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IMAGE ALGEBRA AND RESTORATION

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Abstract

The arrival of image algebras has made it possible to express a vast amount of the heterogeneous material in the image processing literature in a convenient and consistent manner. Furthermore, this type of formulation has proved very fertile and many new ideas have emerged. Image restoration has, however, been studied very much less than enhancement and analysis. We explain briefly the use of one such algebra in this field and show that such tasks as reconstruction from focal series and three-dimensional reconstruction can easily be incorporated.

Key Words: Image algebra, restoration, templates.

Introduction

The topics that are grouped under "image processing" cover an extremely wide range; they interest fields as far apart as forensic science and electron microscopy and new applications constantly arrive, sometimes in the most unexpected quarters: an AFRC research institute has identified "over 90 potential applications for imaging systems in pig production alone" (Schofield, 1990). It is therefore hardly surprising that, although similar methods and algorithms are employed in different fields of application, the notation and vocabulary are so different that their similarity is easily overlooked and exchange of methods and software is a tedious business. In response to this situation, efforts have been made to devise a very open mathematical structure into which most of the themes of image processing could be fitted. A special-purpose image-processing language well-adapted to parallel computer architectures would then translate the mathematics into practice.

The structures that have emerged are the image algebras, of which several have been proposed (Giardina, 1984, 1986; Ritter and Gader, 1987; Ritter et al., 1990; Ritter, 1991; Huang et al., 1989; Dougherty, 1989). That of Huang et al. (1989) is primarily intended for binary images and, although it can be extended to include grey-level images, we shall not consider it further. The algebra of Ritter and colleagues differs from that of Dougherty and Giardina in that the latter regard all arrays as images whereas Ritter distinguishes between images and templates. (This is not the only
difference but the others need not concern us.) The notion of template proves to be very fruitful and we have therefore preferred to discuss restoration in terms of the algebra defined by Ritter and colleagues.

These algebras have so far been used principally for image analysis, with mathematical morphology a major preoccupation, and for enhancement. Such tasks as three-dimensional reconstruction, solution of the phase problem and reconstruction from focal series have attracted less attention. These aspects of restoration are, however, major topics of electron image processing and we consider here their incorporation into the algebraic canon.

**Algebra**

The mathematical structure sought should be as simple as possible, with a limited number of operators and operands. The solution is an algebra in which the main operands have the character of images and the operators are those of everyday arithmetic and set theory: addition, multiplication, maximum and the like. This proves to be rich enough not merely to express most of the existing image processing techniques but also to suggest many novel avenues for exploration.

The operands are, together with value sets (real numbers, complex numbers, integers ...) and coordinate sets, *images* and *templates*. We denote value sets generically by $F$ and coordinate sets by $X, Y, \ldots$ An $F$-valued image $a$ on $X$ is thus of the form

$$a = \{ (x, a(x)) | x \in X \} \quad (1)$$

Explicitly, $a$ consists of pixels, the positions of which are labelled by $x$ and the grey-level or other values are denoted by $a(x)$. For a measured image, $a(x)$ will probably be the grey-level of the pixel but in a transformed image, $a(x)$ may well be complex. Images may be added, multiplied, or the maximum selected but this last requires that the pixel values be real.

Templates are slightly more complicated, for two coordinate sets $X$ and $Y$ are now involved. An $F$-valued template $t$ from $Y$ to $X$ is a function: $t(y)$ is an $F$-valued image on $X$. In order to avoid having to write $t(y)(x)$, it is usual to denote $t(y)$ by $t_y$ and this enables us to denote the weights ("pixel values") of $t_y$ at the various points of $t(y)$ by $t_y(x)$.

In many situations, the template structure shifts bodily, with no change in the weights, as $y$ varies: $t_y(x) = t_{y+z}(x+z)$. Such templates are shift-invariant and can be represented by a simple diagram: Fig. 1 shows such a template in which $y = (x_1, x_2)$ and the weights are $t_y(x_1, x_2) = -12, t_y(x_1 \pm 1, x_2) = t_y(x_1, x_2 \pm 1) = 2, t_y(x_1 \pm 1, x_2 \pm 1) = 1$. The resemblance to isoplanatic point-spread functions is no coincidence.

Nevertheless, there are important examples of templates that are shift-variant.

