A Comparison Theorem for the Topological and Algebraic Classification of Quaternionic Toric 8-Manifolds

Piotr Runge
Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/etd

Part of the Algebra Commons

Recommended Citation
https://digitalcommons.usu.edu/etd/501
A COMPARISON THEOREM FOR THE TOPOLOGICAL AND ALGEBRAIC CLASSIFICATION OF QUATERNIONIC TORIC 8-MANIFOLDS

by

Piotr Runge

A dissertation submitted in partial fulfillment of the requirements for the degree of DOCTOR OF PHILOSOPHY in Mathematical Sciences

Approved:

Dr. Dariusz Wilczynski
Major Professor

Dr. Ian M. Anderson
Committee Member

Dr. LeRoy B. Beasley
Committee Member

Dr. Zhi-Qiang Wang
Committee Member

Dr. James T. Wheeler
Committee Member

Dr. Byron R. Burnham
Dean of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah
2009
ABSTRACT

A Comparison Theorem for the Topological and Algebraic Classification of Quaternionic Toric 8-Manifolds

by

Piotr Runge, Doctor of Philosophy
Utah State University, 2009

Major Professor: Dr. Dariusz Wilczynski
Department: Mathematics and Statistics

In order to discuss topological properties of quaternionic toric 8-manifolds, we introduce the notion of an algebraic morphism in the category of toric spaces. We show that the classification of quaternionic toric 8-manifolds with respect to an algebraic isomorphism is finer than the oriented topological classification. We construct infinite families of quaternionic toric 8-manifolds in the same oriented homeomorphism type but algebraically distinct. To prove that the elements within each family are of the same oriented homeomorphism type, and that we have representatives of all such types of a quaternionic toric 8-manifold, we present and use a method of evaluating the first Pontrjagin class for an arbitrary quaternionic toric 8-manifold.
This dissertation is for my sons, Piotr Jr., Józef, and Karol.
ACKNOWLEDGMENTS

I would like to thank my major professor, Dariusz Wilczynski, for suggesting the project for my research, and for his insights and encouragement.

I want to thank my family and friends, especially my wife, Bożena, and my mother, Stefania, for their patience, love, and support. Also, I want to thank all my teachers and professors that implanted in me the love for mathematics.

But most importantly, I’m grateful to God for the gift of knowledge and for the blessings that He endows me with throughout my life.

Piotr Runge
CONTENTS

<table>
<thead>
<tr>
<th>Section</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>ABSTRACT</td>
<td>iii</td>
</tr>
<tr>
<td>ACKNOWLEDGMENTS</td>
<td>v</td>
</tr>
<tr>
<td>LIST OF TABLES</td>
<td>viii</td>
</tr>
<tr>
<td>LIST OF FIGURES</td>
<td>ix</td>
</tr>
<tr>
<td>1 INTRODUCTION</td>
<td>1</td>
</tr>
<tr>
<td>2 DEFINITIONS AND EXAMPLES</td>
<td>4</td>
</tr>
<tr>
<td>2.1 The definition of a quaternionic toric space</td>
<td>4</td>
</tr>
<tr>
<td>2.2 Algebraic morphism</td>
<td>8</td>
</tr>
<tr>
<td>2.3 The sum of toric spaces</td>
<td>15</td>
</tr>
<tr>
<td>3 A SUMMARY OF FORMER RESULTS, AND THE MAIN RESULT</td>
<td>20</td>
</tr>
<tr>
<td>3.1 Toric manifolds, homology, and cohomology</td>
<td>20</td>
</tr>
<tr>
<td>3.2 Oriented topological classification of quaternionic toric 8-manifolds</td>
<td>24</td>
</tr>
<tr>
<td>3.3 The main result</td>
<td>25</td>
</tr>
<tr>
<td>4 AUTOMORPHISMS OF $F_2$</td>
<td>27</td>
</tr>
<tr>
<td>4.1 Automorphisms of simple form</td>
<td>27</td>
</tr>
<tr>
<td>4.2 Constructions of an element of the preimage under $\pi_*$</td>
<td>29</td>
</tr>
<tr>
<td>4.3 Prebasic elements</td>
<td>32</td>
</tr>
<tr>
<td>5 TOPOLOGICAL INVARIANTS</td>
<td>38</td>
</tr>
<tr>
<td>5.1 Intersection form</td>
<td>39</td>
</tr>
<tr>
<td>5.2 First Pontrjagin class</td>
<td>40</td>
</tr>
<tr>
<td>5.3 Oriented connected sums</td>
<td>43</td>
</tr>
<tr>
<td>5.4 Invariant $q$ for an arbitrary quaternionic toric 8-manifold</td>
<td>44</td>
</tr>
<tr>
<td>6 FAMILIES OF QUATERNIONIC TORIC 8-MANIFOLDS</td>
<td>61</td>
</tr>
<tr>
<td>6.1 The first family</td>
<td>61</td>
</tr>
<tr>
<td>6.2 The action of the Cartesian product of the free group $F_2$</td>
<td>71</td>
</tr>
<tr>
<td>6.3 The second family</td>
<td>73</td>
</tr>
<tr>
<td>6.4 The third family and the proof of the main result</td>
<td>76</td>
</tr>
<tr>
<td>REFERENCES</td>
<td>120</td>
</tr>
<tr>
<td>APPENDICES</td>
<td></td>
</tr>
<tr>
<td>APPENDIX A A SUPPLEMENTARY CALCULATION FOR EXAMPLE 4.11</td>
<td>122</td>
</tr>
<tr>
<td>APPENDIX B A MODIFICATION OF SCOTT’S DISCUSSION OF THE FIRST PONTRJAGIN CLASS</td>
<td>123</td>
</tr>
<tr>
<td>Table</td>
<td>Description</td>
</tr>
<tr>
<td>-------</td>
<td>-----------------------------------------------------------------------------</td>
</tr>
<tr>
<td>6.1</td>
<td>The values of $\lambda_j(E_3)(g(t))$ dependent on the triple $(S, \varepsilon, a_3)$</td>
</tr>
<tr>
<td>6.2</td>
<td>The values of $g(t)$ and $\lambda_j(E_4)(g(t))$ dependent on the quadruple $(S, \varepsilon, a_3, a_4)$</td>
</tr>
<tr>
<td>6.3</td>
<td>The values of $\lambda_j(E_3)(g(t))$ dependent on the triple $(S, \varepsilon, a_3)$</td>
</tr>
<tr>
<td>6.4</td>
<td>The length of $v_1$ dependent on the triple $(S, a_R+2)$ if $</td>
</tr>
<tr>
<td>6.5</td>
<td>The length of $u_1$ dependent on the triple $(S, a_R+2, a_{R+1})$</td>
</tr>
<tr>
<td>6.6</td>
<td>The length of $u_2$ dependent on the triple $(S, a_R+2, a_R)$</td>
</tr>
<tr>
<td>6.7</td>
<td>The length of $v_1$ dependent on the triple $(S, a_R+2)$ if $</td>
</tr>
<tr>
<td>6.8</td>
<td>The lengths of $u_3$ and $u_4$ dependent on $a_{R+2}$</td>
</tr>
<tr>
<td>6.9</td>
<td>The lengths of $u_5$ and $u_6$ dependent on $a_{R+2}$</td>
</tr>
<tr>
<td>6.10</td>
<td>The lengths of $u_7$ and $u_8$ dependent on the pair $(a_{R+2}, a_R)$</td>
</tr>
<tr>
<td>6.11</td>
<td>The lengths of $u_9$ and $u_{10}$ dependent on the pair $(a_{R+2}, a_R)$</td>
</tr>
<tr>
<td>6.12</td>
<td>The lengths of $u_{11}$ and $u_{12}$ dependent on the pair $(a_{R+2}, a_R)$</td>
</tr>
<tr>
<td>6.13</td>
<td>The lengths of $u_{13}$ and $u_{14}$ dependent on the pair $(a_{R+2}, a_R)$</td>
</tr>
<tr>
<td>6.14</td>
<td>The length of $v_2$ dependent on the pair $(i, a_{R+2})$</td>
</tr>
<tr>
<td>6.15</td>
<td>The lengths of $u_{16}$ and $u_{17}$</td>
</tr>
<tr>
<td>6.16</td>
<td>The lengths of $u_d$ $(d = 18, 19, 20, 21)$</td>
</tr>
</tbody>
</table>
# LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>$CL_1$ and $CL_2$.</td>
<td>17</td>
</tr>
<tr>
<td>2.2</td>
<td>$Q'_1$ and $Q'_2$.</td>
<td>17</td>
</tr>
<tr>
<td>2.3</td>
<td>$Q''_1$ and $Q''_2$.</td>
<td>17</td>
</tr>
<tr>
<td>2.4</td>
<td>The sum of $Q_1$ and $Q_2$.</td>
<td>17</td>
</tr>
<tr>
<td>2.5</td>
<td>Notation for the faces of the sum of convex polytopes.</td>
<td>18</td>
</tr>
<tr>
<td>5.1</td>
<td>The algebraic isomorphism $j: \mathcal{X} \rightarrow \mathcal{X}'$.</td>
<td>42</td>
</tr>
<tr>
<td>5.2</td>
<td>The space $\mathcal{X}_k$.</td>
<td>46</td>
</tr>
<tr>
<td>5.3</td>
<td>Notation for the edges of $R$.</td>
<td>50</td>
</tr>
<tr>
<td>5.4</td>
<td>Placement of the faces $F_j$ in $R$.</td>
<td>54</td>
</tr>
<tr>
<td>6.1</td>
<td>$\mathcal{X}(Q_4, \lambda_{0,0})$, and $\mathcal{X}(Q_6, \lambda_{0,1})$.</td>
<td>62</td>
</tr>
<tr>
<td>6.2</td>
<td>$\mathcal{X}(Q_5, \lambda_{5,1})$, and $\mathcal{X}(Q_6, \lambda_{-4,1})$.</td>
<td>63</td>
</tr>
<tr>
<td>6.3</td>
<td>Notation for the faces of $Q = Q_R + Q_2 + Q_R$.</td>
<td>68</td>
</tr>
</tbody>
</table>
CHAPTER 1
INTRODUCTION

Toric manifolds over \( \mathbb{R} \) and \( \mathbb{C} \) are topological analogues of toric varieties in algebraic geometry. They are \( n \)-dimensional (resp. \( 2n \)-dimensional) manifolds with an effective action of the \( n \)-dimensional torus \((S^0)^n\) (resp. \((S^1)^n\)) with the orbit space a simple convex polytope. Such manifolds were studied, for example, in [6] and [1]. In the latter, Davis and Januszkiewicz introduced the notion of a characteristic function that describes a toric manifold associated with a simple convex polytope as a certain topological quotient of the Cartesian product of the polytope and the torus. A similar idea was used by Scott in [7] to define \( 4n \)-dimensional manifolds over the real quaternions \( \mathbb{H} \). He defined quaternionic toric manifolds as topological quotients of the Cartesian product of a simple convex \( n \)-polytope and the quaternionic \( n \)-torus \((S^3)^n\). Since the multiplication in \( \mathbb{H} \) is noncommutative, quaternionic toric manifolds do not admit a quaternionic torus action in general. They are studied from the topological point of view as oriented manifolds.

The results about homology groups and the cohomology ring of quaternionic toric manifolds (see §3 of [7]) are analogous to the results for toric manifolds over \( \mathbb{R} \) and \( \mathbb{C} \) (see sections 3 and 4 of [1]). In particular, the nonzero Betti numbers of toric manifolds are determined by the combinatorics of the polytope (they are related to the \( h \)-vector), while the cohomology ring is a quotient of the face ring of the polytope by an ideal determined by the characteristic function with which the toric manifold is associated. For the topological classification of toric manifolds, one needs to evaluate appropriate characteristic classes. In the case \( n = 2 \) Scott proved that each quaternionic toric 8-manifold is 3-connected and homeomorphic to a manifold which is smooth away from a single point or a smooth manifold. By the theory of Wall [9], such 8-manifolds are characterized, up to an orientation-preserving homeomorphism, by the intersection form and the first Pontrjagin class. The calculation of the intersection form for quaternionic toric 8-manifolds was presented in §4 of [7]. Scott introduced a differential structure on a subset of a quaternionic toric 8-manifold using...
the characteristic function with which the manifold is associated. Using this differential structure and the results of Milnor [4], Scott described a method for evaluating the first Pontrjagin class of some quaternionic toric 8-manifolds defined by characteristic functions satisfying certain strict conditions. A summary of the results presented in [7] can be found in chapters 3 and 5 of this dissertation. We also slightly modify Scott’s discussion of the first Pontrjagin class (see Appendix B) to fit it better in our results.

In chapter 2 of this dissertation, we first define a quaternionic toric space using the characteristic function obtained from a class of homomorphisms of free groups. Then we introduce the notion of an algebraic morphism between toric spaces and present several examples. We also describe geometric properties of algebraic morphisms that can be applied in the discussion of topological properties of toric manifolds. Since algebraic morphisms are also topological embeddings, in the case $n = 2$ one can consider the classification of quaternionic toric 8-manifolds with respect to an orientation-preserving algebraic isomorphism. We shall not approach this classification. In the main result (Theorem 3.14) we show that in every topological type of a quaternionic toric 8-manifold there are infinitely many algebraically distinct elements. This result ensures that the algebraic classification is much finer than the topological classification. To prove the main result, we define infinite families of quaternionic toric 8-manifolds (see chapter 6). We prove that in each family all elements are equivalent topologically but algebraically distinct. The machinery needed in the proof is introduced in chapters 4 and 5. We also use Wall’s result stating that in the case of quaternionic toric 8-manifolds the existence of an isomorphism between fourth homology groups that preserves the invariants is equivalent to the existence of an orientation-preserving homeomorphism. In chapter 5 we present a method of evaluating the first Pontrjagin class for an arbitrary quaternionic toric 8-manifold. This technical result is a generalization of Scott’s discussion mentioned above. For each edge of the polygon with which the manifold is associated, the value of the characteristic function at the given edge is determined by a prebasic element $w$ of the free group $F_2$ (see section 4.3). To evaluate the first Pontrjagin class using the method presented in section 5.4, one needs to find an
automorphism $\psi$ of $F_2$ such that $\psi(x_1) = w$, together with a simple decomposition of $\psi$ (introduced in section 4.1). The method of obtaining both the automorphism $\psi$ and its simple decomposition, for any prebasic element of $F_2$, is presented in chapter 4.
CHAPTER 2
DEFINITIONS AND EXAMPLES

2.1 The definition of a quaternionic toric space

Quaternionic toric spaces were first defined by Scott [7]. Our definition is based on
Scott’s definition, although we make a more formal connection with the homomorphisms of
free groups.

By \( \mathbb{H} \) we denote the normed division algebra of real quaternions. We consider the
presentation of quaternions as vectors in \( \mathbb{R}^4 \) with the Euclidean norm. \( T = S^3 \) will denote
the unit sphere in \( \mathbb{H} \). The multiplication in \( \mathbb{H} \) restricts to a well-defined binary operation
\( T \times T \rightarrow T \) which is a Lie group operation. For a positive integer \( n \), let \( T^n \) denote the
quaternionic \( n \)-torus equipped with the product binary operation. Define \( T^0 \) and \( T^{-1} \) to be
a point. \( F_n \) will denote the free group on the generators \( x_1, x_2, \ldots, x_n \) for a positive integer
\( n \), and the trivial group for \( n = -1, 0 \). The identity element of \( F_n \) will be denoted by 1. Also, throughout the paper, given a set \( A \), \( \mathbb{I}_A \) will denote the identity function from \( A \) to
itself.

Consider the group \( \text{Map}(T^n, T) \) with the multiplication of functions as the group op-
eration. For each generator \( x_i \) of \( F_n \), let the map \( e(x_i): T^n \rightarrow T \) be given by

\[ e(x_i)(t_1, \ldots, t_n) = t_i. \]

Since \( \mathbb{H} \) is a normed division algebra, one can use its algebra multiplication to extend \( e \) to
a homomorphism \( e: F_n \rightarrow \text{Map}(T^n, T) \). Then \( e \) converts a word \( w \in F_n \) to a continuous
map \( e(w): T^n \rightarrow T \), taking the \( n \)-tuple \((t_1, \ldots, t_n)\) to the corresponding product of the \( t_i \)
and \( t_j^{-1} \) in \( T \). Using the map \( e \) we define a functor from the set of homomorphisms of free
groups to the set of maps between quaternionic tori.
Definition 2.1. For \( m, n \in \mathbb{N} \), define \( E: \text{Hom}(F_m, F_n) \to \text{Map}(T^n, T^m) \) by

\[
E(h) = \left( e(h(x_1)), \ldots, e(h(x_m)) \right)
\]

where \( h \in \text{Hom}(F_m, F_n) \).

We will show that \( E \) is a contravariant functor.

Lemma 2.2. If \( h \in \text{Hom}(F_m, F_n) \), then for every \( w \in F_m \),

\[
(e \circ h)(w) = e(w) \circ E(h).
\]

Proof. Since \( h \) and \( e \) are homomorphisms, it is enough to show that the statement holds for \( w = x_i \) (\( 1 \leq i \leq m \)). For each \( i \), let \( w_i = h(x_i) \). Fix \( i \). Then for every \( t \in T^n \),

\[
\begin{align*}
(e(x_i) \circ E(h))(t) &= e(x_i)(e(w_1)(t), \ldots, e(w_m)(t)) = e(w_i)(t) = e(h(x_i))(t) \\
&= ((e \circ h)(x_i))(t).
\end{align*}
\]

\[\square\]

Proposition 2.3. For every \( g \in \text{Hom}(F_l, F_m) \) and \( h \in \text{Hom}(F_m, F_n) \),

\[
E(h \circ g) = E(g) \circ E(h).
\]

Proof. Let \( g \) be given by the equations \( g(x_i) = v_i \) for \( 1 \leq i \leq l \). Then

\[
\begin{align*}
E(h \circ g) &= \left( e((h \circ g)(x_1)), \ldots, e((h \circ g)(x_l)) \right) \\
&= \left( e(h(v_1)), \ldots, e(h(v_l)) \right) \\
&= \left( (e(v_1) \circ E(h)), \ldots, (e(v_l) \circ E(h)) \right) \quad \text{by Lemma 2.2} \\
&= \left( e(v_1)(E(h)), \ldots, e(v_l)(E(h)) \right) \\
&= E(g) \circ E(h).
\end{align*}
\]

\[\square\]

The maps in the range of \( E \) are differentiable. Hence \( E(\text{Aut } F_n) \) is a subgroup of the
group $\text{Diff}(T^n, T^n)$ of self-diffeomorphisms of $T^n$. The elements of $E(\text{Aut} F_n)$ will be called 
\textit{algebraic} diffeomorphisms.

\textbf{Definition 2.4.} For $k \leq n$, let

1. $\iota_{k,n} \in \text{Hom}(F_k, F_n)$ denote the standard inclusion given by the equations $\iota_{k,n}(x_i) = x_i$ for $1 \leq i \leq k$ (by convention, $\iota_{k,n}: \{1\} \to F_n$ is defined by $1 \mapsto 1$ for $k = -1, 0$),

2. $\pi_{n,k} \in \text{Hom}(F_n, F_k)$ be defined by the equations $\pi_{n,k}(x_i) = x_i$ for $1 \leq i \leq k$ and $\pi_{n,k}(x_i) = 1$ for $k + 1 \leq i \leq n$ (by convention, $\pi_{n,k}: F_n \to \{1\}$ is defined by $w \mapsto 1$ for every word $w \in F_n$ for $k = -1, 0$),

3. $P(n, k) = \{E(\sigma \circ \iota_{k,n}) \mid \sigma \in \text{Aut} F_n\}$,

4. $P(k, n) = \{E(\pi_{n,k} \circ \sigma) \mid \sigma \in \text{Aut} F_k\}$,

5. $P(n) = \bigcup_{k \leq n} P(n, k)$.

\textbf{Lemma 2.5.} Let $l \leq m \leq n$. If $\varphi \in P(n, m)$, and $\psi \in P(m, l)$, then $\psi \circ \varphi \in P(n, l)$.

\textit{Proof.} Assume that $\varphi = E(\sigma \circ \iota_{m,n})$ and $\psi = E(\tau \circ \iota_{l,m})$, where $\sigma \in \text{Aut} F_n$ and $\tau \in \text{Aut} F_m$. Then $\psi \circ \varphi = E(\sigma \circ \iota_{m,n} \circ \tau \circ \iota_{l,m})$ by Proposition 2.3. Define $\bar{\tau} \in \text{Aut} F_n$ by the equations $\bar{\tau}(x_i) = (\iota_{m,n} \circ \tau)(x_i)$ if $1 \leq i \leq m$ and $\bar{\tau}(x_i) = x_i$ if $m + 1 \leq i \leq n$. Then $\iota_{m,n} \circ \tau \circ \iota_{l,m} = \bar{\tau} \circ \iota_{l,n}$. Hence $\psi \circ \varphi = E(\sigma \circ \bar{\tau} \circ \iota_{l,n})$. Since $\sigma \circ \bar{\tau} \in \text{Aut} F_n$, $\psi \circ \varphi \in P(n, l)$.

Define a \textit{convex polytope} $Q$ to be a compact intersection of finitely many half spaces in a Euclidean space. $Q$ is homeomorphic to a Euclidean disk $D^n$ for some $n \geq 0$. We call $n$ the \textit{dimension} of $Q$, and $Q$ itself an $n$-polytope. A \textit{proper face} of $Q$ is the intersection of $Q$ with a hyperplane disjoint from the interior of $Q$. Note that each proper face of $Q$ is itself a convex polytope. $Q$ is called the \textit{improper face}. By convention, $\emptyset$ is the \textit{empty face} of $Q$ of dimension $-1$. The set of all faces of $Q$ will be denoted by $\mathcal{P}(Q)$. $\mathcal{P}(Q)$ is a set partially ordered by inclusion (denoted by $\leq$) and graded by the dimension of each element. We call $\mathcal{P}(Q)$ the \textit{graded poset associated with $Q$}. $\mathcal{P}_k(Q)$ will denote the subset of elements of $\mathcal{P}(Q)$ of dimension $k$. 

Definition 2.6. Let \( m \leq n \). A characteristic function of rank \( n \) and height \( h = n - m \) for a graded poset \( \mathcal{P}(Q) \) associated with a convex \( m \)-polytope \( Q \) is a map \( \lambda: \mathcal{P}(Q) \to P(n) \) satisfying the following two conditions:

1. If \( F \in \mathcal{P}_k(Q) \), then \( \lambda(F) \in P(n, k + h) \) for \( k = -1, 0, \ldots, m \);
2. If \( E \in \mathcal{P}_l(Q), F \in \mathcal{P}_k(Q) \) and \( E \leq F \), then there exists \( \lambda_{E,F} \in P(k + h, l + h) \) such that \( \lambda(E) = \lambda_{E,F} \circ \lambda(F) \).

Note that condition (2) in the definition above makes sense by Lemma 2.5. Also, the maps \( \lambda_{E,F} \) are unique. Indeed, assume to the contrary that \( \bar{\lambda} \in P(k + h, l + h) \) has the property \( \lambda(E) = \bar{\lambda} \circ \lambda(F) \). Let \( \lambda(F) = E(\tau \circ \iota_{k+h,n}) \) where \( \tau \in \text{Aut} F_n \). Since \( \lambda_{E,F} \circ \lambda(F) = \bar{\lambda} \circ \lambda(F) \),

\[
\lambda_{E,F} \circ \lambda(F) \circ E(\pi_{n,k+h} \circ \tau^{-1}) = \bar{\lambda} \circ \lambda(F) \circ E(\pi_{n,k+h} \circ \tau^{-1}).
\]

Note that

\[
\lambda(F) \circ E(\pi_{n,k+h} \circ \tau^{-1}) = E(\pi_{n,k+h} \circ \tau^{-1} \circ \tau \circ \iota_{k+h,n}) = 1_{T^{k+h}}
\]

by Lemma 2.3. Hence \( \lambda_{E,F} = \bar{\lambda} \).

We now present the definition of a quaternionic toric space which is the main object of our study.

Definition 2.7. Let \( Q \) be a convex \( m \)-polytope. The toric space \( \mathcal{X}(Q, \lambda) \) of rank \( n \) (where \( m \leq n \)) associated with \( Q \) and a characteristic function \( \lambda \) of rank \( n \) and height \( n - m \) for \( \mathcal{P}(Q) \), is the topological quotient of \( Q \times T^n \) obtained by identifying \((q,s)\) with \((q,t)\) whenever \( q \in \text{int} F \) and \( \lambda(F)(s) = \lambda(F)(t) \).

Note that all identifications of the points of the Cartesian product \( Q \times T^n \) take place in the subset \( \partial Q \times T^n \). In particular, \( \mathcal{X}(Q, \lambda) \) contains a dense subset homeomorphic to \((\text{int} Q) \times T^n \). Hence the dimension of the topological space \( \mathcal{X}(Q, \lambda) \) is \( m + 3n \). Let \( p: \mathcal{X}(Q, \lambda) \to Q \) be defined by \( p([q,t]) = q \). We call \( p \) the projection of \( \mathcal{X}(Q, \lambda) \) onto \( Q \).
For examples of quaternionic toric spaces, we refer the reader to [7], Examples 2.1 – 2.5, pp. 377f.

2.2 Algebraic morphism

Since quaternionic toric spaces are quotient spaces, we would like to consider the maps between the Cartesian products of the form \(Q \times T^n\) that will induce well-behaved maps between the quotients. We first define a well-behaved map between convex polytopes. If \(U\) and \(V\) are convex subsets of \(\mathbb{R}^n\) and \(\mathbb{R}^m\), respectively, the map \(f: U \to V\) is called affine if for every convex linear combination \(\sum_j r_j u_j\), the equality \(f(\sum_j r_j u_j) = \sum_j r_j f(u_j)\) holds.

**Definition 2.8.** Let \(Q, R\) be convex polytopes of dimensions \(n \leq m\), respectively. We call an affine embedding \(f: Q \to R\) order-preserving with shift \(l \in \mathbb{N}\), if there exists a map \(\hat{f}: \mathcal{P}(Q) \to \mathcal{P}(R)\) such that for every \(k \leq n\), \(\hat{f}(\mathcal{P}_k(Q)) \subset \mathcal{P}_{k+l}(R)\).

The elements of \(P(n, k)\) (see Definition 2.4) will serve as maps between the tori.

Consider toric spaces \(\mathcal{X}(Q, \lambda), \mathcal{X}(R, \mu)\) of ranks \(n_1, n_2\), respectively, associated with convex polytopes \(Q, R\) of dimensions \(m_1 \leq m_2\), respectively, and characteristic functions \(\lambda, \mu\) of heights \(h_1, h_2\), respectively. The following definition will provide a condition for a pair of functions \(f: Q \to R\) and \(g \in P(n_1, n_2)\) sufficient to induce a well-defined and well-behaved function between \(\mathcal{X}(Q, \lambda)\) and \(\mathcal{X}(R, \mu)\).

**Definition 2.9.** Under the assumptions above let \(f: Q \to R\) be an order-preserving affine embedding with shift \(l\), and let \(g \in P(n_1, n_2)\). The pair \((f, g)\) satisfies the compatibility condition, if \(h_1 \leq l + h_2\), and for every \(k \leq m_1\), if \(F\) is a face of \(Q\) of dimension \(k\), then there exists \(g_F \in P(k + h_1, k + l + h_2)\) such that \(\mu(\hat{f}(F)) \circ g = g_F \circ \lambda(F)\).

Assume that a pair \((f, g)\) satisfies the compatibility condition. Define the map

\[ j = [f, g]: \mathcal{X}(Q, \lambda) \to \mathcal{X}(R, \mu) \]

by

\[ j([q, t]) = [f(q), g(t)]. \]
To verify that $j$ is well-defined, assume that $[q, t] = [q, s]$ in $\mathcal{X}(Q, \lambda)$ where $q \in \text{int } F$. Then $\lambda(F)(t) = \lambda(F)(s)$ and hence $(g_F \circ \lambda(F))(t) = (g_F \circ \lambda(F))(s)$. Thus $(\mu(\hat{f}(F))(g(t)) = (\mu(\hat{f}(F))(g(s))$ by definition 2.9, implying $[f(q), g(t)] = [f(q), g(s)]$ in $\mathcal{X}(R, \mu)$. Therefore $j$ is a well-defined map between quotient spaces. To verify that $j$ is injective, assume that $[f(q), g(t)] = [f(p), g(s)]$ in $\mathcal{X}(R, \mu)$. Then $f(q) = f(p)$ implying $q = p$, and $(\mu(\hat{f}(F))(g(t)) = (\mu(\hat{f}(F))(g(s))$ where $F$ is the face of $Q$ such that $q \in \text{int } F$. Hence $(g_F \circ \lambda(F))(t) = (g_F \circ \lambda(F))(s)$ by definition 2.9, implying $\lambda(F)(t) = \lambda(F)(s)$ since $g_F$ is injective. Therefore $[q, t] = [p, s]$ in $\mathcal{X}(Q, \lambda)$ and hence $j$ is injective. The properties discussed above imply that the map $j$ is a well-defined topological embedding.

**Definition 2.10.** If the pair $(f, g)$ satisfies the compatibility condition, then the map $[f, g]: \mathcal{X}(Q, \lambda) \rightarrow \mathcal{X}(R, \mu)$ is called an algebraic morphism.

To provide examples of algebraic morphisms, we make the following assumptions for the remainder of this section. Let $\mathcal{X}(Q, \lambda)$ be a toric space where $Q$ is an $m$-polytope and $\lambda$ is of rank $n$ and height $h$.

Even though one can consider the restriction of $\lambda$ to a face $F$ of $Q$, such a restriction is not a characteristic function for $\mathcal{P}(F)$ in general. Still, $\lambda$ provides data sufficient to view $p^{-1}(F)$ as a toric space, and to define an algebraic morphism from that space to $\mathcal{X}(Q, \lambda)$.

**Definition 2.11.** If $F \in \mathcal{P}_k(Q)$, define $L_F(\lambda): \mathcal{P}(F) \rightarrow P(k + h)$ by $L_F(\lambda)(E) = \lambda_{E,F}$ for every $E \in \mathcal{P}(F)$.

**Lemma 2.12.** $L_F(\lambda)$ is a characteristic function of rank $k + h$ and height $h$ for $\mathcal{P}(F)$.

**Proof.** If $E \leq F$ in $\mathcal{P}(F)$, then $E \leq F$ in $\mathcal{P}(Q)$. Hence if $E \in \mathcal{P}_l(F)$, $L_F(\lambda)(E) = \lambda_{E,F} \in P(k + h, l + h)$ by Definition 2.6. Also, if $E_i \in \mathcal{P}_i(F)$ for $i = 1, 2$, and $E_1 \leq E_2$, then

\[
\lambda(E_1) = \lambda_{E_1,F} \circ \lambda(F),
\]

\[
\lambda(E_2) = \lambda_{E_2,F} \circ \lambda(F),
\]
and

$$\lambda(E_1) = \lambda_{E_1, E_2} \circ \lambda(E_2).$$

Thus $\lambda_{E_1, F} = \lambda_{E_1, E_2} \circ \lambda_{E_2, F}$ by Lemma 2.5 and the uniqueness of $\lambda_{E_1, F}$. Hence $L_F(\lambda)(E_1) = \lambda_{E_1, E_2} \circ L_F(\lambda)(E_2)$. Therefore $L_F(\lambda)$ satisfies the conditions of Definition 2.6 and hence is a characteristic function of rank $k + h$ and height $h$ for $P(F)$. \qed

Assume that $\lambda(F) = E(\tau \circ \iota_{k+h,n})$ where $\tau \in \text{Aut} F$ (cf. Definition 2.4). Define $g = E(\pi_{n,k+h} \circ \tau^{-1})$. Then $g \in P(k+h,n)$.

**Definition 2.13.** Let $f$ denote the inclusion $F \subset Q$. Define

$$\kappa_F = [f, g] : \mathcal{X}(F, L_F(\lambda)) \to \mathcal{X}(Q, \lambda).$$

The choice of the automorphism $\tau$ above is not unique. In the following lemma we show that the definition above is independent of that choice.

**Lemma 2.14.** The map $\kappa_F$ does not depend on the choice of the automorphism $\tau$.

**Proof.** Assume that $\omega \in \text{Aut} F$ is another automorphism such that $\lambda(F) = E(\omega \circ \iota_{k+h,n})$. Define $g' = E(\pi_{n,k+h} \circ \omega^{-1})$. Let $[q, t] \in \mathcal{X}(F, L_F(\lambda))$ where $q \in \text{int} E$ and $E \leq F$. Then

\begin{equation}
[f, g][[q, t]] = [f, g'][[q, t]] \iff [q, g(t)] = [q, g'(t)] \\
\text{iff } \lambda(E)(g(t)) = \lambda(E)(g'(t)) \\
\text{iff } (\lambda_{E,F} \circ \lambda(F) \circ g)(t) = (\lambda_{E,F} \circ \lambda(F) \circ g')(t).
\end{equation}

By Proposition 2.3,

$$\lambda(F) \circ g = E(\pi_{n,k+h} \circ \tau^{-1} \circ \iota_{k+h,n}) = E(\pi_{n,k+h} \circ \omega^{-1} \circ \omega \circ \iota_{k+h,n}) = \lambda(F) \circ g'.$$

Hence by (2.1), $[f, g][[q, t]] = [f, g'][[q, t]]$. \qed
Consider the inclusion $f$ of Definition 2.13 as an order-preserving affine embedding with shift 0 (see Definition 2.8). If $E \in \mathcal{P}_1(F)$, then by Proposition 2.3,

$$\lambda(E) \circ g = \lambda_{E,F} \circ \lambda(F) \circ E(\tau^{-1})$$

$$= \lambda_{E,F} \circ E(\tau^{-1} \circ \tau \circ \iota_{k+h,n})$$

$$= \lambda_{E,F} \circ E(1_{F_k+h})$$

$$= E(1_{F_{k+h}}) \circ \lambda_{E,F}$$

$$= 1_{T_{i+h}} \circ L_F(\lambda)(E).$$

Thus the pair $(f, g)$ satisfies the compatibility condition (see Definition 2.9) and hence we have the following

**Proposition 2.15.** The map $\kappa_F$ defined in Definition 2.13 is an algebraic morphism.

It is worth noticing that the image under $\kappa_F$ of $X_\mathcal{L}(F, L_F(\lambda))$ in $X(Q, \lambda)$ is precisely $\pi^{-1}(F)$. The map $[1_F, \lambda(F)]: \pi^{-1}(F) \to X(F, L_F(\lambda))$ is a left inverse of $\kappa_F$.

Note that for a triple of toric spaces $X(Q_i, \lambda_i)$ ($i = 1, 2, 3$), if $j_i = [f_i, g_i]: X(Q_i, \lambda_i) \to X(Q_{i+1}, \lambda_{i+1})$ is an algebraic morphism for $i = 1, 2$, then

$$j_2 \circ j_1 = [f_2 \circ f_1, g_2 \circ g_1]: X(Q_1, \lambda_1) \to X(Q_2, \lambda_2)$$

is an algebraic morphism by Lemma 2.5 and Definition 2.8. Thus we can consider the notion of an algebraic isomorphism.

**Definition 2.16.** An algebraic morphism $j: X \to Y$ is called an algebraic isomorphism if there exists an algebraic morphism $j^{-1}: Y \to X$ such that $j \circ j^{-1} = 1_X$.

In the following proposition we show that algebraic morphisms between toric spaces “induce” algebraic morphisms between the toric spaces associated with corresponding faces. In the discussion of the topological invariants of the toric spaces that are homeomorphic to topological manifolds, inverse images under $p$ of the proper faces of $Q$ will serve as geometric representatives of the generators of homology groups. Algebraic isomorphisms between such
toric spaces induce algebraic isomorphism between the toric spaces associated with proper faces. These induced algebraic isomorphisms will provide a complete understanding of the isomorphisms induced on the homology groups of the original toric spaces (see section 5.2 for details in the case of 8-manifolds). The notation used in the statement of the proposition has been introduced in definitions 2.9 and 2.13.

**Proposition 2.17.** If \( j = [f, g]: \mathcal{X}(Q, \lambda) \to \mathcal{X}(R, \mu) \) is an algebraic morphism, then for every face \( F \) of \( Q \), \( j_F = [f|_F, g_F]: \mathcal{X}(F, L_F(\lambda)) \to \mathcal{X}(\hat{f}(F), L_{\hat{f}(F)}(\mu)) \) is an algebraic morphism, and \( j \circ \kappa_F = \kappa_{\hat{f}(F)} \circ j_F \).

