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Article

Linear Operators That Preserve the Genus of a Graph

LeRoy B. Beasley [1], **Jeong Han Kim** [2] and **Seok-Zun Song** [3,*]

1 Department of Mathematics and Statistics, Utah State University, Logan, UT 84322-3900, USA; leroy.b.beasley@aggiemail.usu.edu
2 School of Computational Sciences, Korean Institute for Advanced Study, Seoul 02455, Korea; jhkim@kias.re.kr
3 Department of Mathematics, Jeju National University, Jeju 63243, Korea
* Correspondence: szsong@jejunu.ac.kr

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Abstract: A graph has genus k if it can be embedded without edge crossings on a smooth orientable surface of genus k and not on one of genus $k-1$. A mapping of the set of graphs on n vertices to itself is called a linear operator if the image of a union of graphs is the union of their images and if it maps the edgeless graph to the edgeless graph. We investigate linear operators on the set of graphs on n vertices that map graphs of genus k to graphs of genus k and graphs of genus $k+1$ to graphs of genus $k+1$. We show that such linear operators are necessarily vertex permutations. Similar results with different restrictions on the genus k preserving operators give the same conclusion.

Keywords: linear operator; genus of a graph; vertex permutation

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1. Introduction

Let $\mathcal{G}_n$ denote the set of all simple graphs on the vertex set $\{v_1, v_2, \cdots, v_n\}$, so that all of our graphs are loopless and determined by their edge set. An operator (mapping) on $\mathcal{G}_n$ is said to be *linear* if it preserves unions (i.e., $T(G \cup H) = T(G) \cup T(H)$), and fixes the empty graph, $\overline{K_n}$, in $\mathcal{G}_n$.

Such operators were considered by Hershkowitz [1] in 1987 and by Pullman [2] in 1985. Linear operators preserving the clique covering number were investigated in [2] and linear operators preserving maximum cycle length were investigated in [1]. Linear operators preserving planarity and chromatic number were investigated in [3].

In this article, we show that if a linear operator $T : \mathcal{G}_n \to \mathcal{G}_n$ preserves the set of graphs of genus k, and also preserves the set of graphs of genus $k+1$, then T is a vertex permutation, that is, there is a permutation, τ, of $\{1, 2, \cdots, n\}$ such that for all indices, $i \neq j$, if $v_i v_j$ is an edge of G then $v_{\tau(i)} v_{\tau(j)}$ is an edge of $T(G)$. Other similar results are presented.

2. Notation, Definitions and Preliminary Results

Let $\mathcal{G}_n$ denote the set of all simple graphs on the vertex set $\{v_1, v_2, \cdots, v_n\}$ so that the union of two graphs is the union of their edge sets. The properties we are investigating are not affected by isolated vertices. Therefore, we can think of each nonempty graph, G, in terms of $\hat{G}$, the subgraph of G obtained by deleting its isolated vertices. We adopt the convention that "G has a property" means "$\hat{G}$ has that property". So that if we say that "G is a 4-cycle", we mean that "$\hat{G}$ is a 4-cycle". Also, "E is an edge" means that $\hat{E} = K_2$. If E is the graph whose edge set consists only of the edge v_iv_j, denote E by $E_{i,j}$. By an abuse of language we say that $E_{i,j}$ (which is a graph) is an edge of G meaning that v_iv_j is an edge of G.

A linear operator $T : \mathcal{G}_n \to \mathcal{G}_n$ is a *vertex permutation* if there is a permutation, τ, of $\{1, 2, \cdots, n\}$ such that for all indices, $i \neq j$, if v_iv_j is an edge of G then $v_\tau(i)v_\tau(j)$ is an edge of $T(G)$, or equivalently, for all $i \neq j$,

$$T(E_{i,j}) = E_{\tau(i),\tau(j)}. \tag{1}$$

When mapping monoids whose addition is union, such as in $\mathcal{G}_n$, if T maps the whole monoid to a single element, then T preserves any set that contains that element, thus, in this case, to seriously investigate any set of preservers, additional conditions must be placed on the operator. See Example 1.

