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Relations Between Theta Functions of Genus One and Two From Geometry

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RELATIONS BETWEEN THETA FUNCTIONS OF GENUS ONE AND TWO FROM GEOMETRY

by

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Abstract. Genus-two curves with special symmetries are related to pairs of genus-one curves by two and three-sheeted ramified coverings. This classical work dates back to early 20th century and is known as Jacobi and Hermite reduction. Jacobians of genus-two curves can be used to construct complex two-dimensional complex projective manifolds known as Kummer surfaces. On the other hand, the defining coordinates and parameters of both elliptic curves and Kummer surfaces can be related to Riemann Theta functions and Siegel Theta functions, respectively. This result goes back to the seminal work of Mumford in the 1980s. We use the geometric relation between elliptic curves and Kummer surfaces to derive functional relations between Theta functions along Humbert varieties of low discriminant.

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1 Introduction

1.1 A Geometric Description

Plane algebraic curves are one dimensional algebraic varieties with important geometric and modular properties that we will discuss in this paper. The algebraic structure of elliptic and hyperelliptic curves over finite fields has applications in cryptography and post quantum computing, including the encryption of digital information. Elliptic and hyperelliptic curves are studied extensively in algebraic geometry, complex analysis, and number theory.

This section provides some basic definitions in algebraic geometry, and provides examples of algebraic curves as well as the relationship between curves of different genus.

Definition 1.1. A *plane algebraic curve* is an algebraic variety of dimension one. Specifically, it is the complex one dimensional solution set to an algebraic equation $c : f(x, y) = 0$ in the complex projective plane $\mathbb{C}P^2$. In this paper we will be studying hyperelliptic curves, or curves of the form $c : y^2 = f(x)$, where $f(x)$ is a polynomial.

Definition 1.2. The *genus* is a topological invariant of a surface, or algebraic curve. Intuitively, the number of holes in a surface corresponds to the genus. The genus g of an algebraic curve $c : y^2 = f(x)$ is related to the homogeneous degree of the polynomial, $\deg(f(x)) = n$.

$$g(c) = \begin{cases} \frac{n-2}{2} & \text{if } n \text{ is even} \\ \frac{n-1}{2} & \text{if } n \text{ is odd} \end{cases}$$

Definition 1.3. Let S and S' be topological spaces. Then a *branched covering*, is a covering map $\pi : S' \to S$, (i.e., a continuous map such that every point in S has an open neighborhood covered by π), except on a small set of points called ramification or branching points.

Theorem 1.4. The *Riemann-Hurwitz formula* describes the relationship between the genus of two surfaces when one is a degree N branched covering of the other. Let $\pi : S' \to S$ be a degree N covering map. Then,

$$2(g(S') - 1) = 2N(g(S) - 1) + \sum_{p \in S'} (e_p - 1)$$

where $g(S')$ and $g(S)$ are the genus of S' and S respectively. For all but finitely many $p \in S'$, the ramification index at p denoted by e_p is 1. At the branching points we have $e_p = 2$. (Thus, the sum in the equation above essentially corresponds to the number of branching points of the cover.) [1,2,4]

Let us look at two examples of plane algebraic curves that will be the focus of the paper.

Example 1.5. (Genus 1 Curve.) A genus one curve, or *elliptic curve*, is a plane algebraic curve which consists of the points satisfying the equation $c^1 : y^2 = f(x)$ where $\deg(f(x)) = 3$ or 4.

We remark that given three points in general position in $\mathbb{C}P^2$, one can use a Möbius transformation, or fractional linear transformation, to move these points to 0, 1, and ∞.

Therefore, taking the four roots of f in general position we can use a Möbius transformation and move three of them to 0, 1, and ∞. The position of the remaining point, λ, is called the modulus of the curve. This fact allows us to write any genus 1 curve in the Legendre normal form

$$c^1 : y^2 = x(x-1)(x-\lambda).$$

We see that the projection of c^1 onto $\mathbb{C}P^1$ is a branched cover with four branching points, $x = 0, 1, \lambda$, and ∞. At these four values of x the fibre is the double point $y^2 = 0$ and at all other values the fibre is comprised of two distinct preimages. So, we call this map a branched *double* (or degree 2) cover of $\mathbb{C}P^1$

As a genus 1 curve is a branched double cover of the genus 0 surface $\mathbb{C}P^1$, we can verify that the Riemann-Hurwitz formula holds true for this mapping.