**Proof.** To prove that \( j_F \) is an algebraic morphism, first note that \( f|_F: F \to \hat{f}(F) \) is an order-preserving affine embedding (with the same shift as \( f \)) as a restriction of \( f \). Let \( E \leq F \). Since \( \lambda \) and \( \mu \) are characteristic functions,

\[
\lambda(E) = \lambda_{E,F} \circ \lambda(F),
\]

and

\[
\mu(\hat{f}(E)) = \mu_{\hat{f}(E), \hat{f}(F)} \circ \mu(\hat{f}(F)).
\]

Also, since the pair \((f, g)\) satisfies the compatibility condition,

\[
g_F \circ \lambda(F) = \mu(\hat{f}(F)) \circ g,
\]

and

\[
g_E \circ \lambda(E) = \mu(\hat{f}(E)) \circ g.
\]

Hence

\[
g_E \circ \lambda_{E,F} \circ \lambda(F) = g_E \circ \lambda(E)
\] by (2.2)
\[
\begin{align*}
&= \mu(\hat{f}(E)) \circ g \\
&= \mu_{f(E),f(F)}(\hat{f}(F)) \circ g \\
&= \mu_{f(E),f(F)}g_F \circ \lambda(F)
\end{align*}
\]

by (2.5)

Since \( \lambda(F) \) is a surjection, the equation above implies

\[
(2.6) \quad g_E \circ \lambda_{E,F} = \mu_{f(E),f(F)}g_F
\]

and hence

\[
(2.7) \quad g_E \circ L_F(\lambda)(E) = L_{f(F)}(\mu(\hat{f}(E))) \circ g_F
\]

by Definition 2.11. Therefore the pair \((f|_F, g_F)\) satisfies the compatibility condition and hence \(j_F\) is an algebraic morphism.

To prove the equation \(j \circ \kappa_F = \kappa_{f(F)} \circ j_F\), let \([q,t] \in \mathcal{X}(F, L_F(\lambda))\) where \(q \in \text{int} E\), \(E \leq F\), and \(F \in \mathcal{P}_k(Q)\). Assume that \(f\) is of shift \(l\), and \(\lambda\) (resp. \(\mu\)) is of rank \(n_1\) (resp. \(n_2\)) and height \(h_1\) (resp. \(h_2\)). Let \(i\) (resp. \(\hat{i}\)) denote the inclusion \(F \subset Q\) (resp. \(\hat{f}(F) \subset R\)).

Let \(\lambda(F) = E(\tau \circ \iota)\) (where \(\tau \in \text{Aut} F_{n_1}\), and \(\iota = \iota_{k+h_1,n_1}\)), and \(\mu(\hat{f}(F)) = E(\hat{\tau} \circ \hat{i})\) (where \(\hat{\tau} \in \text{Aut} F_{n_2}\), and \(\hat{i} = \iota_{k+l+h_2,n_2}\)). Then by Definition 2.13, \(\kappa_F = [i, E(\pi \circ \tau^{-1})]\) (where \(\pi = \pi_{n_1,k+h_1}\), and \(\kappa_{f(F)} = [\hat{i}, E(\hat{\pi} \circ \hat{\tau}^{-1})]\) (where \(\hat{\pi} = \pi_{n_2,k+l+h_2}\)). Since \((f \circ i)(q) = (\hat{i} \circ f|_F)(q) \in \text{int} \hat{f}(E)\), by definition 2.7,

\[
(j \circ \kappa_F)([q,t]) = (\kappa_{f(F)} \circ j_F)([q,t])
\]

if and only if

\[
(2.8) \quad \mu(\hat{f}(E))(g \circ E(\pi \circ \tau^{-1}))(t) = \mu(\hat{f}(E))(E(\hat{\pi} \circ \hat{\tau}^{-1}) \circ g_F)(t). \quad \text{by (2.5)}
\]

We have
\[
\begin{align*}
&= g_E \circ \lambda_{E,F} \circ \pi \circ \tau^{-1} \quad \text{by (2.2)} \\
&= g_E \circ \lambda_{E,F} \circ \pi \circ \tau^{-1} \circ \tau \circ \iota \quad \text{by Proposition 2.3} \\
&= g_E \circ \lambda_{E,F},
\end{align*}
\]

since \(\pi \circ \tau^{-1} \circ \tau \circ \iota = 1_{F_{k+h_1}}\). Also,

\[
\begin{align*}
(2.9) \quad \mu(\hat{f}(E)) \circ E(\hat{\pi} \circ \hat{\tau}^{-1}) &= \mu_{\hat{f}(E),\hat{f}(F)} \circ \mu(\hat{f}(F)) \circ E(\hat{\pi} \circ \hat{\tau}^{-1}) \quad \text{by (2.3)} \\
&= \mu_{\hat{f}(E),\hat{f}(F)} \circ E(\hat{\pi} \circ \hat{\tau}^{-1} \circ \hat{\iota}) \quad \text{by Proposition 2.3} \\
&= \mu_{\hat{f}(E),\hat{f}(F)},
\end{align*}
\]

since \(\hat{\pi} \circ \hat{\tau}^{-1} \circ \hat{\iota} = 1_{F_{k+l+h_2}}\). Therefore equation (2.7) holds by equations (2.8), (2.9), and (2.6) and hence \(j \circ \kappa_F = \kappa_{\hat{f}(F)} \circ j_F\).

The faces of the polytope with which a toric space is associated are not the only subsets whose inverse images under \(p\) are homeomorphic to toric spaces whose characteristic functions are determined by the characteristic function defining the original space. For the next example, we define another type of important subsets of a convex polytope. Let \(V\) be a vertex of a convex \(m\)-polytope \(Q\). Let \(H\) be a hyperplane that separates \(V\) from other vertices of \(Q\). Since the convex \((m-1)\)-polytope \(H \cap Q\) is unique up to an affine homeomorphism, we call it the link of \(V\) in \(Q\) and denote by \(L_V\). We denote the faces of \(L_V\) by \(F \cap L_V\) for all faces \(F\) of \(Q\) such that \(V \leq F\).

**Definition 2.18.** Let \(V \in P_0(Q)\). Define \(R_V(\lambda) : P(L_V) \to P(n)\) by \(R_V(\lambda) (F \cap L_V) = \lambda(F)\) for every \(F \cap L_V \in P(L_V)\).

**Lemma 2.19.** \(R_V(\lambda)\) is a characteristic function of rank \(n\) and height \(h + 1\) for \(P(L_V)\).

**Proof.** If \(F \cap L_V \in P_{k-1}(L_V)\), then \(F \in P_{k}(Q)\). By Definition 2.6, \(R_V(\lambda)(F \cap L_V) \in P(n, k + h) = P(n, (k - 1) + (h + 1))\). Also, if \(E \cap L_V \leq F \cap L_V\) where \(E \cap L_V \in P_{l-1}(L_V)\), then \(E \leq F\) in \(P(Q)\) and \(E \in P_l(Q)\). Thus \(R_V(\lambda)(E \cap L_V) = \lambda_{E,F} \circ R_V(\lambda)(F \cap L_V)\) where...
\( \lambda_{E,F} \in P((k - 1) + (h + 1), (l - 1) + (h + 1)) \). Therefore \( R_V(\lambda) \) satisfies the conditions of Definition 2.6 and hence is a characteristic function of rank \( n \) and height \( h + 1 \) for \( P(L_V) \).

Let \( f \) denote the inclusion \( L_V \subset Q \). Consider \( f \) as an order-preserving affine embedding with shift 1 (see Definition 2.8). If \( F \in P_k(F) \), then

\[
\lambda(F) \circ \mathbb{1}_{T^n} = \mathbb{1}_{T^{k+h}} \circ \lambda(F) = \mathbb{1}_{T^{k+h}} \circ R_V(\lambda)(F \cap L_V).
\]

Thus the pair \((f, \mathbb{1}_{T^n})\) satisfies the compatibility condition (see Definition 2.9) and hence we have the following

**Proposition 2.20.** The map \( [f, \mathbb{1}_{T^n}]: \mathcal{X}(L_V, R_V(\lambda)) \rightarrow \mathcal{X}(Q, \lambda) \) defined above is an algebraic morphism.

The following will be an example of an algebraic isomorphism.

**Example 2.21.** Let \( \sigma \in P(n, n) \). Define \( \lambda': P(Q) \rightarrow P(n) \) by \( \lambda'(F) = \lambda(F) \circ \sigma \) for all \( F \in P(Q) \). If \( F \in P_k(Q) \), then \( \lambda'(F) \in P(n, k + h) \) by Lemma 2.5. Also, if \( E \in P_l(Q) \) and \( E \leq F \), then \( \lambda'(E) = \lambda(E) \circ \sigma = \lambda_{E,F} \circ \lambda(F) \circ \sigma = \lambda_{E,F} \circ \lambda'(F) \). Hence \( \lambda' \) is a characteristic function of rank \( n \) and height \( h \) for \( P(Q) \). Since \( \lambda'(F) \circ \sigma^{-1} = \mathbb{1}_{T^{k+h}} \circ \lambda(F) \) for every face \( F \) of \( Q \), the pair \((\mathbb{1}_Q, \sigma^{-1})\) satisfies the compatibility condition and hence

\[
[\mathbb{1}_Q, \sigma^{-1}]: \mathcal{X}(Q, \lambda) \rightarrow \mathcal{X}(Q, \lambda')
\]

is an algebraic morphism. By a similar argument, \([\mathbb{1}_Q, \sigma]: \mathcal{X}(Q, \lambda') \rightarrow \mathcal{X}(Q, \lambda)\) is an algebraic morphism, and \([\mathbb{1}_Q, \sigma] \circ [\mathbb{1}_Q, \sigma^{-1}] = \mathbb{1}_{\mathcal{X}(Q, \lambda)}\). Therefore \([\mathbb{1}_Q, \sigma^{-1}]\) is an algebraic isomorphism.

### 2.3 The sum of toric spaces

The operation of sum of toric spaces will result in a toric space whose defining data (the polytope and the characteristic function) will be combined from the defining data of the
summands. It will be a generalization of Scott’s construction presented in [7] (see Proposition 4.1, p. 391). One can also compare it to the equivariant connected sums discussion in [1] (§1.11, p. 424).

We begin with the definition of the operation of sum of convex \(n\)-polytopes. For \(i = 1, 2\), let \(Q_i\) be an \(n\)-polytope embedded in \(\mathbb{R}^n\). Assume that for some vertices \(V_1 \in \mathcal{P}_0(Q_1)\) and \(V_2 \in \mathcal{P}_0(Q_2)\), the link \(L_1\) of \(V_1\) in \(Q_1\) and the link \(L_2\) of \(V_2\) in \(Q_2\) are homeomorphic to each other by an order-preserving affine homeomorphism (see Definition 2.8). Let \(f : L_1 \rightarrow L_2\) denote the homeomorphism. Let \(CL_i\) denote the affine cone of \(L_i\) in \(Q_i\) with vertex \(V_i\) (see Figure 2.1). Let \(\circ CL_i = CL_i \setminus L_i\). Then \(Q_i' = Q_i \setminus \circ CL_i\) is itself a convex \(n\)-polytope. Also, \(L_i\) is a face of \(Q_i'\) of codimension 1 (see Figure 2.2). Let \(V_i\) denote the set of all vertices \(V\) of \(Q_i\) such that the line segment \([V, V_i]\) is an edge of \(Q_i\). Since \(V_i \cap \circ CL_i = \emptyset\), the elements of \(V_i\) are vertices of \(Q_i'\). For \(i = 1, 2\), there exists a convex \(n\)-polytope \(Q_i''\) with the following properties.

1. There exists an order-preserving homeomorphism \(h_i : Q_i' \rightarrow Q_i''\).
2. There exists a translation \(t\) of \(\mathbb{R}^n\) such that \(t \circ h_1|_{L_1} = h_2 \circ f\) (in particular, \(h_1(L_1)\) is congruent to \(h_2(L_2)\)).
3. The vertices \(h_i(V),\) where \(V \in V_i,\) span a hyperplane \(H_i\) in \(\mathbb{R}^n\) that is parallel to the hyperplane spanned by the vertices of \(h_i(L_i)\).
4. If \(\bar{H}_i \subset \mathbb{R}^n\) denotes the halfspace with the boundary \(H_i\) such that \(\bar{H}_i \cap h_i(L_i) \neq \emptyset\),

then \(Q_i'' \cap \bar{H}_i\) is congruent to \(h_i(L_i) \times [0, 1]\) (see Figure 2.3).

Let \(t_1 = t|_{h_1(L_1)}\). Consider \(t_1\) to be a gluing function, and define \(R = Q_1'' \cup_{t_1} Q_2''\). Then \(R\) embedded in \(\mathbb{R}^n\) is a convex \(n\)-polytope (see Figure 2.4).

**Definition 2.22.** We call the polytope \(R\) obtained in the construction above the sum of \(Q_1\) and \(Q_2\) and denote it by \(R = Q_1 +_f Q_2\). We say that the sum is performed at the vertices \(V_i\) of \(Q_i\) for \(i = 1, 2\).

Let \(\iota_i\) (for \(i = 1, 2\)) denote the composition of the inclusion of \(Q_i''\) into \(R\) with \(h_i\). There are two types of faces of \(R\). The faces of \(R\) of the first type are of the form \(\iota_i(F)\) for all faces \(F\) of \(Q_i\) such that \(F \cap V_i = \emptyset\) (see Figure 2.5). Let \(t_E = t|_{h_i(E \cap L_i)}\) for every face \(E\).
Fig. 2.1: $CL_1$ and $CL_2$.

Fig. 2.2: $Q'_1$ and $Q'_2$.

Fig. 2.3: $Q''_1$ and $Q''_2$.

Fig. 2.4: The sum of $Q_1$ and $Q_2$. 

$$R = Q_1 + f Q_2$$
of $Q_1$ such that $E \cap V_i \neq \emptyset$. The faces of $R$ of the second type are of the form

$$h_1(E \cap Q'_1) \cup_{t_E} h_2(E' \cap Q'_2)$$

where $E'$ is the face of $Q_2$ such that $f(E \cap L_1) = E' \cap L_2$. We will denote such faces by $E + f E'$ (see Figure 2.5).

We proceed with the discussion about the operation of sum of toric spaces. Let $\mathcal{X}_i = \mathcal{X}(Q_i, \lambda_i)$ (for $i = 1, 2$) be a toric space where $\lambda_i$ is of rank $n$ and height $h$, and such that the values of $\lambda_i$ coincide for all pairs of faces that are glued to each other by $f$. Equivalently, using the notation of Definition 2.18,

$$R_{V_1}(\lambda_1)(E \cap L_1) = R_{V_2}(\lambda_2)(f(E \cap L_1))$$

for all $E \cap L_1 \in \mathcal{P}(L_1)$. Then $j = [f, 1_{T^*}] : \mathcal{X}(L_1, R_{V_1}(\lambda_1)) \rightarrow \mathcal{X}(L_2, R_{V_2}(\lambda_2))$ is an algebraic isomorphism.
Define \( \mu: \mathcal{P}(R) \to P(n) \) in the following way. For all faces of \( R \) of the first type, let

\[
\mu(\iota_i(F)) = \lambda_i(F).
\]

For all faces of \( R \) of the second type, let

\[
\mu(E + f E') = \lambda_1(E).
\]

Then \( \mu \) is a characteristic function of rank \( n \) and height \( h \) for \( \mathcal{P}(R) \).

Let \( \mathcal{X} = \mathcal{X}(R, \mu) \). It follows from the construction that \( \mathcal{X} \) is homeomorphic to

\[
(\mathcal{X}_1 - p^{-1}(\mathcal{C}L_1)) \cup_j (\mathcal{X}_2 - p^{-1}(\mathcal{C}L_2))
\]

Since \( R \) is unique up to an order-preserving affine homeomorphism, \( \mathcal{X} \) is unique up to an algebraic isomorphism.

**Definition 2.23.** We call the toric manifold \( \mathcal{X} \) obtained in the construction above a **sum** of \( \mathcal{X}_1 \) and \( \mathcal{X}_2 \), and write \( \mathcal{X} = \mathcal{X}_1 + j \mathcal{X}_2 \).

Note that if one starts with more than two convex polytopes, and considers consecutive sums that are performed at the given vertex of each polytope not more than once, then the operation is associative, *i.e.,*

\[
(Q_1 + f_1 Q_2) + f_2 Q_3 = Q_1 + f_1 (Q_2 + f_2 Q_3)
\]

(where ‘=’ means ‘affine homeomorphic’). Hence if \( \mathcal{X}_i = \mathcal{X}(Q_i, \lambda_i) \) \( (i = 1, 2, 3) \), and \( j_i = [f_i, \mathbb{1}_{T^n}] \) \( (i = 1, 2) \), then

\[
(\mathcal{X}_1 + j_1 \mathcal{X}_2) + j_2 \mathcal{X}_3 = \mathcal{X}_1 + j_1 (\mathcal{X}_2 + j_2 \mathcal{X}_3)
\]

(where ‘=’ means ‘algebraic isomorphic’).
CHAPTER 3
A SUMMARY OF FORMER RESULTS, AND THE MAIN RESULT

The main objects of our study are quaternionic toric spaces that are homeomorphic to topological 8-manifolds. Scott [7] presented necessary and sufficient conditions that the characteristic function and the polytope defining a quaternionic toric space have to satisfy, to obtain a space homeomorphic to a sphere or to an orientable topological manifold. For the latter, he described the homology groups and the cohomology ring (with integer coefficients). In the case $n = m = 2$ (see Definition 2.7), Scott also discussed topological invariants of the toric spaces homeomorphic to orientable topological 8-manifolds, and presented a topological classification of such spaces. In this chapter we present a summary of those results.

We introduce additional notation that will be used throughout the paper. $A_n$ will denote the abelianization of $F_n$, and $\pi: F_n \rightarrow A_n$ will denote the natural projection. We define the preferred basis for $A_n$ to be $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_n$ where $\bar{x}_i = \pi(x_i)$ for $1 \leq i \leq n$. Given a word $w \in F_n$, $w_{ab}$ will denote the presentation of $\pi(w) \in A_n$ in the preferred basis. The natural map $\pi_\ast: \text{Aut } F_n \rightarrow \text{Aut } A_n$ is defined by $\pi_\ast(\sigma)(\bar{x}_i) = \pi(\sigma(x_i))$ for all $\sigma \in \text{Aut } F_n$ and $1 \leq i \leq n$. Given an automorphism $\sigma \in \text{Aut } F_n$, $M(\sigma)$ will denote the matrix of $\pi_\ast(\sigma)$ in the preferred basis.

3.1 Toric manifolds, homology, and cohomology

We begin with the definitions of basic characteristic functions of height 1 and 0. By Definition 2.6, if $\lambda$ is a characteristic function of height 1 (resp. 0) for the graded poset associated with a convex $m$-polytope $Q$, then the value of $\lambda$ for any vertex (resp. edge) of $Q$ is specified by an element of the free group $F_{m+1}$ (resp. $F_m$). Such element is the value of an automorphism of $F_{m+1}$ (resp. $F_m$) at the generator $x_1$ (cf. Definitions 2.6 and 2.4).

Assume that $\lambda$ is a characteristic function of rank $m + 1$ and height 1 for the graded poset associated with the $m$-simplex $[0, 1, \ldots, m]$. Let $\lambda([i]) = e(w_i)$ where $w_i \in F_{m+1}$ for
$0 \leq i \leq m$.

**Definition 3.1.** Under the assumptions above $\lambda$ is basic, if $\text{Span}\{\pi(w_i)\}_{i=0}^{m} = A_{m+1}$.

We call a convex polytope simple if the link of every vertex (see the paragraph preceding Definition 2.18) is a simplex. As the basic characteristic functions of height 1 are defined only over graded posets associated with simplexes, the basic characteristic functions of height 0 are defined only over graded posets associated with simple polytopes.

Assume that $\lambda$ is a characteristic function of rank $m$ and height 0 for the graded poset associated with a convex $m$-polytope $Q$.

**Definition 3.2.** Under the assumptions above $\lambda$ is basic if $Q$ is simple and $R_V(\lambda)$ is a basic characteristic function for all vertices $V$ of $Q$.

For the definition of $R_V(\lambda)$, see Definition 2.18.

The following theorem and corollary are rephrased statements of Theorem 3.1, p. 379 in [7].

**Theorem 3.3.** Let $\mathcal{X}(Q, \lambda)$ be a quaternionic toric space of rank $n$ where $Q$ is a convex $(n-1)$-polytope and $\lambda$ is of height 1. Then $\mathcal{X}(Q, \lambda)$ is homeomorphic to a sphere if and only if $\lambda$ is basic.

**Corollary 3.4.** Let $\mathcal{X}(Q, \lambda)$ be a toric space of rank $n$ where $Q$ is a convex $n$-polytope and $\lambda$ is of height 0. Then $\mathcal{X}(Q, \lambda)$ is homeomorphic to an orientable topological manifold if and only if $\lambda$ is basic.

From now on, a toric space that is homeomorphic to a topological manifold will be called a toric manifold.

Since toric manifolds are orientable, we’d like to describe a way of orienting them using the orientations of the polytope and the torus. Given a toric manifold $\mathcal{X}(Q, \lambda)$ of rank $n$ and height 0, the preimage under $p$ of $\text{int } Q \times T^n$ is a dense subset in $\mathcal{X}(Q, \lambda)$ (see the paragraph following Definition 2.7). Hence to specify the orientation for $\mathcal{X}(Q, \lambda)$ it is enough to specify the orientations for $Q$ and the torus. We choose one of the orientations for $T$ and take the product orientation for $T^n$ if $n \geq 2$. We also present a method for orienting
the codimension-1 faces of $Q$ (which will be called facets from now on) consistently with the orientation for $Q$.

**Definition 3.5.** Let $Q$ be an oriented simple $n$-polytope and let $E$ be a facet of $Q$. Let $V = (v_1, v_2, \ldots, v_{n-1})$ be an ordered $(n-1)$-tuple of linearly independent vectors in $E$. Let $N(E)$ denote the vector normal to $E$ pointing out of $Q$. The orientation for $E$ specified by $V$ is **consistent with the orientation for $Q$** if and only if the ordered $n$-tuple $(N(E), v_1, v_2, \ldots, v_{n-1})$ specifies the fixed orientation for $Q$.

The following results about homology groups and the cohomology ring of a quaternionic toric manifold, presented in [7], can be compared to the results (presented for example in [1]) for toric manifolds defined over $\mathbb{C}$.

For the remaining part of this section, let $X = X(Q, \lambda)$ be a toric manifold of rank $n$ where $Q$ is an oriented simple convex $n$-polytope and $\lambda$ is basic of height 0. For the discussion of homology and cohomology of $X$, we use a class of codimension-4 cycles defined below.

Let $E$ be a facet of $Q$ with the orientation consistent with the orientation for $Q$. Let $X_E = X(E, L_E(\lambda))$ (see Definition 2.11) and let $\kappa_E : X_E \to X$ be the algebraic morphism defined in Definition 2.13.

**Lemma 3.6.** Under the assumptions above $L_E(\lambda)$ is basic.

By Lemma 3.6 and Corollary 3.4, $X_E$ is a toric manifold. The orientation for $X_E$ is specified. Let $[X_E] \in H_{4(n-1)}(X_E)$ denote the fundamental class of $X_E$ (see [2], sec. 3.3). Let $(\kappa_E)_* : H_{4(n-1)}(X_E) \to H_{4(n-1)}(X)$ denote the map induced by $\kappa_E$.

**Definition 3.7.** For all $E \in \mathcal{P}_{n-1}(Q)$, define $D_E = (\kappa_E)_*([X_E])$.

The cycles $D_E$ ($E \in \mathcal{P}_{n-1}(Q)$) generate $H_{4(n-1)}(X)$. Lower homology groups are generated by the transverse intersections of the cycles $D_E$. Also, the cohomology ring $H^*(X)$ is generated by the degree-4 classes $D_E^*$ which are the Poincaré dual classes of the cycles $D_E$ (see [7], §3).

Let $r$ denote the number of facets of $Q$. Denote these facets by $E_i$ where $1 \leq i \leq r$. If $Q$ is a simple polytope, let $\Delta(Q)$ denote the simplicial complex dual to $Q$. Then the
vertices of $\Delta(Q)$ correspond to the facets of $Q$. Denote these vertices by $v_i$ ($1 \leq i \leq r$). The following definition comes from [8] (see Definition 3.2, p. 267).

**Definition 3.8.** The *face ring* of $\Delta(Q)$ is the quotient $\mathbb{Z}[X_1, \ldots, X_r]/I$, where $I$ is the homogeneous ideal generated by those monomials $X_{i_1} \cdots X_{i_j}$ such that $i_1 < \cdots < i_j$ and \{v_{i_1}, \ldots, v_{i_j}\} \notin \Delta(Q).

For the theorem about the cohomology ring of $\mathcal{X}$, we define the following ideal. Let $A_0 = \text{Hom}(A_n, \mathbb{Z})$ be the space dual to $A_n$. If $E$ is a facet of $Q$, let $\lambda(E) = E(\tau \circ i_{n-1,n})$ where $\tau \in \text{Aut} F_n$ and $\det M(\tau) = 1$. Then $\text{Span}\{\pi(\tau(x_i))\}_{i=1}^{n-1}$ is a unimodular rank-$(n-1)$ sublattice of $A_n$. Let $E^*$ denote the rank-1 dual lattice in $A_n^*$. The bottom row of the matrix $M(\tau)^{-1}$ is then a row vector presentation in the basis dual to the preferred basis for $A_n$ of a primitive generator of $E^*$. Denote this primitive generator by $\bar{E}$. Denote the row vector presentation of $\bar{E}$ in the basis dual to the preferred basis for $A_n$ by $\lambda(E)^*_{ab}$.

Define $J$ to be the homogeneous ideal of $\mathbb{Z}[X_1, \ldots, X_r]$ generated by the $n$ linear relations (for $1 \leq j \leq n$)

$$\sum_{i=1}^{r} \langle \bar{E}_i, \bar{x}_j \rangle \cdot X_i = 0. \quad (3.1)$$

**Theorem 3.9.** The cohomology ring $H^*(\mathcal{X}; \mathbb{Z})$ is isomorphic to

$$\mathbb{Z}[X_1, \ldots, X_r]/(I + J),$$

where the generators $X_i$ are of degree 4.

**Remark 3.10.** In the case $n = 2$, relations (3.1) give us, via Poincaré duality, the relations between the generators $D_{E_i}$ of $H_4(\mathcal{X})$,

$$\sum_{i=1}^{r} \langle \bar{E}_i, \bar{x}_j \rangle \cdot D_{E_i} = 0$$
(for $j = 1, 2$). These relations will play an important role in the discussion about the topological invariants of quaternionic toric 8-manifolds. It is also worth noting that

$$\langle \vec{E}_i, \vec{x}_j \rangle = \lambda(E)_{ab}^* \cdot (x_i)_{ab}$$

for $1 \leq j \leq n$ and $1 \leq i \leq r$.

### 3.2 Oriented topological classification of quaternionic toric 8-manifolds

In this section we present a summary of Scott’s results presented in §4 of [7].

Quaternionic toric 8-manifolds are associated with convex polygons (simple 2-polytopes) and basic characteristic functions of rank 2 and height 0. The parameters $n = m = 2$ (see Definition 2.7) are the only ones that will result in an 8-manifold.

Scott defined two families of quaternionic toric 8-manifolds. Let $Q$ be an oriented $r$-gon, and let $E_i$ ($1 \leq i \leq r$) denote the facets of $Q$ numbered according to the orientation for $Q$. The first family consists of manifolds (type I) defined over an oriented triangle.

**Definition 3.11.** Let $r = 3$, and for each integer $a$ define $\lambda_a$ by the equations

$$\lambda_a(E_1) = e(x_1),$$

$$\lambda_a(E_2) = e(x_2),$$

$$\lambda_a(E_3) = e(x_2^a x_1^{-1} x_2^{-1-a}).$$

Define $\mathcal{X}_a = \mathcal{X}(Q, \lambda_a)$.

The second family consists of manifolds (type II) defined over an oriented square.

**Definition 3.12.** Let $r = 4$, and for each pair of integers $b, c$ define $\lambda_{b,c}$ by the equations

$$\lambda_{b,c}(E_1) = e(x_1),$$

$$\lambda_{b,c}(E_2) = e(x_2),$$

$$\lambda_{b,c}(E_3) = e(x_2^b x_1^{-1} x_2^{-b}).$$
\[ \lambda_{b,c}(E_4) = e(x_2^b x_1^c x_2^{-1} x_1^c x_2^{-b}). \]

Define \( \mathcal{X}_{b,c} = \mathcal{X}(Q, \lambda_{b,c}) \).

The following theorem constitutes the oriented topological classification of quaternionic toric 8-manifolds.

**Theorem 3.13.** Every quaternionic toric 8-manifold is oriented homeomorphic to either a connected sum of \( r - 2 \) manifolds of type I or \( (r - 2)/2 \) manifolds of type II. Moreover, all such connected sums are realized by some quaternionic toric 8-manifold.

Another important result states that every almost smooth (smooth away from a finite number of points) piecewise linear 8-manifold \( M \) with \( H_4(M) = \mathbb{Z} \) (resp. \( H_4(M) = \mathbb{Z}^2 \)) is homeomorphic to a manifold of type I (resp. to a connected sum of two manifolds of type I or homeomorphic to a manifold of type II).

### 3.3 The main result

Since algebraic morphisms (discussed in section 2.2) are topological embeddings, algebraic isomorphisms are homeomorphisms. In the category of toric manifolds, we say that two objects are *algebraically equivalent* if there exists an orientation-preserving algebraic isomorphism between them. The following main result of our research is a comparison theorem for the topological and algebraic classification of quaternionic toric 8-manifolds.

**Theorem 3.14.** Every topological type of a quaternionic toric 8-manifold contains an infinite number of distinct algebraic types of such manifolds.

The remainder of the paper will be devoted to providing the proof of the theorem above. For each oriented homeomorphism type of a quaternionic toric 8-manifold, we will construct in chapter 6 an infinite family of toric manifolds that are of the same homeomorphism type but algebraically distinct. To prove that the elements of the family are in the same type of oriented homeomorphism, one needs to determine the topological invariants described in Theorem 5.1. Algebraic constructions presented in chapter 4 together with additivity
of topological invariants of oriented manifolds with respect to the oriented connected sum operation will be used in section 5.2 to obtain a formula for the evaluation of the first Pontrjagin class of an arbitrary quaternionic toric 8-manifold at the generators of the fourth homology group described in section 3.1. Once the invariants are known, Wall’s theory ensures that in the case of discussed manifolds, the existence of an isomorphism of the fourth homology groups that preserves the invariants is equivalent to the existence of an oriented homeomorphism between two such manifolds (see [9], Corollary on p. 166). Algebraic properties of algebraic morphisms will then be used to show that the elements of each family are algebraically distinct (see Theorem 6.11).
CHAPTER 4

AUTOMORPHISMS OF $F_2$

In this chapter we introduce additional notation and discuss several constructions that will be used to find a useful decomposition of certain values of the characteristic function defining a quaternionic toric 8-manifold.

4.1 Automorphisms of simple form

Definition 4.1. An automorphism $\sigma \in \text{Aut } F_2$ is of simple form if it is defined by the equations $\sigma(x_1) = x_1^a x_2^b x_1^s$, $\sigma(x_2) = x_1$, where $a, b \in \mathbb{Z}$ and $|s| = 1$.

Using the theory of automorphisms of $F_2$ [5], we will show that every automorphism of $F_2$ can be presented as a composition of automorphisms of simple form.

From now on, let $\gamma$ be the automorphism of $F_2$ defined by the equations $\gamma(x_1) = x_2$, $\gamma(x_2) = x_1$. Note that $\gamma$ is of simple form.

Lemma 4.2. The automorphisms of simple form generate the group Aut $F_2$.

Proof. By [5], the group Aut $F_2$ is generated by the automorphisms $\gamma, \gamma_1$ and $\gamma_2$ where $\gamma_1(x_1) = x_1 x_2$, $\gamma_2(x_1) = x_1^{-1}$ and $\gamma_1(x_2) = \gamma_2(x_2) = x_2$. Define $\alpha_1, \alpha_2 \in \text{Aut } F_2$ by the equations $\alpha_1(x_1) = x_2 x_1$, $\alpha_2(x_1) = x_1^{-1}$, and $\alpha_1(x_2) = \alpha_2(x_2) = x_1$. Then $\alpha_1$ and $\alpha_2$ are of simple form. Also, $\gamma_1 = \gamma \circ \alpha_1$ and $\gamma_2 = \gamma \circ \alpha_2$. Since the generators of Aut $F_2$ can be presented as compositions of automorphisms of simple form, the automorphisms of simple form generate the group Aut $F_2$. \qed

Lemma 4.3. If $\alpha \in \text{Aut } F_2$ is of simple form with $\alpha(x_1) = x_1^a x_2^b x_1^s$, then $\alpha^{-1} = \gamma \circ \tilde{\alpha} \circ \gamma$, where $\tilde{\alpha}(x_1) = (x_1^{-a} x_2 x_1^b)^s$, and $\tilde{\alpha}(x_2) = x_1$.

Proof. Let $\beta = \gamma \circ \tilde{\alpha} \circ \gamma$. Then $\beta(x_1) = x_2$, and

$$\beta(x_2) = (\gamma(x_1^{-a} x_2 x_1^{-b}))^s = (x_2^{-a} x_1 x_2^{-b})^s.$$
Hence
\[
(\beta \circ \alpha)(x_1) = \beta(x_1^a x_2^b x_1^a) = x_2^a ((x_2^{-a} x_1 x_2^{-b})^s)^s x_2 = x_1,
\]
\[
(\beta \circ \alpha)(x_2) = x_2,
\]
\[
(\alpha \circ \beta)(x_1) = x_1,
\]
and
\[
(\alpha \circ \beta)(x_2) = \left( (\alpha(x_2))^{-a} (\alpha(x_1))^{-b} \right)^s = (x_1^{-a} x_1^a x_2^b x_1^{-b})^s = x_2.
\]
Therefore \(\beta = \alpha^{-1}\).

By the lemma above, the inverses of automorphisms of simple form can be presented as compositions of such automorphisms. Hence by Lemma 4.2, each automorphism of \(F_2\) can be presented as a composition of the automorphisms of simple form

\[
(4.1) \quad \alpha_1 \circ \alpha_2 \circ \cdots \circ \alpha_k = \prod_{i=1}^{k} \alpha_i.
\]

We call such presentation a simple decomposition. Simple decompositions will play an important role in the calculation of the topological invariants of quaternionic toric 8-manifolds.

Next we introduce some notation for an automorphism of \(F_2\) with a given simple decomposition.

**Definition 4.4.** Let \(\sigma \in \text{Aut} F_2\). Define \(X(\sigma), Y(\sigma)\) to be the coefficients in the presentation \(\pi(\sigma(x_1)) = X(\sigma) \cdot \bar{x}_1 + Y(\sigma) \cdot \bar{x}_2\).

**Definition 4.5.** Given the simple decomposition \(\prod_{i=1}^{k} \alpha_i\) of an automorphism \(\sigma \in \text{Aut} F_2\), define \(X_{-1}(\sigma) = Y_0(\sigma) = 0\), \(X_0(\sigma) = Y_{-1}(\sigma) = 1\), and for all \(1 \leq i \leq k\), define
\[
X_i(\sigma) = X\left(\prod_{j=1}^{i} \alpha_j\right), \quad \text{and} \quad Y_i(\sigma) = Y\left(\prod_{j=1}^{i} \alpha_j\right).
\]
4.2 Constructions of an element of the preimage under $\pi_*$

By [5], the natural map $\pi_* : \text{Aut} F_2 \rightarrow \text{Aut} A_2$ is surjective with kernel the inner automorphisms of $F_2$. In this section we present several constructions of a specific element in the preimage under $\pi_*$ and its simple decomposition.

Let $a \in \text{Aut} A_2$ be given by the matrix

$$M = \begin{bmatrix} k & m \\ l & n \end{bmatrix}$$

in the preferred basis, and let $D = \det M$. Since $a$ is an automorphism, $|D| = 1$.

By the Division Algorithm (see [3], Theorem 6.3, p. 11), for two integers $a, b$ such that $a \neq 0$, there exist unique integers $q, r$ such that $b = qa + r$ where $0 \leq r < |a|$. We call $q$ (resp. $r$) the quotient (resp. the remainder) of the division of $b$ by $a$.

First consider the case $|m| \geq 2$. Let $X_1 = k$, $Y_1 = l$, $X_2 = m$, and $Y_2 = n$. For $i \geq 3$, inductively define $q_{i-2}, X_i$ to be respectively the quotient and the remainder of the division of $X_{i-2}$ by $X_{i-1}$, and define $Y_i = Y_{i-2} - q_{i-2}Y_{i-1}$. Repeat the process until $X_j = 1$ for some $j \geq 3$ (the termination of the process follows from the Division Algorithm since $(k, m) = 1$).