We say that an operator *preserves* a set $\mathcal{X}$ if $A \in \mathcal{X}$ implies that $T(A) \in \mathcal{X}$. The operator *strongly preserves* the set $\mathcal{X}$ if

$$A \in \mathcal{X} \text{ if and only if } T(A) \in \mathcal{X}. \tag{2}$$

Thus, "T strongly preserves the set $\mathcal{X}$" is equivalent to saying "T preserves the set $\mathcal{X}$ and T preserves the complement $\mathcal{G}_n \backslash \mathcal{X}$".

Example 1. *Let $T : \mathcal{G}_{10} \to \mathcal{G}_{10}$ be defined by $T(X) = P$ for all $X \neq \overline{K}$ where P is the Petersen graph, and $T(\overline{K}) = \overline{K}$. Then T is linear and preserves 3-regular graphs since P is 3-regular, but T maps every graph to a 3-regular graph except $\overline{K}$. Clearly T does not strongly preserve any set except $\mathcal{G}_n$.*

Let f be a function on $\mathcal{G}_n$. We say that T *preserves* f if T preserves the set $\{X \in \mathcal{G}_n | f(X) = r\}$ for each r in the image of f. That is

$$T \text{ preserves } f \text{ if and only if, for each } r \text{ in the image of } f,$$

$$T \text{ (strongly) preserves } f^{-1}(r). \tag{3}$$

Let $\varepsilon(G)$ denote the number of edges in G, that is, $\varepsilon(G) = |E(G)|$ where $|X|$ denotes the cardinality of the set X. A graph consisting of k edges sharing a common vertex is called a *k-star*. Note that a 2-star is also a path of two edges.

The following lemma appeared in [3], we include the proof for completeness.

Lemma 1. [3] (Lemma 2.2) *If T is a bijective linear operator on $\mathcal{G}_n$ that preserves the set of 2-stars, then T is a vertex permutation.*

Proof. Suppose that T is bijective, then since $\mathcal{G}_n$ is finite, by the nature of the addition, that being union, T maps edges to edges and preserves ε. Suppose that T preserves 2-stars, then it follows that T preserves $(n-1)$-stars. For $i = 1, 2, \cdots, n$, let $S_i = \bigcup_{k=1, k\neq i}^{n} E_{i,k}$. Then $S_1, S_2, \cdots, S_n$ are the $(n-1)$-stars in $\mathcal{G}_n$. Thus, for each i there is j such that $T(S_i) = S_j$. Let τ be defined by $\tau(i) = j$ if $T(S_i) = S_j$. Since T is bijective, τ is a permutation.

Suppose that i and j are distinct indices so that $E_{i,j}$ is an edge of S_i. Then, $T(E_{i,j})$ is a edge of S_k where $\tau(i) = k$. That is $T(E_{i,j}) = E_{k,\ell}$ for some $\ell \neq k$. But, $E_{i,j} = E_{j,i}$ so that $T(E_{i,j}) = T(E_{j,i}) = E_{\tau(j).m}$ for some m. Then, $\tau(j) = \ell$ because $\tau(i) = k$. Consequently, $T(E_{i,j}) = E_{\tau(i),\tau(j)}$. Then, by Equation (1), T is a vertex permutation. □

If G and H are graphs in $\mathcal{G}_n$ and the edge set of G is contained in the edge set of H, that is $E(G) \subseteq E(H)$, we write $G \sqsubseteq H$. In this case we say that H *dominates* G or G *is dominated by* H.

The genus of a surface, (an orientable smooth surface) is zero if it is a plane or sphere. A surface of genus one is equivalent to a sphere with one "handle", or a torus. The genus of an orientable smooth surface is k if it is equivalent to a sphere with k "handles", or a surface with k holes.

The genus of a graph is the genus of the simplest surface in which it can be embedded (drawn) with no crossings of edges. It is well known that the genus of a graph is zero (i.e., planar) if it is not a subdivision of a K_5, the complete graph on five vertices, or a $K_{3,3}$, the complete bipartite graph whose bipartition consists of two sets each of three members. Let $\gamma(G)$ denote the genus of G for any $G \in \mathcal{G}_n$. That is γ is a function $\gamma : \mathcal{G}_n \to \mathbb{N}$, where $\mathbb{N}$ is the set of nonnegative integers such that $\gamma(G) = k$ if k is the genus of G.

If a graph has genus k then the addition of any edge will at most increase the genus by one, since at most one "handle" must be added to the surface to embed the edge. For more on the genus of a graph and embedding, see [4]. or one of the mathematical web sites such as http://en.wikipedia.org/wiki/Graph_theory, or http://mathworld.wolfram.com/GraphTheory.html.

