) ={}& \frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2 \\ &- \left(\frac{2\sigma_3}{\sqrt{\beta}}\left(|y| - \sqrt{\beta}R_0\right) + \frac{\sigma_3}{\beta^{\frac{1}{4}}}\right)^2 + \frac{2\sigma_3}{\sqrt{\beta}} + \frac{4\sigma_3}{\sqrt{\beta}}\frac{\left(|y| - \sqrt{\beta}R_0\right)}{|y|} + \frac{2\sigma_3}{\beta^{\frac{1}{4}}|y|}. \end{aligned} \tag{3.73}$$

From (3.68) and (3.61), we see that if $\sigma_3 > 0$ is sufficiently small, then we see that as $\beta \to +\infty$,

$$\begin{aligned}
C(y) &\geq \frac{\lambda_1}{\beta} - c_4^2\alpha^{-1}\beta^{-\frac{3}{4}} - c_4^2\beta^{-\frac{3}{4}} + \frac{2\sigma_3}{\sqrt{\beta}} + \beta_{13}c_0^2(R_0 - \beta^{-\frac{1}{2}}|y|)^2 \\
&\quad - 2\left(4\sigma_3^2\left(R_0 - \beta^{-\frac{1}{2}}|y|\right)^2 + \frac{\sigma_3^2}{\sqrt{\beta}}\right) - 2\sigma_3\left((R_0 - \beta^{-\frac{1}{2}}|y|)^2 + |y|^{-2}\right) \\
&> 0 \qquad \text{for} \quad \frac{R_0\sqrt{\beta}}{2} \leq |y| \leq R_0\sqrt{\beta}.
\end{aligned} \tag{3.74}$$

If $0 < \sigma_3 \ll \sigma_i \ll 1$, $i = 1, 2$, then the comparison principle, (3.71), and $v_{1,c}^{\alpha,\beta}(y)\big|_{|y|=\sqrt{\beta}R_0} = 0$ implies that there is a constant $C > 0$ such that if $\frac{\sqrt{\beta}R_0}{2} \leq |y| \leq \sqrt{\beta}R_0$, then

$$v_{1,c}^{\alpha,\beta}(y) \leq C\Upsilon(y) = C\exp\left(-\frac{\sigma_3}{\sqrt{\beta}}\left(|y| - \sqrt{\beta}R_0\right)^2 - \frac{\sigma_3}{\beta^{\frac{1}{4}}}\left(|y| - \frac{\sqrt{\beta}R_0}{2}\right)\right). \tag{3.75}$$

The above estimation also holds for $v_{2,c}^{\alpha,\beta}$. In view of (3.75), we see that

$$\sum_{i=1}^{2} v_{i,c}^{\alpha,\beta} \leq \begin{cases} C\exp\left(-\frac{\sigma_3}{\sqrt{\beta}}\left(|y| - \sqrt{\beta}R_0\right)^2\right) & \text{if } \frac{\sqrt{\beta}R_0}{2} \leq |y| \leq \frac{3\sqrt{\beta}R_0}{4}, \\ C\exp\left(-\frac{\sigma_3 R_0 \beta^{\frac{1}{4}}}{4}\right) & \text{if } \frac{3\sqrt{\beta}R_0}{4} \leq |y| \leq \sqrt{\beta}R_0. \end{cases} \tag{3.76}$$

Let $\Phi(y) = \exp\left(-\sigma_4\sqrt{\beta}\right)$, where $\sigma_4 > 0$ will be determined later. Now we are going to show that there is a constant $C > 0$ satisfying

$$v_{1,c}^{\alpha,\beta}(y) \leq C\Phi(y) \quad \text{for} \quad \frac{3\sqrt{\beta}R_0}{4} \leq |y| \leq \sqrt{\beta}R_0. \tag{3.77}$$

We see that

$$\Delta\Phi - \left(\frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2\right)\Phi(y) = -D(y)\Phi(y),$$

where

$$D(y) = \frac{\lambda_1}{\beta} - \frac{\beta_{11}}{\alpha}(v_{1,c}^{\alpha,\beta})^2 - \beta_{12}(v_{2,c}^{\alpha,\beta})^2 + \beta_{13}(v_3^{\alpha,\beta}(\beta^{-\frac{1}{2}}y))^2. \tag{3.78}$$

From (3.76), we see that as $\beta \to +\infty$,

$$\begin{aligned}
D(y) &\geq \frac{\lambda_1}{\beta} - c\exp\left(-\frac{\sigma_3 R_0\beta^{\frac{1}{4}}}{2}\right) + \beta_{13}c_0^2(R_0 - \beta^{-\frac{1}{2}}|y|)^2 \\
&> 0 \qquad \text{for} \quad \frac{3R_0\sqrt{\beta}}{4} \leq |y| \leq R_0\sqrt{\beta}.
\end{aligned} \tag{3.79}$$