For our present purposes, the most useful operation involving an image $a$ and a template $t$ is generalized convolution:

$$a \odot t = \{ (y, b(y)) | b(y) = \Sigma a(x) t_y(x) \} \quad (2)$$

The sum is taken over $X$ but in practice, $t_y(x)$ will usually be much smaller than $a$. For shift-invariant templates, this is merely discrete convolution but if $t_y$ alters as $y$ varies, a more complicated combination is generated.

These definitions can be extended to allow images, templates and operators to be multi-valued. Thus if its pixel values are complex, we might regard an image as two-valued. If we wish to associate some characteristic or label with each pixel, then we can associate one value of a multi-valued image with its grey-level and the others with quantities characterizing the labels; the latter might, for example, describe chemical composition, or directionality or membership of a class. The operators need to be multi-valued as well in order to manipulate the different components of the image (or template) appropriately. Grey-level values will probably require arithmetic operators but the others we have mentioned are more likely to be linked by logic operators. Formally, we regard a multi-valued image $a$ as a stack of component images $a_1 \ldots a_n$. If $o$ denotes an ordered set of operations, $o = (o_1, o_2, \ldots, o_n)$, then

$$a \circ o = \{ (x, c(x)) | c(x) = (a_1 o_1 b_1, a_2 o_2 b_2 \ldots a_n o_n b_n) \} \quad (3)$$

180
Fig. 1 - A shift-invariant template with "target point" $y = (x_1, x_2)$

Again, $o$ may be unary; for example, with

$n = 2$ and $o = (\sin, \cos), \quad o \mathbf{a} = (\sin a_1, \cos a_2)$ (4)

The operators connecting images and templates are generalized in the same way.

The above definition of $\mathbf{a} \circ \mathbf{b}$ is restrictive in the sense that only combinations involving the same components of different images are permitted ($a_j \circ_j b_j$). We have proposed a simple extension of this definition (Hawkes, 1992) that allows components to be mixed, which is necessary if these components represent the real and imaginary parts of a complex image. We then write

$$a \circ \mathbf{b} = \{ (x, c(x)) \mid c(x) = (a \circ_1 b, a \circ_2 b, \ldots, a \circ_n b) \}$$ (5)

In the unary case, this could also be used to distribute real and imaginary parts correctly. For the logarithm, for example, we know that $\ln \mathbf{a} = \ln |\mathbf{a}| + i \arg \mathbf{a}$ and if $\mathbf{a}$ is stacked as $(\text{Re} \mathbf{a}, \text{Im} \mathbf{a}) = (a_1, a_2)$, the operator $\circ = \ln \mathbf{a}$ will create $(\circ_1 a, \circ_2 a)$ where $\circ_1 = \ln \text{mod}$ and $\circ_2 = \arg$. (The unary operator $\circ = \ln \mathbf{a}$ would create $(\ln a_1, \ln a_2)$.)

Restoration

Electron image restoration is primarily concerned with recovery of the electron wavefunction from focal series for weakly scattering specimens or from other data sets for the general case (the so-called phase problem) or with reconstruction of three-dimensional structure from projections. We consider these briefly in turn. Another example, not considered here, is the deconvolution of the probe current density in scanning instruments.

Focal series.

The data set here consists of $n$ images recorded at different defocus values but otherwise in principle the same. The corrected wavefunction is the inverse Fourier transform of a weighted sum of the Fourier transforms of these images or the sum of the convolutions of the images with the corresponding response functions. The various forms of the weights that have been proposed, notably by Schiske (1973) and Kirkland et al. (1980) are not important here. If the $n$ images are denoted by $a_j, j = 1, 2, \ldots, n$ and $A_j$ is the Fourier transform of $a_j$, then the reconstructed image $\hat{a}$ with transform $\hat{A}$ is given by

$$\hat{A} = \sum_{j=1}^{n} A_j W_j$$ (6)

or

$$\hat{a} = \sum_{j=1}^{n} a_j * w_j$$ (7)

in which $W_j$ are the appropriate filters or weights and $w_j$ their inverse transforms. Suppose now that $\mathbf{a}$ is a multi-valued image.

$$\mathbf{a} = (a_1, a_2, \ldots, a_n)$$ (8)

and likewise for $A, W$ and $w$. Then the image algebra expression for restoration from focal series is simply

$$\hat{\mathbf{a}} = \sum_{j} p_j (\mathbf{a} \oplus w)$$ (9)

in which $p_j$ is a projection function that in general deletes components from a multi-valued image and here selects the $j$-th component.