3. Genus Preservers

Let k be a nonnegative integer, and let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator that preserves genus k, and that preserves genus $k+1$. Since n is finite, $\mathcal{G}_n$ is finite, so there is an integer d such that $L = T^d$ is idempotent. Then, L being a power of T also preserves both genus k and genus $k+1$.

Lemma 2. *L is the identity operator on $\mathcal{G}_n$.*

Proof. Since L is linear, $L(\overline{K}) = \overline{K}$. Suppose that $L(X) = \overline{K}$ and E is an edge, $E \sqsubseteq X$. Then, by the linearity of L, $L(E) = \overline{K}$. Choose a graph Z whose genus is $k+1$ and dominates E, and such that the genus of $Z \setminus E$ is k. This is always possible, since given any graph of genus $k+1$, delete edges until arriving at a graph of genus k. Permute vertices so that the last edge deleted is the edge E. But then, $T(Z) = T(Z \setminus E)$ so that $k+1 = \gamma(Z) = \gamma(T(Z)) = \gamma(T(Z \setminus E)) = \gamma(Z \setminus E) = k$, a contradiction. Therefore, $X = \overline{K}$. That is,

$$L(X) = \overline{K} \text{ if and only if } X = \overline{K}. \tag{4}$$

Suppose that E is an arbitrary edge graph. Then, $\varepsilon(T(E)) \geq 1$ since $T(E) \neq \overline{K}$. Suppose that F is an edge, $F \neq E$, and that $F \sqsubseteq L(E)$. It follows that $L(F) \sqsubseteq L(E)$ since L is idempotent. Now, let Z be a graph whose genus is $k+1$ and which dominates both E and F, and such that the genus of $Z \setminus E$ is k. Let $Y = Z \setminus (E \cup F)$. Then $Y + F = Z \setminus E$, so that $\gamma(Y + F) = k$ and $\gamma(Y \cup F \cup E) = k+1$ but $L(Y \cup F \cup E) = L(Y) \cup L(E) \cup L(F) \sqsubseteq L(Y) \cup L(E) = L(Y \cup E)$ since $L(F) \sqsubseteq L(E)$. So that, $k+1 = \gamma(Y \cup F \cup E) = \gamma(L(Y \cup F \cup E) = \gamma(L(Y \cup E) = \gamma(Y \cup E) = k$, a contradiction. Thus, $L(E) = E$. That is, by linearity, L is the identity operator. □

Lemma 3. *T is bijective.*

Proof. Recall that $L = T^d$ is the identity. We may assume that $d > 1$ in the definition, for if $p = 1$, by Lemma 2 T is the identity and the lemma follows. Equation (4) implies that

$$T(X) = \overline{K} \text{ if and only if } X = \overline{K}. \tag{5}$$

Suppose that E is an edge and $S = T(E)$. Then Lemma 2 implies that $T^{d-1}(S) = L(E) = E$. Let F be any edge dominated by S. Then $T^{d-1}(F) = E$, so $T(E) = L(F) = F$, by Lemma 2. We now have that T maps edge graphs to edge graphs in $\mathcal{G}_n$.

Suppose that for some edges E and F that $T(E) = T(F)$. Then $T^d(E) = T^d(F)$, and hence, $E = F$ by Lemma 2. Thus T is bijective on the set of edge graphs of $\mathcal{G}_n$ and by the linearity of T, T is bijective on $\mathcal{G}_n$. □

By the finiteness of $\mathcal{G}_n$, we have

Corollary 1. *T^{-1} exists and preserves genus k and genus $k+1$.*

We now define a function that gives the minimum n such that $\mathcal{G}_n$ has a vertex of genus k. That is, let $\varphi : \mathbb{N} \to \mathbb{N}$ such that $n = \varphi(k)$ if and only if $\mathcal{G}_n$ contains a graph of genus k and $\mathcal{G}_{n-1}$ does not. For example, $\varphi(1) = 5$ since K_5 has genus 1 and every graph on four vertices is planar.