We see that $g(c) = 1, g(\mathbb{C}P^1) = 0, N = 2$ and we have 4 branching points. Substituting this into the Riemann-Hurwitz formula we see that,

$$2(1-1) = 2(2(0-1)) + 4$$

clearly a true statement. $\square$

Example 1.6. (Genus 2 Curve.) A genus two curve is a plane algebraic curve which consists of the points satisfying the equation $c^2 : y^2 = f(x)$ where $\deg(f(x)) =$ 5 or 6. Similarly, a genus 2 curve with moduli λ_1, λ_2 and λ_3 can be written in Rosenhain normal form,

$$c^2 : y^2 = x(x-1)(x-\lambda_1)(x-\lambda_2)(x-\lambda_3).$$

As in the genus one case, the genus 2 curve c^2 can be presented as a branched double cover of $\mathbb{C}P^1$, now with six branching points, $x = 0, 1, \lambda_1, \lambda_2, \lambda_3$, and ∞. It is easy to verify that the Riemann-Hurwitz formula holds for this example. $\square$

1.2 The Relation Between Curves of Genus 1 and 2

These examples show that there are branched covering maps from both genus 1 and genus 2 curves onto a genus 0 curve. The next natural question is what is the relation between genus 2 curves and genus 1 curves, and is there a branched covering between them? If we could construct such a covering map, we know that the Riemann-Hurwitz formula would impose certain constraints. Let S be a curve of genus 2 and S' be a curve of genus 1. Then the Riemann-Hurwitz formula tells us that

$$2(g(S') - 1) = 2N(g(S) - 1) + \sum_{p \in S'} (e_p - 1)$$
$$\Longrightarrow 2(2-1) = 2 = 2N(1-1) + \sum_{p \in S'} (e_p - 1) = \sum_{p \in S'} (e_p - 1)$$

We see in this formula that the degree of the map vanishes, and the number of branching points must necessarily be 2. So a covering map from a genus 2 curve to a genus 1 curve can be of any degree, but must have two branching points. A well known example of one such map is due to Carl Gustav Jacobi (1804- 1851).

Example 1.7. (Jacobi Reduction, $N = 2$.) We will consider a special class of genus 2 curves in this example. Take a genus 2 curve where the moduli of the curve are a^2, b^2, and a^2b^2. In Rosenhain normal form, the equation of this curve is,

$$c_{sp}^2 : y^2 = x(x-1)(x-a^2)(x-b^2)(x-a^2b^2).$$

There is a well known transformation (due to Jacobi) that is a degree 2 covering map reducing the genus 2 curve to a pair of genus 1 curves or elliptic curves [4]. Observe that if we set

$$\eta_\pm = \frac{y}{\lambda_\pm}$$
$$\lambda_\pm = \frac{(x-a^2)^2(x-b^2)^2}{\sqrt{(1-a^2)(1-b^2)}(x \pm ab)}$$
$$\xi = \frac{(1-a^2)(1-b^2)x}{(x-a^2)(x-b^2)}$$
$$\lambda_\pm^2 = -\frac{(a \pm b)^2}{(1-a)^2(1-b)^2}$$

then c^2 reduces to a pair of elliptic curves in the coordinates (η_+, ξ) and (η_-, ξ),

$$\epsilon_\pm : \eta_\pm^2 = \xi(\xi - 1)(1 - \lambda_\pm^2 \xi)$$

□

From this example we see that we have two double covers.