Define $q_{j-1} = X_{j-1}$, $q_j = Y_j$, $s_1 = 1$, $s_2 = Y_{j-1} - X_{j-1}Y_j$, and $s_i = 1$ for $3 \leq i \leq j$. Since $|X_1Y_2 - X_2Y_1| = 1$, inductively for $i \geq 3$,

$$|X_{i-1}Y_i - X_iY_{i-1}| = |X_{i-1}(Y_{i-2} - q_{i-2}Y_{i-1}) - (X_{i-2} - q_{i-2}X_{i-1})Y_{i-1}|$$

$$= |X_{i-1}Y_{i-2} - X_{i-1}q_{i-2}Y_{i-1} - X_{i-2}Y_{i-1} + q_{i-2}X_{i-1}Y_{i-1}|$$

$$= |X_{i-1}Y_{i-2} - X_{i-2}Y_{i-1}| = 1.$$ 

In particular, $|s_2| = |1 \cdot Y_{j-1} - X_{j-1}Y_j| = 1$. For $1 \leq i \leq j$, define $\alpha_i \in \text{Aut} F_2$ by the equations $\alpha_i(x_1) = x_2^{s_j}x_1^{q_j-i+1}$, $\alpha_i(x_2) = x_1$. Then all $\alpha_i$ are of simple form. Let

$$\sigma = \gamma \circ \prod_{i=1}^{j} \alpha_i.$$
Proposition 4.6. Under the assumptions above, \( \pi_*(\sigma) = a \).

Proof. By equation (4.2),

\[
M(\sigma) = \begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
Y_j & 1 \\
X_j & 0
\end{bmatrix}
\begin{bmatrix}
q_{j-1} & 1 \\
s_2 & 0
\end{bmatrix}
\prod_{i=2}^{j-1} \begin{bmatrix}
q_{j-i} & 1 \\
1 & 0
\end{bmatrix}.
\]

Note that

\[
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\begin{bmatrix}
Y_j & 1 \\
X_j & 0
\end{bmatrix}
\begin{bmatrix}
q_{j-1} & 1 \\
s_2 & 0
\end{bmatrix}
= \begin{bmatrix}
X_j & 0 \\
Y_j & 1
\end{bmatrix}
\begin{bmatrix}
q_{j-1}X_j & X_j \\
q_{j-1}Y_j + s_2 & Y_j
\end{bmatrix}
= \begin{bmatrix}
X_{j-1} & X_j \\
Y_{j-1} & Y_j
\end{bmatrix}.
\]

From that stage, inductively from \( h = j - 2 \) descending to \( h = 1 \), multiplying on the right by the matrix \( \begin{bmatrix} q_h & 1 \\ 1 & 0 \end{bmatrix} \) will give us

\[
\begin{bmatrix}
X_{h+1} & X_{h+2} \\
Y_{h+1} & Y_{h+2}
\end{bmatrix}
\begin{bmatrix}
q_h & 1 \\
1 & 0
\end{bmatrix}
= \begin{bmatrix}
q_hX_{h+1} + X_{h+2} & X_{h+1} \\
q_hY_{h+1} + Y_{h+2} & Y_{h+1}
\end{bmatrix}
= \begin{bmatrix}
X_h & X_{h+1} \\
Y_h & Y_{h+1}
\end{bmatrix}.
\]

Therefore

\[
M(\sigma) = \begin{bmatrix}
X_1 & X_2 \\
Y_1 & Y_2
\end{bmatrix}
= \begin{bmatrix} k & m \\ l & n \end{bmatrix} = M
\]

and hence \( \pi_*(\sigma) = a \).

In the second case consider \( |m| = 1 \). Let \( r = kD \) and \( s = -mD \). Define \( \alpha_1, \alpha_2 \in \text{Aut} F_2 \) by the equations \( \alpha_1(x_1) = x_1^nx_2^m, \alpha_2(x_1) = (x_1^{-r}x_2)^s, \alpha_1(x_2) = \alpha_2(x_2) = x_1 \). Since \( |m| = 1 \) and \( |s| = 1 \), \( \alpha_1 \) and \( \alpha_2 \) are of simple form. Define \( \sigma = \gamma \circ \alpha_1 \circ \alpha_2 \).

Proposition 4.7. Under the assumptions above, \( \pi_*(\sigma) = a \).
Proof. Since
\[ M(\alpha_1) = \begin{bmatrix} n & 1 \\ m & 0 \end{bmatrix}, \quad \text{and} \quad M(\alpha_2) = \begin{bmatrix} -sr & 1 \\ s & 0 \end{bmatrix}, \]
we obtain
\[ M(\sigma) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} n & 1 \\ m & 0 \end{bmatrix} \begin{bmatrix} -sr & 1 \\ s & 0 \end{bmatrix} = \begin{bmatrix} m & 0 \\ s & 0 \end{bmatrix} \begin{bmatrix} -sr & 1 \\ s & 0 \end{bmatrix} = \begin{bmatrix} -msr & m \\ -s(nr - 1) & n \end{bmatrix}. \]
Since
\[ -s(nr - 1) = mD(nkD - 1) = m(nk - D) = m(nk - kn + ml) = l, \]
and
\[ -msr = mmDkD = k, \]
we obtain
\[ M(\sigma) = \begin{bmatrix} k & m \\ l & n \end{bmatrix} = M. \]
Therefore \( \pi_*(\sigma) = a. \]

In the final third case, consider \( m = 0. \) Define \( \alpha_1, \alpha_2 \in \text{Aut} F_2 \) by the equations
\[ \alpha_1(x_1) = x_2^n, \quad \alpha_2(x_1) = (x_1^{kn}x_2)^k, \quad \alpha_1(x_2) = \alpha_2(x_2) = x_1. \]
Since \(|k| = |n| = 1, \alpha_1 \) and \( \alpha_2 \) are of simple form. Define \( \sigma = \alpha_1 \circ \alpha_2. \) Then
\[ M(\sigma) = \begin{bmatrix} 0 & 1 \\ n & 0 \end{bmatrix} \begin{bmatrix} ln & 1 \\ k & 0 \end{bmatrix} = \begin{bmatrix} k & 0 \\ l & n \end{bmatrix} = M \]
and hence \( \pi_*(\sigma) = a. \)

We apply one of the constructions in the following
Example 4.8. Let $a \in \text{Aut} \ A_2$ be given by the matrix $M = \begin{bmatrix} 8 & 5 \\ 3 & 2 \end{bmatrix}$ in the preferred basis for $A_2$. Since $|5| \geq 2$, we apply the first case of the discussion above. Let $X_1 = 8$, $X_2 = 5$, $Y_1 = 3$, and $Y_2 = 2$. Since $8 = 1 \cdot 5 + 3$, let $q_1 = 1$, $X_3 = 3$, and $Y_3 = 3 - 1 \cdot 2 = 1$. Since $5 = 1 \cdot 3 + 2$, let $q_2 = 1$, $X_4 = 2$, and $Y_4 = 2 - 1 \cdot 1 = 1$. Since $3 = 1 \cdot 2 + 1$, let $q_3 = 1$, $X_5 = 1$, and $Y_5 = 1 - 1 \cdot 1 = 0$. Since $X_5 = 1$, we stop the inductive process, and define $q_4 = 2$, $q_5 = 0$, $s_2 = 1 - 2 \cdot 0 = 1$, and $s_i = 1$ for $i = 1, 3, 4, 5$. For $1 \leq i \leq 5$, define $\alpha_i(x_1) = x_2$, $\alpha_i(x_2) = x_2 x_1^2$, $\alpha_i(x_1) = x_2 x_1 (i = 3, 4, 5)$, and $\alpha_i(x_2) = x_1^3$, and $\pi_*(\sigma) = a$.

4.3 Prebasic elements

By Definition 2.6, the values of the characteristic function of rank 2 and height 0 for the edges of a polygon (the only “important” values of such characteristic functions) are of the form $e(\psi(x_1))$ for some automorphism $\psi \in \text{Aut} \ F_2$. We call an element $w \in F_2$ prebasic if there exists an automorphism $\psi \in \text{Aut} \ F_2$ such that $w = \psi(x_1)$. Note that if a word $w$ is prebasic then $\pi(w)$ is a primitive element of $A_2$.

In this section we will present a verification tool for prebasic elements, and given a prebasic word $w$, a construction of a simple decomposition of an automorphism $\psi \in \text{Aut} \ F_2$
such that $\psi(x_1) = w$. Obtaining an automorphism $\psi$ and its simple decomposition will be an important step in the calculation of one of the topological invariants of an arbitrary quaternionic toric 8-manifold. This calculation will be a key tool in the proof of Theorem 3.14.

The discussion will be based on the following theorem (due to Wilczynski in this generality).

**Theorem 4.9.** If $u, v \in F_2$ are prebasic elements with $\pi(u) = \pi(v)$, then there exists an inner automorphism $\varphi \in \text{Aut} F_2$ such that $\varphi(u) = v$.

**Proof.** The proof uses the fact that the natural map $\pi_* : \text{Aut} F_2 \to \text{Aut} A_2$ is surjective with kernel the inner automorphisms of $F_2$ [5].

Suppose first that $u = x_1$ and $\psi(u) = v$ for some $\psi \in \text{Aut} F_2$. Let $\psi_* = \pi_*(\psi)$. Since $\psi_*(\bar{x}_1) = \bar{x}_1$, $M(\psi) = \begin{bmatrix} 1 & m \\ 0 & n \end{bmatrix}$, with $|n| = 1$. Consider the element $\psi' \in \text{Aut} F_2$ defined by the equations $\psi'(x_1) = x_1$, $\psi'(x_2) = x_1^m x_2^n$. Then $\psi_* = \psi'_*$ and hence $\psi = \varphi \circ \psi'$ for some inner automorphism $\varphi$. Thus $v = \psi(u) = \varphi(\psi'(u)) = \varphi(u)$, as required.

In the general case, $\pi(u)$ is a primitive element of $A$. Since $\pi_*$ is surjective, there exists an element $f \in \text{Aut} F_2$ such that $\pi(f(u)) = \pi(x_1)$. Then $\pi(f(v)) = \pi(f(u)) = \pi(x_1)$ holds as well. Therefore there exist inner automorphisms $\varphi_1, \varphi_2$ of $F_2$ such that $\varphi_1(x_1) = f(u)$ and $\varphi_2(x_1) = f(v)$. Then $\varphi = f^{-1} \circ (\varphi_2 \circ \varphi_1^{-1}) \circ f$ has the required property. \hfill \Box

The following corollary provides a verification tool for prebasic elements.

**Corollary 4.10.** Let $w$ be an element of $F_2$ such that $\pi(w)$ is primitive in $A_2$. Let $\sigma \in \text{Aut} F_2$ satisfy $\pi(\sigma(x_1)) = \pi(w)$. Then $w$ is prebasic if and only if $\sigma^{-1}(w) = vx_1v^{-1}$ for some element $v \in F_2$.

**Proof.** First note that $\pi(\sigma^{-1}(w)) = \pi(x_1)$, and $x_1$ is a prebasic element of $F_2$. Hence if $\sigma^{-1}(w)$ is not of the form $vx_1v^{-1}$ for some $v \in F_2$, then by the contrapositive of Theorem
4.9, \( w \) is not prebasic. Assume that such \( v \) exists, and \( \sigma^{-1}(w) = vx_1v^{-1} \). Let \( \tau \in \text{Aut} F_2 \) be defined by the equations \( \tau(x_p) = vx_pv^{-1} (p = 1, 2) \). Then

\[
(\sigma \circ \tau)(x_1) = \sigma(vx_1v^{-1}) = (\sigma \circ \sigma^{-1})(w) = w
\]

and hence \( w \) is prebasic. \( \square \)

Note that given the reduced form of an element \( w \) of \( F_2 \) it is easy to verify if it is of the form \( vx_1v^{-1} \) for some \( v \) in \( F_2 \). Also, if \( w \) is of the desired form, then one can easily read the reduced form of \( v \) from the reduced form of \( w \).

Assume that \( w \) is a prebasic element of \( F_2 \). We now present a construction of an automorphism \( \psi \in \text{Aut} F_2 \) such that \( \psi(x_1) = w \) and its simple decomposition. Assume that \( w_{ab} = \begin{bmatrix} k \\ l \end{bmatrix} \) for some relatively prime integers \( k \) and \( l \). By Bézout’s Lemma (see [3], Theorem 6.5, p. 11) there exist integers \( m', n \) such that \( kn + m'l = 1 \). Let \( m = -m' \). Then

\( kn - ml = 1 \). Hence there exists an automorphism \( a \in \text{Aut} A_2 \) whose matrix in the preferred basis is

\[
M = \begin{bmatrix} k & m \\ l & n \end{bmatrix}.
\]

Using one of the constructions described is section 4.2, we obtain an automorphism \( \sigma \in \text{Aut} F_2 \) such that \( \pi_*(\sigma) = a \). Also, \( \sigma \) has a certain simple decomposition

\[
\sigma = \prod_{i=1}^{r} \alpha_i
\]

described in the construction. Since \( \pi(\sigma(x_1)) = \pi(w) \), by Corollary 4.10, \( \sigma^{-1}(w) = vx_1v^{-1} \) for some element \( v \in F_2 \). Assume that the reduced form of \( v \) is

\[
(\prod_{i=1}^{s-1} (x_2^a x_1^b)) x_2^a
\]
for some $s \in \mathbb{N}$, where $a_i \neq 0$ for $2 \leq i \leq s$, and $b_i \neq 0$ for $1 \leq i \leq s - 1$. Define $2s - 1$ inner automorphisms of $F_2$ by the equations

\[
\beta'_{2i-1}(x_p) = x_2^{a_i}x_p x_2^{-a_i} \quad \text{for } 1 \leq i \leq s,
\]
\[
\beta'_{2i}(x_p) = x_1^{b_i}x_p x_1^{-b_i} \quad \text{for } 1 \leq i \leq s - 1,
\]

where $p = 1, 2$. Then

\[
\left( \prod_{j=1}^{2s-1} \beta'_j \right)(x_1) = vx_1v^{-1} = \sigma^{-1}(w).
\]

Let $\beta_{2i-1} = \gamma \circ \beta'_{2i-1}$ for $1 \leq i \leq s$, and $\beta_{2i} = \beta'_{2i} \circ \gamma$ for $1 \leq i \leq s - 1$. Then all the $\beta_j$ ($1 \leq j \leq 2s - 1$) are of simple form, and

\[
\prod_{j=1}^{2s-1} \beta'_j = \beta'_1 \circ \prod_{i=1}^{s-1} (\beta'_{2i} \circ \beta'_{2i+1}) = \gamma \circ \beta_1 \circ \prod_{i=1}^{s-1} (\beta_{2i} \circ \gamma \circ \beta_{2i+1})
\]
\[
= \gamma \circ \beta_1 \circ \prod_{i=1}^{s-1} (\beta_{2i} \circ \beta_{2i+1})
\]
\[
= \gamma \circ \prod_{j=1}^{2s-1} \beta_j.
\]

Let

\[
(4.3) \quad \tau = \gamma \circ \prod_{j=1}^{2s} \beta_j.
\]

Then $\tau(x_1) = \sigma^{-1}(w)$. Define $\psi = \sigma \circ \tau$. Then

\[
\psi(x_1) = (\sigma \circ \tau)(x_1) = (\sigma \circ \sigma^{-1})(w) = w.
\]

From the construction above we obtain the simple decomposition

\[
(\prod_{i=1}^r \alpha_i) \circ \gamma \circ (\prod_{j=1}^{2s-1} \beta_j)
\]
of the automorphism \( \psi \).

In the following example we will verify that a certain word \( w \) is prebasic, and apply the construction discussed above to obtain an automorphism \( \psi \) of \( F_2 \) such that \( \psi(x_1) = w \).

**Example 4.11.** Let \( w = x_1^{-1}x_2^{-1}x_1^{-1}x_2x_1^2x_2x_1^3x_2x_1^2x_2x_1x_2 \). Since \( w_{ab} = \left[ \begin{array}{cc} 8 & 5 \\ 3 & 2 \end{array} \right] \), let \( a \in \text{Aut} A_2 \) be given by the matrix \( \left[ \begin{array}{cc} 8 & 5 \\ 3 & 2 \end{array} \right] \) in the preferred basis for \( A_2 \). We use the automorphism \( \sigma \) obtained in Example 4.8 with the simple decomposition \( \sigma = \prod_{i=2}^{5} \alpha_i \). To verify that \( w \) is prebasic, we need to show that there exists a word \( v \in F_2 \) such that \( \sigma^{-1}(w) = vx_1v^{-1} \).

Using the notation of Lemma 4.3, and the simple decomposition of \( \sigma \),

\[
\sigma^{-1} = \gamma \circ \hat{\alpha}_5 \circ \hat{\alpha}_4 \circ \hat{\alpha}_3 \circ \hat{\alpha}_2 \circ \gamma,
\]

where \( \hat{\alpha}_2(x_1) = x_2x_1^{-2} \), \( \hat{\alpha}_i(x_1) = x_2x_1^{-1} \) for \( i = 3, 4, 5 \), and \( \hat{\alpha}_i(x_2) = x_1 \) for \( 1 \leq i \leq 5 \). Since

\[
(\hat{\alpha}_5 \circ \hat{\alpha}_4 \circ \hat{\alpha}_3 \circ \hat{\alpha}_2)(x_1) = (\hat{\alpha}_5 \circ \hat{\alpha}_4 \circ \hat{\alpha}_3)(x_2x_1^{-2}) = (\hat{\alpha}_5 \circ \hat{\alpha}_4)(x_1) \cdot ((\hat{\alpha}_5 \circ \hat{\alpha}_4)(x_2x_1^{-1}))^{-2}
\]

\[
= \hat{\alpha}_5(x_2x_1^{-1}) \cdot (\hat{\alpha}_5(x_1(x_2x_1^{-1})))^{-2}
\]

\[
= x_1(x_2x_1^{-1})^{-1} \cdot (\hat{\alpha}_5(x_2x_1^{-1}))^{-2} = x_1^2 x_2^{-1}(x_2x_1^{-1} x_2x_1^{-1})^{-2}
\]

\[
= x_1^2 x_2^{-1} x_1 x_2^{-1} x_1 x_2^{-1} x_2 x_1^{-1}
\]

and

\[
(\hat{\alpha}_5 \circ \hat{\alpha}_4 \circ \hat{\alpha}_3 \circ \hat{\alpha}_2)(x_2) = (\hat{\alpha}_5 \circ \hat{\alpha}_4 \circ \hat{\alpha}_3)(x_1) = (\hat{\alpha}_5 \circ \hat{\alpha}_4)(x_2x_1^{-1})
\]

\[
= \hat{\alpha}_5(x_1) \cdot \hat{\alpha}_5((x_2x_1^{-1})^{-1}) = x_2x_1^{-1} \cdot \hat{\alpha}_5(x_2x_1^{-1})
\]

\[
= x_2x_1^{-1} x_2x_1^{-1} x_1^{-1} = x_2x_1^{-1} x_2x_1^{-1} x_1 x_2^{-1}
\]

the equations defining \( \sigma^{-1} \) are

\[
\sigma^{-1}(x_1) = x_1 x_2^{-1} x_1 x_2^{-2}
\]
\[ \sigma^{-1}(x_2) = x_2^3 x_1 x_2^{-1} x_1^{-1} x_2^{-1} x_2 x_1^2 x_2^{-1} x_2 x_1^{-1}. \]

Thus (see Appendix A for details)
\[ \sigma^{-1}(w) = (x_1 x_2^{-1} x_1^{-1} x_2) x_1 (x_1 x_2^{-1} x_1 x_2^{-1} x_1 x_2^{-1})^{-1}. \]

Let \( v = x_1 x_2^{-1} x_1^{-1} x_2 x_2^{-1} \). Then \( \sigma^{-1}(w) = vx_1v^{-1} \). We have verified that \( w \) is prebasic.

Let \( \beta_1 = \gamma \), and for \( 2 \leq j \leq 7 \), define the automorphisms \( \beta_j \in \text{Aut} F_2 \) by the equations
\[ \beta_j(x_1) = x_1^{a_j} x_2 x_1^{-a_j} \] where \( a_2 = a_4 = a_7 = 1 \), \( a_3 = a_5 = a_6 = -1 \), and \( \beta_j(x_2) = x_1 \) for all \( j \).

Define
\[ \psi = (\prod_{i=2}^{5} \alpha_i) \circ \gamma \circ (\prod_{j=2}^{7} \beta_j) = (\prod_{i=2}^{5} \alpha_i) \circ (\prod_{j=2}^{7} \beta_j). \]

Then \( \psi(x_1) = w \), and the equation above provides a simple decomposition of the automorphism \( \psi \).
Scott showed in [7] that quaternionic toric 8-manifolds are 3-connected. The following is a summary of Wall’s classification theorem for 3-connected 8-manifolds (see [9], pp. 167f, 171, 174, 179).

**Theorem 5.1.** A 3-connected 8-manifold $M$ which is smooth away from a single point is characterized (topologically) by the intersection form $\cap: H_4(M) \otimes H_4(M) \to \mathbb{Z}$ and an element $q(M) \in H^4(M)$, which is one half of the first Pontrjagin class $p_1(M)$. Also, these invariants are subject to the relation $\langle q(M), x \rangle \equiv x \cap x \pmod{2}$.

In this chapter we discuss the intersection form and the invariant $q$ of an arbitrary quaternionic toric 8-manifold. First we introduce additional notation that will be used for the remainder of the paper.

Unless otherwise specified, by $X = X(Q, \lambda)$ denote a quaternionic toric 8-manifold, where $Q$ is an oriented $r$-gon with edges $E_i$ ($1 \leq i \leq r$) ordered according to the orientation for $Q$ and oriented consistently with the orientation for $Q$ (see Definition 3.5). Let

- $\lambda(E_i) = e(\tau_i(x_1))$ where $\tau_i \in \text{Aut} F_2$ for $1 \leq i \leq r$,
- $X_i(\mathcal{X}) = X(\tau_i)$, and $Y_i(\mathcal{X}) = Y(\tau_i)$ (see Definition 4.4),
- $\xi_i(\mathcal{X}) = X_i(\mathcal{X})Y_{i+1}(\mathcal{X}) - X_{i+1}(\mathcal{X})Y_i(\mathcal{X})$ for $1 \leq i \leq r-1$, and $\xi_r(\mathcal{X}) = X_r(\mathcal{X})Y_1(\mathcal{X}) - X_1(\mathcal{X})Y_r(\mathcal{X})$,
- $\chi_1(\mathcal{X}) = X_r(\mathcal{X})Y_2(\mathcal{X}) - X_2(\mathcal{X})Y_r(\mathcal{X})$, and $\chi_i(\mathcal{X}) = X_{i-1}(\mathcal{X})Y_{i+1}(\mathcal{X}) - X_{i+1}(\mathcal{X})Y_{i-1}(\mathcal{X})$ for $2 \leq i \leq r-1$,
- $[\mathcal{X}] \in H_8(\mathcal{X})$ denote the fundamental class of $\mathcal{X}$,
- $D_i(\mathcal{X}) = D_{E_i} \in H_4(\mathcal{X})$ for $1 \leq i \leq r$ (see Definition 3.7),
- $D^*_i(\mathcal{X}) \in H^4(\mathcal{X})$ denote the Poincaré duals of the cycles $D_i(\mathcal{X})$ for $1 \leq i \leq r$. 


Whenever it will be clear what toric manifold is considered, $\mathcal{X}$ will be omitted in the notation.

In sections 5.1 and 5.2 we present slight modifications of Scott’s discussion presented in §4 of [7]. In sections 5.3 and 5.4 we focus on the method of evaluating the invariant $q$ for an arbitrary quaternionic toric 8-manifold.

### 5.1 Intersection form

The discussion of the intersection form of quaternionic toric 8-manifolds presented by Scott in [7] is complete, but he presents the matrix of the intersection form (in a certain basis) only for toric manifolds satisfying the assumption $\xi_i(\mathcal{X}) = 1$ for all $1 \leq i \leq r - 1$, where $\mathcal{X}$ is associated with an $r$-gon. We omit this assumption to present the matrix of the intersection form (in the same basis) for an arbitrary toric 8-manifold.

Equations (4) and (5) on page 388 in [7] imply

$$
\xi_{i-1}D_{i-1} \cap D_i = \xi_i D_i \cap D_{i+1}
$$

and

$$
\xi_{i-1}D_i \cap D_i = -\chi_i D_i \cap D_{i+1}
$$

for $2 \leq i \leq r - 1$. Define

$$
\varepsilon(\mathcal{X}) = D_1 \cap D_2 = \langle D_1^* \cup D_2^*, [\mathcal{X}] \rangle.
$$

By equations (5.1) and (5.3),

$$
D_i \cap D_{i+1} = \xi_i \xi_i \varepsilon
$$
for $2 \leq i \leq r - 2$. Equations (5.4) and (5.2) imply

$$D_i \cap D_i = -\xi_1 \xi_{i-1} \xi_i \chi_i \varepsilon$$

for $2 \leq i \leq r - 1$. Therefore we have the following

**Proposition 5.2.** With respect to the basis $(D_2, D_3, \ldots, D_{r-1})$ for $H_4(X)$, the matrix of the intersection form of $X$ is

$$
\begin{bmatrix}
-\xi_1 \xi_2 \chi_2 & \xi_2 & & \\
\xi_2 & -\xi_2 \xi_3 \chi_3 & \xi_3 & \\
& \xi_3 & -\xi_3 \xi_4 \chi_4 & \xi_4 \\
&& \ddots & \\
& & \xi_{r-3} & -\xi_{r-3} \xi_{r-2} \chi_{r-2} & \xi_{r-2} \\
& & & \xi_{r-2} & -\xi_{r-2} \xi_{r-1} \chi_{r-1}
\end{bmatrix}.
$$

5.2 First Pontrjagin class

In the discussion about the first Pontrjagin class for quaternionic toric 8-manifolds (see [7], pp. 389ff), Scott presented a method of evaluating the invariant at the generator $D_i$ $(2 \leq i \leq r - 1)$ provided that the characteristic function satisfies the following condition. Let $w_k = \tau_k(x_1)$ $(1 \leq k \leq r)$. Then both pairs $\{w_{i-1}, w_i\}$ and $\{w_i, w_{i+1}\}$ generate $F_2$. To satisfy the extra assumption on $\lambda$ that $\xi_k(X) = 1$ for all $1 \leq k \leq r - 1$, Scott assumes that there exist $a, b \in \mathbb{Z}$ such that $w_{i+1} = w_i^a w_{i-1}^b$. At the end of the discussion, Scott obtains the clutching function $y \mapsto (x \mapsto y^a \bar{x} y^b)$ for the normal bundle of $D_i$. Then he applies Milnor’s result (see [4], Lemma 3, p. 402) to this bundle. Actually, in his result, Milnor does not consider the vector bundle with conjugation. Hence we slightly modify Scott’s discussion by starting with the assumption $w_{i+1} = w_i^a w_{i-1}^b$, to eventually obtain a vector bundle to which Milnor’s result will be applied explicitly. The details of the modification are presented in Appendix B. As a consequence of the modification, we obtain the part of the result of Proposition 5.3 where $s = 1$. 
Proposition 5.3. If for some $2 \leq i \leq r - 1$, the pair $\{w_{i-1}, w_i\}$ generates $F_i$, and $w_{i+1} = w_i^a w_{i-1}^b w_i^b$ for some $a, b \in \mathbb{Z}$ and $|s| = 1$, then $\langle p_1(X), D_i \rangle = 2(a - b)$.

To prove the result for the case $w_{i+1} = w_i^a w_{i-1}^b w_i^b$, we first discuss the maps on the homology groups induced by algebraic isomorphisms between toric 8-manifolds.

Let $X = X(Q, \lambda), Y = X(R, \mu)$ be toric 8-manifolds. Let $j = [f, g]: X \to Y$ be an algebraic isomorphism where the shift of $f$ is 0. Then the image $f(F)$ of any face $F \in \mathcal{P}_k(Q)$, is a face in $\mathcal{P}_k(R)$. Consider the induced map $j_*: H_4(X) \to H_4(Y)$. Let $F$ be a facet of $Q$. Then $D_F(X)$ is a generator of $H_4(X)$. We obtain the image of $D_F(X)$ under $j_*$ using Proposition 2.17:

$$j_*(D_F(X)) = j_*((\kappa_F)_*([X_F])) = (\kappa_{f(F)})_*((j_*)_*([X_F]))$$

by Proposition 2.17

$$= \begin{cases} (\kappa_{f(F)})_*([Y_{f(F)}]) & \text{if } j_F \text{ is orientation-preserving}, \\ (\kappa_{f(F)})_*([-Y_{f(F)}]) & \text{if } j_F \text{ is orientation-reversing}. \end{cases}$$

Therefore by Definition 3.7,

$$j_*(D_F) = \begin{cases} D_{f(F)}(Y) & \text{if } j_F \text{ is orientation-preserving}, \\ -D_{f(F)}(Y) & \text{if } j_F \text{ is orientation-reversing}. \end{cases}$$

(5.5)

We return to the evaluation of the first Pontrjagin class at the generator $D_i$ in the case $w_{i+1} = w_i^a w_{i-1}^b w_i^b$. Let $\lambda': \mathcal{P}(Q) \to \mathcal{P}(2)$ be defined by

$$\lambda'(F) = \begin{cases} \lambda(F) & \text{if } F \neq E_{i-1}, \\ e(w_{i-1}^{-1}) & \text{if } F = E_{i-1}. \end{cases}$$

Then $\lambda'$ is a basic characteristic function. Let $X' = X(Q, \lambda')$, and let $j = [1_Q, 1_{T^2}]: X \to X'$. Then $j$ is an algebraic morphism with $j_F = 1_T$ for $F \neq E_{i-1}$ and $j_{E_{i-1}}$ with rule of assignment $t \mapsto t^{-1}$. Actually, $j$ is an algebraic isomorphism with $j^{-1} = [1_Q, 1_{T^2}]$. By equation (5.5), $j_*(D_j(X)) = D_j(X')$ for $j \neq i - 1$, and $j_*(D_{i-1}(X)) = -D_{i-1}(X')$. Hence the matrix
of $j_*$ in bases $(D_2(X), D_3(X), \ldots, D_{r-1}(X))$ for $H_4(X)$ and $(D_2(X'), D_3(X'), \ldots, D_{r-1}(X'))$ for $H_4(X')$, is the identity $(r-2) \times (r-2)$-matrix with the $(i-2)$-nd column multiplied by $-1$. Therefore
\[
\langle p_1(X), D_i(X) \rangle = \langle p_1(X'), D_i(X') \rangle.
\]
If we let $\tilde{w}_{i-1} = w_{i-1}^{-1}$, then $w_{i+1} = w_i^a \tilde{w}_{i-1} w_i^b$ in $X'$ (see Figure 5.1) and hence
\[
\langle p_1(X'), D_i(X') \rangle = 2(a-b).
\]
Therefore
\[
\langle p_1(X), D_i(X) \rangle = 2(a-b).
\]
We have completed the proof of the part of the result of Proposition 5.3 where $s = -1$.

Scott introduced in [7] the notion of a nice quaternionic toric 8-manifold (see the paragraph preceding Proposition 4.1, p. 391). We rephrase Scott’s definition, considering a property of the characteristic function rather than a property of the toric manifold.

**Definition 5.4.** We call $\lambda$ generative at the edge $E_i$ $(1 \leq i \leq r)$ if both pairs $\{w_{i-1}, w_i\}$, $\{w_i, w_{i+1}\}$ generate $F_2$.

The following is a corollary of Proposition 5.3 (compare it also to [7], Theorem 4.3, p. 392).

**Corollary 5.5.** If for some $2 \leq i \leq r-1$, $\tau_i = \tau_{i-1} \circ \alpha_1$ and $\tau_{i+1} = \tau_i \circ \alpha_2$, where $\alpha_j$ are

\[
\begin{align*}
Q \xrightarrow{j} Q \quad \text{where} \quad w_i^a w_{i-1}^{-1} w_i^b = w_i^a \tilde{w}_{i-1} w_i^b
\end{align*}
\]

Fig. 5.1: The algebraic isomorphism $j: X \to X'$. 
of simple form (cf. Definition 4.1) for \( j = 1, 2 \), then \( \lambda \) is generative at the edge \( E_i \). Also, if \( \alpha_2(x_1) = x_1^a x_2^b \), then \( \langle p_1(\mathcal{X}), D_i \rangle = 2(a - b) \).

**Proof.** As before, let \( w_j = \tau_j(x_1) \) for \( j = i - 1, i, i + 1 \). Since \( x_1 = \alpha_1(x_2) \), \( w_{i-1} = (\tau_{i-1} \circ \alpha_1)(x_2) = \tau_i(x_2) \) and hence the pair \( \{w_{i-1}, w_i\} \) generates \( F_2 \). Similarly, \( w_i = (\tau_i \circ \alpha_2)(x_2) = \tau_{i+1}(x_2) \) and hence the pair \( \{w_i, w_{i+1}\} \) generates \( F_2 \). Thus \( \lambda \) is generative at the edge \( E_i \).

Furthermore,

\[
w_{i+1} = (\tau_i \circ \alpha_2)(x_1) = (\tau_i(x_1))^a (\tau_i(x_2))^a (\tau_i(x_1))^b = w_i^a w_{i-1}^a w_i^b
\]

and hence \( \langle p_1(\mathcal{X}), D_i \rangle = 2(a - b) \) by Proposition 5.3. \( \square \)

The corollary above will serve as the key calculation tool in the discussion about the invariant \( q \) for an arbitrary quaternionic toric 8-manifold.

### 5.3 Oriented connected sums

Recall the discussion of the operation of sum of toric spaces in section 2.3. Using the notation of that discussion, assume that \( Q_i \) \( (i = 1, 2) \) is a simple oriented convex \( n \)-polytope. Then \( Q'_i \) inherits the orientation from \( Q_i \). Choose the orientation of the facets of \( Q'_i \) (in particular, \( L_i \)) to be consistent with the orientation for \( Q'_i \) (see Definition 3.5). Let \( h_i \) be orientation-preserving, and let \( f : L_1 \to L_2 \) be orientation-reversing. Then \( Q_1 +_f Q_2 \) has the orientation consistent with the orientations for \( Q_i \). Furthermore, let \( \lambda_i \) be basic of height 0. Then \( R_{V_i}(\lambda_i) \) is basic of height 1 (by Definition 3.2). Under these assumptions, \( \mathcal{X}(L_i, R_{V_i}(\lambda_i)) \) is homeomorphic to \( S^{4n-1} \) by Theorem 3.3, and \( j = [f, 1_T^n] : \mathcal{X}(L_1, R_{V_1}(\lambda_1)) \to \mathcal{X}(L_2, R_{V_2}(\lambda_2)) \) is orientation-reversing. Hence \( \mathcal{X}_1 +_j \mathcal{X}_2 \) is homeomorphic to the oriented connected sum \( \mathcal{X}_1 \# \mathcal{X}_2 \).

If \( Q_i \) is a polygon, then \( L_i \) is a line segment. There exists only one orientation-reversing affine homeomorphism between oriented line segments. Hence we will omit \( f \) in the notation of sum of oriented convex polygons and \( j \) in the notation of sum of quaternionic toric 8-manifolds.
5.4 Invariant \( q \) for an arbitrary quaternionic toric 8-manifold

The following theorem, together with the construction discussed in section 4.3, will allow one to evaluate the invariant \( q \in H^4(\mathcal{X}) \) at basic elements \( D_2, D_3, \ldots, D_{r-1} \) of \( H_4(\mathcal{X}) \).

From now on, \( \mathbb{N}_r \) will denote the set of positive integers less than or equal to \( r \). In addition to the assumptions about \( \mathcal{X} \) introduced in the beginning of the chapter, for all \( k \in \mathbb{N}_r \), let

- \( \alpha_{k,1} = \alpha_{k,2} = \gamma \) (the automorphism \( \gamma \) was defined prior to Lemma 4.2 in section 4.1),
- \( \prod_{j=1}^{l_k} \alpha_{k,j} \) be a simple decomposition (see equation (4.1)) of \( \tau_k \) for some \( l_k \geq 2 \),
- \( q_{k,j}(\mathcal{X}) = a_{k,j} - b_{k,j} \) where \( \alpha_{k,j}(x_1) = x_1^{a_{k,j}} x_2^{b_{k,j}} x_1^{b_{k,j}} \) for all \( 1 \leq j \leq l_k \) (in particular, \( q_{k,1}(\mathcal{X}) = q_{k,2}(\mathcal{X}) = 0 \)),
- \( X_{k,j}(\mathcal{X}) = X_j(\tau_k) \) and \( Y_{k,j}(\mathcal{X}) = Y_j(\tau_k) \) for all \( 1 \leq j \leq l_k \) (see Definition 4.5),
- \( S_k(\mathcal{X}) = \sum_{2 \leq j \leq l_k-1} (Y_k(\mathcal{X}) X_{k,j}(\mathcal{X}) - X_k(\mathcal{X}) Y_{k,j}(\mathcal{X})) q_{k,j+1}(\mathcal{X}) \).

As before, \( \mathcal{X} \) will be often omitted to simplify notation.