From [5] we have

Lemma 4. [5] (Corollary 4.1) *Let k be a nonnegative integer. Let $n \geq 6$ if $k = 0$ or 1, and $n \geq 8 + \varphi(k-1)$ if $k \geq 2$. Then, $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator that strongly preserves genus k if and only if T is a vertex permutation.*

Lemma 5. *Let k be a nonnegative integer. If $k = 0$ or 1, let $n \geq 6$ and for $k \geq 2$ let $n \geq 8 + \varphi(k-1)$. Then T is bijective and preserves genus k if and only if T is a vertex permutation,*

Proof. If T is a vertex permutation, then T is bijective and preserves genus k since the genus of a graph is independent of the labeling of the vertices.

If $k = 0$, the lemma follows from Lemma 4 since $\mathcal{G}_n$ is a finite set. If $k = 1$, then T strongly preserves the set of planar graphs since T is bijective and if the image of a planar graph (genus 0) is not planar it must have genus at least 2. But by adding edges to the planar graph we get a graph of genus 1 mapped to a graph of genus at least 2, a contradiction. Thus the lemma again follows from Lemma 4.

Suppose that T is bijective and preserves genus k. Due to the nature of the addition, that being union, T preserves the set of edges. That is, T maps edges to edges and is bijective on that set.

Suppose that T maps a pair of parallel edges, say E and F, to a 2-star. Then there is an edge Q such that $E \cup F \cup Q$ is a graph of three edges, at least two of which are parallel, and $T(E \cup F \cup Q)$ is a 3-cycle. For notational convenience and without loss of generality we may assume that $T(Q)$ is the edge $\{v_1, v_2\}$, $T(F)$ is the edge $\{v_2, v_3\}$, and $T(E)$ is the edge $\{v_1, v_3\}$. Let Q be the edge $\{u, w\}$.

Let H be a graph of genus $k-1$ on $n-8$ vertices that dominates edge Q and neither edge E nor F. By appending a K_5 at each of the vertices u and w we may assume that E is an edge in one of the K_5's and F is an edge in the other. Call this graph G. Since appending the two K_5's adds four vertices each to the graph, $G \in \mathcal{G}_n$. Let $Z = G \setminus (E \cup F)$. Then the genus of Z is $k-1$, the genus of $Z \cup E$ is k, as is the genus of $Z \cup F$, and the genus of $G = Z \cup E \cup F$ is $k+1$. Further, because T strongly preserves genus k, we can deduce that the genus of $T(Z)$ is at most $k-1$, the genus of $T(Z \cup E)$ is k and the genus of $T(G)$ is at least $k+1$. In fact, since the addition of a single edge can increase the genus by at most one, we have that the genus of $T(Z)$ is $k-1$ and the genus of $T(G)$ is $k+1$.

Embed the graph $T(Z)$ in a surface of genus $k-1$. Thus, the edge $T(Q)$ is drawn on this surface, append a handle on the surface that goes from any face adjacent to the vertex v_3 to a face adjacent to vertex v_1 and vertex v_2, and to the edge $T(Q)$. Use this handle to draw the edge $T(F)$ without edge crossings from vertex v_2 to vertex v_3. On this surface (of genus k) draw the edge $T(E)$ by beginning at vertex v_1, proceed parallel to the edge $T(Q)$ in the face that the new handle was inserted to the handle then parallel to the edge $T(F)$ (on the handle) to vertex v_3. Drawing this edge does not require crossing any edge. Thus, we have embedded $T(G)$ in a surface of genus k, a contradiction since the genus of $T(G)$ is $k+1$ and T^{-1} preserves genus k by Corollary 1. Thus, T maps parallel pairs to parallel pairs, or equivalently, since T is bijective, T maps 2-stars to 2-stars.

The lemma now follows from Lemma 1. □

Lemma 6. *Let $k \geq 1$ be a nonnegative integer and $n \geq 6$ for $k = 1$ and $n \geq 8 + \varphi(k-1)$ for $k \geq 2$. If $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator that preserves genus k and maps edges to edges, then T is a vertex permutation.*