$$\pi_+ : c_{sp}^2 \to \epsilon_+, \qquad \pi_- : c_{sp}^2 \to \epsilon_-$$

As previously noted, by the Riemann-Hurwitz formula we know that each of these mapping must have exactly 2 branching points. We can verify that this is indeed the case for the mapping described above. The local coordinate x on the genus 2 curve can be related to the local coordinate ξ on the two genus 1 curves by using the map and carrying out a Taylor series expansion about local points x_0 and $\xi(x_0) = \xi_0$. About $x_0 = \pm ab$ we see that

$$\xi - \xi_0 = (x \pm ab)^2 + (\text{higer order terms})...$$

The first term in the Taylor series is quadratic. Because at these points we have two values of x being mapped to one point in ξ, the degree of this covering map is indeed 2; in other words a branched double covering. We can verify that $x_0 = \pm ab$ are the only two branching points by expanding about some other fixed local point, say $x = x_0$. We see that

$$\xi - \xi_0 = (x - x_0) + (\text{higer order terms})...$$

the first term in the series is linear. So we the map is double cover with two branching points $x_0 = \pm ab$, as required by the Riemann-Hurwitz formula.

We also note that because the degree of the map is not in constraint by the Riemann-Hurwitz formula, we could theoretically construct such a branched cover with two branching points of any degree. There is a known analogue to this in degree three known as Hermite reduction [5]. Because of the complexity of these maps, it is not clear if there are known higher order covers.

2 Modular Description

2.1 Genus 1 or Elliptic Curves.

We will now give a modular description of curves of genus 1. To do this we must introduce the properties of elliptic functions and discuss their relationship to genus 1 curves.

Definition 2.1. Let α, $\beta \in \mathbb{R}$ be linearly independent and non-trivial. A function f is called doubly periodic with periods α and β if for all $z \in \mathbb{C}$ we have $f(z+\alpha) = f(z)$ and $f(z+\beta) = f(z)$.

Given a doubly periodic function g, with periods a and b, let $\tau = \frac{a}{b}$, where we label a and b such that $\text{im}(\tau) > 0$. Then by a scaling argument, we see that $f(z) = g(bz)$ is a doubly periodic function with period 1 and τ. It is therefore sufficient only to consider double periodic functions with periods 1 and τ [3].

Definition 2.2. The lattice points of a doubly periodic function are the collection of points $\Lambda = \{n + m\tau \in \mathbb{C} | n, m \in \mathbb{Z}\}$. We say that two points $z_1, z_2 \in \mathbb{C}/\Lambda$ are congruent if $z_1 = z_2 + n + m\tau$ for some $m, n \in \mathbb{Z}$.

Definition 2.3. The fundamental domain of a doubly periodic function is the "parallelogram"-shaped region $D = \{z = x + \tau y | x, y \in [0, 1)\}$.

Remark 2.4. We can tessellate the fundamental domain over the entire complex plane by shifting the fundamental domain by linear combinations of the two periods. Consequently, we can partition the complex plane into translations of the fundamental domain by elements in the lattice Λ. Therefore, a doubly period function is completely determined by its values in any shifted fundamental domain [4].

Theorem 2.5. $\mathbb{C}/\Lambda$ is homeomorphic to the complex 1 dimensional torus $\mathbb{T}^1$.

Definition 2.6. Elliptic functions are doubly periodic and meromorphic over the complex plane.

Definition 2.7. The Weierstrass $\wp$-function [6] is an elliptic function defined by

$$\wp(z) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \Big(\frac{1}{z + n + m\tau} - \frac{1}{(n + m\tau)^2} \Big).$$

The functions $\wp(z)$ and its derivative $\wp'(z)$ parameterize the algebraic curve of genus one, $c^1 : y^2 = 4x^3 - g_2 x - g_3$. This parameterization provides a group homomorphism between the complex torus $\mathbb{C}/\Lambda$ and c^1, the genus one curve†,

$$\phi : \mathbb{C}/\Lambda \to c^1, \qquad \phi(z) = (\wp(z), \wp'(z)).$$

Though this map does provide a homomorphism between the complex torus and the genus 1 curve, we are also interested in the two torsion points which relate to the 4 branching points of the genus 1 curve discussed in the previous section. In this case, the homomorphism between the complex torus and the genus 1 curve maps the coordinates of the torus to expressions of Jacobi ϑ-functions, and the modulus of the curve to Jacobi ϑ-constants. These expressions are discussed explicitly in Remark 3.1.