**Theorem 5.6.** Under the assumptions above, for all \( 2 \leq i \leq r - 1 \),

\[
\langle q(\mathcal{X}), D_i \rangle = \xi_{i-1} S_{i-1} - \xi_{i-1} \xi_i \chi_i S_i + \xi_i S_{i+1}.
\]

Note that if one is given a set of elements \( w_k \) \( (1 \leq k \leq r) \) of \( F_2 \) specifying the values of the characteristic function \( \lambda \) of height 0 defining a toric space of rank 2 associated with an \( r \)-gon and \( \lambda \), then one can easily verify if the space is a toric manifold. If it is, then one can use the construction described in section 4.3 for each \( w_k \) to obtain an automorphism \( \tau_k \in \text{Aut} \ F_2 \) such that \( \tau_k(x_1) = w_k \), and its simple decomposition. This data is sufficient to use Theorem 5.6. Hence the theorem provides a method of evaluating the invariant \( q \) for an arbitrary quaternionic toric 8-manifold.

The proof of Theorem 5.6 will be preceded by the following discussion. We will possibly use indices greater than \( r \) for the edges of \( Q \). It will be understood to treat them cyclically.
For every $k \in \mathbb{N}_r$, denote the vertex $E_{k-1} \cap E_k$ of $Q$ by $V_k$. Let $Q_k$ ($k \in \mathbb{N}_r$) denote an oriented $(l_{k-1} + l_k - 1)$-gon with edges $E_{k,i}$ ($1 \leq i \leq l_{k-1} + l_k - 1$) numbered according to the orientation of $Q_k$ and oriented consistently with the orientation for $Q_k$. For every $i \in \mathbb{N}_{l_{k-1}+l_k-1}$, denote the vertex $E_{k,i-1} \cap E_{k,i}$ of $Q_k$ by $V_{k,i}$. Define $\mathcal{X}_k = \mathcal{X}(Q_k, \lambda_k)$ to be the toric space associated with $Q_k$ and the characteristic function $\lambda_k$ of rank 2 and height zero defined by

\[
(5.6) \quad \lambda_k(E_{k,i}) = \begin{cases} 
    e\left( \prod_{j=1}^{l_{k-1}-i+1} \alpha_{k-1,j}(x_1) \right) & \text{for } 1 \leq i \leq l_{k-1}, \\
    e\left( \prod_{j=1}^{l_{k-1}-(i+1)+1} \alpha_{k-1,j}(x_1) \right) & \text{for } l_{k-1} + 1 \leq i \leq l_{k-1} + l_k - 1.
\end{cases}
\]

Note that $\lambda_k(E_{k,1}) = \lambda(E_{k-1})$ and $\lambda_k(E_{k,l_{k-1}+l_k-1}) = \lambda(E_k)$. See Figure 5.2 for a picture of $Q_k$ with the words defining $\lambda_k$ at the edges of $Q_k$.

To show that the spaces $\mathcal{X}_k$ are manifolds, we prove that all characteristic functions $\lambda_k$ are basic (cf. Corollary 3.4).

**Lemma 5.7.** The characteristic function $\lambda_k$ is basic for all $k \in \mathbb{N}_r$.

**Proof.** First note that $\prod_{j=1}^{l_{k-1}-i+1} \alpha_{k-1,j}$ is an automorphism of $F_2$ for $1 \leq i \leq l_{k-1}$, and

\[
\prod_{j=1}^{l_{k-1}-(i+1)+1} \alpha_{k-1,j}(x_1) = \prod_{j=1}^{l_{k-1}-i+1} \alpha_{k-1,j}(x_2)
\]

for $1 \leq i \leq l_{k-1} - 1$. Hence by equation (5.6), $|\xi_i(\mathcal{X}_k)| = 1$ for $1 \leq i \leq l_{k-1} - 1$. Similarly, $\prod_{j=1}^{l_{k-1}+l_k-1} \alpha_{k,j}$ is an automorphism of $F_2$ for $l_{k-1} + 1 \leq i \leq l_{k-1} + l_k - 1$, and

\[
\prod_{j=1}^{i-(l_{k-1}+1)+1} \alpha_{k,j}(x_1) = \prod_{j=1}^{i-l_{k-1}+1} \alpha_{k,j}(x_2)
\]

for $l_{k-1} + 2 \leq i \leq l_{k-1} + l_k - 1$. Thus $|\xi_i(\mathcal{X}_k)| = 1$ for $l_{k-1} + 1 \leq i \leq l_{k-1} + l_k - 2$. $|\xi_{l_k}(\mathcal{X}_k)| = 1$ since $\lambda_k(E_{k,l_{k-1}+1}) = e(x_2)$ and $\lambda_k(E_{k,l_{k-1}+1}) = e(x_1)$. Finally, $|\xi_{l_{k-1}+l_k-1}(\mathcal{X}_k)| = 1$ since
Fig. 5.2: The space $\mathcal{X}_k$. 
\[ \xi_{k-1+l_{k-1}}(\mathcal{X}_k) = -\xi_{k-1}(\mathcal{X}) \] and \[ |\xi_{k-1}(\mathcal{X})| = 1. \]

Having determined that the spaces \( \mathcal{X}_k \) are toric 8-manifolds, we will evaluate the invariant \( q(\mathcal{X}_k) \) at basic elements \( D_i \in H_4(\mathcal{X}_k) \) (for \( 2 \leq i \leq l_{k-1} + l_k - 2 \)). To accomplish this, we show that \( \lambda_k \) is generative (see Definition 5.4) at all edges \( E_{k,i} \) for \( 2 \leq i \leq l_{k-1} + l_k - 2 \), and use Corollary 5.5 and Theorem 5.1.

Recall that \( \lambda_k(E_{k,1}) = e(\tau_{k-1}(x_1)) \). Using the notation and the result of Lemma 4.3,

\[
\lambda_k(E_{k,2}) = e((\tau_{k-1} \circ \alpha_{k-1,i_{k-1}}^{-1})(x_1)) = e((\tau_{k-1} \circ \gamma \circ \tilde{\alpha}_{k-1,i_{k-1}}^{-1} \circ \gamma)(x_1)) = e((\tau_{k-1} \circ \gamma)(x_1)).
\]

Furthermore, for \( 3 \leq i \leq l_{k-1} \),

\[
\lambda_k(E_{k,i}) = e\left( \left( \tau_{k-1} \circ \prod_{j=0}^{i-2} \alpha_{k-1,i_{k-1}-j}^{-1} \right)(x_1) \right) = e\left( \left( \tau_{k-1} \circ \gamma \circ \prod_{j=0}^{i-2} \tilde{\alpha}_{k-1,i_{k-1}-j}^{-1} \circ \gamma \right)(x_1) \right) = e\left( \left( \tau_{k-1} \circ \gamma \circ \prod_{j=0}^{i-3} \tilde{\alpha}_{k-1,i_{k-1}-j}^{-1} \right)(x_1) \right).
\]

Hence \( \lambda_k \) is generative at the edge \( E_{k,i} \) for \( 2 \leq i \leq l_{k-1} - 1 \). By Lemma 4.3, \( \tilde{\alpha}_{k,j}(x_1) = (x_1^{-a_{k,j}} x_2^{-b_{k,j}})^{s_{k,j}} \). Therefore

\[
\langle q(\mathcal{X}_k), D_i \rangle = b_{k-1,l_{k-1}+2-i} - a_{k-1,l_{k-1}+2-i} = -q_{k-1,l_{k-1}+2-i}
\]

by Corollary 5.5 and Theorem 5.1. Also,

\[
\lambda_k(E_{k,l_{k-1}-1}) = e((\alpha_{k-1,1} \circ \alpha_{k-1,2})(x_1)) = e(x_1),
\]
\[
\lambda_k(E_{k,l_{k-1}}) = e(\alpha_{k-1,1}(x_1)) = e(\gamma(x_1)),
\]
\[
\lambda_k(E_{k,l_{k-1}+1}) = e((\alpha_{k,1} \circ \alpha_{k,2})(x_1)) = e((\gamma \circ \gamma)(x_1)).
\]
Hence $\lambda$ is generative at the edge $E_{k,l_{k-1}}$, and

$$\langle q(\mathcal{X}_k), D_{l_k} \rangle = 0.$$

Finally, the definition of $\lambda_k$ (see equation (5.6)) implies that this characteristic function is also generative at the edge $E_{k,i}$ for $l_{k-1} + 1 \leq i \leq l_k - l_{k-1} + l_k - 2$, and

$$\langle q(\mathcal{X}_k), D_i \rangle = -q_{k,i-l_{k-1}+2}$$

for such $i$. We summarize the calculations above in the following

**Lemma 5.8.** For all $k \in \mathbb{N}_r$,

$$\langle q(\mathcal{X}_k), D_i \rangle = \begin{cases} -q_{k-1,l_{k-1}-i+2} & \text{for } 2 \leq i \leq l_{k-1}, \\ q_{k,i-l_k+2} & \text{for } l_{k-1} + 1 \leq i \leq l_{k-1} + l_k - 2. \end{cases}$$

In particular, $\langle q(\mathcal{X}_k), D_{l_{k-1}} \rangle = -q_{k-1,2} = 0$.

Next we define a quaternionic toric 8-manifold as the sum of $\mathcal{X}$ and some of the $\mathcal{X}_k$.

**Definition 5.9.** For any subset $K = \{k_1, \ldots, k_m\}$ of $\mathbb{N}_r$ such that $1 \leq k_1 < k_2 < \cdots < k_m \leq r$, define

$$\mathcal{X}_K = \mathcal{X} + \sum_{k \in K} \mathcal{X}_k$$

where each $\mathcal{X}_k$ is added to $\mathcal{X}$ at the vertex $V_k$ of $Q$ and $V_{k,1}$ of $Q_k$. We will say that $\mathcal{X}_K$ is the result of *completing* $\mathcal{X}$ at the vertices $V_{k_1}, \ldots, V_{k_m}$.

Using notation of the discussion of the operation of sum of toric spaces (presented in section 2.3), let

- $R = Q + \sum_{k \in K} Q_k$,
- $Q' = Q - \bigcup_{k \in K} \overline{\text{CL}}_k$,
- $\iota_Q : Q' \to R$ denote the inclusion of a summand,
• $Q'_k = Q_k - CL$ (for all $k \in K$), where $L$ denotes the link of the vertex $V_{k,1}$ in $Q_k$,

• $\iota_k: Q'_k \to R$ denote the inclusion of a summand.

We distinguish three types of edges of $R$. The edges of $R$ of the first type are of the form $\iota_Q(E_i)$ for all $1 \leq i \leq r$ such that $E_i \cap \bigcup_{k \in K} V_k = \emptyset$ (see Figure 5.3). The edges of $R$ of the second type are of the form $\iota_k(E_{k,i})$ for all $k \in K$ and $2 \leq i \leq l_{k-1} + l_k - 2$ (see Figure 5.3). The edges of $R$ of the third type are of one of the following forms

• $E_i + E_{i+1,1}$ for all $i \in \mathbb{N}_r - K$ such that $i + 1 \in K$,

• $E_{i,l_i} + E_i + E_{i+1,1}$ for all $i \in K$ such that $i + 1 \in K$, or

• $E_i$ for all $i \in K$ such that $i + 1 \notin K$ (see Figure 5.3).

Let $\iota_X: H_4(\mathcal{X}) \to H_4(\mathcal{X}_K)$ denote the inclusion of a direct summand. Then

$$D_{\iota_Q(E_i)} = \iota_X(D_{E_i})$$

and hence

$$(5.7) \quad \left\langle q(\mathcal{X}_K), D_{\iota_Q(E_i)} \right\rangle = \left\langle q(\mathcal{X}), D_{E_i} \right\rangle$$

for all faces of $R$ of the first type. For all $k \in K$, let $\iota_{\mathcal{X}_k}: H_4(\mathcal{X}_k) \to H_4(\mathcal{X}_K)$ denote the inclusion of a direct summand. Then

$$D_{\iota_k(E_{k,i})} = \iota_{\mathcal{X}_k}(D_{E_{k,i}})$$

and hence

$$(5.8) \quad \left\langle q(\mathcal{X}_K), D_{\iota_k(E_{k,i})} \right\rangle = \left\langle q(\mathcal{X}_k), D_{E_{k,i}} \right\rangle$$

for all faces of $R$ of the second type.

We proceed with the proof of Theorem 5.6.
Fig. 5.3: Notation for the edges of $R$. 
Proof of Theorem 5.6. Fix $i$. Let $K = \mathbb{N}_r - \{i, i+1\}$. Consider the toric manifold $\mathcal{X}_K$ (see Definition 5.9) associated with the polygon $R = Q + \sum_{k \in K} Q_k$. Denote the characteristic function defining $\mathcal{X}_K$ by $\mu$. Denote the faces of $R$ in the following way. Let

$$F_i = t_Q(E_i),$$
$$F_{k,j} = t_k(E_{k,j}) \quad \text{for all } k \in K \text{ and } 2 \leq j \leq l_{k-1} + l_k - 2,$$
$$F_j = \begin{cases} E_{i+1} + E_{i+2,1} & \text{for } j = i + 1, \\ E_{j,l_{j-1}+l_{j-1}+1} + E_j + E_{j+1,1} & \text{for } 1 \leq j \leq i - 2 \text{ and } i + 2 \leq j \leq r, \\ E_{i-1,l_{i-2}+l_{i-1}+1} + E_{i-1} & \text{for } j = i - 1. \end{cases}$$

Then $\mu(F_j) = \lambda(E_j)$ for all $j \in \mathbb{N}_r$, and $\mu(F_{k,j}) = \lambda_k(E_{k,j})$ for all $k \in K$ and $2 \leq j \leq l_{k-1} + l_k - 2$. Also, $F_i$ is a face of $R$ of the first type, so

$$\langle q(\mathcal{X}_K), D_{F_i} \rangle = \langle q(\mathcal{X}), D_i \rangle \quad \text{(5.9)}$$

by equation (5.7). To determine the value of $\langle q(\mathcal{X}_K), D_{F_i} \rangle$, we will present the generator $D_{F_i}$ as a combination of the remaining generators of $H_4(\mathcal{X}_K)$ and use linearity of the invariant $q(\mathcal{X}_K)$. Using notation introduced in the discussion preceding Theorem 3.9, the definitions of $\lambda$, $\lambda_k$ (see equation (5.6)) and $\mu$ imply that

$$\mu(F_j)_{ab}^* = [-Y_j(\mathcal{X}), X_j(\mathcal{X})]$$

for all $j \in \mathbb{N}_r$, and for all $k \in K$,

$$\mu(F_{k,j})_{ab} = \begin{cases} [-Y_{k-1,l_{k-1}+1-j}(\mathcal{X}), X_{k-1,l_{k-1}+1-j}(\mathcal{X})] & \text{for } 2 \leq j \leq l_{k-1}, \\ [-Y_{k,j-l_{k-1}+1}(\mathcal{X}), X_{k,j-l_{k-1}+1}(\mathcal{X})] & \text{for } l_{k-1} + 1 \leq j \leq l_{k-1} + l_k - 2. \end{cases}$$
From now on, $\mathcal{X}$ will be omitted in the notation. Define $K_+ = K \cup \{i + 1\}$. The relations between the generators of $H_4(\mathcal{X}_K)$ (see Remark 3.10) are

\begin{align}
(5.10) \quad - \left( Y_i D_{F_i} + \sum_{j \in K_+} Y_j D_{F_j} \right)
+ \sum_{k \in K} \left( \sum_{j=2}^{l_k} Y_{k-1,l_k-1+1-j} D_{F_{k,j}} + \sum_{j=l_k+1+1}^{l_k+2} Y_{k,j-l_k-1+1} D_{F_{k,j}} \right) = 0,
\end{align}

and

\begin{align}
(5.11) \quad X_i D_{F_i} + \sum_{j \in K_+} X_j D_{F_j}
+ \sum_{k \in K} \left( \sum_{j=2}^{l_k} X_{k-1,l_k-1+1-j} D_{F_{k,j}} + \sum_{j=l_k+1+1}^{l_k+2} X_{k,j-l_k-1+1} D_{F_{k,j}} \right) = 0.
\end{align}

Add equation (5.10) multiplied by $X_{i+1}$ to equation (5.11) multiplied by $Y_{i+1}$ to obtain

\begin{align}
(X_i Y_{i+1} - Y_i X_{i+1}) D_{F_i} + \sum_{j=1}^{i-2} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j}
+ (X_{i+1} - Y_{i+1} X_{i+1}) D_{F_{i+1}} + \sum_{j=i+2}^{r} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j}
+ \sum_{k \in K} \left( \sum_{j=2}^{l_k} (Y_{i+1} X_{k-1,l_k-1+1-j} - X_{i+1} Y_{k-1,l_k-1+1-j}) D_{F_{k,j}}
+ \sum_{j=l_k+1+1}^{l_k+2} (Y_{i+1} X_{k,j-l_k-1+1} - X_{i+1} Y_{k,j-l_k-1+1}) D_{F_{k,j}} \right) = 0.
\end{align}

The equation above simplifies to

\begin{align}
\xi_i D_{F_i} + \sum_{j=1}^{i-2} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j} + \chi_i D_{F_{i+1}} + \sum_{j=i+2}^{r} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j}
+ \sum_{k \in K} \left( \sum_{j=2}^{l_k} (Y_{i+1} X_{k-1,l_k-1+1-j} - X_{i+1} Y_{k-1,l_k-1+1-j}) D_{F_{k,j}} \right).
\end{align}
\begin{align*}
&\sum_{j=l_{k-1}+1}^{l_k-l_k-2} (Y_{i+1}X_{k,j-l_{k-1}+1} - X_{i+1}Y_{k,j-l_{k-1}+1}) D_{F_{k,j}} = 0.
\end{align*}

Solve the equation above for \( D_{F_i} \) to obtain

\begin{align*}
D_{F_i} &= -\xi_i \left( \sum_{j=1}^{i-2} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j} + \sum_{j=i+2}^{r} (X_j Y_{i+1} - Y_j X_{i+1}) D_{F_j} \\
&\quad + \sum_{k \in K} \left( \sum_{j=2}^{l_{k-1}} (Y_{i+1}X_{k-lk-1+1,j} - X_{i+1}Y_{k-lk-1+1,j}) D_{F_{k,j}} \\
&\quad + \sum_{j=l_{k-1}+1}^{l_k-l_k-2} (Y_{i+1}X_{k,j-lk-1+1} - X_{i+1}Y_{k,j-lk-1+1}) D_{F_{k,j}} \right) \right) - \xi_i \chi_i D_{F_{i-1}}.
\end{align*}

Since \( q(\mathcal{X}_K) \) is linear, the equation above implies that

\begin{equation}
\langle q(\mathcal{X}_K), D_{F_{i}} \rangle = -\xi_i \left( \sum_{j=1}^{i-2} (X_j Y_{i+1} - Y_j X_{i+1}) \langle q(\mathcal{X}_K), D_{F_j} \rangle \\
+ \sum_{j=i+2}^{r} (X_j Y_{i+1} - Y_j X_{i+1}) \langle q(\mathcal{X}_K), D_{F_j} \rangle \\
+ \sum_{k \in K} \left( \sum_{j=2}^{l_{k-1}} (Y_{i+1}X_{k-lk-1+1,j} - X_{i+1}Y_{k-lk-1+1,j}) \langle q(\mathcal{X}_K), D_{F_{k,j}} \rangle \\
+ \sum_{j=l_{k-1}+1}^{l_k-l_k-2} (Y_{i+1}X_{k,j-lk-1+1} - X_{i+1}Y_{k,j-lk-1+1}) \langle q(\mathcal{X}_K), D_{F_{k,j}} \rangle \right) \right) \\
- \xi_i \chi_i \langle q(\mathcal{X}_K), D_{F_{i-1}} \rangle.
\end{equation}

Note that for all \( j \in K - \{i - 1\} \), the faces \( F_j \) are positioned in the sequence \( F_{j, l_{j-1}+l_j-2}, F_j, F_{j+1,2} \) with respect to the direction in \( \partial R \) given by the orientation of \( R \) (see Figure 5.4).
Since

$$\mu(F_{j,l-1+l-2}) = e((\tau_j \circ \alpha_j^{-1})(x_1)),$$

$$\mu(F_j) = e((\tau_j \circ \alpha_j^{-1} \circ \alpha_j)(x_1)),$$

$$\mu(F_{j+1,2}) = e((\tau_j \circ \alpha_j^{-1} \circ \alpha_j \circ \gamma \circ \tilde{\alpha}_j^{-1} \circ \gamma)(x_1)) = e((\tau_j \circ \alpha_j^{-1} \circ \alpha_j \circ \gamma \circ \tilde{\alpha}_j^{-1} \circ \gamma)(x_1)) = e((\tau_j \circ \alpha_j^{-1} \circ \alpha_j \circ \gamma)(x_1)),$$

\(\mu\) is generative at the edge \(F_j\) by Corollary 5.5, and \(\langle q(\mathcal{X}_K), D_{F_j} \rangle = 0\). Note that \(\mu\) is actually generative at all edges of \(R\) except for \(F_{i-1}, F_i, \) and \(F_{i+1}\). By Lemma 5.8, and equation (5.8), the expression for \(\langle q(\mathcal{X}_K), D_{F_i} \rangle\) above (equation (5.12)) simplifies to

\[
\langle q(\mathcal{X}_K), D_{F_i} \rangle = -\xi_i \sum_{k \in K} \left( - \sum_{j=2}^{l_{k-1}} (Y_{i+1}X_{k,l_{k-1}+1-j} - X_{i+1}Y_{k,l_{k-1}+1-j}) q_{k-1,l_{k-1}-j+2} \right.
\]

\[
+ \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} (Y_{i+1}X_{k,j-l_{k-1}+1} - X_{i+1}Y_{k,j-l_{k-1}+1}) q_{k,j-l_{k-1}+2} \right)
\]

\[
- \xi_i \chi_i \langle q(\mathcal{X}_K), D_{F_{i-1}} \rangle.
\]
After further simplifications (see Appendix C for details), by equation (5.9) we obtain the following formula for \( \langle q(X), D_i \rangle \)

\[
\langle q(X), D_i \rangle = -\xi_i \sum_{m=2}^{l_i-1} (Y_{i+1}X_{i-1,m} - X_{i+1}Y_{i-1,m})q_{i-1,m+1} + \xi_i Y_{i+1}
\]

\[
= -\xi_i \langle q(X), D_{F_{i-1}} \rangle.
\]  

(5.14)

To complete the calculation for \( \langle q(X), D_i \rangle \), we need to find the value of \( \langle q(X_K), D_{F_{i-1}} \rangle \).

Consider the toric manifold \( X_{K_+} \). Since the operation of sum of toric spaces is associative, \( X_{K_+} = X_K + X_{i+1} \) where the sum is performed at the vertices \( F_i \cap F_{i+1} \) of \( R \) and \( V_{i+1} \) of \( Q_{i+1} \). Then \( X_{K_+} \) is associated with the polygon \( W = R + Q_{i+1} = Q + \sum_{k \in K_+} Q_k \). Denote the characteristic function defining \( X_{K_+} \) by \( \nu \). Denote the faces of \( W \) in the way analogous to the notation of the faces of \( R \). Let

\[
G_{k,j} = \iota_k(E_{k,j}) \quad \text{for all } k \in K_+ \text{ and } 2 \leq j \leq l_{k-1} + l_k - 2,
\]

\[
G_j = \begin{cases} 
E_i + E_{i+1,1} & \text{for } j = i, \\
E_{j-1} + E_{j-1} + E_{j-1} + E_{j-1,1} & \text{for } 1 \leq j \leq i - 2 \text{ and } i + 1 \leq j \leq r, \\
E_{l_{i-1},1} + l_{i-1} + 1 + E_{l_{i-1}} & \text{for } j = i - 1.
\end{cases}
\]

Then \( \nu(G_j) = \lambda(E_j) \) for all \( j \in \mathbb{N}_r \), and \( \nu(G_{k,j}) = \lambda_k(E_{k,j}) \) for all \( k \in K_+ \) and \( 2 \leq j \leq l_{k-1} + l_k - 2 \). If we consider \( X_{K_+} \) as the sum \( X_K + X_{i+1} \), then using notation of the discussion of the operation of sum of toric spaces, \( \iota_R(F_{i-1}) = G_{i-1} \) and hence \( G_{i-1} \) is a face of \( W \) of the first type in this context. Thus

\[
\langle q(X_K), D_{F_{i-1}} \rangle = \langle q(X_{K_+}), D_{G_{i-1}} \rangle \quad \text{by equation } (5.7).
\]

(5.15)
invariant \( q(\mathcal{X}_{K^+}) \). The definitions of \( \lambda, \lambda_k \) (see equation (5.6)) and \( \nu \) imply that

\[
\nu(G_j)^* = [-Y_j(\mathcal{X}), X_j(\mathcal{X})]
\]

for all \( j \in \mathbb{N}_r \), and for all \( k \in K_+ \),

\[
\nu(G_{k,j})^* = \begin{cases} 
[Y_{k-1,l_{k-1}+1-j}(\mathcal{X}), X_{k-1,l_{k-1}+1-j}(\mathcal{X})] & \text{for } 2 \leq j \leq l_{k-1}, \\
[Y_{k,j-l_{k-1}+1}(\mathcal{X}), X_{k,j-l_{k-1}+1}(\mathcal{X})] & \text{for } l_{k-1} + 1 \leq j \leq l_{k-1} + l_k - 2.
\end{cases}
\]

From now on, \( \mathcal{X} \) will be omitted in the notation. The relations between the generators of \( H_{4}(\mathcal{X}_{K^+}) \) are

\[
(5.16) \quad -\left( \sum_{j \in \mathbb{N}_r} Y_j D_{G_j} + \sum_{k \in K_+} \left( \sum_{j=2}^{l_{k-1}} Y_{k-1,l_{k-1}+1-j} D_{G_{k,j}} + \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} Y_{k,j-l_{k-1}+1} D_{G_{k,j}} \right) \right) = 0,
\]

and

\[
(5.17) \quad \sum_{j \in \mathbb{N}_r} x_j D_{G_j} + \sum_{k \in K_+} \left( \sum_{j=2}^{l_{k-1}} X_{k-1,l_{k-1}+1-j} D_{G_{k,j}} + \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} X_{k,j-l_{k-1}+1} D_{G_{k,j}} \right) = 0.
\]

Add equation (5.16) multiplied by \( X_i \) to equation (5.17) multiplied by \( Y_i \) to obtain

\[
\xi_{i-1} D_{G_{i-1}} + \sum_{j=1}^{i-3} (X_j Y_i - Y_j X_i) D_{G_j} + \chi_{i-1} D_{G_{i-2}} + \sum_{j=i+1}^{r} (X_j Y_i - Y_j X_i) D_{G_j}
\]

\[
+ \sum_{k \in K_+} \left( \sum_{j=2}^{l_{k-1}} (Y_{i}X_{k-1,l_{k-1}+1-j} - X_{i}Y_{k-1,l_{k-1}+1-j}) D_{G_{k,j}} + \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} (Y_{i}X_{k,j-l_{k-1}+1} - X_{i}Y_{k,j-l_{k-1}+1}) D_{G_{k,j}} \right) = 0.
\]

Solve the equation above for \( D_{G_{i-1}} \) to obtain

\[
D_{G_{i-1}} = -\xi_{i-1} \left( \sum_{j=1}^{i-3} (X_j Y_i - Y_j X_i) D_{G_j} + \sum_{j=i+1}^{r} (X_j Y_i - Y_j X_i) D_{G_j} \right)
\]
Since $q(X_{K_+})$ is linear, the equation above implies that

$$
\langle q(X_{K_+}), D_{G_{i-1}} \rangle = -\xi_{i-1} \left( \sum_{j=1}^{i-3} (X_j Y_i - Y_j X_i) \langle q(X_{K_+}), D_{G_j} \rangle + \sum_{j=i+1}^n (X_j Y_i - Y_j X_i) \langle q(X_{K_+}), D_{G_j} \rangle 
+ \sum_{k \in K_+}^{l_{k-1}} \left( \sum_{j=2}^{l_{k-1}+1} (Y_i X_{k-1,l_{k-1}+1-j} - X_i Y_{k-1,l_{k-1}+1-j}) \langle q(X_{K_+}), D_{G_{k,j}} \rangle 
+ \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} (Y_i X_{k,j-l_{k-1}+1} - X_i Y_{k,j-l_{k-1}+1}) \langle q(X_{K_+}), D_{G_{k,j}} \rangle \right) 
- \xi_{i-1} \chi_{i-1} \langle q(X_{K_+}), D_{G_{i-2}} \rangle \right).
$$

By an argument analogous to the one used to evaluate $\langle q(X_K), D_{F_j} \rangle$ for all $j \in K - \{i-1\}$, $\nu$ is generative at the edge $G_j$ and $\langle q(X_{K_+}), D_{G_j} \rangle = 0$ for all $j \in K_+ - \{i-1\}$. Note that $\nu$ is actually generative at all edges of $W$ except for $G_{i-1}$ and $G_i$. By Lemma 5.8, and equation (5.8), the expression for $\langle q(X_{K_+}), D_{G_{i-1}} \rangle$ above simplifies to

$$
(5.18)
\langle q(X_{K_+}), D_{G_{i-1}} \rangle = -\xi_{i-1} \sum_{k \in K_+}^{l_{k-1}-1} \left( - \sum_{j=2}^{l_{k-1}+l_k-2} (Y_i X_{k-1,l_{k-1}+1-j} - X_i Y_{k-1,l_{k-1}+1-j}) q_{k-1,l_{k-1}+j-2} \right)
+ \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} (Y_i X_{k,j-l_{k-1}+1} - X_i Y_{k,j-l_{k-1}+1}) q_{k,j-l_{k-1}+2}.
$$
After further simplifications (see Appendix C for details), by equation (5.15) we obtain the following formula:

\[
(5.19) \quad \langle q(X_K), D_{F_{i-1}} \rangle = -\xi_i \sum_{m=2}^{l_{i-1}-1} (Y_i X_{i-1,m} - X_i Y_{i-1,m}) q_{i-1,m+1} + \xi_i S_i.
\]

Substitute the value of \(\langle q(X_K), D_{F_{i-1}} \rangle\) calculated above in equation (5.14) to obtain

\[
(5.20) \quad \langle q(X), D_i \rangle = -\xi_i \sum_{m=2}^{l_{i-1}-1} (Y_{i+1} X_{i-1,m} - X_{i+1} Y_{i-1,m}) q_{i-1,m+1} + \xi_i S_i + \xi_i \sum_{m=2}^{l_{i-1}-1} (Y_i X_{i-1,m} - X_i Y_{i-1,m}) q_{i-1,m+1} - \xi_i \chi \xi_i S_i.
\]

Note that

\[
\xi_i \chi Y_i - Y_{i+1} = \xi_i (X_i Y_i - \xi_i Y_{i+1})
\]

\[
= \xi_i (X_{i-1} Y_{i+1} - X_{i+1} Y_{i-1} - X_{i-1} Y_i Y_{i+1} + X_i Y_{i-1} Y_{i+1})
\]

\[
= Y_{i-1} \xi_i,
\]

and

\[
\xi_i \chi X_i - X_{i+1} = \xi_i (X_i Y_i - \xi_i X_{i+1})
\]

\[
= \xi_i (X_{i-1} Y_{i+1} - X_{i+1} Y_{i-1} - X_{i-1} Y_i X_{i+1} + X_i Y_{i-1} X_{i+1})
\]

\[
= \xi_i X_{i-1} \xi_i.
\]

Therefore

\[
\xi_i \chi \xi_i \sum_{m=2}^{l_{i-1}-1} (Y_i X_{i-1,m} - X_i Y_{i-1,m}) q_{i-1,m+1}
\]

\[
- \xi_i \sum_{m=2}^{l_{i-1}-1} (Y_{i+1} X_{i-1,m} - X_{i+1} Y_{i-1,m}) q_{i-1,m+1}
\]
\[
\begin{align*}
&= \sum_{m=2}^{l_i-1} \left( (\xi_{i-1}Y_i - Y_{i+1})X_{i-1,m} - (\xi_{i-1}X_i - X_{i+1})Y_{i-1,m} \right) q_{i-1,m+1} \\
&= \sum_{m=2}^{l_i-1} (\xi_{i-1}Y_{i-1} - \xi_iX_{i-1,m} - \xi_{i-1}X_{i-1}X_{i-1,m}) q_{i-1,m+1} \\
&= \xi_{i-1} \sum_{m=2}^{l_i-1} (Y_{i-1}X_{i-1,m} - X_{i-1}Y_{i-1,m}) q_{i-1,m+1} \\
&= \xi_{i-1} S_{i-1}
\end{align*}
\]

and hence equation (5.20) simplifies to

\[
\langle q(X), D_i \rangle = \xi_{i-1} S_{i-1} - \xi_{i-1} \xi_i S_i + \xi_i S_{i+1}.
\]

The following example will illustrate the application of Theorem 5.6.

**Example 5.10.** Let \( Q \) be an oriented square. Let \( w_1 = x_1^5x_2x_1^{-2} \), \( w_2 = x_2^2x_1^{-1}x_2x_3^3x_2^{-2} \), \( w_3 = x_1x_2x_1^2x_2x_1^1 \), \( w_4 = x_1^{-1}x_2^{-1}x_1^{-1}x_2x_3^3x_2x_3^3x_2x_2x_1 \), and let \( \lambda(E_k) = e(w_k) \) for \( k \in \mathbb{N} \).

Then \( X_1 = 3, Y_1 = 1, X_2 = 2, Y_2 = 1, X_3 = 5, Y_3 = 2, X_4 = 8, \) and \( Y_4 = 3 \). Hence \( \xi_1 = 1 \), and \( \xi_2 = \xi_3 = \xi_4 = -1 \). Thus \( \lambda \) is basic. Also, \( \chi_2 = 1 \) and \( \chi_3 = -2 \). Let \( \alpha_{1,3}(x_1) = x_1^5x_2x_1^{-2} \), \( \alpha_{2,3}(x_1) = x_2 \), \( \alpha_{2,4}(x_1) = x_2^2x_1^{-1}x_2 \), \( \alpha_{2,5}(x_1) = x_1^{-1}x_2x_3^3 \), \( \alpha_{3,3}(x_1) = x_2 \), \( \alpha_{3,4}(x_1) = x_2^2x_1^{-1}x_2 \), \( \alpha_{4,4}(x_1) = x_2 \), \( \alpha_{4,5}(x_1) = x_2 \), \( \alpha_{4,6}(x_1) = x_2 \), and for \( 7 \leq j \leq 12 \), let \( \alpha_{4,j}(x_1) = x_1^{a_{4,j}}x_2^{-1}x_1^{-1} \alpha_{4,j} \) where \( a_{4,7} = a_{4,9} = a_{4,12} = 1 \) and \( a_{4,8} = a_{4,10} = a_{4,11} = -1 \). For \( 1 \leq k \leq 4 \), define \( \tau_k = \prod_{j=1}^{l_k} \alpha_{k,j} \) where \( l_1 = 3, l_2 = 5, l_3 = 4, \) and \( l_4 = 12 \). Then \( w_k = \tau_k(x_1) \) for all \( k \) (for \( k = 4 \), refer to Example 4.11). Therefore \( q_{1,3} = 7, q_{2,3} = 0, q_{2,4} = 4, q_{2,5} = -4, q_{3,3} = q_{3,4} = q_{4,3} = q_{4,8} = q_{4,10} = q_{4,11} = -2, q_{4,4} = q_{4,5} = q_{4,6} = -1, \) and \( q_{4,7} = q_{4,9} = q_{4,12} = 2 \). Also, \( X_{k,2} = 1 \) and \( Y_{k,2} = 0 \) for all \( k, X_{2,3} = 0, Y_{2,3} = 1, X_{2,4} = 1, Y_{2,4} = 0, X_{3,3} = 2, Y_{3,3} = 1, X_{4,3} = 2, Y_{4,3} = 1, X_{4,4} = 3, Y_{4,4} = 1, X_{4,5} = 5 \) and \( Y_{4,5} = 2 \) for \( j = 5, 7, 9, 11, X_{4,5} = 8 \) and \( Y_{4,5} = 3 \) for \( j = 6, 8, 10 \). Thus

\[
S_1 = (1 \cdot 1 - 3 \cdot 0) \cdot 7 = 7,
\]

\[
S_2 = (1 \cdot 1 - 2 \cdot 0) \cdot 0 + (1 \cdot 0 - 3 \cdot 1) \cdot 4 + (1 \cdot 1 - 3 \cdot 0) \cdot (-4) = -16,
\]
\[ S_3 = (2 \cdot 1 - 5 \cdot 0) \cdot (-2) + (2 \cdot 2 - 5 \cdot 1) \cdot (-2) = -2, \]

\[ S_4 = (3 \cdot 1 - 8 \cdot 0) \cdot (-2) + (3 \cdot 2 - 8 \cdot 1) \cdot (-1) + (3 \cdot 3 - 8 \cdot 1) \cdot (-1) + (3 \cdot 5 - 8 \cdot 2) \cdot (-1) \]
\[ + (3 \cdot 8 - 8 \cdot 3) \cdot 2 + (3 \cdot 5 - 8 \cdot 2) \cdot (-2) + (3 \cdot 8 - 8 \cdot 3) \cdot 2 \]
\[ + (3 \cdot 5 - 8 \cdot 2) \cdot (-2) + (3 \cdot 8 - 8 \cdot 3) \cdot (-2) + (3 \cdot 5 - 8 \cdot 2) \cdot 2 \]
\[ = -2 \]

and hence
\[ \langle g(X), D_2 \rangle = 1 \cdot 7 - 1 \cdot (-1) \cdot 1 \cdot (-16) + (-1) \cdot (-2) = -7, \]

and
\[ \langle g(X), D_3 \rangle = -1 \cdot (-16) - (-1) \cdot (-1) \cdot (-2) \cdot (-2) + (-1) \cdot (-2) = 14. \]
CHAPTER 6
FAMILIES OF QUATERNIONIC TORIC 8-MANIFOLDS

In this chapter we will define families of quaternionic toric 8-manifolds whose elements will satisfy certain properties, to eventually obtain families that will prove the statement of the main result of the paper (Theorem 3.14).