If $0 < \sigma_4 \ll \sigma_i \ll 1$, $i = 1, 2, 3$, then the comparison principle, (3.76), and $v_{1,c}^{\alpha,\beta}(y)\big|_{|y|=\sqrt{\beta}R_0} = 0$ implies that there is a constant $C > 0$ such that if $\frac{3\sqrt{\beta}R_0}{4} \leq |y| \leq \sqrt{\beta}R_0$, then

$$v_{1,c}^{\alpha,\beta}(y) \leq C\Phi(y) = C\exp\left(-\sigma_4\sqrt{\beta}\right). \tag{3.80}$$

□

Together with (3.71) and (3.75), by choosing small positive constants $0 < c \ll \sigma_4 \ll \sigma_i \ll 1$, $i = 1, 2, 3$, we see that (3.56) also holds for $n = 3$.

Now, we prove the following convergence result for the first and second components.

PROPOSITION 12. (i) *If* $n = 1, 2$, *there is a positive least energy solution* $(v_{1,b}, v_{2,b})$ *of* (1.5) *such that* $(v_{1,b}^{\alpha,\beta}, v_{2,b}^{\alpha,\beta}) \to (v_{1,b}, v_{2,b})$ *in* $[H^1(\Omega_b^\beta)]^2$, *up to a subsequence, as* $\alpha\#\beta^{1+\delta_n} \to \infty$.

(ii) *If* $n = 3$, *there is a positive least energy solution* $(v_{1,c}, v_{2,c})$ *of* (1.11) *such that* $(v_{1,c}^{\alpha,\beta}, v_{2,c}^{\alpha,\beta}) \to (v_{1,c}, v_{2,c})$ *in* $[H^1(\Omega_c^\beta)]^2$, *up to a subsequence, as* $\alpha\#\beta^{1+\delta_n} \to \infty$.

Proof. In view of Propositions 5 and 8, it holds that for $n = 1, 2$,

(3.81)
$$\begin{aligned}
&\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} 2\beta_{12}|v_{1,b}^{\alpha,\beta} v_{2,b}^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|)^{\frac{n-1}{2}}|^2_{2,\Omega_b^\beta}|S^{n-1}| \\
&= \lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\Bigg[\bigg(\left|\nabla v_{1,b}^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|)^{\frac{n-1}{2}}\right|^2_{2,\Omega_b^\beta} + \left|\nabla v_{2,b}^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|)^{\frac{n-1}{2}}\right|^2_{2,\Omega_b^\beta} \\
&+ \sum_{i=1}^{2}\beta_{i3}\int_{\Omega_b^\beta}\sqrt{\beta}(v_3^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|))^2(v_{i,b}^{\alpha,\beta})^2\,(R_0 - \beta^{-\frac14}|y|))^{n-1}dy\bigg)|S^{n-1}| \\
&+ \beta^{\frac54-\frac n2}\bigg\{\sum_{i=1}^{2}\bigg(|\nabla v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + \frac{\lambda_i}{\beta}|v_{i,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + \beta_{i3}\int_{\Omega_c^\beta}(v_3^{\alpha,\beta}(\beta^{-\frac12}y))^2(v_{i,c}^{\alpha,\beta})^2\,dy\bigg)\bigg\}\Bigg] \\
&\le 4M_b|S^{n-1}|R_0^{n-1},
\end{aligned}$$

and for $n = 3$,

(3.82)
$$\begin{aligned}
&\lim_{\alpha\#\beta^{1+\delta_n}\to\infty} 2\beta_{12}|v_{1,c}^{\alpha,\beta} v_{2,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} \\
&= \lim_{\alpha\#\beta^{1+\delta_n}\to\infty}\Bigg[\bigg(|\nabla v_{1,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + |\nabla v_{2,c}^{\alpha,\beta}|^2_{2,\Omega_c^\beta} + \sum_{i=1}^{2}\beta_{i3}\int_{\Omega_c^\beta}(v_3^{\alpha,\beta}(\beta^{-\frac12}y))^2(v_{i,c}^{\alpha,\beta})^2\,dy\bigg) \\
&+ \beta^{\frac n2-\frac54}|S^{n-1}|\bigg\{\sum_{i=1}^{2}\bigg(\left|\nabla v_{i,b}^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|)^{\frac{n-1}{2}}\right|^2_{2,\Omega_b^\beta} + \frac{\lambda_i}{\sqrt{\beta}}|v_{i,b}^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|)^{\frac{n-1}{2}}|^2_{2,\Omega_b^\beta}\bigg) \\
&+ \sum_{i=1}^{2}\beta_{i3}\int_{\Omega_b^\beta}\sqrt{\beta}(v_3^{\alpha,\beta}(R_0 - \beta^{-\frac14}|y|))^2(v_{i,b}^{\alpha,\beta})^2(R_0 - \beta^{-\frac14}|y|)^{n-1}dy\bigg\}\Bigg] \\
&\le 4M_c.
\end{aligned}$$