Definition 2.8. The Jacobi ϑ-function is function of an elliptic variable, $z \in \mathbb{C}$, and modular variable $\tau \in \mathbb{H}$ the upper half plane. The function is defined in the following way,

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} \exp\big(\pi i n^2 \tau + 2\pi i n z\big).$$

We call $\vartheta(0;\tau)$ a Jacobi ϑ-constant. These constants are modular forms, specifically modular forms of half integer weight with characteristic [6].

Definition 2.9. The Jacobi ϑ-function defined above is often considered along the three auxiliary or half-period ϑ-functions. These functions are usually defined using the following

†remarks at the end of this section describe the group structure on a genus 1 or elliptic curve.

conventions,

$$\vartheta_{00}(z;\tau) = \vartheta(z;\tau)$$
$$\vartheta_{01}(z;\tau) = \vartheta(z+\frac{1}{2};\tau)$$
$$\vartheta_{10}(z;\tau) = \exp\left(\frac{1}{4}\pi i\tau + \pi iz\right)\vartheta(z+\frac{1}{2};\tau)$$
$$\vartheta_{11}(z;\tau) = \exp\left(\frac{1}{4}\pi i\tau + \pi i(z+\frac{1}{2})\right)\vartheta(z+\frac{1}{2}\tau+\frac{1}{2};\tau)$$

Remark 2.10. The symmetries of the torus result in many special transformation properties of $\vartheta(z;\tau)$ under the action of the modular group $\mathrm{sl}(2,\mathbb{Z})$ on τ in the usual manner. This group is generated by $\tau \mapsto \tau+1$ and $\tau \mapsto -\frac{1}{\tau}$. Identities for $\tau \mapsto \tau+1$, geometrically are the consequence of the holomorphic isomorphic tori resulting from transforming τ to $\tau+1$ in the fundamental domain $\mathbb{C}/\Lambda$, and has the same effect as adding $\frac{1}{2}$ to z. For the transformation $t \mapsto -\frac{1}{\tau}$,

$$\vartheta\left(\frac{z}{\tau};-\frac{1}{\tau}\right) = \alpha\vartheta(z;\tau)$$

where $\alpha = (-i\tau)^{\frac{1}{2}}exp(\frac{\pi}{\tau}iz^2)$, [6].

The complex torus $\mathbb{T}^1$ is an algebraic torus, and therefore a commutative affine group. The mapping between the complex torus and the genus 1 curve by $\wp(z)$ and the Jacobi ϑ-function, induce a group structure on the points of the genus 1 curve. When equipped with the group structure we call the genus 1 curve an elliptic curve. The binary operation, or group law, for elliptic curves is commutative, so it is usually referred to as 'addition'.

Geometrically, to add two points say A and B on the curve, first take the straight line through them. If the line intersects the curve, it can only do so at one other point, say R. Now, we draw a line through the point R and the point at infinity, our neutral element (essentially reflecting the point across the x-axis). Where is point at which the line through R and the neutral element intersect the curve is called $A+B$. In the case where the straight line through A and B does not intersect the elliptic curve at another point, we say that the intersection is on the *point at infinity.* The point at infinity is invariant under reflection across the x-axis, and it is not difficult to show that it serves as the identity element in this abelian group. The inverse of a point A is then described as its reflection across the x-axis. Associativity of this binary operation can also be shown [4].

2.2 Genus 2 Curves.

In contrast to the case of genus 1 curves, algebraic curves of genus 2 are topologically inequivalent to the complex 2-torus $\mathbb{T}^2$; in fact, not all such tori are even algebraic. Consequently, we cannot express coordinates on these curves a Jacobi ϑ-functions. We can however construct an algebraic 2-torus at the cost of going up one complex dimension. We do this by taking the Jacobian of the genus 2 curve.

Definition 2.11. The Jacobian is a standard construction in algebraic geometry. It takes a genus 2 curve, c^2, a complex 1 dimensional curve in $\mathbb{C}P^2$ and constructs a complex 2 dimensional algebraic torus, $Jac(c^2)$, now in $\mathbb{C}P^2$.