6.1 The first family

By the classification of nondegenerate quadratic forms, rank, signature and parity of a matrix of such form, specify the isomorphism type of the form. The elements of the first family will have all possible isomorphism types of the intersection form.

Let \( R \geq 1, S \) be an integer such that \(|S| \leq R\) and \( S \equiv R \pmod{2}\), and let \( P \in \{0, 1\}\). For each such triple we define the toric manifold \( X(Q_R, \lambda_{S,P}) \), where \( Q_R \) is an oriented \((R+2)\)-gon with edges \( E_1, E_2, \ldots, E_{R+2} \) numbered according to the orientation for \( Q_R \), and \( \lambda_{S,P} \) is a basic characteristic function of rank 2 and height 0, defined as follows.

**Definition 6.1.** Let \( R \geq 2 \) be even. Define

\[
\lambda_{0,P}(E_i) = e(\gamma^{i-1}(x_1))
\]

for all \( 1 \leq i \leq R \) and \( i = R+2 \). Define

\[
\lambda_{0,P}(E_{R+1}) = e(x_1 x_2^P).
\]

**Definition 6.2.** Let \( R \geq 1 \) and \( S \neq 0 \). Define

\[
\lambda_{S,1}(E_i) = e(\gamma^{i-1}(x_1))
\]
for all $1 \leq i \leq R - |S| + 2$. Define

$$
\lambda_{S,1}(E_{R-|S|+2}) = \begin{cases} 
  e(x_1^j x_2) & \text{if } S > 0, \\
  e(x_1^{-j} x_2) & \text{if } S < 0
\end{cases}
$$

for all $1 \leq j \leq |S|$.

For examples of spaces defined in the definitions above, see Figures 6.1 and 6.2.

In the following proposition we gather properties of the invariants of the intersection form of toric manifolds defined above. Let $I(\mathcal{X}(Q_R, \lambda_{S,P}))$ denote the matrix of the intersection form of $\mathcal{X}(Q_R, \lambda_{S,P})$ in the basis $(D_2, D_3, \ldots, D_{R+1})$ for $H_4(\mathcal{X}(Q_R, \lambda_{S,P}))$.

**Proposition 6.3.** (1) $I(\mathcal{X}(Q_R, \lambda_{S,P}))$ is of rank $R$, parity $P$, and the absolute value of the signature of $I(\mathcal{X}(Q_R, \lambda_{S,P}))$ is equal to $|S|$.

(2) The signature of $I(\mathcal{X}(Q_R, \lambda_{S,1}))$ is the opposite of the signature of $I(\mathcal{X}(Q_R, \lambda_{-S,1}))$.

(3) If $R_i \equiv S \pmod{2}$ for $i = 1, 2$, then the signatures of $I(\mathcal{X}(Q_R, \lambda_{S,1}))$ are the same for $i = 1, 2$.

The proof of the proposition above will be preceded by the list of numbers $\xi_i$ and $\chi_i$ for the manifolds $\mathcal{X}(Q_R, \lambda_{S,P})$. Let $\bar{S}$ denote the sign of $S$ if $S \neq 0$ and let $\bar{S} = -1$ if $S = 0.$

![Fig. 6.1: $\mathcal{X}(Q_4, \lambda_{0,0})$, and $\mathcal{X}(Q_6, \lambda_{0,1})$](image)
From Definitions 6.1 and 6.2 it follows that for all considered triples \((R, S, P)\),

\[
\xi_i(\mathcal{X}(Q_R, \lambda_{S,P})) = \begin{cases} 
(-1)^{i+1} & \text{for } 1 \leq i \leq R - |S| + 1, \\
-S & \text{for } R - |S| + 2 \leq i \leq R + 1.
\end{cases}
\]

(6.1)

If \(S = 0\),

\[
\chi_i(\mathcal{X}(Q_R, \lambda_{0,P})) = \begin{cases} 
0 & \text{for } 2 \leq i \leq R - 1 \text{ and } i = R + 1, \\
P & \text{for } i = R.
\end{cases}
\]

(6.2)

If \(S \neq 0\),

\[
\chi_i(\mathcal{X}(Q_R, \lambda_{S,1})) = \begin{cases} 
0 & \text{for } 2 \leq i \leq R - |S| + 1, \\
1 & \text{for } i = R - |S| + 2, \\
-2S & \text{for } R - |S| + 3 \leq i \leq R + 1.
\end{cases}
\]

(6.3)

Proof of Proposition 6.3. For (1), the statement about the rank is obvious. By Proposition 5.2, and equations (6.1), (6.2) and (6.3), the matrix

\[
M_{R,S,P} = \varepsilon(\mathcal{X}(Q_R, \lambda_{S,P})) \cdot I(\mathcal{X}(Q_R, \lambda_{S,P}))
\]
has the following description. If \( S \neq 0 \), then

\[
(M_{R,S,1})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - |S|, \\
\bar{S} & \text{for } i = R - |S| + 1, \\
2\bar{S} & \text{for } R - |S| + 2 \leq i \leq R,
\end{cases}
\]

and

\[
(M_{R,S,1})_{i,i+1} = (M_{R,S,1})_{i+1,i} = \begin{cases} 
(-1)^i & \text{for } 1 \leq i \leq R - |S|, \\
-\bar{S} & \text{for } R - |S| + 1 \leq i \leq R - 1.
\end{cases}
\]

All other entries are 0. If \( S = 0 \), then

\[
(M_{R,0,P})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - 2 \text{ and } i = R, \\
P & \text{for } i = R - 1,
\end{cases}
\]

and

\[
(M_{R,0,P})_{i,i+1} = (M_{R,0,P})_{i+1,i} = (-1)^i
\]

for \( 1 \leq i \leq R - 1 \). All other entries are 0.

For all considered triples \((R, S, P)\), define the upper-triangular matrix \( N_{R,S,P} \in GL(R, \mathbb{Z}) \) in the following way. All entries on the main diagonal are 1. All entries above the main diagonal are 1 except for the \( i \)-th column and \( i \)-th row for all even \( 2 \leq i \leq R - |S| \) such that \( Y_{i-1}(\mathcal{X}(Q_R, \lambda_{S,P})) - X_i(\mathcal{X}(Q_R, \lambda_{S,P})) \neq 1 \), whose entries above the main diagonal are 0. Define

\[
A_{R,S,P} = (N_{R,S,P})^T \cdot M_{R,S,P},
\]

and

\[
B_{R,S,P} = A_{R,S,P} \cdot N_{R,S,P}.
\]
Then if $S \neq 0$,

$$(A_{R,S,1})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - |S|, \\
S & \text{for } R - |S| + 1 \leq i \leq R,
\end{cases}$$

and

$$(A_{R,S,1})_{i,i+1} = \begin{cases} 
(-1)^i & \text{for } 1 \leq i \leq R - |S|, \\
-\bar{S} & \text{for } R - |S| + 1 \leq i \leq R - 1,
\end{cases}$$

and

$$(A_{R,S,1})_{i+1,i} = \begin{cases} 
-1 & \text{for odd } 1 \leq i \leq R - |S|, \\
0 & \text{for even } 2 \leq i \leq R - |S| \text{ and all } R - |S| + 1 \leq i \leq R - 1.
\end{cases}$$

All other entries are 0. If $S = 0$, then

$$(A_{R,0,P})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - 2, \\
P & \text{for } i = R - 1, \\
-P & \text{for } i = R,
\end{cases}$$

$$(A_{R,0,P})_{i,i+1} = (-1)^i$$

for $1 \leq i \leq R - 1$, and

$$(A_{R,0,P})_{i+1,i} = \begin{cases} 
-1 & \text{for odd } 1 \leq i \leq R - 3, \\
0 & \text{for even } 2 \leq i \leq R - 2, \\
P - 1 & \text{for } i = R - 1.
\end{cases}$$

All other entries are 0.
Furthermore, if $S \neq 0$, then

$$(B_{R,S,1})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - |S|, \\
S & \text{for } R - |S| + 1 \leq i \leq R,
\end{cases}$$

and

$$(B_{R,S,1})_{i,i+1} = (B_{R,S,1})_{i+1,i} = \begin{cases} 
-1 & \text{for odd } 1 \leq i \leq R - |S|, \\
0 & \text{for even } 2 \leq i \leq R - |S| \text{ and all } R - |S| + 1 \leq i \leq R - 1.
\end{cases}$$

All other entries are 0. If $S = 0$, then

$$(B_{R,0,P})_{i,i} = \begin{cases} 
0 & \text{for } 1 \leq i \leq R - 2, \\
P & \text{for } i = R - 1, \\
-P & \text{for } i = R,
\end{cases}$$

and

$$(B_{R,0,P})_{i,i+1} = (B_{R,0,P})_{i+1,i} = \begin{cases} 
-1 & \text{for odd } 1 \leq i \leq R - 3, \\
0 & \text{for even } 2 \leq i \leq R - 2, \\
P - 1 & \text{for } i = R - 1.
\end{cases}$$

All other entries are 0.

Hence if $S \neq 0$, then $B_{R,S,1}$ is a direct sum of $\frac{R-|S|}{2}$ matrices $-\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and the matrix $\bar{S} \cdot I_{|S|}$, where $I_{|S|}$ denotes the $|S| \times |S|$ identity matrix. Also, $B_{R,0,P}$ is a direct sum of $\frac{R}{2} - P$ matrices $-\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P$ matrices $\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Note that $I(\mathcal{X}(QR, \lambda_{S,P}))$ is of the same isomorphism type as $B_{R,S,P}$. From the theory of nondegenerate quadratic forms it follows that for all considered triples $(R, S, P)$, $I(\mathcal{X}(QR, \lambda_{S,P}))$ is of rank $R$, parity $P$, and the
absolute value of the signature of $I(\mathcal{X}(Q_R, \lambda_{S,P}))$ is equal to $|S|$.

For (2), we use additivity of the signature of oriented topological manifolds (by definition, the signature of the matrix of the intersection form of such manifold) with respect to the oriented connected sum operation. It is enough to consider the case $S > 0$. By (1), the absolute value of the signature of both $\mathcal{X}(Q_R, \lambda_{S,1})$ and $\mathcal{X}(Q_R, \lambda_{-S,1})$ is equal to $S$, for any considered $R$. Fix $R$ and $S > 0$. Define

$$\mathcal{Y} = \mathcal{X}(Q_R, \lambda_{S,1}) + \mathcal{X}(Q_2, \lambda_{0,0}) + \mathcal{X}(Q_R, \lambda_{-S,1})$$

where the first part of the sum is performed at the vertices $E_1 \cap E_2$ of $Q_R$ and $E_1 \cap E_4$ of $Q_2$, and the second part of the sum is performed at the vertices $E_2 \cap E_3$ of $Q_2$ and $E_1 \cap E_2$ of $Q_R$. Then $\mathcal{Y}$ is defined over an oriented $(2R + 4)$-gon $Q = Q_R + Q_2 + Q_R$. Denote the edges of $Q$ by $F_1, F_2, \ldots, F_{2R+4}$, ordered according to the orientation of $Q$, where $F_1$ is the sum of the faces $E_3$ of $Q_2$ and $E_1$ of the third summand $Q_R$ (see Figure 6.3). Then

$$\xi_i(\mathcal{Y}) = \begin{cases} 
\xi_i(\mathcal{X}(Q_R, \lambda_{S,1})) & \text{for } 2 \leq i \leq R + 1, \\
-1 & \text{for } i = R + 2, \\
\xi_{i-R-2}(\mathcal{X}(Q_R, \lambda_{-S,1})) & \text{for } R + 3 \leq i \leq 2R + 3,
\end{cases}$$

and

$$\chi_i(\mathcal{Y}) = \begin{cases} 
\chi_i(\mathcal{X}(Q_R, \lambda_{S,1})) & \text{for } 1 \leq i \leq R + 1, \\
-1 & \text{for } i = R + 2, \\
S & \text{for } i = R + 3, \\
\chi_{i-R-2}(\mathcal{X}(Q_R, \lambda_{-S,1})) & \text{for } R + 4 \leq i \leq 2R + 3.
\end{cases}$$
Fig. 6.3: Notation for the faces of $Q = Q_R + Q_2 + Q_R$. 

$F_2 = E_2 + E_4$ 

$F_1 = E_3 + E_1$ 

$F_{R+3} = E_1 + E_1$ 

$F_{R+4} = E_2 + E_2$ 

$Q_R + Q_2 + Q_R = Q_R + Q_2 + Q_R$
Let $I(Y)$ denote the matrix of the intersection form of $Y$ in the basis $(D_2, D_3, \ldots, D_{2R+3})$ for $H_4(Y)$. By Proposition 5.2, the $(2R + 2) \times (2R + 2)$ matrix

$$M = \varepsilon(Y) \cdot I(Y)$$

has the following description. The upper left $R \times R$ block of $M$ is the matrix $M_{R,S,1}$. The bottom right $R \times R$ block of $M$ is the matrix $M_{R,-S,1}$. The middle $2 \times 2$ block of $M$ is the matrix $\begin{bmatrix} 1 & -1 \\ -1 & S \end{bmatrix}$. The only other nonzero entries are $M_{R,R+1} = M_{R+1,R} = -1$, and $M_{R+2,R+3} = M_{R+3,R+2} = 1$. Define the upper-triangular matrix $N \in GL(2R + 2, \mathbb{Q})$ in the following way. All entries on the main diagonal are 1. All entries above the main diagonal are 1 except for the $i$-th column and $i$-th row for all even $2 \leq i \leq R - |S|$ and all even (resp. odd) $R + 4 \leq i \leq 2R - |S| + 2$ if $S$ is even (resp. odd), whose entries above the main diagonal are 0. Replace all nonzero entries above the main diagonal in the $(R + 1)$-st column by $\frac{S}{2}$, and replace all entries above the main diagonal in the $(R + 1)$-st row by 0. Then the upper left $R \times R$ block of $N$ is equal to the matrix $N_{R,S,1}$ (defined in the proof of (1)), and the bottom right $R \times R$ block of $N$ is equal to the matrix $N_{R,-S,1} = N_{R,S,1}$. Define

$$A = N^T \cdot M.$$

Then $A$ has the following description. The upper left $R \times R$ block of $A$ is the matrix $A_{R,S,1}$. The bottom right $R \times R$ block of $A$ is the matrix $A_{R,-S,1}$. The middle $2 \times 2$ block of $A$ is the matrix $\begin{bmatrix} 0 & -1 \\ -1 & \frac{S}{2} \end{bmatrix}$. The only other nonzero entries are $A_{R,R+1} = -1$, and $A_{R+2,R+3} = 1$. Define

$$B = A \cdot N.$$

Then $B$ has the following description. The upper left $R \times R$ block of $B$ is the matrix $B_{R,S,1}$. The bottom right $R \times R$ block of $B$ is the matrix $B_{R,-S,1}$. The middle $2 \times 2$ block of $B$ is the matrix $-\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. All other entries are 0. From the theory of nondegenerate quadratic forms it follows that the signature of $I(Y)$ is equal to 0.
Recall that topologically the sum

$$\mathcal{X}(Q_R, \lambda_{S,1}) + \mathcal{X}(Q_2, \lambda_{0,0}) + \mathcal{X}(Q_R, \lambda_{-S,1})$$

is a manifold homeomorphic by an orientation-preserving homeomorphism to the oriented connected sum

$$\mathcal{X}(Q_R, \lambda_{S,1}) \# \mathcal{X}(Q_2, \lambda_{0,0}) \# \mathcal{X}(Q_R, \lambda_{-S,1}).$$

By additivity of the signature of oriented topological manifolds with respect to the oriented connected sum operation, the signature of $\mathcal{Y}$ is equal to the sum of signatures of $\mathcal{X}(Q_R, \lambda_{S,1})$, $\mathcal{X}(Q_2, \lambda_{0,0})$ and $\mathcal{X}(Q_R, \lambda_{-S,1})$. By (1), the signature of $\mathcal{X}(Q_2, \lambda_{0,0})$ is equal to 0, and the absolute value of the signatures of both $\mathcal{X}(Q_R, \lambda_{S,1})$ and $\mathcal{X}(Q_R, \lambda_{-S,1})$ is equal to $S \neq 0$. Hence the signature of $(\mathcal{X}(Q_R, \lambda_{S,1}))$ is the opposite of the signature of $(\mathcal{X}(Q_R, \lambda_{-S,1}))$.

For (3), note that if $S \neq 0$ then

$$\mathcal{X}(Q_{R_1}, \lambda_{S,P}) + \mathcal{X}(Q_2, \lambda_{0,0}) + \mathcal{X}(Q_{R_2}, \lambda_{S,P})$$

$$= \mathcal{X}(Q_{|S|}, \lambda_{S,P}) + \mathcal{X}(Q_{R_1+R_2-2|S|+2}, \lambda_{0,0}) + \mathcal{X}(Q_{|S|}, \lambda_{S,P}).$$

By (1), the signatures of both $I(\mathcal{X}(Q_2, \lambda_{0,0}))$ and $I(\mathcal{X}(Q_{R_1+R_2-2|S|+2}, \lambda_{0,0}))$ are equal to 0. Hence the signatures of $I(\mathcal{X}(Q_{R_1}, \lambda_{S,P}))$ and $I(\mathcal{X}(Q_{R_2}, \lambda_{S,P}))$ are opposites of each other if and only if $S = 0$.

The statements of Proposition 6.3 allow us to formulate the following

**Definition 6.4.** Define $\mathcal{X}_{R,S,P}$ to be either the oriented toric manifold $\mathcal{X}(Q_R, \lambda_{S,P})$ or $\mathcal{X}(Q_R, \lambda_{-S,P})$, depending on which manifold’s signature of the matrix of the intersection form is equal to $S$.

The first family consists of toric manifolds $\mathcal{X}_{R,S,P}$ for all considered triples $(R, S, P)$.
6.2 The action of the Cartesian product of the free group $F_2$

To define the remaining families, for each integer $r \geq 3$, define an action of the Cartesian product $F^r_2$ on the set of quaternionic toric 8-manifolds associated with the oriented $r$-gon $Q$ in the following way. Let $\mathcal{X} = \mathcal{X}(Q, \lambda)$ be a toric 8-manifold. Denote the edges of $Q$ by $E_1, E_2, \ldots, E_r$ according to the orientation of $Q$. Assume that $\lambda(E_k) = e(w_k)$ ($1 \leq k \leq r$) for a prebasic element $w_k \in F_2$. Let $v = (v_1, v_2, \ldots, v_r) \in F^r_2$. Define $v\lambda : \mathcal{P}(Q) \rightarrow P(2)$ by

$$v\lambda(E_k) = e(v_kw_kv_k^{-1}),$$

for all $1 \leq k \leq r$, and $v\lambda(F) = \lambda(F)$ for all remaining faces of $Q$. Then $v\lambda$ is a basic characteristic function of rank 2 and height 0. Define $v\mathcal{X} = \mathcal{X}(Q, v\lambda)$. Then $v\mathcal{X}$ is a quaternionic toric 8-manifold. The following lemma provides a relation between the numbers $S_k(\mathcal{X})$ and $S_k(v\mathcal{X})$.

**Lemma 6.5.** Under the assumptions above,

$$S_k(v\mathcal{X}) = S_k(\mathcal{X}) + 2\det [(v_k)_{ab}, (w_k)_{ab}],$$

for every $1 \leq k \leq r$.

**Proof.** Fix $k$. We use the assumptions of Theorem 5.6 for $\mathcal{X}$. Assume that the reduced form of $v_k$ is

$$\left(\prod_{i=1}^{s-1} (x_2^{a_i}x_1^{b_i})x_2^{a_s}\right),$$

for some $s \in \mathbb{N}$, where $a_i \neq 0$ for $2 \leq i \leq s - 1$, and $b_i \neq 0$ for $1 \leq i \leq s - 1$. As in the discussion following Corollary 4.10, let $\tau \in \text{Aut } F_2$ be defined by equation (4.3). Then

$$(\tau \circ \tau_k)(x_1) = v_kw_kv_k^{-1}.$$
Hence we obtain the simple decomposition

$$
\gamma \circ \gamma \circ \gamma \circ \left( \prod_{j=1}^{2s-1} \beta_j \right) \circ \left( \prod_{j=1}^{l_k} \alpha_{k,j} \right)
$$

of the automorphism $\tau_k$ of $F_2$ such that $v\lambda(E_k) = e(\tau_k(x_1))$. This simple decomposition satisfies the assumptions of Theorem 5.6 for $v\mathcal{X}$. Note that $X_k(v\mathcal{X}) = X_k(\mathcal{X})$ and $Y_k(v\mathcal{X}) = Y_k(\mathcal{X})$. Also, for every $1 \leq m \leq l_k$, $X_{k,m+2s+2}(v\mathcal{X}) = X_{k,m}(\mathcal{X})$, $Y_{k,m+2s+2}(v\mathcal{X}) = Y_{k,m}(\mathcal{X})$, and $q_{k,m+2s+2}(v\mathcal{X}) = q_{k,m}(\mathcal{X})$. Furthermore, for every $1 \leq i \leq s + 1$, $X_{k,2i-1}(v\mathcal{X}) = 0$, $Y_{k,2i-1}(v\mathcal{X}) = 1$, $X_{k,2i}(v\mathcal{X}) = 1$, and $Y_{k,2i}(v\mathcal{X}) = 0$. We also have $q_{k,i}(v\mathcal{X}) = 0$ for $i = 1, 2, 3$, $q_{k,2i+2}(v\mathcal{X}) = 2a_i$ for $1 \leq i \leq s$, and $q_{k,2i+3}(v\mathcal{X}) = 2b_i$ for $1 \leq i \leq s - 1$. Hence

$$
S_k(v\mathcal{X}) = \sum_{i=2}^{l_k+2s+1} (Y_k(v\mathcal{X})X_{k,i}(v\mathcal{X}) - X_k(v\mathcal{X})Y_{k,i}(v\mathcal{X})) q_{k,i+1}(v\mathcal{X})
$$

$$
= \sum_{i=1}^{s-1} (Y_k(\mathcal{X}) \cdot 2b_i) - \sum_{i=1}^{s} (X_k(\mathcal{X}) \cdot 2a_i)
$$

$$
+ \sum_{2 \leq m \leq l_k - 1} \left( Y_k(\mathcal{X})X_{k,m}(\mathcal{X}) - X_k(\mathcal{X})Y_{k,m}(\mathcal{X}) \right) q_{k,m+1}(\mathcal{X})
$$

$$
= 2 \left( \sum_{i=1}^{s-1} b_i Y_k(\mathcal{X}) - \sum_{i=1}^{s} a_i X_k(\mathcal{X}) \right) + S_k(\mathcal{X})
$$

$$
= S_k(\mathcal{X}) + 2 \det \left[ (v_k)_{ab}, (w_k)_{ab} \right].
$$

Note that $\xi_i(v\mathcal{X}) = \xi_i(\mathcal{X})$ for all $1 \leq k \leq r - 1$, and $\chi_i(v\mathcal{X}) = \chi_i(\mathcal{X})$ for all $2 \leq k \leq r - 1$. Hence by Theorem 5.6, the relation between the invariants $q(\mathcal{X})$ and $q(v\mathcal{X})$ depends on the relation between the numbers $S_k(\mathcal{X})$ and $S_k(v\mathcal{X})$ ($1 \leq k \leq r$) described in the lemma above.

The method used to prove part (3) of Proposition 6.3 can also be used to prove the following generalization.

**Proposition 6.6.** Under the assumptions of Proposition 6.3, if $R_i \equiv S \pmod{2}$ for $i = 1, 2$, and $v_1, v_2$ are elements of $F_2^{R_i+2}$ with the first two components equal to 1, then the signatures of $I(v_i \mathcal{X}(Q_{R_i}, \lambda_{S,i}))$ are the same for $i = 1, 2$. 
6.3 The second family

Each element of the first family (see section 6.1) will now be used as a “building block” for another family of quaternionic toric 8-manifolds. We will use notation introduced in section 6.1.

Definition 6.7. For each manifold \( X_{R,S,P} \), let \( V_{R,S,P} \) be a subgroup of \( F_{2}^{R+2} \) of elements of the form

\[
(1, 1, x_{i_3}^{a_3}, x_{i_4}^{a_4}, \ldots, x_{i_{R+1}}^{a_{R+1}}, x_{1}^{a_{R+2}})
\]

where \( a_j \in \mathbb{Z} \) for all \( 3 \leq j \leq R+2 \), and for all \( 3 \leq k \leq R+1 \), either \( i_k = 1 \) if \( Y_k(X_{R,S,P}) = 1 \) or \( i_k = 2 \) if \( Y_k(X_{R,S,P}) = 0 \).

The second family consists of toric manifolds \( vX_{R,S,P} \) where for each considered triple \((R, S, P)\), \( v \in V_{R,S,P} \).

Theorem 6.8. Every quaternionic toric 8-manifold is homeomorphic, by an orientation preserving homeomorphism, to at least one of the toric manifolds \( vX_{R,S,P} \) (\( v \in V_{R,S,P} \)).

Proof. Consider an arbitrary quaternionic toric 8-manifold \( X \). If the matrix of the intersection form of \( X \) has rank \( R \), signature \( S' \) and parity \( P \), then by the theory of nondegenerate quadratic forms, Proposition 6.3, and Definition 6.4, \( H_4(X) \) has a basis \( (\beta_1, \beta_2, \ldots, \beta_R) \) with respect to which the matrix of the intersection form of \( X \) is equal to the matrix of the intersection form of \( X_{R,S',P} \) in the basis \( (D_2, D_3, \ldots, D_{R+1}) \) for \( H_4(X_{R,S,P}) \). Then the isomorphism between the fourth homology groups of \( X \) and \( X_{R,S',P} \) defined by \( \beta_i \mapsto D_{i+1} (1 \leq i \leq R) \) preserves the intersection form. The matrices of intersection forms of all manifolds \( vX_{R,S',P} \) (\( v \in V_{R,S',P} \)) in the bases \( (D_2(vV_{R,S',P}), D_3(vV_{R,S',P}), \ldots, D_{R+1}(vV_{R,S',P})) \) are equal to each other by Proposition 6.6. Hence it is enough to show that all possible matrices of the invariant \( q \) (obtained by evaluation of \( q \) on each element of the base of \( H_4 \)) are obtained by appropriate choice of \( v \).

Define \( a_1 = a_2 = 0 \). First consider the case \( S' = 0 \). Then \( S_i(X_{R,0,P}) = 0 \) for all \( 1 \leq i \leq R \) and \( i = R+2 \), and \( S_{R+1}(X_{R,0,P}) = P \). Also, \( X_{2i}(X_{R,0,P}) = 0 \) and \( Y_{2i}(X_{R,0,P}) = 1 \).
for $1 \leq i \leq \frac{R}{2} + 1$, $X_{2i-1}(X_{R,0,P}) = 1$ and $Y_{2i-1}(X_{R,0,P}) = 0$ for $1 \leq i \leq \frac{R}{2}$, $X_{R+1}(X_{R,0,P}) = 1$ and $Y_{R+1}(X_{R,0,P}) = P$. Hence by Lemma 6.5 and Definition 6.7,

$$S_i(vX_{R,0,P}) = \begin{cases} 2(-1)^ia_i & \text{if } 1 \leq i \leq R \text{ or } i = R + 2, \\ P + 2(-1)^{P-1}a_{R+1} & \text{if } i = R + 1. \end{cases}$$

Therefore by equations (6.1) and (6.2), and Theorem 5.6, we obtain

$$\langle q(vX_{R,0,P}), D_i \rangle = \begin{cases} 2(a_{i+1} - a_{i-1}) & \text{if } 2 \leq i \leq R - 1 \\ 2((-1)^P a_{R+1} + Pa_R - a_{R-1}) - P & \text{if } i = R. \end{cases}$$

(6.4)

Now consider the case $S' \neq 0$. Then $X_{R,S',P} = X(Q_{R+2}, \lambda_{S,P})$ where $S$ is equal to either $S'$ or $-S'$ (see Definition 6.4). Then $S_i(X_{R,S',1}) = 0$ for all $1 \leq i \leq R - |S| + 2$, and for all $1 \leq j \leq |S|$, $S_{R-|S|+2+j}(X_{R,S',1}) = \bar{S}j$ (where $\bar{S}$ denotes the sign of $S$). Also, $X_{2i}(X_{R,S',1}) = 0$ and $Y_{2i}(X_{R,S',1}) = 1$ for $1 \leq i \leq \frac{R-|S|}{2} + 1$, $X_{2i-1}(X_{R,S',1}) = 1$ and $Y_{2i-1}(X_{R,S',1}) = 0$ for $1 \leq i \leq \frac{R-|S|}{2} + 1$, and for all $1 \leq j \leq |S|$, $X_{R-|S|+2+j}(X_{R,S',1}) = \bar{S} \cdot j$ and $Y_{R-|S|+2+j}(X_{R,S',1}) = 1$. Hence by Lemma 6.5 and Definition 6.7,

$$S_i(vX_{R,S',1}) = \begin{cases} 2(-1)^ia_i & \text{if } 1 \leq i \leq R - |S| + 2, \\ \bar{S}(i - R + |S| - 2) + 2a_i & \text{if } R - |S| + 3 \leq i \leq R + 2. \end{cases}$$

Therefore by equations (6.1) and (6.3), and Theorem 5.6, we obtain

$$\langle q(vX_{R,S',1}), D_i \rangle = 2(a_{i+1} - a_{i-1})$$

if $2 \leq i \leq R - |S| + 1$. If $i = R - |S| + 2$, then

$$\langle q(vX_{R,S',1}), D_i \rangle = (-1)^i \cdot 2 \cdot (-1)^{i-1}a_{i-1} - (-1)^i \cdot (-S) \cdot 1 \cdot 2(-1)^ia_i + (-S) \cdot (\bar{S} + 2a_{i+1})$$

$$= -2a_{i-1} + 2\bar{S}a_i - 2a_{i+1}$$

$$= 2(-a_{i-1} + \bar{S}a_i - a_{i+1}) - 1.$$
If \( i = R - |S| + 3 \), then

\[
\langle g(v\mathcal{X}_{R,S'}), D_i \rangle = -\bar{S} \cdot 2 \cdot (-1)^{i-1}a_{i-1} - (-\bar{S}) \cdot (-2\bar{S}) \cdot (\bar{S} + 2a_i) \\
+ (-\bar{S}) \cdot (2\bar{S} + 2a_{i+1}) \\
= -2\bar{S}a_{i-1} + 2 + 4\bar{S}a_i - 2 - 2\bar{S}a_{i+1} \\
= 2\bar{S}(-a_{i-1} + 2a_i - a_{i+1}) .
\]

Also, for all \( R - |S| + 4 \leq i \leq R + 1 \),

\[
\langle g(v\mathcal{X}_{R,S'}), D_i \rangle = -\bar{S} \cdot (\bar{S}(i - R + |S| - 3) + 2a_{i-1}) \\
- (-\bar{S}) \cdot (-\bar{S}) \cdot (\bar{S}(i - R + |S| - 2) + 2a_i) \\
+ (-\bar{S}) \cdot (\bar{S}(i - R + |S| - 1) + 2a_{i+1}) \\
= -i + R - |S| + 3 - 2\bar{S}a_{i-1} + 2i - 2R + 2|S| - 4 + 4\bar{S}a_i \\
- i + R - |S| + 1 - 2\bar{S}a_{i+1} \\
= 2\bar{S}(-a_{i-1} + 2a_i - a_{i+1}) .
\]

Summarizing,

\[
(6.5) \quad \langle g(v\mathcal{X}_{R,S'}), D_i \rangle = \begin{cases} 
2(a_{i+1} - a_{i-1}) & \text{if } 2 \leq i \leq R - |S| + 1, \\
2(-a_{i-1} + \bar{S}a_i - a_{i+1}) - 1 & \text{if } i = R - |S| + 2, \\
2\bar{S}(-a_{i-1} + 2a_i - a_{i+1}) & \text{if } R - |S| + 3 \leq i \leq R + 1.
\end{cases}
\]

Define \( f : \mathbb{Z}^R \to \mathbb{Z}^R \) in the bases \((k_2, k_3, \ldots, k_{R+1})\) for the domain and \((a_3, a_4, \ldots, a_{R+2})\) for the range, by

\[
f(k_i) = \begin{cases} 
a_{i+1} - a_{i-1} & \text{if } 2 \leq i \leq R - 1 \text{ or } i = R + 1, \\
(-1)^P a_{R+1} + Pa_R - a_{R-1} & \text{if } i = R
\end{cases}
\]
if $S' = 0$, and

$$f(k_i) = \begin{cases} 
    a_{i+1} - a_{i-1} & \text{if } 2 \leq i \leq R - |S| + 1, \\
    -a_{i-1} + \bar{S}a_i - a_{i+1} & \text{if } i = R - |S| + 2, \\
    \bar{S}(-a_{i-1} + 2a_i - a_{i+1}) & \text{if } R - |S| + 3 \leq i \leq R + 1
\end{cases}$$

if $S' \neq 0$. Then in all considered cases, $f$ is a homomorphism. Also, the matrix of $f$ in considered bases is upper triangular with all diagonal entries equal to $\pm 1$. Hence $f$ is an isomorphism. Therefore by equations (6.4) and (6.5), every possible matrix of the invariant $q$ can be obtained by appropriate choice of $v$. By the theory of Wall [9], $\mathcal{X}$ is homeomorphic, by an orientation preserving homeomorphism, to $v\mathcal{X}_{R,S',P}$. 

As mentioned in the outline of the proof of the main result (Theorem 3.14), for each topological type of quaternionic toric $8$-manifold, we will define an infinite family of manifolds that are of distinct algebraic types. The second family defined above provides representatives of all topological types of quaternionic toric $8$-manifold.

### 6.4 The third family and the proof of the main result

Actually, in this section we will define a family of infinite families. For each element of the second family, we define infinitely many quaternionic toric $8$-manifolds in the following way. For every $j \in \mathbb{N}$, define an element of $F^{R+2}_2$ by

$$z_j = (1, 1, \ldots, 1, (x_1x_2x_1^{-1}x_2^{-1})^j).$$

For each quadruple $(R, S, P, v)$, the third family consists of toric manifolds $z_jv\mathcal{X}_{R,S,P}$ ($j \in \mathbb{N}$) where for each considered triple $(R, S, P)$, $v \in V_{R,S,P}$.

**Proposition 6.9.** For every quadruple $(R, S, P, v)$, with $v \in V_{R,S,P}$, the toric manifolds $z_jv\mathcal{X}_{R,S,P}$ ($j \in \mathbb{N}$) are all topologically equivalent by an orientation preserving homeomorphism.
Proof. For a fixed triple \((R, S, P)\), \(X_{R,S,P} = X(Q_{R+2}, \lambda_{S',P})\) where \(S'\) is equal to either \(S\) or 
\(-S\) by definition 6.4. Then for every \(v \in V_{R,S,P}\), and every \(j \in \mathbb{N}\), the numbers \(\xi_i(z_jvX_{R,S,P})\)
\((1 \leq i \leq R + 1)\), \(\chi_i(z_jvX_{R,S,P})\) \((2 \leq i \leq R + 1)\), and \(S_i(z_jvX_{R,S,P})\) \((1 \leq i \leq R + 2)\) are the
same by Proposition 6.6 (for the numbers \(S_i\) see Lemma 6.5). These numbers determine
the matrices of the intersection form, and of the invariant \(q\) in basis \((D_2, D_3, \ldots, D_{R+1})\) of
the fourth homology group of each manifold (see Proposition 5.2 and Theorem 5.6). Hence
the matrices of the corresponding topological invariants of toric manifolds \(z_jvX_{R,S,P}\) are
the same for all \(j \in \mathbb{N}\). Thus by the theory of Wall [9], the existence of the isomorphism
of the fourth homology groups between any two such manifolds, defined by \(D_i \mapsto D_i\) for
all \(2 \leq i \leq R + 1\), that preserves the invariants, implies the existence of an orientation
preserving homeomorphism between the manifolds. \(\square\)

There are only two elements of \(P(1, 1)\), \(t \mapsto t\) and \(t \mapsto t^{-1}\). The following proposition
is therefore a consequence of Definition 2.10.