Another extension of image algebra, which we
have not yet mentioned, concerns the nature of the pixel. We have described an F-valued image \( a \) on \( X \) thus

\[
\mathbf{a} = \{ (x, a(x)) \mid x \in X \} \quad (10)
\]

and said that \((x, a(x))\) represents a pixel located at \(x\) and having grey-level (or other) value \(a(x)\). Wilson (1990) has enquired whether it is useful to consider other interpretations of this description and in particular, the consequences of studying images, each of whose "pixels" is itself an image. This idea makes it possible to write the above expression for \( \mathbf{a} \) (or \( \mathbf{A} \)) even more simply: \( a \) is a row vector, each of whose elements is a member of the focal series and \( w \) is likewise a column vector. Then we can either form the everyday matrix product of \( A \) and \( W \), interpreting \( A_j W_j \) as the direct product \((A_j W_j)_{pq} = A_{jp} W_{jp}\) or else allow matrix products of the form \( a \circ w \), for which

\[
(a \circ w)_{pq} = \sum_r a_{pr} w_{rp} \quad (11)
\]

in which case the real-space focal-series reconstruction becomes simply \( \mathbf{a} = \mathbf{a} \circ \mathbf{w} \).

The phase problem

We have dealt with this explicitly elsewhere (Hawkes, 1991a) and we merely state that a very compact form of the various iterative algorithms of the Gerchberg-Saxton type can be found, both in their original form (1972, 1973) and in the extensions proposed by Fienup (e.g. 1984) and elaborated by many others.

Three-dimensional reconstruction

This involves several very different kinds of calculation, which are themselves governed by the nature of the specimen being reconstructed. In general, after collection of a tilt series, the images may require unbending, classification, enhancement or other forms of pre-processing before such procedures as filtered back-projection or iterative reconstruction can be applied. The basic image algebra is now in the process of being embedded in a more general structure, which makes the solution of eigenvalue equations, required during classification by correspondence analysis, easier to incorporate. Filtered back-projection can be written immediately as an image-template convolution. The use of projection onto convex sets to minimize the damage due to the absence of "missing-cone" data is already in algebraic form (this is also discussed briefly in Hawkes, 1991b).

Conclusions

The foregoing material can give no more than a very superficial idea of the attractions of image algebra, many aspects of which have been excluded here. Furthermore, despite the long accounts of the basic algebra and of many applications and some extensions, the major contributors regard it as far from complete: "..., we have made no serious attempts to extend the algebra to the symbolic domain. In particular, high level image operations which employ tools from such diverse areas as knowledge representation, graph theory and surface representation have not been considered. Furthermore, the mathematics associated with the image algebra and its implications for image processing is, in itself, largely uncharted territory" (Ritter et al., 1990). Although the present discussion has been confined to the translation of existing electron image restoration algorithms into algebraic terms, we believe that there are much more exciting reasons for interest in image algebra than uniformity of expression, valuable though this is. Once an algorithm or procedure has been written in the compact form that image algebra frequently confers on it, it is impossible not to enquire what happens if we alter or generalize some element of the expression. If the latter contains images and templates, we naturally enquire what happens if we use other templates of even other image-template operators. What changes must we make if different value sets are involved (typically complex quantities instead of real)? The field of complex numbers is an extension of the field of real numbers; are any of the other extensions of interest? Davidson (1992) has shown how very intimate is the connection between minimax algebra and image algebra and the
implications of this for mathematical morphology; what implications will it have in other branches of image processing? Image algebra, which began as a unifying force, is proving to be a source of new ideas as well as an elegant receptacle for those we already know.

References


Discussion with Reviewers

J.K. Weiss: It appears that this algebra would be ideal for the multivariate statistical analysis of so-called "spectrum images" currently being performed by Bonnet and others. Have you considered the structure of the image algebra for these cases where the F-valued set now has entire spectra at each F-value with some relationships between each of the F-values?

Author: I am in fact working on just this case now.

J.P. Davey: One shift-invariant template with an exact local (sufficiently far from the edges) inverse is a simple shift operation. If the rule applied at the edges of the image is to assume that the image is
cyclically continued (if there are n pixels in a line, then for a(0) we take the value of a(n), and for a(n+1) we take a(1), and so on), then if the image is convoluted with the shift template and subsequently with its inverse, the original image is restored everywhere. Are there other templates, also having exact local inverses, for which this rule would fail to restore the image near the edges? 

Author: I think not, though there may be pathological cases. All the steps in the proposed procedure are linear and so one should be able to retrace one’s steps (neglecting noise). But this certainly merits closer examination.