We define

$$\check{v}^{\alpha,\beta}(y) = (v_1^{\alpha,\beta}(y), v_2^{\alpha,\beta}(y)) = \begin{cases} (v_{1,b}^{\alpha,\beta}(y), v_{2,b}^{\alpha,\beta}(y)) \text{ if } n = 1, 2, \\ (v_{1,c}^{\alpha,\beta}(y), v_{2,c}^{\alpha,\beta}(y)) \text{ if } n = 3, \end{cases} \tag{3.83}$$

and

$$\Omega^\beta = \begin{cases} \Omega_b^\beta \text{ if } n = 1, 2, \\ \Omega_c^\beta \text{ if } n = 3. \end{cases} \tag{3.84}$$

From Proposition 8, we see that $\check{v}^{\alpha,\beta}$ is uniformly bounded in $[H^1(\Omega^\beta)]^2$. Thus we can choose a subsequence α_j, β_j of α, β with $\lim_{j\to\infty}\alpha_j\#\beta_j^{1+\delta_n} = \infty$ such that there is $\check{v} = (v_1, v_2)$ satisfying $\check{v}^j \equiv \check{v}^{\alpha_j,\beta_j} \rightharpoonup \check{v}$ weakly in $[H^1(\Omega^{\beta_j})]^2$ and strongly in

$[L^4_{\text{loc}}(\Omega^{\beta_j})]^2$ as $j \to \infty$. From Propositions 3 and 7, we recall that $u_3^{\alpha_j,\beta_j} = v_3^{\alpha_j,\beta_j} \to U_3$ in $C^1(\overline{\{x \in \Omega \mid |x| \geq R\}})$ for $n = 1, 2$ and in $C(\Omega)$ for $n = 3$ as $j \to \infty$ where U_3 is a least energy solution of (1.4). Then by using test functions, it is easy to see that $\breve{v}$ is a weak solution of (1.5) for $n = 1, 2$ and (1.11) for $n = 3$. In view of Proposition 10, we have $v_1 \neq 0$ and $v_2 \neq 0$. By (3.81) and (3.82), we get that if $n = 1, 2$,

$$\begin{aligned}
&4M_b|S^{n-1}|R_0^{n-1}\\
&\leq \sum_{i=1}^{2}\left(\Big|\frac{dv_i}{dt}\Big|^2_{2,\mathbb{R}^1_+} + \beta_{i3}\int_{\mathbb{R}^1_+}\Big(\frac{dU_3(R_0)}{dr}\Big)^2 v_i^2 t^2 dt\right)|S^{n-1}|R_0^{n-1}\\
&\leq \lim_{j\to\infty}\sum_{i=1}^{2}\left(\Big|\nabla v_{i,b}^{\alpha_j,\beta_j}(R_0 - \beta_j^{-\frac14}|y|)^{\frac{n-1}{2}}\Big|^2_{2,\Omega_b^{\beta_j}}\right.\\
&\left.\quad + \beta_{i3}\int_{\Omega_b^{\beta_j}}\sqrt{\beta_j}(v_3^{\alpha_j,\beta_j}(R_0 - \beta_j^{-\frac14}|y|))^2(v_{i,b}^{\alpha_j,\beta_j})^2\,(R_0 - \beta_j^{-\frac14}|y|))^{n-1}dy\right)|S^{n-1}|\\
&\leq 4M_b|S^{n-1}|R_0^{n-1},
\end{aligned} \tag{3.85}$$

and if $n = 3$,

$$\begin{aligned}
4M_c &\leq \sum_{i=1}^{2}\left(|\nabla v_i|^2_{2,\mathbb{R}^n} + \beta_{i3}\int_{\mathbb{R}^n}(U_3(0))^2(v_i)^2\,dy\right)\\
&\leq \lim_{j\to\infty}\sum_{i=1}^{2}\left(|\nabla v_{i,c}^{\alpha_j,\beta_j}|^2_{2,\Omega_c^{\beta_j}} + \beta_{i3}\int_{\Omega_c^{\beta_j}}(v_3^{\alpha_j,\beta_j}(\beta_j^{-\frac12}y))^2(v_{i,c}^{\alpha_j,\beta_j})^2\,dy\right)\\
&\leq 4M_c.
\end{aligned} \tag{3.86}$$