**Proposition 6.10.** Let \(j = [f, g] : X(Q, \lambda) \to X(R, \mu)\) be an algebraic isomorphism
between quaternionic toric 8-manifolds. Then for every edge \(F\) of \(Q\),

\[
\mu(f(F))(g(t)) = (\lambda(F)(t))^\varepsilon,
\]

where \(\varepsilon\) is equal to either 1 or \(-1\).

The result of the proposition above will be the key tool in the proof of the following

**Theorem 6.11.** For every quadruple \((R, S, P, v)\), with \(v \in V_{R,S,P}\), the toric manifolds
\(z_jvX_{R,S,P}\) \((j \in \mathbb{N})\) are all algebraically distinct.

Proof. For the purpose of the proof, for any word \(w \in F_2\), let \(L(w)\) denote the number of
appearances of nonzero powers of the generators \(x_1, x_2\) in the reduced form of \(w\). We call
\(L(w)\) the *length* of \(w\). By definition, let \(L(e(w)(t)) = L(w)\). For a fixed triple \((R, S', P)\),
\(X_{R,S',P} = X(Q_{R+2}, \lambda_{S,P})\) where \(S\) is equal to either \(S'\) or \(-S'\) by definition 6.4. Then
for every \(v \in V_{R,S',P}\), and every \(j \in \mathbb{N}\), \(z_jvX_{R,S',P} = X(Q_{R+2}, z_jv\lambda_{S,P})\) by Proposition


6.6. In addition to the triple \((R, S', P)\), fix \(v \in V_{R, S', P}\). Let \(0 \leq i < j\) be integers. Let \(\mathcal{X}_i = z_i v \mathcal{X}_{R, S', P}\) and \(\mathcal{X}_j = z_j v \mathcal{X}_{R, S', P}\). Denote the characteristic functions defining \(\mathcal{X}_i\) and \(\mathcal{X}_j\) by \(\lambda_i\) and \(\lambda_j\) respectively. Assume to the contrary that there exists an algebraic isomorphism \([f, g]: \mathcal{X}_i \to \mathcal{X}_j\). Since \(\lambda_j(E_1)(t) = t_1\) and \(\lambda_j(E_2)(t) = t_2\), by Proposition 6.10,

\[
g(t) = \left(\frac{\lambda_i(f^{-1}(E_1))}{\lambda_i(f^{-1}(E_2))}(t)\right)^{\pm 1},
\]

where the relation between the \(\pm\) signs on both coordinates is determined by the condition that \([f, g]\) is orientation-preserving. For the remainder of the proof, define \(a_1 = a_2 = 0\). Also, let \(\bar{S}\) denote the sign of \(S\) if \(S \neq 0\) and let \(\bar{S} = -1\) if \(S = 0\). Define

\[
\begin{align*}
v_1 &= z_i v \lambda_{S, P}(E_{R+2})(t) = \left(\begin{array}{c} t_1 t_2 t_1^{-1} t_2^{-1} \end{array}\right)^i t_1^{a_{R+2}}(t_1 t_2) t_1^{-a_{R+2}}(t_2 t_1 t_2^{-1} t_1^{-1})^i, \\
v_2 &= z_i v \lambda_{0, P}(E_{R+2})(t) = \left(\begin{array}{c} t_1 t_2 t_1^{-1} t_2^{-1} \end{array}\right)^i t_1^{a_{R+2}} t_2 t_1^{-a_{R+2}}(t_2 t_1 t_2^{-1} t_1^{-1})^i, \\
v_3 &= z_j v \lambda_{S, P}(E_{R+2})(t) = \left(\begin{array}{c} t_1 t_2 t_1^{-1} t_2^{-1} \end{array}\right)^j t_1^{a_{R+2}}(t_1 t_2) t_1^{-a_{R+2}}(t_2 t_1 t_2^{-1} t_1^{-1})^j, \\
v_4 &= z_j v \lambda_{0, P}(E_{R+2})(t) = \left(\begin{array}{c} t_1 t_2 t_1^{-1} t_2^{-1} \end{array}\right)^j t_1^{a_{R+2}} t_2 t_1^{-a_{R+2}}(t_2 t_1 t_2^{-1} t_1^{-1})^j.
\end{align*}
\]

Throughout the proof there will be multiple references to the words defined above. To simplify notation, if \(\lambda(F) = e(w)\), we define \((\lambda(F)(t))_{ab} = w_{ab}\).

There are two types of an affine homeomorphism \(f: Q \to Q\). It can be either orientation-preserving or orientation-reversing. We will discuss the two cases separately.

If \(f\) is orientation-preserving, then \(f\) is defined by \(E_{k+l} \mapsto E_k\) for some integer \(0 \leq l \leq R + 1\), and all \(1 \leq k \leq R + 2\). Let \(\varepsilon = \pm 1\). Since \(f\) is orientation-preserving, so should be \(g\) (for \([f, g]\) to be orientation-preserving). Since \(f(E_{l+1}) = E_1\) and \(f(E_{l+2}) = E_2\), by equation
(6.6), the value of $g(t)$ is dependent on $l$ and is equal to either

(6.8a)  $$(t_2^{a_{l+1}}t_1^{a_{l+1}}, t_1^{a_{l+2}}t_2^{a_{l+2}})$$

if $0 \leq l \leq R - |S| - 2$ and $l$ even, or $l = R - |S|$ and $S \neq 0$, or

(6.8b)  $$(t_2^l, (t_1t_2^{-1}t_2^{-1})^i t_1^{a_3}t_1^S - t_1^{-a_3}(t_2t_1^{-1}t_2^{-1})^i)$$

if $l = R = |S| = 1$, or

(6.8c)  $$(t_1^{a_{l+1}}t_2^{a_{l+1}}, t_1^{a_{l+2}}t_2^{a_{l+2}})$$

if $1 \leq l \leq R - |S| - 3$ and $l$ odd, or $l = R - |S| - 1$ and $P \neq |S| + 1$, or

(6.8d)  $$(t_1^{a_R}t_2^{a_R}, t_1^{a_R+1}(t_1t_2)^{-a_{R+1}})$$

if $l = R - 1$ and $P = |S| + 1$, or

(6.8e)  $$(t_1^{a_{R+2}}t_2^{a_{R+2}}, t_1^{a_{R+3}}(t_1t_2)^{-a_{R+2}}t_1^{-a_{R+3}})$$

if $l = R - |S| + 1$ and $|S| \geq 2$, or

(6.8f)  $$(t_1^{a_{l+1}}(t_1^{S(l-R+|S|-1)}t_2^{-1}t_2^{-1})^i t_1^{a_{l+1}}t_1^{a_{l+2}}(t_1^{S(l-R+|S|)}t_2^{-1})^i t_1^{-a_{l+2}})$$

if $R - |S| + 2 \leq l \leq R - 1$, or

(6.8g)  $$(t_1^{a_{R+1}}t_1^{S-S^{l-R+|S|-1}}t_2^l t_1^{a_{R+1}}, t_1^{a_{R+2}}t_1^{S(t_1t_2^{-1}t_2^{-1})^i t_1^{a_{R+2}}t_1^{-a_{R+2}}(t_1t_2^{-1}t_2^{-1})^i})$$

if $l = R \geq 2$ and $P = 1$, or

(6.8h)  $$(t_2^{a_{l+1}}t_1^{-a_{l+1}}t_2^{-1}t_1^{-1})^i t_1^{a_{l+2}}t_2^{a_{l+2}}(t_1t_2^{-1}t_2^{-1})^i$$
if \( l = R \) and \( S = P = 0 \), or

\[
(t_1^2 t_2^{-1} t_2^{-1})^i i t_1^{aR+2} (t_1^S t_2)^e t_1^{-aR+2} (t_2 t_1^2 t_2^{-1} t_1^{-1})^i, t_1^{-e})
\]

if \( l = R + 1 \). We will now consider each of the cases listed above to show that existence of the algebraic isomorphism \([f, g]\) contradicts either some basic properties of automorphisms of \( F_2 \), or the statement of Proposition 6.10.

If \( l = 0 \), then \( f(E_{R+2}) = E_{R+2} \). Since \( i < j \), by definition of \( z_j \),

\[
L(\lambda_i(E_{R+2})(t)) < L(\lambda_j(E_{R+2})(t)).
\]

By equation (6.8a), if \( l = 0 \), \( g(t) = (t_1^e, t_2^{e}) \). Thus

\[
L(\lambda_j(E_{R+2})(g(t))) = L(\lambda_j(E_{R+2})(t)).
\]

Hence

\[
(\lambda_i(E_{R+2})(t))^{\pm 1} \neq \lambda_j(E_{R+2})(g(t)),
\]
contradicting the statement of Proposition 6.10.

Consider the case \( l = R = |S| = 1 \). By equation (6.8b), if \( i \geq 1 \) then \( g \) is not an algebraic diffeomorphism because

\[
L((t_1 t_2^{-1} t_2^{-1})^i i t_1^{a3} (t_1^S t_2)^{-S e} t_1^{-a3} (t_2 t_1^2 t_2^{-1} t_1^{-1})) \geq 5.
\]

If \( i = 0 \), then \( g(t) = (t_2^e, t_1^{a3} (t_1^S t_2)^{-S e} t_1^{-a3}) \). So \( g \) is an algebraic diffeomorphism if and only if \( a_3 = -S \) or \( a_3 = 0 \). We have

\[
g(t) = \begin{cases} 
(t_2^e, (t_2^S t_1)^{-S e}) & \text{if } a_3 = -S, \\
(t_2^e, (t_1^S t_2)^{-S e}) & \text{if } a_3 = 0.
\end{cases}
\]
Also,
\[
\lambda_j(E_3)(t) = \begin{cases} 
(t_1 t_2 t_1^{-1} t_2^{-1})^j t_2 t_1^S(t_2 t_1^{-1} t_2^{-1})^j & \text{if } a_3 = -S, \\
(t_1 t_2 t_1^{-1} t_2^{-1})^j t_1^S t_2 (t_2 t_1^{-1} t_2^{-1})^j & \text{if } a_3 = 0.
\end{cases}
\]

Thus the value of \(\lambda_j(E_3)(g(t))\) is equal to either
\[
(t_2^S (t_2 t_1^S)^{-S} t_2^{-\varepsilon} (t_2 t_1^S)^{S \varepsilon})^j (t_2 t_1^S)^{-S} t_2^{-\varepsilon} ((t_2 t_1^S)^{-S} t_2^{-\varepsilon} (t_2 t_1^S)^{S} t_2^{-\varepsilon})^j
\]
if \(a_3 = -S\), or
\[
(t_2^S (t_1 t_2^S)^{-S} t_2^{-\varepsilon} (t_1 t_2^S)^{S \varepsilon})^j t_1^S t_2 (t_1 t_2^S)^{-S} t_2^{-\varepsilon} ((t_1 t_2^S)^{-S} t_2^{-\varepsilon} (t_1 t_2^S)^{S} t_2^{-\varepsilon})^j
\]
if \(a_3 = 0\). These values, dependent on the triple \((S, \varepsilon, a_3)\), are gathered in table 6.1. Recall that \(\lambda_0(E_1)(t) = t_1\). Since \(f(E_1) = E_3\), by Proposition 6.10 we get
\[
(6.9) \quad t_1^{\pm 1} = \lambda_j(E_3)(g(t)).
\]

Since \(j \geq 1\), using the information from Table 6.1,
\[
L\left(\lambda_j(E_3)(g(t))\right) \geq 5.
\]

Table 6.1: The values of \(\lambda_j(E_3)(g(t))\) dependent on the triple \((S, \varepsilon, a_3)\)

<table>
<thead>
<tr>
<th>((S, \varepsilon, a_3))</th>
<th>(\lambda_j(E_3)(g(t)))</th>
</tr>
</thead>
<tbody>
<tr>
<td>((1, 1, -1))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j t_1^{-1} (t_1^{-1} t_2 t_1^{-1})^j)</td>
</tr>
<tr>
<td>((1, 1, 0))</td>
<td>((t_1^{-1} t_2^{-1} t_1 t_2^{-1})^j t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
<tr>
<td>((-1, 1, 1))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j t_1^{-1} t_2^{-1} t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
<tr>
<td>((-1, 1, -1))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j - t_2^{-1} t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
<tr>
<td>((-1, 1, 0))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
<tr>
<td>((-1, -1, 1))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j + t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
<tr>
<td>((-1, -1, 0))</td>
<td>((t_2 t_1^{-1} t_2^{-1})^j t_1 t_2 (t_2 t_1^{-1} t_2^{-1})^j)</td>
</tr>
</tbody>
</table>
This inequality contradicts equation (6.9) and hence the statement of Proposition 6.10.

Consider the case $2 \leq l \leq R - |S| - 2$ and $l$ even, or $l = R - |S|$ and $S \neq 0$. By equation (6.8a),

$$M(g(t)) = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$ 

Also, $(\lambda_i(E_l)(t))_{ab} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, and $(\lambda_j(E_{R+2})(t))_{ab} = \begin{bmatrix} S \\ 1 \end{bmatrix}$. Thus

$$M(g(t))(\lambda_j(E_{R+2})(t))_{ab} = \varepsilon \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} S \\ 1 \end{bmatrix} = \varepsilon S.$$ 

Since $f(E_l) = E_{R+2}$, by Proposition 6.10 we get

$$\varepsilon \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon S,$$

which is a true statement if and only if $S = 0$. Let $S = 0$. Then $g$ is an algebraic diffeomorphism if and only if $a_{l+1} = 0$ or $a_{l+2} = 0$. We have

$$g(t) = \begin{cases} (t_1^{a_{l+1}}, t_2^{a_{l+2}}) & \text{if } a_{l+1} = 0, \\ (t_1^{a_{l+1}}, t_2^{a_{l+2}}) & \text{if } a_{l+2} = 0. \end{cases}$$

Also, $\lambda_i(E_l)(t) = t_1^{a_{l+1}} + t_2^{a_{l+2}}$, and $\lambda_j(E_{R+2})(t) = v_4$ (see equation (6.7d)). Thus the value of $\lambda_j(E_{R+2})(g(t))$ is equal to either

$$t_1^{a_{l+1} + 2}(t_1^{a_{l+1}} + t_2^{a_{l+2}})^2 t_1^{a_{l+1} + 2} t_2^{a_{l+2}} (t_1^{a_{l+1}} + t_2^{a_{l+2}})^2 t_1^{a_{l+1} + 2}$$

if $a_{l+1} = 0$, or

$$t_2^{a_{l+1} + 2}(t_1^{a_{l+1}} + t_2^{a_{l+2}})^2 t_1^{a_{l+1} + 2} t_2^{a_{l+2}} (t_1^{a_{l+1}} + t_2^{a_{l+2}})^2 t_1^{a_{l+1} + 2}.$$
if \( a_{l+2} = 0 \). Since \( f(E_l) = E_{R+2} \), by Proposition 6.10 we get

\[(6.10) \quad t_1^{a_l} t_2^{\pm 1} t_1^{-a_l} = \lambda_j(E_{R+2})(g(t)).\]

Since \( j \geq 1 \),

\[L\left(\lambda_j(E_{R+2})(g(t))\right) \geq 5.\]

This inequality contradicts equation (6.10) and hence the statement of Proposition 6.10.

Consider the case \( 1 \leq l \leq R - |S| - 3 \) and \( l \) odd, or \( l = R - |S| - 1 \) and \( P \neq |S| + 1 \). By equation (6.8c),

\[M(g(t)) = \varepsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.\]

Also, \( (\lambda_i(E_l)(t))_{ab} = \left[ \begin{smallmatrix} 1 \\ 0 \end{smallmatrix} \right] \), and \( (\lambda_j(E_{R+2})(t))_{ab} = \left[ \begin{smallmatrix} S \\ 1 \end{smallmatrix} \right] \). Thus

\[M(g(t)) (\lambda_j(E_{R+2})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} S \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\ S \end{bmatrix}.\]

Since \( f(E_l) = E_{R+2} \), by Proposition 6.10 we get

\[\pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\ S \end{bmatrix},\]

which is a true statement if and only if \( S = 0 \). Let \( S = 0 \). Then \( g \) is an algebraic diffeomorphism if and only if \( a_{l+1} = 0 \) or \( a_{l+2} = 0 \) (see equation (6.8c)). We have

\[g(t) = \begin{cases} (t_2^{\varepsilon} t_2^{a_{l+2}} t_1^{\varepsilon} t_2^{-a_{l+2}}) & \text{if } a_{l+1} = 0, \\ (t_1^{a_{l+1}} t_2^{\varepsilon} t_1^{a_{l+1}}) & \text{if } a_{l+2} = 0. \end{cases}\]
Also, $\lambda_i(E_l)(t) = t_2^{a_l} t_1 t_2^{-a_l}$, and $\lambda_j(E_{R+2})(t) = v_4$ (see equation (6.7d)). Thus the value of $\lambda_j(E_{R+2})(g(t))$ is equal to either

$$t_2^{a_l+1} t_1 t_2^{-a_l}$$

if $a_{l+1} = 0$, or

$$t_1^{a_{l+1}} (t_2^{1+1} t_2^{-1} t_1) t_2^{a_{R+2}} t_1 t_2^{-a_{R+2}} (t_1 t_2^{1+1} t_1) t_2^{a_{l+1}}$$

if $a_{l+2} = 0$. Since $f(E_l) = E_{R+2}$, by Proposition 6.10 we get

$$(6.11) \quad t_2^{a_l} t_1^{1} t_2^{-a_l} = \lambda_j(E_{R+2})(g(t)).$$

Since $j \geq 1$,

$$L\left(\lambda_j(E_{R+2})(g(t))\right) \geq 5.$$ 

This inequality contradicts equation (6.11) and hence the statement of Proposition 6.10.

Consider the case $l = R - 1$, and $P = |S| + 1$ (i.e., $S = 0$ and $P = 1$). By equation (6.8d),

$$M(g(t)) = \varepsilon \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}.$$ 

Also, $(\lambda_i(E_{R-2})(t))_{ab} = [0 \ 1]$, and $(\lambda_j(E_{R+1})(t))_{ab} = [1 \ 1]$. Thus

$$M(g(t)) (\lambda_j(E_{R+1})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$ 

Since $f(E_{R-2}) = E_{R+1}$, by Proposition 6.10 we get

$$\pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

a false statement that contradicts the statement of the proposition.
Consider the case \( I = R - |S| + 1 \) and \(|S| \geq 2\). By equation (6.8e),

\[
M(g(t)) = \varepsilon \begin{bmatrix}
0 & -1 \\
1 & -\bar{S}
\end{bmatrix}.
\]

Also, \((\lambda_i(E_I)(t))_{ab} = [\frac{1}{0}]\), and \((\lambda_j(E_{R+2})(t))_{ab} = [\frac{S}{1}]\). Thus

\[
M(g(t))(\lambda_j(E_{R+2})(t))_{ab} = \varepsilon \begin{bmatrix}
0 & -1 \\
1 & -\bar{S}
\end{bmatrix} \begin{bmatrix}
S \\
1
\end{bmatrix} = \varepsilon \begin{bmatrix}
-1 \\
S - \bar{S}
\end{bmatrix}.
\]

Since \( f(E_I) = E_{R+2} \), by Proposition 6.10 we get

\[
\pm \begin{bmatrix}
1 \\
0
\end{bmatrix} = \varepsilon \begin{bmatrix}
-1 \\
S - \bar{S}
\end{bmatrix},
\]

a false statement (because \(|S| \geq 2\)) that contradicts the statement of the proposition.

Consider the case \( R - |S| + 2 \leq l \leq R - 1 \). By equation (6.8f),

\[
M(g(t)) = \varepsilon \begin{bmatrix}
\bar{S}(l - R + |S| - 1) & -l + R - |S| \\
1 & -\bar{S}
\end{bmatrix}.
\]

Also, \((\lambda_i(E_{I+3})(t))_{ab} = [\frac{\bar{S}(l-R+|S|+1)}{1}]\), and \((\lambda_j(E_3)(t))_{ab} = [\frac{1}{0}]\). Thus

\[
M(g(t))(\lambda_j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix}
\bar{S}(l - R + |S| - 1) & -l + R - |S| \\
1 & -\bar{S}
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \varepsilon \begin{bmatrix}
\bar{S}(l - R + |S| - 1) \\
1
\end{bmatrix}.
\]
Since \( f(E_l) = E_{R+2} \), by Proposition 6.10 we get

\[
\pm \begin{bmatrix}
\bar{S}(l - R + |S| + 1) \\
1
\end{bmatrix} = \varepsilon \begin{bmatrix}
\bar{S}(l - R + |S| - 1) \\
1
\end{bmatrix},
\]

a false statement that contradicts the statement of the proposition.

Consider the case \( l = R \geq 2 \) and \( P = 1 \). By equation (6.8g),

\[
M(g(t)) = \varepsilon \begin{bmatrix}
S - \bar{S} - |S| \\
1 - \bar{S}
\end{bmatrix}.
\]

Also, \((\lambda_i(E_2)(t))_{ab} = [0 1]\).

\[
(z_j v \lambda_{S,1}(E_4)(t))_{ab} = \begin{cases}
\begin{bmatrix} 2\bar{S} \\ 1 \end{bmatrix} & \text{if } |S| = R, \\
\begin{bmatrix} 0 \\ 1 \end{bmatrix} & \text{if } |S| \neq R,
\end{cases}
\]

\((z_i v \lambda_{0,1}(E_{R-1})(t))_{ab} = [1 0]\), and \((z_j v \lambda_{0,1}(E_{R+1})(t))_{ab} = [1 1]\). Thus if \(|S| = R\),

\[
M(g(t)) (z_j v \lambda_{S,1}(E_4)(t))_{ab} = \varepsilon \begin{bmatrix}
S - \bar{S} - |S| \\
1 - \bar{S}
\end{bmatrix} \begin{bmatrix} 2\bar{S} \\
1 \end{bmatrix} = \varepsilon \begin{bmatrix} |S| - 2 \\
\bar{S}
\end{bmatrix},
\]

and if \(|S| \neq R\),

\[
M(g(t)) (z_j v \lambda_{S,1}(E_4)(t))_{ab} = \varepsilon \begin{bmatrix}
S - \bar{S} - |S| \\
1 - \bar{S}
\end{bmatrix} \begin{bmatrix} 0 \\
1 \end{bmatrix} = \varepsilon \begin{bmatrix} -|S| \\
-\bar{S}
\end{bmatrix}.
\]

Also,

\[
M(g(t)) (z_j v \lambda_{0,1}(E_{R+1})(t))_{ab} = \varepsilon \begin{bmatrix} 1 \\
1 \\
1 \\
1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\
2 \end{bmatrix}.
\]
Since \( f(E_2) = E_4 \), by Proposition 6.10 we get

\[
\pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{cases} 
\varepsilon \left[ \frac{|S|-2}{S} \right] & \text{if } |S| = R, \\
\varepsilon \left[ -\frac{|S|}{-S} \right] & \text{if } |S| \neq R.
\end{cases}
\]

The equation above is a true statement if and only if \( S = 0 \) or \( |S| = R = 2 \). If \( S = 0 \), then by Proposition 6.10 we get

\[
\pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 2 \end{bmatrix}
\]

since \( f(E_{R-1}) = E_{R+1} \). This is a false statement that contradicts the statement of Proposition 6.10. If \( |S| = R = 2 \), then

\[
g(t) = (t_1^{a_3}t_2^S)\varepsilon^{-a_3}t_1^{-a_3}(t_1t_2t_1^{-1}t_2^{-1})^{i_1}t_1^{a_4}(t_1^S)\varepsilon^{-a_4}t_1^{-a_4}(t_2t_1^{-1}t_1^{-1})^{i_1}t_1^{a_3}t_2^{-1}t_1^{-1}).
\]

Also, \( \lambda_i(E_1)(t) = t_1 \), and \( \lambda_j(E_3)(t) = t_1^{a_3+S}t_2^{-a_3} \). Thus the value of \( \lambda_j(E_3)(g(t)) \) is equal to

\[
t_1^{a_3}t_2^S\varepsilon^{(a_3+S)}t_1^{-a_3}(t_1t_2t_1^{-1}t_2^{-1})^{i_1}t_1^{a_4}(t_1^S)\varepsilon^{-a_4}t_1^{-a_4}(t_2t_1^{-1}t_1^{-1})^{i_1}t_1^{a_3}t_2^{-1}t_1^{-1}).
\]

Since \( f(E_1) = E_3 \), by Proposition 6.10 we get

\[
t_1^{\pm 1} = \lambda_j(E_3)(g(t)).
\]

The equation above is a true statement if and only if \( i = 0 \) and with the following values for the quadruple \( (S, \varepsilon, a_3, a_4) \): \( \pm(2, 1, 0, 0), \pm(2, 1, -1, -2), \pm(2, -1, 0, -1), \) and \( \pm(2, -1, -1, -1) \). Recall that if \( |S| = R = 2 \), \( \lambda_i(E_2)(t) = t_2 \), and

\[
\lambda_j(E_4)(t) = (t_1t_2t_1^{-1}t_2^{-1})^{j}t_1^{a_4+S}t_2^{-a_4}(t_2t_1^{-1}t_1^{-1})^{j}.
\]
Since \( f(E_2) = E_4 \), by Proposition 6.10 we get

\[
(6.12) \quad t_2^{\pm 1} = \lambda_j(E_4)(g(t)).
\]

The values of \( g(t) \) and \( \lambda_j(E_4)(g(t)) \) are gathered in Table 6.2. Since \( j \geq 1 \), for all considered values of the quadruple \((S, \varepsilon, a_3, a_4)\),

\[
L\left(\lambda_j(E_4)(g(t))\right) \geq 5.
\]

This inequality contradicts equation (6.12) and hence the statement of Proposition 6.10.

The proof of the case \( l = R \geq 2 \) and \( P = 1 \) is now completed.

Consider the case \( l = R \) and \( S = P = 0 \). By equation (6.8h), if \( g \) is an algebraic diffeomorphism, then \( \bar{g} : T^2 \to T^2 \) defined by

\[
\bar{g}(t) = (t_1, t_2^a t_1^{-1} t_2^{a_1} t_1^{a_2} t_2^{a_3} t_1^{a_4} a_{R+2} e^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1}) t_2^{a_{R+1}})
\]

is an algebraic diffeomorphism. If \( i \geq 1 \), then

\[
L\left(t_2^{a_{R+1}} (t_1 t_2 t_1^{-1} t_2^{-1}) t_1^{a_{R+2}} t_2^{a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1}) t_2^{a_{R+1}} \right) \geq 7,
\]

Table 6.2: The values of \( g(t) \) and \( \lambda_j(E_4)(g(t)) \) dependent on the quadruple \((S, \varepsilon, a_3, a_4)\)
and \( \bar{g} \) is not an algebraic diffeomorphism. If \( i = 0 \), then \( \bar{g}(t) = (t_1^\varepsilon, t_2^{-a_{R+1}}t_1^{a_{R+2}}t_2^{-a_{R+2}}t_2^{a_{R+1}}) \).

So \( \bar{g} \) is an algebraic diffeomorphism if and only if \( a_{R+1} = 0 \) or \( a_{R+2} = 0 \). Then

\[
g(t) = \begin{cases} 
(t_1^{a_{R+2}}t_2^{-a_{R+2}}, t_1) & \text{if } a_{R+1} = 0, \\
(t_2^{a_{R+1}}t_2^{-a_{R+1}}, t_2) & \text{if } a_{R+2} = 0.
\end{cases}
\]

Also, \( \lambda_0(E_R)(t) = t_1^{a_{R+1}}t_2^{-a_R} \), and

\[
\lambda_j(E_{R+2})(t) = \begin{cases} 
(t_1t_2^{-1}t_2^{-1}t_1^{a_{R+2}}t_2^{-a_{R+2}}(t_2t_1^{-1}t_1^{-1})^j) & \text{if } a_{R+1} = 0, \\
(t_1t_2^{-1}t_2^{-1}t_2(t_2t_1^{-1}t_1^{-1})^j) & \text{if } a_{R+2} = 0.
\end{cases}
\]

Thus the value of \( \lambda_j(E_{R+2})(g(t)) \) is equal to either

\[
t_2^{a_{R+2}}(t_2^{-1}t_1^{-1}t_1^{a_{R+2}}t_2^{-a_{R+2}}(t_2t_1^{-1}t_1^{-1})^j) t_2^{-a_{R+2}}
\]

if \( a_{R+1} = 0 \), or

\[
t_2^{a_{R+1}}(t_1^{a_{R+1}}t_2^{-1}t_1^{-1}t_2^{-1}t_2^{a_{R+2}}(t_2t_1^{-1}t_1^{-1})^j) t_2^{-a_{R+1}}
\]

if \( a_{R+2} = 0 \). Since \( f(E_R) = E_{R+2} \), by Proposition 6.10 we get

\[
(6.13) \quad t_1^{a_R}t_2^{a_{R+1}}t_1^{-a_R} = \lambda_j(E_{R+2})(g(t)).
\]

Since \( j \geq 1 \),

\[
L\left(\lambda_j(E_{R+2})(g(t))\right) \geq 7.
\]

This inequality contradicts equation (6.13) and hence the statement of Proposition 6.10.

The proof of the case \( l = R \) and \( S = P = 0 \) is now completed.

Consider the case \( l = R + 1 \). By equation (6.8i),

\[
M(g(t)) = \varepsilon \begin{bmatrix} S & -1 \\ 1 & 0 \end{bmatrix}.
\]
Also, \((\lambda_i(E_2)(t))_{ab} = [\varnothing, 1]\),

\[
(\lambda_j(E_3)(t))_{ab} = \begin{cases} 
  [\bar{S}] & \text{if } |S| = R, \\
  [1] & \text{if } 1 \leq |S| \leq R - 1,
\end{cases}
\]

\((z_i v \lambda_0 P(E_{R+1})(t))_{ab} = [\varnothing, \frac{1}{n}]\), and \((z_j v \lambda_0 P(E_{R+2})(t))_{ab} = [\varnothing, 1]\). Thus if \(|S| = R,

\[
M(g(t))(\lambda_j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix} S & -1 \\
1 & 0 \end{bmatrix} \begin{bmatrix} \bar{S} \\
1 \end{bmatrix} = \varepsilon \begin{bmatrix} R - 1 \\
\bar{S} \end{bmatrix},
\]

and if \(1 \leq |S| \leq R - 1,

\[
M(g(t))(\lambda_j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix} S & -1 \\
1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\
0 \end{bmatrix} = \varepsilon \begin{bmatrix} S \\
1 \end{bmatrix}.
\]

Also,

\[
M(g(t))(z_j v \lambda_0 P(E_{R+2})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & -1 \\
1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\
1 \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\
0 \end{bmatrix}.
\]

Since \(f(E_2) = E_3\), by Proposition 6.10 we get

\[
\pm \begin{bmatrix} 0 \\
1 \end{bmatrix} = \begin{cases} 
  \varepsilon \left[ \frac{R - 1}{\bar{S}} \right] & \text{if } |S| = R, \\
  \varepsilon \left[ \frac{S}{1} \right] & \text{if } 1 \leq |S| \leq R - 1.
\end{cases}
\]

The equation above is a true statement if and only if \(R = 1\). Let \(R = |S| = 1\) and \(l = 2\). If \(i \geq 1\), then

\[
L((t_1 t_2 t_1^{-1} t_2^{-1}) i t_1 a_3 (t_1^S t_2) \varepsilon t_1^{-a_3} (t_2 t_1 t_2^{-1} t_1^{-1}) i) \geq 5,
\]

so \(g\) (see equation (6.8i)) is an algebraic diffeomorphism only if \(i = 0\). Then \(g(t) = (t_1 a_3 (t_1^S t_2) \varepsilon t_1^{-a_3}, t_1^{-\varepsilon})\). Recall that \(\lambda_i(E_2)(t) = t_2\), and

\[
\lambda_j(E_3)(t) = (t_1 t_2 t_1^{-1} t_2^{-1}) j t_1^{a_3 + S} t_2^{t_1^{-a_3}} (t_1 t_2 t_1^{-1} t_2^{-1})^{-j}.
\]
Thus

\[
\lambda_j(E_3)(g(t)) = \left( t_1^{a_3} (t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3} \right) j t_1^{a_3} (t_1^S t_2)^{\varepsilon (a_3 + S)} t_1^{-\varepsilon} (t_1^S t_2)^{-\varepsilon} t_1^{-a_3} \right) - j \\
= t_1^{a_3} ((t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3}) j (t_1^S t_2)^{\varepsilon a_3 -\varepsilon k} t_1^{S e -\varepsilon k} \\
= \left( t_1^{a_3} ((t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3}) j (t_1^S t_2)^{\varepsilon a_3 -\varepsilon k} \right) t_1^{S e -\varepsilon k} \\
= \left( t_1^{a_3} ((t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3}) j (t_1^S t_2)^{\varepsilon a_3 -\varepsilon k} \right) - 1,
\]

where \( k = (S + \varepsilon)/2 \). Since \( f(E_2) = E_3 \), by Proposition 6.10 we get

\[(6.14) \quad t_2^{\pm 1} = \lambda_j(E_3)(g(t)).\]

Since the value of \( t_1^{a_3} ((t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3}) j (t_1^S t_2)^{\varepsilon a_3 -\varepsilon k} \) is equal to

\[
\begin{cases}
  t_1^{a_3} (t_1 t_2 t_1^{-1} t_2^{-1} t_1)^j (t_1 t_2)^{a_3} t_1 & \text{if } S = \varepsilon = 1, \\
  t_1^{a_3} (t_2^{-1} t_1 t_2^{-1} t_1)^j (t_1 t_2)^{-a_3} & \text{if } S = -\varepsilon = 1, \\
  t_1^{a_3} (t_2^{-1} t_1^{-1} t_2^{-1} t_1)^j (t_1^{-1} t_2)^{a_3} & \text{if } S = -\varepsilon = -1, \\
  t_1^{a_3} (t_2^{-1} t_1 t_2^{-1} t_1)^j (t_2^{-1} t_1)^{a_3 -1} & \text{if } S = \varepsilon = -1,
\end{cases}
\]

and \( j \geq 1 \),

\[
L\left( t_1^{a_3} ((t_1^S t_2)^{-\varepsilon} t_1^e (t_1^S t_2)^{-\varepsilon} t_1^{-a_3}) j (t_1^S t_2)^{\varepsilon a_3 -\varepsilon k} \right) \geq 3.
\]

Hence

\[
L\left( \lambda_j(E_3)(g(t)) \right) \geq 5.
\]

This inequality contradicts equation (6.14) and hence the statement of Proposition 6.10.
Since \( f(E_{R+1}) = E_{R+2} \), if \( S = 0 \), by Proposition 6.10 we get
\[
\pm \begin{bmatrix} 1 \\ P \end{bmatrix} = \varepsilon \begin{bmatrix} -1 \\ 0 \end{bmatrix}.
\]
The equation above is a true statement if and only if \( P = 0 \). Let \( S = P = 0 \). If \( i \geq 1 \), then
\[
L((t_1t_2t_1^{-1}t_2^{-1})^j t_1^{aR+2} t_2^{a-R^2}(t_2t_1t_1^{-1}t_2^{-1})^i) \geq 7,
\]
and \( g \) is not an algebraic diffeomorphism (see equation (6.8h)). If \( i = 0 \), then \( g(t) = (t_1^{aR+2} t_2^{a-R^2}, t_1^{-j}) \). Also, \( \lambda_0(E_{R+1})(t) = t_2^{aR+1} t_1^{a-R^1} \), and \( \lambda_j(E_{R+2})(t) = v_4 \) (see equation (6.7d)). Thus the value of \( \lambda_j(E_{R+2})(g(t)) \) is equal to
\[
(t_1^{aR+2} t_2^{a-R^2} t_1^{-j} t_2^{-i}) t_1^{aR+2} t_2^{a-R^2} t_1^{-j} t_2^{a-R^2} (t_1^{aR+2} t_2^{a-R^2} t_1^{-j} t_2^{a-R^2})^j
= t_1^{aR+2} t_2^{a-R^2} t_1^{-j} t_2^{-i} (t_1^{aR+2} t_2^{a-R^2} t_1^{-j} t_2^{a-R^2})^j t_1^{-a-R^2}.
\]
Since \( f(E_{R+1}) = E_{R+2} \), by Proposition 6.10 we get
\[
(6.15) \quad t_2^{a-R+1} t_1^{a-R+1} = \lambda_j(E_{R+2})(g(t)).
\]
Since \( j \geq 1 \),
\[
L\left( \lambda_j(E_{R+2})(g(t)) \right) \geq 7.
\]
This inequality contradicts equation (6.15) and hence the statement of Proposition 6.10. The proof of the case \( l = R + 1 \) is now completed. The proof of the case where \( f \) is orientation-preserving is now completed as well.