By analogy, we associate the coordinates on $Jac(c^2)$ as higher order ϑ-functions called Siegel ϑ-functions. The exact relation between these the coordinates of the Jacobian and Siegel ϑ-function is discussed in Section 3.1.

Definition 2.12. Siegel ϑ-functions are generalizations of the Riemann ϑ-function to the Siegel upper-half space. They are functions an elliptic variable $\vec{z} \in \mathbb{C}^2$ and a modular matrix $\bar{\tau}$ where

$$\bar{\tau} = \begin{bmatrix} \tau_{11} & z \\ z & \tau_{22} \end{bmatrix}$$

which is derived from the period matrix of unique holomorphic one forms integrated along the 4 periods of the algebraic genus 2 curve. Like Jacobi ϑ-functions, these have similar transformation properties under $\mathrm{sl}(2, \mathbb{Z})$.

Now consider again the algebraic curve of genus 2 discussed in Example 1.7, $c^2_{sp} : y^2 = y^2 = x(x-1)(x-a^2)(x-b^2)(x-a^2b^2)$. Given the covering maps of Jacobi reduction are degree 2 covers, $Jac(c^2_{sp})$, has modular matrix

$$\bar{\tau} = \begin{bmatrix} \tau_{11} & \frac{1}{2} \\ \frac{1}{2} & \tau_{22} \end{bmatrix}$$

The coordinates on $Jac(c^2_{sp})$ can be expressed in terms of Siegel ϑ-functions $\vartheta(\vec{z}; \bar{\tau})$, where $\vec{z} \in \mathbb{C}^2/\Lambda^2$, where Λ^2 is the lattice $\langle \mathbb{1}, \bar{\tau} \rangle$ and the moduli of this curve can be expressed in terms of Siegel ϑ-constants of the form $\vartheta(\vec{0}; \bar{\tau})$. The reason for this is discussed in Section 3.1.

3 Relation Between ϑ-Functions Using Geometry.

We are interested in studying the interrelation between Jacobi and Siegel ϑ-functions. Given the geometric interrelation between the special set of algebraic curves of genus 2 with symmetry c^2_{sp}, and genus 1 curve c^2 discussed in Example 1.7, and the fact the the coordinates of c^1 and $Jac(c^2)$ are expressions of Jacobi and Siegel ϑ-functions respectively, we can use this geometric relation to derive relationships between these two classes of ϑ-functions.

From the previous section, we know that we can construct an abelian surface from the genus 2 curve, namely $Jac(c^2)$ an algebraic complex 2 dimensional torus. In order to study $Jac(c^2)$, we take a bi-rational model of it, the symmetric square, $Sym^2(c^2)$. Take two copies of the genus 2 curve, say c_1 and c_2, and two general points on them say $p_1 \in c_1$ and $p_2 \in c_2$. Using the projection map defined in this example, $\pi_\pm : c_i \to \epsilon^i_\pm$ $i = 1, 2$ we see can project p_1 and p_2 to $\pi_+(p_1) \in \epsilon^1_+$ and $\pi_+(p_1) \in \epsilon^2_+$. As these points sit on two copies of the same genus 1 or elliptic curve we can add these points together using the elliptic group law described in Section 2.1 and get

$$\pi_+(p_1) + \pi_+(p_2) = u_+ \in \epsilon_+$$

similarly for the projection π_-, we have

$$\pi_-(p_1) + \pi_-(p_2) = u_- \in \epsilon_-.$$

Remark 3.1. The elliptic curves $\epsilon_\pm$ can be realized as the intersection of two quadric surfaces in $\mathbb{C}P^3$, with coordinates $[x_{00} : x_{01} : x_{10} : x_{11}] \in \mathbb{C}P^3$ on ϵ_+ and $[y_{00} : y_{01} : y_{10} : y_{11}] \in \mathbb{C}P^3$ on ϵ_-. These coordinates are the auxiliary Jacobi ϑ-functions, with a modular variable that corresponds to the modular parameter of the elliptic curve, say τ_+ and τ_- and more complicated elliptic variables u_+ and u_- resulting from the bi-rational model described above,