Proof. By Lemma 5, we only need show that T is bijective. Since T maps edges to edges, $T(X) = \overline{K}$ if and only if $X = \overline{K}$. Now, suppose that $T(E) = T(F)$ for some distinct edges E and F. Let Z be a graph of minimum size (number of edges) that is of genus k. By a vertex permutation we may assume that Z dominates $E \cup F$. Since T maps edges to edges, $\varepsilon(T(G)) \leq \varepsilon(G)$ for every $G \in \mathcal{G}_n$. Now, $\gamma(G \setminus F) = k - 1$ so $\gamma(T(G \setminus F)) \leq k - 1$. Since $T(E) = T(F)$, it follows that $T(G) = T((G \setminus (E \cup F)) \cup E \cup F) = T(G \setminus (E \cup F)) \cup T(E) \cup T(F) = T(G \setminus (E \cup F)) \cup T(E) = T(G \setminus F)$. But, $k = \gamma(T(G)) = \gamma(T(G \setminus F)) \leq k - 1$ a contradiction. Thus, T is bijective. □

Note that if $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator defined by $T(X) = E_{i,j}$ for some fixed $i \neq j$ for all $X \neq \overline{K}$ (and $T(\overline{K}) = \overline{K}$) then T satisfies the hypothesis of Lemma 6 for $k = 0$. Thus the lemma is not true for planar graphs ($k = 0$).

Lemma 7. *Let k be a nonnegative integer. If $k = 0$ or 1, let $n \geq 6$ and for $k \geq 2$ let $n \geq 8 + \varphi(k-1)$. If $T : \mathcal{G}_n \to \mathcal{G}_n$ is a linear operator that preserves genus k and such that $T(K_n) = K_n$, then T is a vertex permutation.*

Proof. Suppose there is an edge whose image has more than one edge, say $\varepsilon(T(F)) \geq 2$. Let Z be a graph whose genus is $k+1$ and has the minimum number of edges possible for a graph of genus $k+1$. Let $\varepsilon(Z) = q$, so that any graph with fewer than q edges has genus at most k. By a vertex permutation, we may assume that $\varepsilon(Z \cap T(F)) \geq 2$. Let $F_1 = F$. Since $T(K_n) = K_n$, there is some edge F_2 such that $\varepsilon(Z \cap T(F_1 \cup F_2)) \geq 3$. Continuing in this way there are ℓ edges $F_1, F_2, \cdots, F_\ell$ such that $\ell < q$ and $T(F_1 \cup F_2 \cup \cdots, \cup F_\ell) \sqsupseteq Z$. Then, however, $\gamma(F_1 \cup F_2 \cup \cdots, \cup F_\ell) \leq k$ while $\gamma(T(F_1 \cup F_2 \cup \cdots, \cup F_\ell)) \geq k+1$ since $T(F_1 \cup F_2 \cup \cdots, \cup F_\ell) \sqsupseteq Z$, a contradiction. Thus, the image of an edge is either an edge or $\overline{K}$.

Suppose that $T(F) = \overline{K}$ for some edge F. Then, since $T(K_n) = K_n$, there is some edge whose image dominates more than one edge, a contradiction by the paragraph above. Thus, the image of an edge is an edge. and since $T(K_n) = K_n$, it follows that T is bijective on the set of edges, and hence bijective on $\mathcal{G}_n$. By Lemma 6, T is a vertex permutation. □

Theorem 1. *Let k be a nonnegative integer. Let $n \geq 6$ if $k = 0$ or 1, and $n \geq 8 + \varphi(k-1)$ if $k \geq 2$. Let $T : \mathcal{G}_n \to \mathcal{G}_n$ be a linear operator. Then, the following are equivalent:*

(1) *T strongly preserves genus k;*
(2) *T is bijective and preserves genus k;*
(3) *T preserves genus k and genus $k+1$;*
(4) *T preserves genus k and $T(K_n) = K_n$;*
(5) *T preserves genus k and T maps edges to edges, for $k \geq 1$;*
(6) *T is a vertex permutation.*

Proof. Clearly, (6) implies all the other five statements.

By Lemma 4, we have (1) implies (6). By Lemma 5, (2) implies (6).

Suppose that T preserves genus k and genus $k+1$. By Lemma 3, T is bijective, and hence, by Lemma 5, T is a vertex permutation. Thus, (3) implies (6).

By Lemma 7, (4) implies (6), and by Lemma 6 we have (5) implies (6). We now have established all equivalences. □

4. Conclusions

In this paper, we investigated the linear operators that preserve the genus of a graph. Thus we proved that such linear operators are necessarily vertex permutations. We also obtained some characterizations of the linear operators that preserve the genus of a graph, say, the linear operator that preserves consecutive two genus of a graph. For further research, we conjecture that the linear operator which preserves any two genus of a graph should be the vertex permutation.

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