If \( f \) is orientation-reversing, then \( f \) is defined by \( E_{l-k} \to E_k \) for some integer \( 1 \leq l \leq R + 2 \), and all \( 1 \leq k \leq R + 2 \). Again, let \( \varepsilon = \pm 1 \). Since \( f \) is orientation-reversing, so should be \( g \) (for \( [f, g] \) to be orientation-preserving). Since \( f(E_{l-1}) = E_1 \) and \( f(E_{l-2}) = E_2 \), by
equation (6.6), the value of $g(t)$ is dependent on $l$ and is equal to either

(6.16a) \[ (t_1 t_2 t_1^{-1} t_2^{-1})^i t_1^{a_{R+2}} (t_1 t_2)^{\epsilon} t_1^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1})^i t_1^{a_{R+1}} (t_1 S - \tilde{S}) t_1^{-a_{R+1}} \]

if $l = P = 1$ and $R \geq 2$, or

(6.16b) \[ (t_1 t_2 t_1^{-1} t_2^{-1})^i t_1^{a_{R+2}} t_2^{\epsilon} t_1^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1})^i t_2^{a_{R+1}} t_1^{-a_{R+1}} \]

if $l = 1$ and $S = P = 0$, or

(6.16c) \[ (t_1 t_2 t_1^{-1} t_2^{-1})^i t_1^{a_{R+1}} (t_1 t_2)^{\epsilon} (t_2 t_1 t_2^{-1} t_1^{-1})^i t_2^{-S\epsilon} \]

if $l = R = |S| = 1$, or

(6.16d) \[ (t_1^{a_{l-1}} t_2^{\epsilon} t_1^{-a_{l-2}})^i t_1^{a_{R+2}} (t_1 t_2)^{\epsilon} t_1^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1})^i \]

if $l = 2$, or

(6.16e) \[ (t_1^{a_{l-1}} t_2^{\epsilon} t_1^{-a_{l-2}})^i t_1^{a_{R+2}} (t_1 t_2)^{\epsilon} t_1^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1})^i \]

if $3 \leq l \leq R - |S| + 1$ and $l$ odd, or $l = R - |S| + 3$ and $S \neq 0$, or

(6.16f) \[ (t_1^{a_{R-|S|+3}} (t_1 t_2)^{\epsilon} t_1^{-a_{R-|S|+3}} t_1^{a_{R-|S|+2}} t_2^{-S\epsilon} t_1^{-a_{R-|S|+2}} \]

if $l = R - |S| + 4$ and $|S| \geq 2$, or

(6.16g) \[ (t_1^{a_{l-1}} t_2^{\epsilon} t_1^{-a_{l-2}})^i t_1^{a_{R+2}} (t_1 t_2)^{\epsilon} t_1^{-a_{R+2}} (t_2 t_1 t_2^{-1} t_1^{-1})^i \]

if $4 \leq l \leq R - |S|$ and $l$ even, or $l = R - |S| + 2$ and $P \neq |S| + 1$, or

(6.16h) \[ (t_1^{a_{R+1}} (t_1 t_2)^{\epsilon} t_1^{-a_{R+1}} t_1^{a_{R}} t_2^{\epsilon} t_1^{-a_{R}}) \]
if \( l = R + 2 \) and \( P = |S| + 1 \), or
\[
(6.16i) \quad (t_1^{a_l-1}(t_1^{S(l-R+|S|-3)}t_2)\varepsilon^{a_l-1}, t_1^{a_l-2}(t_1^{S(l-R+|S|-4)}t_2)\varepsilon^{a_l-2})
\]
if \( R-|S|+5 \leq l \leq R+2 \). Similarly to the case where \( f \) is orientation-preserving, we will now consider each of the cases listed above to show that existence of the algebraic isomorphism \([f,g]\) contradicts either some basic properties of automorphisms of \( F_2 \), or the statement of Proposition 6.10.

Consider the case \( l = P = 1 \) and \( R \geq 2 \). By equation (6.16a),
\[
M(g(t)) = \varepsilon \begin{bmatrix} S & -|S| + 1 \\ 1 & -\bar{S} \end{bmatrix}.
\]
First, let \( S \neq 0 \). Then
\[
(\lambda_\i(E_R)(t))_{ab} = \begin{cases} \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } |S| = 1, \\ \begin{bmatrix} S(|S|-2) \\ 1 \end{bmatrix} & \text{if } |S| \geq 2, \end{cases}
\]
and
\[
(\lambda_\j(E_3)(t))_{ab} = \begin{cases} \begin{bmatrix} \bar{S} \\ 1 \end{bmatrix} & \text{if } |S| = R, \\ \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \text{if } |S| \neq R. \end{cases}
\]
Thus if \( |S| = R \),
\[
M(g(t))(\lambda_\j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix} S & -|S| + 1 \\ 1 & -\bar{S} \end{bmatrix} \begin{bmatrix} \bar{S} \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix},
\]
and if \( 1 \leq |S| \leq R-1 \),
\[
M(g(t))(\lambda_\j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix} S & -|S| + 1 \\ 1 & -\bar{S} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} S \\ 1 \end{bmatrix}.
\]
Since \( f(E_R) = E_3 \), by Proposition 6.10, if \(|S| = 1\), we get

\[
\pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} S \\ 1 \end{bmatrix},
\]

while if \(|S| \geq 2\), we get either

\[
\pm \begin{bmatrix} \tilde{S}(|S| - 2) \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}
\]

if \(|S| = R\) or

\[
\pm \begin{bmatrix} \tilde{S}(|S| - 2) \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} S \\ 1 \end{bmatrix}
\]

if \(|S| \neq R\). All three equations above are false statements (in the third equation we obtain \(\tilde{S}(|S| - 2) = S\) which is a false statement since \(|\tilde{S}| = 1\)) that contradict the statement of Proposition 6.10.

If \(S = 0\), then \((\lambda_i(E_2)(t))_{ab} = [\frac{1}{1}]\), and \((\lambda_j(E_{R+1})(t))_{ab} = [\frac{1}{1}]\). Thus

\[
M(g(t))(\lambda_j(E_{R+1})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 2 \end{bmatrix}.
\]

Since \(f(E_2) = E_{R+1}\), by Proposition 6.10 we get

\[
\pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 2 \end{bmatrix},
\]

a false statement that contradicts the statement of the proposition.
Consider the case $l = 1$ and $S = P = 0$. By equation (6.16b), if $g$ is an algebraic diffeomorphism, then $\bar{g} : T^2 \to T^2$ defined by

$$\bar{g}(t) = (t_2^{-a_R+1}(t_1^{-1}t_2^{-1})^{t_1^{a_R+2}t_2^{a_R+2}}(t_2^{-1}t_1^{-1})^{t_1^{a_R+1}t_2^\varepsilon})$$

is an algebraic diffeomorphism. If $i \geq 1$, then

$$L(t_2^{-a_R+1}(t_1^{-1}t_2^{-1})^{t_1^{a_R+1}t_2^{a_R+2}}(t_2^{-1}t_1^{-1})^{t_1^{a_R+1}t_2^\varepsilon}) \geq 7,$$

and $\bar{g}$ is not an algebraic diffeomorphism. If $i = 0$, then $\bar{g}(t) = (t_2^{-a_R+1}t_1^{a_R+2}t_2^{a_R+2}, t_1^\varepsilon)$. So $\bar{g}$ is an algebraic diffeomorphism if and only if $a_{R+1} = 0$ or $a_{R+2} = 0$. Let $i = 0$. Then

$$g(t) = \begin{cases} 
(t_2^{a_R+1}t_1^{-1}t_2^{-1}, t_1^\varepsilon) & \text{if } a_{R+1} = 0, \\
(t_2^{a_R+1}t_1^{-1}t_1^\varepsilon) & \text{if } a_{R+2} = 0.
\end{cases}$$

Also, $\lambda_0(E_1)(t) = t_1$, and $\lambda_j(E_{R+2})(t) = v_4$ (see equation (6.7d)). Thus

$$\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
(t_2^{a_{R+2}}t_1^{a_{R+2}}t_2^{a_R+2}t_1^{a_R+2}t_2^{a_R+2}t_1^{a_{R+2}}) & \text{if } a_{R+1} = 0, \\
(t_2^{a_{R+1}}t_1^{a_R+2}t_1^{a_R+1}t_2^{a_R+1}t_1^{a_R+1}t_2^{a_R+1}) & \text{if } a_{R+2} = 0.
\end{cases}$$

Since $f(E_1) = E_{R+2}$, by Proposition 6.10 we get

$$t_1^{\pm 1} = \lambda_j(E_{R+2})(g(t)).$$

Since $j \geq 1$,

$$L \left( \lambda_j(E_{R+2})(g(t)) \right) \geq 7.$$ 

This inequality contradicts equation (6.17) and hence the statement of Proposition 6.10. The proof of the case $l = 1$ and $S = P = 0$ is now completed.
Consider the case $l = R = |S| = 1$. If $i \neq 0$, then by equation (6.16c) $g$ is not an algebraic diffeomorphism because

$$L((t_1 t_2 t_1^{-1} t_2^{-1})^i t_1^{a_3} (t_1^S t_2^S t_1^{-a_3} t_2^{-S})^j) \geq 5.$$ 

Let $i = 0$. Then $g(t) = (t_1^{a_3} (t_1^S t_2^S t_1^{-a_3} t_2^{-S}))$ So $g$ is an algebraic diffeomorphism if and only if $a_3 = 0$ or $a_3 = -S$. We have

$$g(t) = \begin{cases} (t_1^S t_2^S t_1^{-a_3} t_2^{-S}), & \text{if } a_3 = 0, \\ (t_2^S t_1^S t_2^{-a_3} t_1^{-S}), & \text{if } a_3 = -S. \end{cases}$$

Also,

$$\lambda_j(E_3)(t) = \begin{cases} (t_1 t_2 t_1^{-1} t_2^{-1})^j t_1^{a_3} t_2^S t_2^{-1} t_1^{-1} t_2^{-1})^j & \text{if } a_3 = 0, \\ (t_1 t_2 t_1^{-1} t_2^{-1})^j t_2^S t_1^S t_2^{-1} t_1^{-1} t_2^{-1})^j & \text{if } a_3 = -S. \end{cases}$$

Thus the value of $\lambda_j(E_3)(g(t))$ is equal to either

$$(t_1^S t_2^S t_1^{-a_3} t_2^{-S})^j (t_1^S t_2^S t_1^{-a_3} t_2^{-S})^j (t_2^S t_1^S t_2^{-a_3} t_1^{-S})^j (t_2^S t_1^S t_2^{-a_3} t_1^{-S})^j$$

if $a_3 = 0$, or

$$(t_2^S t_1^S t_2^{-a_3} t_1^{-S})^j (t_2^S t_1^S t_2^{-a_3} t_1^{-S})^j (t_1^S t_2^S t_1^{-a_3} t_2^{-S})^j (t_1^S t_2^S t_1^{-a_3} t_2^{-S})^j$$

if $a_3 = -S$. These values, dependent on the triple $(S, \varepsilon, a_3)$, are gathered in table 6.3.

Recall that $\lambda_0(E_1)(t) = t_1$. Since $f(E_1) = E_3$, by Proposition 6.10 we get

$$(6.18) \quad t_1^{\pm 1} = \lambda_j(E_3)(g(t)).$$

Since $j \geq 1$, using the information from table 6.3,

$$L\left(\lambda_j(E_3)(g(t))\right) \geq 5.$$
This inequality contradicts equation (6.18) and hence the statement of Proposition 6.10.

Consider the case \( l = 2 \). By equation (6.16d),

\[
M(g(t)) = \varepsilon \begin{bmatrix}
1 & -S \\
0 & -1
\end{bmatrix}.
\]

If \( 1 \leq |S| \leq R - 1 \), \( (\lambda_i(E_{R+1})(t))_{ab} = \left[ \hat{S}(|S|-1) \right] \), and \( (\lambda_j(E_3)(t))_{ab} = [\frac{1}{0}] \). Thus

\[
M(g(t))(\lambda_j(E_3)(t))_{ab} = \varepsilon \begin{bmatrix}
1 & -S \\
0 & -1
\end{bmatrix} \begin{bmatrix}
1 \\
0
\end{bmatrix} = \varepsilon \begin{bmatrix}
1 \\
0
\end{bmatrix}.
\]

Since \( f(E_{R+1}) = E_3 \), by Proposition 6.10 we get

\[
\pm \left[ \hat{S}(|S|-1) \right] = \varepsilon \begin{bmatrix}
1 \\
0
\end{bmatrix},
\]

a false statement that contradicts the statement of the proposition.

If \( i \geq 1 \), then

\[
L((t_1t_2^{-1}t_1^{-1}t_2^{-1})^i t_1^{a_{R+2}}(t_1^{S}t_2)^{-\varepsilon} t_1^{-a_{R+2}}(t_2t_1^{-1}t_2^{-1})^i) \geq 5,
\]

so \( g \) is an algebraic diffeomorphism if and only if \( i = 0 \) (see equation (6.16d)). Let \( i = 0 \).
Recall that $\lambda_0(E_2) = t_2$, and $\lambda_j(E_{R+2}) = v_3$ (see equation (6.7c)). Since $f(E_2) = E_{R+2}$, by Proposition 6.10 we get

\begin{equation}
(6.19) \quad t_2^{\pm 1} = \lambda_j(E_{R+2})(g(t)).
\end{equation}

If $|S| = R$, then the value of $\lambda_j(E_{R+2})(g(t))$ is equal to either

- if $S = R$ and $\varepsilon = 1$, or

\begin{align*}
t_1^{a_{R+2}}(t_1 t_2^{-1} t_1^{-1} t_2)^j t_1^{a_{R+2}+R} t_2^{-1} t_1^{a_{R+2}+R} (t_2 t_1^{-1} t_1^{-1})^j t_1^{a_{R+2}+R} - R
\end{align*}

if $S = R$ and $\varepsilon = -1$, or

\begin{align*}
t_1^{a_{R+2}}(t_1 t_2^{-1} t_1^{-1} t_2)^j t_1^{a_{R+2}+R} t_2^{-1} t_1^{a_{R+2}+R} (t_2 t_1^{-1} t_1^{-1})^j t_1^{a_{R+2}+R} - R
\end{align*}

if $S = -R$ and $\varepsilon = 1$, or

\begin{align*}
t_1^{a_{R+2}}(t_1 t_2^{-1} t_1^{-1} t_2)^j t_1^{a_{R+2}+R} t_2^{-1} t_1^{a_{R+2}+R} (t_2 t_1^{-1} t_1^{-1})^j t_1^{a_{R+2}+R} + R
\end{align*}

if $S = -R$ and $\varepsilon = -1$. In all four cases above,

\begin{equation*}
L \left( \lambda_j(E_{R+2})(g(t)) \right) \geq 5
\end{equation*}

since $R \geq 1$ and $j \geq 1$. The inequality above contradicts equation (6.19) and hence the statement of Proposition 6.10. If $S = 0$, then

\begin{align*}
\lambda_j(E_{R+2})(g(t)) = t_1^{a_{R+2}}(t_1 t_2^{-1} t_1^{-1} t_2)^j t_1^{a_{R+2}+R} t_2^{-1} t_1^{a_{R+2}+R} (t_2 t_1^{-1} t_1^{-1})^j t_1^{a_{R+2}+R}.
\end{align*}
Since \( j \geq 1 \),
\[
L\left( \lambda_j(E_{R+2})(g(t)) \right) \geq 5.
\]
This inequality also contradicts equation (6.19) and hence the statement of Proposition 6.10. The proof of the case \( l = 2 \) is now completed.

Consider the case \( 3 \leq l \leq R - |S| + 1 \) and \( l \) odd. By equation (6.16e),
\[
M(g(t)) = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.
\]
Let \( 1 \leq |S| \leq R - 1 \). Then \((\lambda_i(E_{R+2})(t) )_{ab} = \begin{bmatrix} S \\ 1 \end{bmatrix}\), and \((\lambda_j(E_l)(t) )_{ab} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\). Thus
\[
M(g(t))(\lambda_j(E_l)(t) )_{ab} = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} 0 \\ 1 \end{bmatrix}.
\]
Since \( f(E_{R+2}) = E_t \), by Proposition 6.10 we get
\[
\pm \begin{bmatrix} S \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 0 \\ 1 \end{bmatrix},
\]
a false statement (since \( S \neq 0 \)) that contradicts the statement of the proposition. Let \( S = 0 \).

First consider \( l = 3 \). Then by equation (6.16e), \( g(t) = (t_2^a, t_1^a) \). Hence
\[
L(\lambda_j(E_{R+2})(g(t)) ) = L(\lambda_j(E_{R+2})(t)) .
\]
Since \( \lambda_j(E_{R+2})(t) = v_4 \) (see equation (6.7d)), and \( j \geq 1 \),
\[
(6.20) \quad L(\lambda_j(E_{R+2})(g(t)) ) \geq 7.
\]
Recall that
\[
\lambda_i(E_3)(t) = \begin{cases} \begin{bmatrix} t_1^{a_3} t_1 t_2^{a_3} \\ t_2^{a_3} t_1 t_2^{-a_3} \end{bmatrix} & \text{if } R = 2 \text{ and } P = 1, \\ \begin{bmatrix} t_2^{a_3} t_1 t_2^{-a_3} \end{bmatrix} & \text{otherwise.} \end{cases}
\]
Therefore

\[(6.21) \quad L(\lambda_i(E_3)(t)) \leq 3.\]

Since \(f(E_3) = E_{R+2}\), by Proposition 6.10 we get

\[(\lambda_i(E_3)(t))^{\pm 1} = \lambda_j(E_{R+2})(g(t)).\]

By inequalities (6.20) and (6.21), the equation above is a false statement that contradicts the statement of Proposition 6.10.

With \(S\) still equal to 0, let \(5 \leq l \leq R - 1\) and \(l\) odd, or \(l = R + 1\) and \(P = 0\). Then \(\lambda_i(E_l)(t) = t^{a_l} t_1 t_2^{-a_l}\), and \(\lambda_j(E_{R+2})(t) = v_4\) (see equation (6.7d)). Since \(f(E_l) = E_{R+2}\), by Proposition 6.10 we get

\[(6.22) \quad t^{a_l} t_1^{\pm 1} t_2^{-a_l} = \lambda_j(E_{R+2})(g(t)).\]

By equation (6.16e), if \(g\) is an algebraic diffeomorphism, then \(\bar{g}: T^2 \to T^2\) defined by

\[
\bar{g}(t) = (t^{\varepsilon - a_l - 2} t_1^{a_l - 1} t^{\varepsilon - a_l - 1} t_2^{a_l - 2}, t_1^{\varepsilon})
\]

is an algebraic diffeomorphism, which is true if and only if \(a_l - 2 = 0\) or \(a_l - 1 = 0\). Since

\[
g(t) = \begin{cases} 
(t^{\varepsilon - a_l - 2} t_1^{a_l - 1} t^{\varepsilon - a_l - 1} t_2^{a_l - 2}, t_1^{\varepsilon}) & \text{if } a_l - 1 = 0, \\
(t_1^{a_l - 1} t^{\varepsilon - a_l - 1} t_2^{a_l - 2}, t_1^{\varepsilon}) & \text{if } a_l - 2 = 0,
\end{cases}
\]

we have

\[
\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
t^{a_l - 2} (t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 1} t_1^{-\varepsilon}) j t^{\varepsilon} t_1^{a_l - 2} t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 2} (t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 2} j t^{\varepsilon} t_1^{a_l - 2}) t_1^{a_l - 1} t_2^{a_l - 2} & \text{if } a_l - 1 = 0, \\
t^{a_l - 1} (t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 1} t_1^{-\varepsilon}) j t^{\varepsilon} t_1^{a_l - 2} t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 2} (t^{\varepsilon} t_1^{a_l - 1} t_2^{\varepsilon - a_l - 2} j t^{\varepsilon} t_1^{a_l - 2}) t_1^{a_l - 1} t_2^{a_l - 2} & \text{if } a_l - 2 = 0.
\end{cases}
\]
Thus
\[ L\left(\lambda_j(E_{R+2})(g(t))\right) \geq 5 \]
since \( j \geq 1 \). The inequality above contradicts equation (6.22) and hence the statement of Proposition 6.10.

With \( S = 0 \), let \( P = 1 \), \( l = R + 1 \), and \( R \geq 4 \). Then \((\lambda_i(E_{R+2})(t))_{ab} = [0]\), and \((\lambda_j(E_{R+1})(t))_{ab} = [1]\). Thus
\[ M(g(t))(\lambda_j(E_{R+1})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \]

Since \( f(E_{R+2}) = E_{R+1} \), by Proposition 6.10 we get
\[ \pm \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \]
a false statement that contradicts the statement of the proposition. The proof of the case \( 3 \leq l \leq R - |S| + 1 \) and \( l \) odd is now completed.

Consider the case \( l = R - |S| + 3 \) and \( S \neq 0 \). By equation (6.16e),
\[ M(g(t)) = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}. \]

Also, \((\lambda_i(E_{R+2})(t))_{ab} = [S]_{1}\), and \((\lambda_j(E_{R-|S|+3})(t))_{ab} = [\bar{S}]_{1}\). Thus
\[ M(g(t))(\lambda_j(E_{R-|S|+3})(t))_{ab} = \varepsilon \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \bar{S} \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ \bar{S} \end{bmatrix}. \]

Since \( f(E_{R+2}) = E_{R-|S|+3} \), by Proposition 6.10 we get
\[ \pm \begin{bmatrix} S \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ \bar{S} \end{bmatrix}, \]
which is a true statement if and only if $|S| = 1$. First, let $R = |S| = 1$ and $l = 3$. Then $f(E_3) = E_3$. Hence by Proposition 6.10,

$$103$$

(6.23) \hspace{1cm} \left( \lambda_i(E_3)(t) \right)^{\pm 1} = \lambda_j(E_3)(g(t))$$

Since $i < j$, by definition of $z_j$,

$$L\left( \lambda_i(E_3)(t) \right) < L\left( \lambda_j(E_3)(t) \right).$$

By equation (6.16e), $g(t) = (t_2^a, t_1^a)$. Hence

$$L\left( \lambda_j(E_3)(g(t)) \right) = L(\lambda_j(E_3)(t)).$$

Therefore

$$\left( \lambda_i(E_3)(t) \right)^{\pm 1} \neq \lambda_j(E_3)(g(t)),$$

contradicting equation (6.23) and hence the statement of Proposition 6.10.

Now let $R \geq 2$, $|S| = 1$ and $l = R + 2$. Then $\lambda_i(E_{R+2})(t) = v_1$ (see equation (6.7a)), and $\lambda_j(E_{R+2})(t) = v_3$ (see equation (6.7c)). Since $f(E_{R+2}) = E_{R+2}$, by Proposition 6.10 we get

$$103$$

(6.24) \hspace{1cm} v_1^{\pm 1} = \lambda_j(E_{R+2})(g(t)).$$

The values of the length of $v_1$ when $|S| = 1$, dependent on the values of $S$, $i$, and $a_{R+2}$, are gathered in table 6.4. By equation (6.16e), $g$ is an algebraic diffeomorphism if and only if $a_R = 0$ or $a_{R+1} = 0$. We have

$$g(t) = \begin{cases} 
(t_1^{a_R+1} t_2^{-a_R+1}, t_1^a) & \text{if } a_R = 0, \\
(t_2^{a_R} t_1^{-a_R}, t_2^a) & \text{if } a_{R+1} = 0.
\end{cases}$$
Table 6.4: The length of \( v_1 \) dependent on the triple \((S, i, a_{R+2})\) if \(|S| = 1\)

<table>
<thead>
<tr>
<th>( a_{R+2} )</th>
<th>( S = 1 )</th>
<th>( S = -1 )</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( L(v_1) ) if ( i = 0 )</td>
<td>( L(v_1) ) if ( i &gt; 0 )</td>
</tr>
<tr>
<td>( \leq -2 )</td>
<td>3</td>
<td>8i + 3</td>
</tr>
<tr>
<td>-1</td>
<td>2</td>
<td>8i - 3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>8i + 1</td>
</tr>
<tr>
<td>( \geq 1 )</td>
<td>3</td>
<td>8i + 3</td>
</tr>
</tbody>
</table>

Let

\[
u_1 = t_1^{a_{R+1}} \left( t_2^{\epsilon_1} t_2^{\epsilon_1 - \epsilon} t_1^{\epsilon_1} t_1^{\epsilon_1 - \epsilon} \right) t_2^{(a_{R+2} + S)} t_1^{\epsilon_1 - \epsilon a_{R+2}} t_2^{(a_{R+2} + S)} t_1^{\epsilon_1 - \epsilon a_{R+2}} t_2^{a_{R+1}},
\]

and

\[
u_2 = t_2^{a_R} \left( t_2^{\epsilon_2} t_2^{\epsilon_2 - \epsilon} t_1^{\epsilon_2} t_1^{\epsilon_2 - \epsilon} \right) t_2^{(a_{R+2} + S)} t_1^{\epsilon_2 - \epsilon a_{R+2}} t_2^{(a_{R+2} + S)} t_1^{\epsilon_2 - \epsilon a_{R+2}} t_2^{a_R}.
\]

Then

\[
\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
u_1 & \text{if } a_R = 0, \\ \nu_2 & \text{if } a_{R+1} = 0. \end{cases}
\]

The values of the length of \( u_1 \), dependent on the values of \( S, a_{R+1}, \) and \( a_{R+2} \), are gathered in table 6.5. The values of the length of \( u_2 \), dependent on the values of \( S, a_{R+2}, \) and \( a_R \), are gathered in table 6.6. Since \( 0 \leq i < j \), for all values of \( i, j, S, a_R, a_{R+1}, a_{R+2} \), we have \( L(v_1^{\pm 1}) < L(u_d) \) \((d = 1, 2)\). Hence \( L(v_1^{\pm 1}) < L\left(\lambda_j(E_{R+2})(g(t))\right)\), contradicting equation (6.24) and hence the statement of Proposition 6.10. The proof of the case \( l = R - |S| + 3 \) and \( S \neq 0 \) is now completed.

Consider the case \( l = R - |S| + 4 \) and \(|S| \geq 2\). By equation (6.16f),

\[
M(g(t)) = \varepsilon \begin{bmatrix} \bar{S} & 0 \\ 1 & -\bar{S} \end{bmatrix}.
\]
Table 6.5: The length of $u_1$ dependent on the triple $(S, a_{R+2}, a_{R+1})$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$a_{R+2}$</th>
<th>$a_{R+1}$</th>
<th>$L(u_1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−1</td>
<td>0</td>
<td>$8j - 3$</td>
</tr>
<tr>
<td>1</td>
<td>−1</td>
<td>$\neq 0$</td>
<td>$8j - 1$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>0</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\neq 0$</td>
<td>$8j + 3$</td>
</tr>
<tr>
<td>1</td>
<td>$\leq -2$ or $\geq 1$</td>
<td>0</td>
<td>$8j + 3$</td>
</tr>
<tr>
<td>1</td>
<td>$\leq -2$ or $\geq 1$</td>
<td>$\neq 0$</td>
<td>$8j + 5$</td>
</tr>
<tr>
<td>−1</td>
<td>0</td>
<td>0</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>−1</td>
<td>0</td>
<td>$\neq 0$</td>
<td>$8j + 3$</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>0</td>
<td>$8j - 1$</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>$\neq 0$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>−1</td>
<td>$\leq -1$ or $\geq 2$</td>
<td>0</td>
<td>$8j + 3$</td>
</tr>
<tr>
<td>−1</td>
<td>$\leq -1$ or $\geq 2$</td>
<td>$\neq 0$</td>
<td>$8j + 5$</td>
</tr>
</tbody>
</table>

Table 6.6: The length of $u_2$ dependent on the triple $(S, a_{R+2}, a_R)$

<table>
<thead>
<tr>
<th>$S$</th>
<th>$a_{R+2}$</th>
<th>$a_R$</th>
<th>$L(u_2)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>−1</td>
<td>$-\varepsilon$</td>
<td>$8j - 5$</td>
</tr>
<tr>
<td>1</td>
<td>−1</td>
<td>$\neq -\varepsilon$</td>
<td>$8j - 3$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$-\varepsilon$</td>
<td>$8j - 1$</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$\neq -\varepsilon$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>1</td>
<td>$\leq -2$ or $\geq 1$</td>
<td>$-\varepsilon$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>1</td>
<td>$\leq -2$ or $\geq 1$</td>
<td>$\neq -\varepsilon$</td>
<td>$8j + 3$</td>
</tr>
<tr>
<td>−1</td>
<td>0</td>
<td>$-\varepsilon$</td>
<td>$8j - 1$</td>
</tr>
<tr>
<td>−1</td>
<td>0</td>
<td>$\neq -\varepsilon$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>$-\varepsilon$</td>
<td>$8j - 3$</td>
</tr>
<tr>
<td>−1</td>
<td>1</td>
<td>$\neq -\varepsilon$</td>
<td>$8j - 1$</td>
</tr>
<tr>
<td>−1</td>
<td>$\leq -1$ or $\geq 2$</td>
<td>$-\varepsilon$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>−1</td>
<td>$\leq -1$ or $\geq 2$</td>
<td>$\neq -\varepsilon$</td>
<td>$8j + 3$</td>
</tr>
</tbody>
</table>
Also, \((\lambda_i(E_{R+2}(t)))_{ab} = [S_1]\), and \((\lambda_j(E_{R-|S|+4}(t)))_{ab} = [\frac{2S}{1}]\). Thus

\[
M(g(t))(\lambda_j(E_{R-|S|+4})(t))_{ab} = \varepsilon \begin{bmatrix}
\tilde{S} & 0 \\
1 & -\tilde{S}
\end{bmatrix} \begin{bmatrix}
2S \\
1
\end{bmatrix} = \varepsilon \begin{bmatrix}
2 \\
\tilde{S}
\end{bmatrix}.
\]

Since \(f(E_{R+2}) = E_{R-|S|+4}\), by Proposition 6.10 we get

\[
\pm \begin{bmatrix}
S \\
1
\end{bmatrix} = \varepsilon \begin{bmatrix}
2 \\
\tilde{S}
\end{bmatrix},
\]

which is a true statement if and only if \(|S| = 2\). Let \(|S| = 2\) and \(l = R + 2\). Then \(\lambda_i(E_{R+2})(t) = v_1\) (see equation (6.7a)), and \(\lambda_j(E_{R+2})(t) = v_3\) (see equation (6.7c)). Since \(f(E_{R+2}) = E_{R+2}\), by Proposition 6.10 we get

\[
(6.25) \quad v_1^{\pm 1} = \lambda_j(E_{R+2})(g(t)).
\]

The values of the length of \(v_1\) when \(|S| = 2\), dependent on the values of \(S, i,\) and \(a_{R+2}\), are gathered in table 6.7. Define

\[
u_3 = (t_1 t_2^{-1} t_1^{-1} t_2^{-1})^j(t_1 t_2)^{a_{R+2}+2} t_2^{-1}(t_1 t_2)^{-a_{R+2}}(t_2^{-1} t_1 t_2)^{-1} t_1^{-1} t_2^{-1} t_1^{-1} t_2^{-1},
\]

\[
u_4 = (t_2^{-1} t_1^{-1} t_2^{-1} t_1^{-1})^j(t_2^{-1} t_1^{-1})^{a_{R+2}+2} t_2(-t_2^{-1} t_1^{-1})^{-a_{R+2}}(t_1^{-1} t_2^{-1} t_1^{-1} t_2^{-1})^j,
\]

\[
u_5 = (t_1^{-1} t_2 t_1^{-1} t_2^{-1})^j(t_1^{-1} t_1 t_2^{-1} t_2^{-1} t_1^{-1} t_2^{-1})^{-a_{R+2}}(t_1^{-1} t_1^{-1} t_2^{-1} t_1^{-1} t_2^{-1} t_1^{-1} t_2^{-1})^j.
\]

<table>
<thead>
<tr>
<th>(a_{R+2})</th>
<th>(S = -2)</th>
<th>(S = 2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(L(v_1)) if (i = 0)</td>
<td>(L(v_1)) if (i &gt; 0)</td>
<td>(a_{R+2})</td>
</tr>
<tr>
<td>(\leq -1)</td>
<td>3</td>
<td>8i + 3</td>
</tr>
<tr>
<td>0</td>
<td>2</td>
<td>8i + 1</td>
</tr>
<tr>
<td>1</td>
<td>3</td>
<td>8i + 3</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>8i - 1</td>
</tr>
<tr>
<td>(\geq 3)</td>
<td>3</td>
<td>8i + 3</td>
</tr>
</tbody>
</table>
and

\[ u_6 = t_2^{-1}(t_2^{-1}t_1^{-1}t_2)^j t_2(t_2^{-1}t_1)^{a_{R+2}-2}t_2^{-1}(t_2^{-1}t_1)^{-a_{R+2}}t_2^{-1}(t_2^{-1}t_1t_1^{-1})^j t_2. \]

We will analyze lengths of the words defined above. These calculations will then be used to complete the proof of this case. Note that the value of \( u_3 \) is equal to either

\[ (t_1t_2^{-1}t_1^{-1}t_2)^j t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1)^{-a_{R+2}-3}t_2^{-1}(t_1t_2)^{-a_{R+2}}(t_2^{-1}t_1t_2^{-1})^j \]

if \( a_{R+2} \leq -3 \), or

\[ (t_1t_2^{-1}t_1^{-1}t_2)^j t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1t_2^{-1})^j \]

if \( a_{R+2} = -2 \), or

\[ (t_1t_2^{-1}t_1^{-1}t_2)^j t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1t_2^{-1})^j \]

if \( a_{R+2} = -1 \), or

\[ (t_1t_2^{-1}t_1^{-1}t_2)^j t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1t_2^{-1})^j \]

if \( a_{R+2} = 0 \), or

\[ (t_1t_2^{-1}t_1^{-1}t_2)^j (t_1t_2)^{a_{R+2}+1}t_1(t_2^{-1}t_1^{-1})^{a_{R+2}}(t_2^{-1}t_1t_2^{-1})^j \]

if \( a_{R+2} \geq 1 \). The value of \( u_4 \) is equal to either

\[ (t_2^{-1}t_1^{-1}t_2t_1)^j t_2^{-1}t_1^{-1}t_2^{-1}t_1t_2(t_2^{-1}t_1t_2)^{-a_{R+2}-3}t_1^{-1} \]

\[ (t_2^{-1}t_1^{-1})^{-a_{R+2}-2}t_2^{-1}t_1^{-2}t_1t_2(t_1t_2^{-1}t_1t_2)^j \]

if \( a_{R+2} \leq -3 \), or

\[ (t_2^{-1}t_1^{-1}t_2t_1)^j t_2^{-1}t_1^{-3}t_2^{-1}t_1t_2(t_1t_2^{-1}t_1t_2)^j \]

if \( a_{R+2} = -2 \), or

\[ (t_2^{-1}t_1^{-1}t_2t_1)^j t_2^{-1}t_1^{-3}t_2^{-1}t_1t_2(t_1t_2^{-1}t_1t_2)^j \]
if $a_{R+2} = -1$, or

\[(t_2^{-1}t_1^{-1}t_2^{-1}t_1j_t_2^{-1}t_1^{-1}t_2^{-1}t_1^{-1}t_2(t_1^{-1}t_2^{-1}t_1t_2)^j)\]

if $a_{R+2} = 0$, or

\[(t_2^{-1}t_1^{-1}t_2^{-1}t_1j(t_2^{-1}t_1^{-1})a_{R+2}+2t_2(t_1t_2)^a_{R+2}(t_1^{-1}t_2^{-1}t_1t_2)^j)\]

if $a_{R+2} \geq 1$. The value of $u_5$ is equal to either

\[(t_1^{-1}t_2t_1t_2^{-1})^{-1}_j^{-1}_1^{-1}t_2t_1t_2^{-2}t_1t_2^{-1}t_1(t_2t_1^{-1}t_2^{-1}t_1)^j^{-1}_1\]

if $a_{R+2} \leq -1$, or

\[(t_1^{-1}t_2t_1t_2^{-1})^{-1}_j^{-1}_1^{-1}t_2t_1t_2^{-2}t_1t_2^{-1}t_1(t_2t_1^{-1}t_2^{-1}t_1)^j^{-1}_1\]

if $a_{R+2} = 0$, or

\[(t_1^{-1}_1t_2t_1t_2^{-1})^{-1}_j^{-1}_1^{-1}t_2t_1t_2^{-2}t_1t_2^{-1}t_1(t_2t_1^{-1}t_2^{-1}t_1)^j\]

if $a_{R+2} = 1$, or

\[(t_1^{-1}t_2t_1t_2^{-1})^{-1}_j^{-1}_1^{-1}t_2t_1t_2^{-2}t_1(t_2t_1^{-1}t_2^{-1}t_1)^j\]

if $a_{R+2} = 2$, or

\[(t_1^{-1}_1t_2t_1t_2^{-1})^{-1}_j^{-1}_1^{-1}t_2t_1t_2^{-2}t_1(t_2t_1^{-1}t_2^{-1}t_1)^j\]

if $a_{R+2} \geq 3$. Finally, the value of $u_6$ is equal to either

\[t_2^{-1}(t_1^{-1}t_2^{-1}t_1^{-1}t_2)^j^{-1}_1^{-1}t_2^{-1}t_1^{-1}t_2^{-1}(t_1^{-1}t_2^{-1}t_1t_2)^j^{-1}_1^{-1}t_2^{-1}(t_2^{-1}t_1^{-1}t_2^{-1}t_1)^j^{-1}_2\]
if $a_{R+2} \leq -1$, or

$$t_2^{-1}(t_1 t_2^{-1} t_1^{-1} t_2) j^{-1} t_1 t_2^{-1} t_1^{-1} t_2 t_1^{-1} t_2^{-2} t_1 t_2^{-1} (t_2^{-1} t_1 t_2^{-1} t_1^{-1} t_2) t_2$$

if $a_{R+2} = 0$, or

$$t_2^{-1}(t_1 t_2^{-1} t_1^{-1} t_2) j^{-1} t_1 t_2^{-1} t_1^{-3} (t_2^{-1} t_1 t_2^{-1} t_1^{-1} t_2) t_2$$

if $a_{R+2} = 1$, or

$$t_2^{-1}(t_1 t_2^{-1} t_1^{-1} t_2) j^{-1} t_1 t_2^{-1} t_1^{-1} (t_2^{-1} t_1 t_2^{-1} t_1^{-1} t_2) t_2$$

if $a_{R+2} = 2$, or

$$t_2^{-1}(t_1 t_2^{-1} t_1^{-1} t_2) j^{-1} t_1 t_2^{-1} t_1^{-4} (t_2^{-1} t_1 t_2^{-1} t_1^{-1} t_2) t_2$$

if $a_{R+2} \geq 3$. The values of the length of $u_d$ ($d = 3, 4, 5, 6)$, dependent on the value of $a_{R+2}$, are gathered in tables 6.8 and 6.9. By equation (6.16f), $g$ is an algebraic diffeomorphism if and only if $\bar{g}: T^2 \rightarrow T^2$ defined by

$$\bar{g}(t) = (t_1 ^{a_{R+1} - a_R} (t_2 ^{S})^e, t_1 ^{a_{R+1} - a_R}, t_2 ^{-S} t_1 ^{-a_R})$$

is an algebraic diffeomorphism, which is true if and only if $a_{R+1} = a_R$ or $a_{R+1} = a_R - S$.