$$x_{00} = \vartheta_{00}(u_+;\tau_+), \quad x_{01} = \vartheta_{01}(u_+;\tau_+), \quad x_{10} = \vartheta_{10}(u_+;\tau_+), \quad x_{11} = \vartheta_{11}(u_+;\tau_+)$$
$$y_{00} = \vartheta_{00}(u_-;\tau_-), \quad y_{01} = \vartheta_{01}(u_-;\tau_-), \quad y_{10} = \vartheta_{10}(u_-;\tau_-), \quad y_{11} = \vartheta_{11}(u_-;\tau_-)$$

This bi-rational model is singular, in other words the algebraic equation defining the space has singular points where the tangent space is not well defined. This leads us to a discussion of K3 surfaces and their elliptic fibration.

3.1 Elliptic Fibration of the Kummer Surface.

On the complex 1 dimensional torus $\mathbb{C}/\Lambda$, there is a natural involution, namely multiplication by -1, $z \mapsto -z$. This is not however, a free involution. By inspection of the quotient space, we see that this involution has four fixed points, $z = 0, \frac{1}{2}, \frac{\tau}{2}, \frac{1}{2} + \frac{\tau}{2}$.

Now, consider the general complex 2 dimensional torus formed from the quotient $\mathbb{C}^2/\Lambda$. By analogy, we have $z \mapsto -z$ projecting to an involution on the 2 torus, but now with 16 fixed points because in each complex dimension we have 4 fixed points. Dividing by these fixed points, we construct a singular space. We can resolve these singularities by introducing multiple charts with smooth transition functions (a common tactic in algebraic geometry). This resolution of the singularities results in a smooth variety, a special K3 surface called a Kummer surface.

A Kummer surface carries and elliptic fibration with section (a smooth point at ∞ in each fibre). We describe such a fibration using a Weierstrass model where each point in the fibre is associated with a complex 1 dimensional torus or elliptic curve given in Weierstrass normal form

$$Y^2 = 4X^3 - g_2(T)X - g_3(T)$$

where X and Y are coordinates on the individual fibre and T is the base coordinate. If the torus in the fibre was always smooth the the elliptic fibration would look like $\mathbb{T}^1 \times \mathbb{C}P^1$, however this is clearly not a K3 surface (since the two do not have the same Euler characteristic). So, in order to represent the K3 surface as an elliptic fibration, we must allow the fibration to have singular points, or points in the fibre where the torus or elliptic curve is singular.

Remark 3.2. This elliptic fibration is significant in our discussion of ϑ-functions because the coordinates in of the Weierstrass model Y, X, and T can be considered Siegel ϑ-functions.

3.2 Relating $Kum(Jac(c^2))$ to $Kum(\epsilon_+ \times \epsilon_-)$.

We can now specialize this construction the the genus 2 curve with symmetry $c^2_{sp} : y^2 = x(x-1)(x-a^2)(x-b^2)(x-a^2b^2)$, discussed in Example 1.7. We already know that we can construct an abelian surface from this curve $Jac(c^2_{sp})$, and using the description above we can construct $Kum(Jac(c^2_{sp}))$.

Now, taking the pair of elliptic curves ϵ_+ and ϵ_- which are covered by c^2_{sp} in Example 1.7, we can construct another algebraic 2 torus, namely the product of these elliptic curves, $\epsilon_+ \times \epsilon_-$. The product of $[x_{00} : x_{01} : x_{10} : x_{11}] \sim [\frac{x_{00}}{x_{11}} : \frac{x_{01}}{x_{11}} : \frac{x_{10}}{x_{11}} : 1]$ the coordinates on ϵ_+ and $[y_{00} : y_{01} : y_{10} : y_{11}] \sim [\frac{y_{00}}{y_{11}} : \frac{y_{01}}{y_{11}} : \frac{y_{10}}{y_{11}} : 1]$ the coordinates on ϵ_-, result in the coordinates on $\epsilon_+ \times \epsilon_-$,

$$Y_1 = \frac{x_{00}y_{00}}{x_{11}y_{11}}, \qquad Y_2 = \frac{x_{01}y_{01}}{x_{11}y_{11}}, \qquad Y_3 = \frac{x_{10}y_{10}}{x_{11}y_{11}}.$$

Therefore, we have written the coordinates on the algebraic 2 torus $\epsilon_+ \times \epsilon_-$ as rational expressions of Jacobi ϑ-functions.