Together with the exponential decay of $\breve{v}^j$ in Proposition 11, we note that not only does $\breve{v}^j$ weekly converge to $\breve{v}$ in $[H^1(\Omega^{\beta_j})]^2$, but also the $[H^1(\Omega^{\beta_j})]^2$-norm of $\breve{v}^j$ converges to the $[H^1(\Omega^{\beta_j})]^2$-norm of $\breve{v}$. Therefore, $\breve{v}^j$ strongly converges to $\breve{v}$ in $[H^1(\Omega^{\beta_j})]^2$. Moreover, (3.85) and (3.86) imply $\breve{v}$ is a positive least energy solution of (1.5) for $n = 1, 2$ and (1.11) for $n = 3$. Now we complete the proof of Proposition 12. ∎

3.4. Proof of Theorem 1. Completion of Proof for Theorem 1. In the preceding propositions, we have proved all results in Theorem 1 except (1.17). We recall the notations K and G in (3.14) and (3.15). In view of Proposition 3 and Lemma 2(ii), we see that

$$\begin{aligned}
c_{\alpha,\beta} = I(\vec{u}^{\alpha,\beta}) &= \frac{\|u_3^{\alpha,\beta}\|^4_{\lambda_3,\Omega}}{4\beta_{33}|u_3^{\alpha,\beta}|^4_{4,\Omega}} + \frac{K(\vec{u}^{\alpha,\beta})}{4\beta_{33}|u_3^{\alpha,\beta}|^4_{4,\Omega}G(\vec{u}^{\alpha,\beta})}\\
&\geq L + \frac{K(\vec{u}^{\alpha,\beta})}{4\beta_{33}|u_3^{\alpha,\beta}|^4_{4,\Omega}G(\vec{u}^{\alpha,\beta})},
\end{aligned} \tag{3.87}$$

here we used $L = \inf_{u\in H^1_0(\Omega)}\frac{\|u\|^4_{\lambda_3,\Omega}}{4\beta_{33}|u|^4_{4,\Omega}}$. By (3.3) and Proposition 3(iii), we get

$$\begin{aligned}
&I(\vec{u}^{\alpha,\beta})\\
&\geq L + \frac{(L + o(1))(\|u_1^{\alpha,\beta}\|^2_{\lambda_1,\Omega} + \|u_2^{\alpha,\beta}\|^2_{\lambda_2,\Omega} + \beta\beta_{13}|u_1^{\alpha,\beta}u_3^{\alpha,\beta}|^2_{2,\Omega} + \beta\beta_{23}|u_2^{\alpha,\beta}u_3^{\alpha,\beta}|^2_{2,\Omega})^2}{G(\vec{u}^{\alpha,\beta})}.
\end{aligned} \tag{3.88}$$

From Proposition 3(iii), we also note that

$$
\begin{aligned}
G(\vec{u}^{\alpha,\beta}) &= (\beta_{11}|u_1^{\alpha,\beta}|_{4,\Omega}^4 + \beta_{22}|u_2^{\alpha,\beta}|_{4,\Omega}^4 + 2\alpha\beta_{12}|u_1^{\alpha,\beta}u_2^{\alpha,\beta}|_{2,\Omega}^2)\beta_{33}|u_3^{\alpha,\beta}|_{4,\Omega}^4 \\
&\quad - (\beta\beta_{13}|u_1^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2 + \beta\beta_{23}|u_2^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2)^2 \\
&= (\beta_{11}|u_1^{\alpha,\beta}|_{4,\Omega}^4 + \beta_{22}|u_2^{\alpha,\beta}|_{4,\Omega}^4 + 2\alpha\beta_{12}|u_1^{\alpha,\beta}u_2^{\alpha,\beta}|_{2,\Omega}^2)(4L + o(1)) \\
&\quad - (\beta\beta_{13}|u_1^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2 + \beta\beta_{23}|u_2^{\alpha,\beta}u_3^{\alpha,\beta}|_{2,\Omega}^2)^2.
\end{aligned} \tag{3.89}
$$

Using the exponential decay and the convergence in Propositions 11 and 12 for the scaling of $u_i^{\alpha,\beta}$ depending on the space dimension n, we get by the same argument with Proposition 4 that

$$
\begin{aligned}
c_{\alpha,\beta} &\geq \min\left\{L + \frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1} + o(1)), L + \frac{\beta^{2-\frac{n}{2}}}{\alpha}(M_c + o(1))\right\} \\
&= \begin{cases} L + \frac{\beta^{\frac{3}{4}}}{\alpha}(M_b|S^{n-1}|R_0^{n-1} + o(1)) & \text{as } \alpha\#\beta^{\frac{3}{4}} \to \infty \quad \text{if} \quad n = 1, 2, \\ L + \frac{\beta^{\frac{1}{2}}}{\alpha}(M_c + o(1)) & \text{as } \alpha\#\beta^{\frac{1}{2}} \to \infty \quad \text{if} \quad n = 3. \end{cases}
\end{aligned}
$$

Since we have already the opposite estimate in Proposition 4, we complete the proof of Theorem 1. ∎

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