First, consider the case $a_{R+1} = a_R$. Then $g(t) = (t_1 ^{a_{R+1} - a_R} (t_2 ^{S})^e, t_1 ^{a_{R+1} - a_R}, t_2 ^{-S} t_1 ^{-a_R})$, and the

<table>
<thead>
<tr>
<th>$a_{R+2}$</th>
<th>$L(u_3)$</th>
<th>$L(u_4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq -3$</td>
<td>$8j - 4a_{R+2} - 9$</td>
<td>$8j - 4a_{R+2} - 7$</td>
</tr>
<tr>
<td>$-2$</td>
<td>$8j - 5$</td>
<td>$8j - 3$</td>
</tr>
<tr>
<td>$-1$</td>
<td>$8j - 1$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>$0$</td>
<td>$8j + 3$</td>
<td>$8j + 5$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$8j + 4a_{R+2} + 3$</td>
<td>$8j + 4a_{R+2} + 5$</td>
</tr>
</tbody>
</table>
value of \( \lambda_j(E_{R+2})(g(t)) \) is equal to

\[
\begin{align*}
\lambda_j(E_{R+2})(g(t)) &= \begin{cases} 
  t_1^{a_R} (t_1^{S_1})^{e} t_2^{S_2} (t_1^{S_1})^{-e} t_2^{S_2}) (t_1^{S_1})^{e} t_2^{S_2}) t_2^{S_2} 
  & \text{if } S = 2 \text{ and } \epsilon = 1, \\
  t_1^{a_R} t_1^{-a_R} t_2^{a_R} & \text{if } S = 2 \text{ and } \epsilon = -1, \\
  t_1^{a_R} u_5 t_1^{-a_R} & \text{if } S = -2 \text{ and } \epsilon = 1, \\
  t_1^{a_R} u_6 t_1^{-a_R} & \text{if } S = -2 \text{ and } \epsilon = -1.
\end{cases}
\end{align*}
\]

Define \( u_d = t_1^{a_R} u_{d-4} t_1^{-a_R} \) for \( d = 7, 8, 9, 10 \). The values of the length of \( u_d \), dependent on the values of \( a_R \) and \( a_{R+2} \), calculated using the values from tables 6.8 and 6.9, are gathered in tables 6.10 and 6.11. Since \( 0 \leq i < j \), using the information gathered in tables 6.7, 6.10,
Define \( u_{i11} = t_1^{aR} u_4(t_1^{-1}, t_2^{-1}) t_1^{-aR} \), \( u_{12} = t_1^{aR} u_3(t_1^{-1}, t_2^{-1}) t_1^{-aR} \), \( u_{13} = t_1^{aR} u_6(t_1^{-1}, t_2^{-1}) t_1^{-aR} \), and \( u_{14} = t_1^{aR} u_5(t_1^{-1}, t_2^{-1}) t_1^{-aR} \). Note that \( L(u_{d}(t_1^{-1}, t_2^{-1})) = L(u_d) \) for \( d = 3, 4, 5, 6 \). The values of the length of \( u_d \) \((d = 11, 12, 13, 14)\), dependent on the values of \( a_R \) and \( a_{R+2} \), calculated using the values from tables 6.8 and 6.9, are gathered in tables 6.12 and 6.13.
The proof of the case \( i, j, a \), using the information gathered in tables 6.7, 6.12, 6.13, for all values of \( i, j, a, a_{R+2} \), we have \( L(u_1^{\pm 1}) < L(u_d) \) \((d = 11, 12, 13, 14)\) and hence

\[
L(u_1^{\pm 1}) < L\left(\lambda_j(E_{R+2})(g(t))\right).
\]

This inequality contradicts equation (6.25) and hence the statement of Proposition 6.10.

The proof of the case \( l = R - |S| + 4 \) and \( |S| \geq 2 \) is now completed.

Consider the case \( 4 \leq l \leq R - |S| \) and \( l \) even, or \( l = R - |S| + 2 \) and \( P \neq |S| + 1 \). By equation (6.16g),

\[
M(g(t)) = \varepsilon \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix}.
\]

If \( S \neq 0 \), then \((\lambda_i(E_{R+2})(t))_{ab} = [S]_1\), and \((\lambda_j(E_l)(t))_{ab} = [0]_1\). Thus

\[
M(g(t))(\lambda_j(E_l)(t))_{ab} = \varepsilon \begin{bmatrix}
1 & 0 \\
0 & -1
\end{bmatrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} 0 \\ -1 \end{bmatrix}.
\]
Since \( f(E_{R+2}) = E_t \), by Proposition 6.10 we get

\[
\pm \begin{bmatrix} S \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 0 \\ -1 \end{bmatrix},
\]

a false statement (since \( S \neq 0 \)) that contradicts the statement of the proposition. By equation (6.16g), \( g \) is an algebraic diffeomorphism if and only if \( a_{l-2} = 0 \) or \( a_{l-1} = 0 \). Let \( S = 0 \). If \( 4 \leq l \leq R \) and \( l \) even, then \( \lambda_i(E_t)(t) = t_1^{a_l} t_2 t_1^{-a_l} \), and \( \lambda_j(E_{R+2})(t) = v_4 \) (see equation (6.7d)). Since \( f(E_t) = E_{R+2} \), by Proposition 6.10 we get

(6.26) \[ t_1^{a_l} t_2^{\pm 1} t_1^{-a_l} = \lambda_j(E_{R+2})(g(t)). \]

We have

\[
\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
 t_2^{a_{l-1}} (t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}) \frac{\varepsilon}{t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} (t_2^{\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon}) t_2^{-a_{l-1}} & \text{if } a_{l-2} = 0, \\
 t_1^{a_{l-2}} (t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}) \frac{\varepsilon}{t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} (t_2^{\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon}) t_1^{-a_{l-2}} & \text{if } a_{l-1} = 0.
\end{cases}
\]

Since \( j \geq 1 \),

\[
L \left( t_2^{a_{l-1}} (t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}) \frac{\varepsilon}{t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} (t_2^{\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon}) t_2^{-a_{l-1}} \right) \geq 7,
\]

and

\[
L \left( t_1^{a_{l-2}} (t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}) \frac{\varepsilon}{t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon}} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} (t_2^{\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon} t_1^{\varepsilon} t_2^{-\varepsilon}) t_1^{-a_{l-2}} \right) \geq 5.
\]

The inequalities above contradict equation (6.26) and hence the statement of Proposition 6.10.

Let \( S = P = 0 \) and \( l = R + 2 \). Then \( \lambda_i(E_{R+2})(t) = v_2 \) (see equation (6.7b)), and \( \lambda_j(E_{R+2})(t) = v_4 \) (see equation (6.7d)), Since \( f(E_{R+2}) = E_{R+2} \), by Proposition 6.10 we get

(6.27) \[ v_2^{\pm 1} = \lambda_j(E_{R+2})(g(t)). \]
Define
\[ u_{15} = (t_1^{\varepsilon} t_2^{\varepsilon} t_1^{\varepsilon} t_2^{\varepsilon}) t_1^{\varepsilon a_R+2} t_2^{\varepsilon} t_1^{\varepsilon a_R+2} (t_2^{\varepsilon} t_1^{\varepsilon} t_2^{\varepsilon} t_1^{\varepsilon}) j. \]

Then
\[
\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
    t_2^{a_R+1} u_{15}^{a_R+1} & \text{if } a_R = 0, \\
    t_2^{a_R} u_{15}^{a_R} & \text{if } a_{R+1} = 0.
\end{cases}
\]

Let \( u_{16} = t_2^{a_R+1} u_{15}^{a_R+1} \) and \( u_{17} = t_2^{a_R} u_{15}^{a_R} \). The values of the length of \( v_2 \) dependent on the values of \( i \) and \( a_{R+2} \) are gathered in table 6.14. The values of lengths of \( u_{16} \) and \( u_{17} \) are gathered in table 6.15. Since \( 0 \leq i < j \), using the information gathered in tables 6.14, 6.15, for all values of \( i, j, a_R, a_{R+1}, a_{R+2} \), we have \( L(v_2^\pm) < L(u_d) \) \((d = 16, 17)\) and hence
\[
L(v_2^\pm) < L(\lambda_j(E_{R+2})(g(t))).
\]

This inequality contradicts equation (6.27) and hence the statement of Proposition 6.10.

The proof of the case \( 4 \leq l \leq R - |S| \) and \( l \) even, or \( l = R - |S| + 2 \) and \( P \neq |S| + 1 \) is now completed.

Consider the case \( l = R + 2 \) and \( P = |S| + 1 \) \((i.e., \ S = 0 \ and \ P = 1)\). By equation (6.16h),
\[
M(g(t)) = \varepsilon \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}.
\]

Table 6.14: The length of \( v_2 \) dependent on the pair \((i, a_{R+2})\)

<table>
<thead>
<tr>
<th>( a_{R+2} )</th>
<th>( L(v_2) ) if ( i = 0 )</th>
<th>( L(v_2) ) if ( i &gt; 0 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>( 8i - 1 )</td>
</tr>
<tr>
<td>( \neq 0 )</td>
<td>3</td>
<td>( 8i + 3 )</td>
</tr>
</tbody>
</table>
Define the value of \( a \) if and only if a false statement that contradicts the statement of the proposition. Since \( \lambda \) is not equal to 0, let \( \lambda = 4 \) if \( a = 8 \) or \( \lambda = 8 \) if \( a = 1 \), then:

\[
M(g(t))(\lambda_j(E_{R-1})(t))_{ab} = \varepsilon \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\]

Since \( f(E_3) = E_{R-1} \), by Proposition 6.10 we get

\[
\pm \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 1 \end{bmatrix},
\]

a false statement that contradicts the statement of the proposition.

Let \( R = 2 \) and \( l = 4 \). Then \( g(t) = (t_1^{a_3}(t_1 t_2)^{\varepsilon}t_1^{-a_3}, t_2^{-\varepsilon}) \). So \( g \) is an algebraic diffeomorphism if and only if \( a_3 = -1 \) or \( a_3 = 0 \). Also, \( \lambda_i(E_4)(t) = v_2 \) (see equation (6.7b)), and \( \lambda_j(E_4)(t) = v_3 \) (see equation (6.7d)). Since \( f(E_4) = E_4 \), by Proposition 6.10 we get

\[
(6.28) \quad v_2^{\pm 1} = \lambda_j(E_4)(g(t)).
\]

The value of \( \lambda_j(E_4)(g(t)) \) is equal to

\[
(t_1^{a_3}(t_1 t_2)^{\varepsilon}t_1^{-a_3}, t_2^{-\varepsilon})(t_1^{a_3}(t_1 t_2)^{-\varepsilon}t_1^{a_3})(t_1 t_2)^{\varepsilon}t_1^{-a_3}t_2^{-\varepsilon})j \]

\[
t_1^{a_3}(t_1 t_2)^{-\varepsilon}t_1^{a_3}(t_1 t_2)^{-\varepsilon}t_1^{a_3}t_2^{\varepsilon}t_1^{a_3}t_2^{-\varepsilon}.
\]

Define

\[
u_{18} = (t_2 t_1 t_2^{-1} t_1^{-1})j t_2(t_1 t_2)^{a_1} t_1(t_1 t_2)^{-a_4} t_2^{-1}(t_1 t_2 t_1^{-1} t_2^{-1})j,
\]

\[
u_{19} = (t_1^{-1} t_2 t_1^{-1} t_2^{-1})j t_1^{-1}(t_1 t_2)^{-a_4} t_2(t_1 t_2)^{a_1} t_1(t_2 t_1^{-1} t_2^{-1})j.
\]
\[ u_{20} = (t_1^{-1} t_1^{-1} t_2^{-1} t_2^{-1} j (t_1 t_2)^{a_4} t_2^{-1} (t_1 t_2)^{-a_4} (t_2^{-1} t_1 t_2^{-1})^{-1})^j, \]

and

\[ u_{21} = (t_2^{-1} t_1^{-1} t_2^{-1} t_1^{-1} j (t_1 t_2)^{-a_4} t_2 (t_1 t_2)^{a_4} (t_1^{-1} t_2^{-1} t_2^{-1})^j. \]

Then

\[
\lambda_j(E_{R+2})(g(t)) = \begin{cases} 
  u_{18} & \text{if } a_3 = -1 \text{ and } \varepsilon = 1, \\
  u_{19} & \text{if } a_3 = -1 \text{ and } \varepsilon = -1, \\
  u_{20} & \text{if } a_3 = 0 \text{ and } \varepsilon = 1, \\
  u_{21} & \text{if } a_3 = 0 \text{ and } \varepsilon = -1.
\end{cases}
\]

We will analyze lengths of the words \( u_d \) \((d = 18, 19, 20, 21)\) depending on the value of \( a_4 \). the value of \( u_{18} \) is equal to either

\[
(t_2^{-1} t_1^{-1})^{-1} t_2^{-1} t_2^{-1} t_1^{-2} (t_2^{-1} t_1^{-1})^{-a_4-1} t_2^{-1} (t_1 t_2)^{-a_4-1} t_2^{-1} t_1^{-1} t_2 t_2^{-1} (t_1 t_2^{-1} t_1^{-1})^{-1}
\]

if \( a_4 \leq -1 \), or

\[
(t_2^{-1} t_1^{-1})^{-1} t_2^{-1} (t_1 t_2^{-1} t_2^{-1})^j
\]

if \( a_4 = 0 \), or

\[
(t_2^{-1} t_1^{-1})^j t_2 (t_1 t_2)^{a_4-1} t_1^{-1} (t_2^{-1} t_1^{-1})^{a_4-1} t_2 t_2^{-1} (t_1 t_2^{-1} t_2^{-1})^j
\]

if \( a_4 \geq 1 \). The value of \( u_{19} \) is equal to either

\[
(t_1^{-1} t_2^{-1} t_1^{-1})^{-1} t_1^{-1} t_2^{-1} t_2^{-2} t_2 (t_1 t_2)^{-a_4-1} t_1^{-1} (t_2^{-1} t_1^{-1})^{-a_4-2} t_2^{-1} t_1^{-2} t_2^{-1} t_1 (t_2^{-1} t_2^{-1} t_1^{-1})^{-1}
\]

if \( a_4 \leq -2 \), or

\[
(t_1^{-1} t_2^{-1} t_1^{-1})^{-1} t_1^{-1} t_2^{-1} t_2^{-1} t_2^{-1} t_1 (t_2^{-1} t_2^{-1} t_1)^{j-1}
\]
if \( a_4 = -1 \), or
\[
(t_1^{-1}t_2^{-1}t_2^{-1}t_2^{-1})^{j-1}t_1^{-1}t_2t_1(t_2^{-1}t_2^{-1}t_1)^j
\]
if \( a_4 = 0 \), or
\[
(t_1^{-1}t_2^{-1}t_2^{-1}t_2^{-1})^{j}t_1^{-1}(t_2^{-1}t_1^{-1})a_4^{-1}t_2(t_1t_2)^a_4^{-1}t_1(t_2^{-1}t_2^{-1}t_1)^j
\]
if \( a_4 \geq 1 \). The value of \( u_{20} \) is equal to either
\[
(t_1t_2^{-1}t_1^{-1}t_2)^{j-1}t_1t_2^{-1}t_1^{-2}(t_2^{-1}t_1^{-1})^{-a_4-1}t_1^{-1}(t_2t_1)^{-a_4-1}t_1^2t_2t_1^{-1}(t_2^{-1}t_1t_2^{-1})^{j-1}
\]
if \( a_4 \leq -1 \), or
\[
(t_1t_2^{-1}t_1^{-1}t_2)^{j-1}t_1t_2^{-1}t_1^{-1}(t_2^{-1}t_1t_2^{-1})^{j}
\]
if \( a_4 = 0 \), or
\[
(t_1t_2^{-1}t_1^{-1}t_2)(t_1t_2)^{a_4-1}t_1(t_2^{-1}t_1^{-1})a_4(t_2^{-1}t_1t_2^{-1})^{j}
\]
if \( a_{R+2} \geq 1 \). Finally, the value of \( u_{21} \) is equal to either
\[
(t_2^{-1}t_1^{-1}t_2^{-1}t_1)^{j-1}t_2^{-1}t_1^{-1}t_2t_1^2(t_1t_2)^{-a_4-1}t_1^{-1}(t_2^{-1}t_1^{-1})^{-a_4-2}t_2^{-1}t_1^{-2}t_2^{-1}t_1t_2(t_2^{-1}t_2^{-1}t_1t_2)^{j-1}
\]
if \( a_4 \leq -2 \), or
\[
(t_2^{-1}t_1^{-1}t_2t_1)^{j-1}t_2^{-1}t_1^{-1}t_2t_1^2t_2^2t_1^{-2}t_1t_2(t_1^{-1}t_2^{-1}t_1t_2)^{j-1}
\]
if \( a_4 = -1 \), or
\[
(t_2^{-1}t_1^{-1}t_2t_1)^{j}(t_2^{-1}t_1^{-1})a_4t_2(t_1t_2)^{a_4}(t_1^{-1}t_2^{-1}t_1t_2)^j
\]
if \( a_4 \geq 0 \). The values of the length of \( u_d \) \((d = 18, 19, 20, 21)\), dependent on the value of \( a_4 \), are gathered in table 6.16. Since \( 0 \leq i < j \), using the information gathered in tables 6.14, 6.16, for all values of \( i, j, a_3, a_4 \), we have \( L(v_2^{\pm 1}) < L(u_d) \) \((d = 18, 19, 20, 21)\) and hence
\[
L(v_2^{\pm 1}) < L\left(\lambda_j(E_4)(g(t))\right).
\]
Table 6.16: The lengths of $u_d$ ($d = 18, 19, 20, 21$)

<table>
<thead>
<tr>
<th>$a_4$</th>
<th>$L(u_{18})$</th>
<th>$L(u_{19})$</th>
<th>$L(u_{20})$</th>
<th>$L(u_{21})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\leq -2$</td>
<td>$8j - 4a_4 - 3$</td>
<td>$8j - 4a_4 - 5$</td>
<td>$8j - 4a_4 - 5$</td>
<td>$8j - 4a_4 - 3$</td>
</tr>
<tr>
<td>$-1$ or $0$</td>
<td>$8j + 1$</td>
<td>$8j - 1$</td>
<td>$8j - 1$</td>
<td>$8j + 1$</td>
</tr>
<tr>
<td>$\geq 1$</td>
<td>$8j + 4a_4 + 1$</td>
<td>$8j + 4a_4 - 1$</td>
<td>$8j + 4a_4 - 1$</td>
<td>$8j + 4a_4 + 1$</td>
</tr>
</tbody>
</table>

This inequality contradicts equation (6.28) and hence the statement of Proposition 6.10.

The proof of the case $l = R + 2$ and $P = |S| + 1$ is now completed.

Consider the final case $R - |S| + 5 \leq l \leq R + 2$. By equation (6.16i),

$$M(g(t)) = \varepsilon \begin{bmatrix} \bar{S}(l - R + |S| - 3) & -l + R - |S| + 4 \\ 1 & -\bar{S} \end{bmatrix}.$$  

Also, $(\lambda_i(E_{l-3})(t))_{ab} = \begin{bmatrix} \bar{S}(l-R+|S|-5) \end{bmatrix}$, and

$$(\lambda_j(E_3)(t))_{ab} = \begin{cases} \begin{bmatrix} \bar{S} \end{bmatrix} & \text{if } |S| = R, \\ \begin{bmatrix} 1 \end{bmatrix} & \text{if } 3 \leq |S| \leq R - 1. \end{cases}$$

Thus the value of $M(g(t))(\lambda_j(E_3)(t))_{ab}$ is equal to either

$$\varepsilon \begin{bmatrix} \bar{S}(l - 3) & -l + 4 \\ 1 & -\bar{S} \end{bmatrix} \begin{bmatrix} \bar{S} \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

if $|S| = R$, or

$$\varepsilon \begin{bmatrix} \bar{S}(l - R + |S| - 3) & -l + R - |S| + 4 \\ 1 & -\bar{S} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \varepsilon \begin{bmatrix} \bar{S}(l - R + |S| - 3) \\ 1 \end{bmatrix}$$
if $3 \leq |S| \leq R - 1$. Since $f(E_{l-3}) = E_3$, by Proposition 6.10, we get either

$$\pm \begin{bmatrix} S(l - 5) \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

if $|S| = R$, or

$$\pm \begin{bmatrix} \hat{S}(l - R + |S| - 3) \\ 1 \end{bmatrix} = \varepsilon \begin{bmatrix} \hat{S}(l - R + |S| - 5) \\ 1 \end{bmatrix}$$

if $3 \leq |S| \leq R - 1$. Both equations above are false statements that contradict the statement of Proposition 6.10. This concludes the proof of the case where $f$ is orientation-reversing.

We have contradicted existence of an algebraic isomorphism between $\mathcal{X}_i$ and $\mathcal{X}_j$ in all possible cases. The proof of the theorem is therefore completed.

Note that the results of Theorem 6.8, Proposition 6.9, and Theorem 6.11 constitute the proof of the main result of the paper (Theorem 3.14).
REFERENCES


APPENDICES
APPENDIX A
A SUPPLEMENTARY CALCULATION FOR EXAMPLE 4.11

Recall that

\[ \sigma^{-1}(x_1) = x_1x_2^{-1}x_1^{-2} \]
\[ \sigma^{-1}(x_2) = x_2x_1^{-1}x_2^{-1}x_1x_2^{-1}x_2^{-1}x_1 \]

Then

\[ \sigma^{-1}(w) = (x_1x_2^{-1}x_1^{-2})^{-1}(x_2x_1^{-1}x_2^{-1}x_1x_2^{-1}x_2^{-1}x_1^{-1})^{-1}\]
\[ = x_2^{-2}x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1 \]
\[ = x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1 \]
\[ = (x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1) \]
\[ = (x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1x_2^{-1}x_1^{-1}x_2^{-1}x_1)^{-1} \]
We refer the reader to Scott’s discussion presented in §4 of [7]. Our modification starts with replacing Scott’s assumption about the word \( w_{i+1} \) (see p. 390 in [7]). We let

\[ w_{i+1} = w_i^a w_{i-1}^b w_i^c. \]

The concept of Scott’s discussion remains the same, so we just outline the changes required by the modification mentioned above. We let \( v_{i+1} \) be of the form \( w_i^{c} w_i^{d} w_i^{e} w_i^{f} \). Then with respect to the chosen trivializations, \( \partial_1 \) is diffeomorphic to the manifold

\[ T \times D^4 \cup_f D^4 \times T, \tag{B.1} \]

where \( f \) is the diffeomorphism of \( T^2 \) with rule of assignment

\[ (t_1, t_2) \mapsto (t_2^a t_1^b t_2^c, t_2). \]

Similarly, \( \partial_2 \) is diffeomorphic to

\[ T \times D^4 \cup_g D^4 \times T, \]

where \( g \) is the diffeomorphism of \( T^2 \) with rule of assignment

\[ (t_1, t_2) \mapsto (t_2^a t_1^b t_2^c, t_2). \]

Since the map \( \phi: D^4 \times T \to D^4 \times T \) with rule of assignment \( (x, y) \mapsto (y^{-a} x y^{-b}, y) \) is a diffeomorphism, we can replace \( f \) in (B.1) with \( \phi \circ f \) without changing the diffeotype. Hence \( \partial_1 \) is diffeomorphic to \( S^7 \), and the identical argument works for \( \partial_2 \). Then the neighborhood
of $D_i$ is diffeomorphic to the manifold obtained by gluing two polydisks $D^4 \times D^4$ together along the diffeomorphism $\tilde{\phi}: D^4 \times T \to D^4 \times T$ with rule of assignment

$$(x, y) \mapsto (y, y^a x y^b).$$

$D_i$ sits inside this manifold as the embedded 4-sphere

$$\{0\} \times D^4 \cup_{\tilde{\phi}|\{0\} \times T} D^4 \times \{0\}$$

and $\tilde{\phi}$ determines the clutching function

$$y \mapsto (x \mapsto y^a x y^b)$$

for its normal bundle. Now, this oriented normal bundle is precisely the vector bundle discussed by Milnor, where the first Pontrjagin class is found to be $\pm 2(a - b)\iota$ with $\iota$ being the generator for $H^4(S^4)$ (see [4], Lemma 3, p. 402). By choosing the appropriate generator for $\pi_3(SO)$, we obtain

$$\langle p_1(X), D_i \rangle = 2(a - b).$$
APPENDIX C
SUPPLEMENTARY CALCULATIONS FOR THE PROOF OF THEOREM 5.6

First we simplify the sums in equation (5.13) to obtain equation (5.14).

\[
\sum_{k \in K} \left( - \sum_{j=2}^{l_{k-1} - 1} \left( Y_{i+1} X_{k-1,l_{k-1}+1-j} - X_{i+1} Y_{k-1,l_{k-1}+1-j} \right) q_{k-1,l_{k-1}-j+2} \right. \\
\left. \quad + \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} \left( Y_{i+1} X_{k,j-l_{k-1}+1} - X_{i+1} Y_{k,j-l_{k-1}+1} \right) q_{k,j-l_{k-1}+2} \right)
\]

\[
= \sum_{k=1}^{i-1} \left( - \sum_{j=2}^{l_{k-1} - 1} \left( Y_{i+1} X_{k-1,l_{k-1}+1-j} - X_{i+1} Y_{k-1,l_{k-1}+1-j} \right) q_{k-1,l_{k-1}-j+2} \right. \\
\left. \quad + \sum_{k=i+2}^{r} \left( \sum_{j=l_{k-1}+1}^{l_{k-1}-1} \left( Y_{i+1} X_{k,j-l_{k-1}+1} - X_{i+1} Y_{k,j-l_{k-1}+1} \right) q_{k,j-l_{k-1}+2} \right) 
\quad + \sum_{k=i+2}^{r} \left( - \sum_{j=2}^{l_{k-1} - 1} \left( Y_{i+1} X_{k-1,l_{k-1}+1-j} - X_{i+1} Y_{k-1,l_{k-1}+1-j} \right) q_{k-1,l_{k-1}-j+2} \right. \\
\left. \quad \left. \quad + \sum_{j=l_{k-1}+1}^{l_{k-1}+l_k-2} \left( Y_{i+1} X_{k,j-l_{k-1}+1} - X_{i+1} Y_{k,j-l_{k-1}+1} \right) q_{k,j-l_{k-1}+2} \right) 
\quad + \sum_{j=l_{i-2}+1}^{l_{i-1}-1} \left( Y_{i+1} X_{i-1,j-l_{i-2}+1} - X_{i+1} Y_{i-1,j-l_{i-2}+1} \right) q_{i-1,j-l_{i-2}+2} \right)
\quad - \sum_{j=2}^{l_{i+1} - 1} \left( Y_{i+1} X_{i+1,l_{i+1}+1-j} - X_{i+1} Y_{i+1,l_{i+1}+1-j} \right) q_{i+1,l_{i+1}-j+2} \\
\left. \quad + \sum_{k=i+3}^{r} \left( - \sum_{j=2}^{l_{k-1} - 1} \left( Y_{i+1} X_{k-1,l_{k-1}+1-j} - X_{i+1} Y_{k-1,l_{k-1}+1-j} \right) q_{k-1,l_{k-1}-j+2} \right) \right)
\]
Next we simplify the sums in equation (5.18) to obtain equation (5.19).

\[
\begin{align*}
&\sum_{k=i+2}^{r-1} \left( \sum_{j=l_{k-1}+1}^{l_k-2} \left( Y_{i+1}X_{k,j-l_{k-1}+1} - X_{i+1}Y_{k,j-l_{k-1}+1} \right) q_{k,j-l_{k-1}+2} \right) \\
&+ \sum_{j=l_{r-1}+1}^{l_r-2} \left( Y_{i+1}X_{r,j-l_{r-1}+1} - X_{i+1}Y_{r,j-l_{r-1}+1} \right) q_{r,j-l_{r-1}+2} \\
= &\sum_{m=2}^{i-2} \left( \sum_{k=1}^{l_k-1} \left( Y_{i+1}X_{k,m} - X_{i+1}Y_{k,m} \right) q_{k,m+1} \right) \\
&+ \sum_{j=2}^{i-2} \left( \sum_{m=2}^{l_k-1} \left( -Y_{i+1}X_{k,m} + X_{i+1}Y_{k,m} \right) q_{k,m+1} \right) \\
&+ \sum_{k=i+2}^{r-1} \left( \sum_{m=2}^{l_k-1} \left( Y_{i+1}X_{k,m} - X_{i+1}Y_{k,m} \right) q_{k,m+1} \right) \\
&+ \sum_{m=2}^{i-1} \left( Y_{i+1}X_{i-1,m} - X_{i+1}Y_{i-1,m} \right) q_{i-1,m+1} \\
&- \sum_{m=2}^{r-1} \left( \sum_{k=i+2}^{l_k-1} \left( Y_{i+1}X_{k,m} - X_{i+1}Y_{k,m} \right) q_{k,m+1} \right) \\
&+ \sum_{m=2}^{i-2} \left( \sum_{k=1}^{l_k-1} \left( -Y_{i+1}X_{k,m} + X_{i+1}Y_{k,m} \right) q_{k,m+1} \right) \\
&+ \sum_{m=2}^{i-1} \left( Y_{i+1}X_{r,m} - r_{i+1}Y_{r,m} \right) q_{r,m+1} \\
= &\sum_{m=2}^{i-1} \left( Y_{i+1}X_{i-1,m} - X_{i+1}Y_{i-1,m} \right) q_{i-1,m+1} - S_{i+1}
\end{align*}
\]
\begin{align*}
  &+ \sum_{k=1}^{i-1} \left( \sum_{j=l_{k-1}+1}^{l_k-2} (Y_{i,k,j} - X_i Y_{k,j} q_{k,j}) q_{k,j} - l_{k-1} + 1 \right) \\
  &+ \sum_{k=i+1}^{r} \left( - \sum_{j=2}^{l_{k-1}+1-j} (Y_{i,k,j} - X_i Y_{k,j} q_{k,j}) q_{k,j} - l_{k-1} + j + 2 \right) \\
  &+ \sum_{k=i+1}^{r} \left( \sum_{j=l_{k-1}+1}^{l_k-2} (Y_{i,k,j} - X_i Y_{k,j} q_{k,j}) q_{k,j} - l_{k-1} + 2 \right) \\
  &= - \sum_{j=2}^{l_{i-1}+l_{i-2}+1-j} (Y_{i,k,l_{i-1}+1-j} - X_i Y_{i,l_{i-1}+1-j} q_{i,l_{i-1}+j} + 2) \\
  &+ \sum_{k=i+2}^{r} \left( - \sum_{j=2}^{l_{k-1}+1-j} (Y_{i,k,j} - X_i Y_{k,j} q_{k,j}) q_{k,j} - l_{k-1} + j + 2 \right) \\
  &+ \sum_{k=i+1}^{r} \left( \sum_{j=l_{k-1}+1}^{l_k-2} (Y_{i,k,j} - X_i Y_{k,j} q_{k,j}) q_{k,j} - l_{k-1} + 2 \right) \\
  &+ \sum_{j=l_{r-1}+1}^{l_{r-1}+l_{r-2}+1+l_{r-1}+1} (Y_{i,k,l_{r-1}+1} - X_i Y_{r,j-l_{r-1}+1} q_{r,j-l_{r-1}+2}) \\
  &= - \sum_{m=2}^{i-2} (Y_{i,k,l_{m-1}} - X_i Y_{k,m} q_{k,m} + 1) \\
  &+ \sum_{k=1}^{i-2} \left( - \sum_{m=2}^{l_{k-1}} (Y_{i,k,m} - X_i Y_{k,m} q_{k,m}) q_{k,m} + 1 \right) \\
  &+ \sum_{k=1}^{i-2} \left( \sum_{m=2}^{l_{k-1}} (Y_{i,k,m} - X_i Y_{k,m} q_{k,m}) q_{k,m} + 1 \right) \\
  &+ \sum_{m=2}^{l_{i-1}+l_{i-2}+1} (Y_{i,k,l_{i-1}+1} - X_i Y_{i-1,m} q_{i-1,m} + 1) \\
  &+ \sum_{m=2}^{l_{i-1}+l_{i-2}+1} (Y_{i,k,l_{i-1}+1} - X_i Y_{i-1,m} q_{i-1,m} + 1) \\
  &+ \sum_{m=2}^{l_{i-1}+l_{i-2}+1} (Y_{i,k,l_{i-1}+1} - X_i Y_{i-1,m} q_{i-1,m} + 1)
\end{align*}
\[- \sum_{m=2}^{l_i-1} (Y_i X_{i,m} - X_i Y_{i,m}) q_{i,m+1} \]
\[+ \sum_{k=i+1}^{r-1} \left( - \sum_{m=2}^{l_k-1} (Y_k X_{k,m} - X_k Y_{k,m}) q_{k,m+1} \right) \]
\[+ \sum_{k=i+1}^{r-1} \sum_{m=2}^{l_k-1} (Y_k X_{k,m} - X_k Y_{k,m}) q_{k,m+1} \]
\[+ \sum_{m=2}^{l_r-1} (Y_r X_{r,m} - X_r Y_{r,m}) q_{r,m+1} \]
\[= \sum_{m=2}^{l_{i-1}-1} (Y_i X_{i-1,m} - X_i Y_{i-1,m}) q_{i-1,m+1} - S_i. \]
CURRICULUM VITAE

Piotr Runge
(November 2009)

Education

• MS in Mathematics (with a teaching certificate), Adam Mickiewicz University, Poznan, Poland. (6/01).

• PhD in Mathematical Sciences, Utah State University, Logan, Utah. (expected 12/09).

Experience

• Mathematics Lecturer, Utah State University Tooele Regional Campus, Tooele, Utah (8/08 – Present).

• Graduate Teaching Assistant, Utah State University Dept. of Mathematics & Statistics, Logan, Utah (8/01 – 5/08).

• Mathematics Tutor, Adam Mickiewicz University, Poznan, Poland (9/96 – 6/01).

• Teaching Practice (high-school), XV Liceum Ogólnokształcące im. prof. Wiktora Degi, Poznan, Poland (9/00).

• Teaching Practice (middle-school), Szkola Podstawowa nr 17 im. Ignacego Kraszewskiego, Poznan, Poland (9/99).

Honors

• Academic Achievement Award, Utah State University Dept. of Mathematics & Statistics, Logan, Utah (4/03).

• Graduate Teaching Award, Utah State University Dept. of Mathematics & Statistics, Logan, Utah (4/07).