We can similarly construct the Kummer surface $Kum(\epsilon_+ \times \epsilon_-)$.

Theorem 3.3. There is an isogeny (a group homomorphism of finite kernel) between $\epsilon_+ \times \epsilon_-$ and $Jac(c^2_{sp})$, which results in a bi-rational map between the abelian surfaces $Kum(Jac(c^2_{sp}))$ and $Kum(\epsilon_+ \times \epsilon_-)$.

The isogeny and resulting bi-rational map define the relations between the coordinates Y, X, and T on $Jac(c^2_{sp})$ which are expressions of Siegel theta functions by Remark 3.2 and the coordinates Y_1, Y_2 and Y_3 on $\epsilon_+ \times \epsilon_-$, which are expressions of Jacobi ϑ-functions as described above. Solving for these expressions we see that

$$\begin{aligned}
T &= -\frac{a^2b^2(Y_1 - Y_2 + 1)}{Y_1 + Y_2 - 1} \\
X &= -\frac{a^2b^2(Y_3(a^2b^2 - a^2 - b^2 + 1) - Y_2(a^2 + b^2) + a^2b^2 + 1}{Y_1 + Y_2 - 1} \\
Y &= \frac{Y_3(a^2b^2 - a^2 - b_1^2) + Y_2(a^2 + b^2) - a^2b^2 - 1}{Y_1 + Y_2 - 1}
\end{aligned}$$

These equations define an explicit relationship between certain Jacobi and Siegel ϑ-functions.

4 Future Directions.

4.1 Higher Degree Covering Maps.

In this paper, we used the degree 2 covering map from Jacobi reduction to study the relation between a class of specialized genus 2 curves and a pair of elliptic curves. We know that the degree of the covering map between such curves is not in constraint by the Riemann-Hurwitz formula, so we could preform similar analyses using known higher order covering maps. Hermite reduction is an example of a degree three mapping. In this case the coordinates on $Jac(c^2)$ would again be expressions of Siegel ϑ-functions, but now with modular matrix,

$$\bar{\tau} = \begin{bmatrix} \tau_{11} & \frac{1}{3} \\ \frac{1}{3} & \tau_{22} \end{bmatrix}.$$

4.2 Higher Genus Curves.

We could potentially conduct similar analysis for curves of a higher genus, though this could potentially be a very challenging task. The Jacobian of a hyperelliptic curve of genus g is bi-rationally equivalent to g copies of the hyperelliptic curve, invariant under the interchange of copies. We might analyze the feasibility of such a project by investigating the relationship that the Riemann-Hurwitz formula imposes on curves of genus g in terms of curves of genus g' [2,4].

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Relations Between Theta Functions of Genus One and Two from Geometry.
Reflective Writing Section.
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As an honors student I have participated in research over the course of my entire undergraduate education. During my first year at school, as part of an REU (Research Experience for Undergraduates) I began working with Dr. Ian Anderson to develop a digital database of integrable systems and their properties. Under his tutelage I combed the literature on integrable systems and soliton theory to find as many integrable systems as I could find and then verify or derive from scratch their properties using the software we developed. This research project was a great experience that taught me a lot about mathematics research, but for my capstone project I wanted to delve deeper.

The summer before my senior year, Shantel Spatig, graduate student working with Dr. Andreas Malmendier in the math department, gave me a paper on elliptic curves and the soliton solution of the Korteweg-de Vries equation. I was particularly intrigued by this, since in previous research projects I had studied many properties of the Korteweg-de Vries equation among other integrable systems. While reading this paper, I immediately became enamored by elliptic curves and their unique and amazing structure. The fall of 2017 I asked Dr. Malmendier if he would consider working with me on a capstone project studying more about elliptic and hyperelliptic curves.

I quickly realized that this was not going to be an easy project. Much of the mathematics needed to study these curves was far beyond what I had learned in my courses as an undergraduate. To supplement my understanding, I enrolled in three graduate courses on matrix theory, lie algebras, representation theory, as well as courses on advanced algebra, differential geometry, and analysis. But even these additional courses did not provide sufficient background to understand all the material. Dr. Malmendier spent many hours working with me in his office and collaborating over email and skype. He provided me with supplemental reading to help provide additional background. Under his tutelage, I was exposed to mathematics that most undergraduates never encounter during the course of their degree. Dr. Malmendier definitely pushed the limits of my mathematical abilities as an undergraduate.

I had the opportunity to present my research at three different conferences, each with a unique audience. Consequently, each talk was specifically tailored to meet the level of understanding of that audience. Preparing for and presenting my research at these conferences was particularly nerve wracking. Not only did I have to face all the gaps in my understanding, but I was typically presenting to a room full of professors and graduate students. Because I was so intimidated by the audience and the potential questions they may ask, I spent countless hours and late nights studying and preparing for these conferences. This was some of the most productive time of my research. As I prepared for these conferences I was able to solidify and deepen my understanding.

I first had the opportunity to present my research at the Joint Math Meetings in San Diego last January. The Joint Math Meetings constitute the largest mathematics conference in the world, with over 6,400 mathematicians in attendance. Top researchers from every field of mathematics join together to present their research and network. Talks ranged across every discipline of mathematics: mathematics education, applied mathematics and numerical analysis, big data and statistics, pure mathematics and many others. During my time at the conference, I attended as many plenary lectures as I could. I listened to 'celebrities' in the mathematics world explain their cutting-edge research. Many people whose names appear in papers I had been studying were speaking at the conference or in attendance. It was a unique experience to see the way these great minds think about mathematics and explain their research. More than anything, attending the Joint Math Meetings impressed upon me the great sense of community among mathematicians across all disciplines of research.

I also presented my research at the MAA Intermountain Section Annual Conference at USU. Students and faculty from math departments throughout the intermountain region met at USU. I was able to meet other undergraduates and learn about their research projects. As part of the conference they hosted a game of Jeopardy and other activities for the undergraduates that provided a unique experience to socialize with students of other universities. Overall, I feel that the presentation aspect of my capstone project served to not only deepen my research experience, but also broaden my experience across disciplines.

As I have worked on my capstone many of my future goals have become solidified. My work in elliptic curve has opened future doors of employment and education. This summer I will be working for the National Security Agency as an applied research mathematician. I will be using the same research skills I have developed during my capstone, now with applications in encryption and signal analysis. In the fall I plan on returning to USU to complete a master's degree in mathematics, working on even more research with Dr. Malmendier in algebraic geometry. The relationship I have developed with him as a mentor on the capstone project will serve to accelerate the research I will do as part of a master's degree. We hope to publish our work in a significant journal before the end of the year. Upon completion of my master's degree, I plan to pursue a PhD in mathematics at a top research institution.

My honors capstone project has been a challenging but rewarding experience. The writing aspect of the capstone appeared daunting at first, and there were definitely times that I did not know how I could complete it and at the same time do well in my course work. The sense of accomplishment in finishing this project has definitely been worth it. The final product will serve as a starting point for future papers that Dr. Malmendier and I hope to publish. Above all I have come to gain a deep passion for studying algebraic geometry. To an incoming honors student looking for a project, I would encourage them to go for a project that pushes them far beyond their current capabilities. With the help of a good mentor, I think they will be surprised how much they can grow and learn during their capstone project.

Biography.

Thomas Hill earned a BS in Mathematics from Utah State University, with a minor in statistics. He has conducted research with Dr. Ian Anderson on integrable systems theory that resulted in a Goldwater Scholarship. Most recently he has been conducting research in algebraic geometry with Dr. Andreas Malmendier. He is interning as an Applied Research Mathematician with the National Security Agency. He plans to return to USU for a masters and then pursue a PhD in mathematics at a top research institution.