$SU(3) \times SU(2) \times U(1)$: The residual symmetry of conformal gravity.

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$SU(4) \times SU(2)$ is shown to characterize the possible choices of spacetime metric constructible from the gauge fields of the conformal group. When this symmetry is broken by the choice of metric, exactly the $SU(3) \times SU(2) \times U(1)$ symmetry of the standard model remains.

The conformal group consists of Lorentz transformations, translations, inverse translations and dilations. Treated in the standard way, conformal symmetry fails as a unifying gauge group: dilations make the mass spectrum continuous, and the inverse translational gauge field is auxiliary. The theory reduces back to the Poincaré group.

These conclusions hold if the gauge field of translations is identified with the spacetime metric. However, because of the presence of the inverse translations, there are 3 distinct metrics of definite Weyl weight constructible from the conformal gauge fields. Minimally extending the symmetry to include a vierbein for each of these 3 possible metrics, a different spacetime metric arises for every choice of inner product of the vierbeins. The covering group of the compact part of the minimal transitive group classifying these inner products is $SU(4)$. An additional $SU(2)$ symmetry classifies the antisymmetric parts of the vierbein product.

If the metric is chosen as the gauge field of the translations in the standard way, the $SU(4) \times SU(2)$ symmetry is broken to $SU(3) \times SU(2) \times U(1)$.

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1 Motivation

There are two points crucial to the understanding of gravity and its relationship to the standard model of the remaining forces [1]. These observations motivate the subsequent search for an additional symmetry and ultimately lead to a natural extension of the conformal algebra.

1. Conformal symmetry must underlie our mathematical description of nature.

2. Naive conformal gravity fails.

The first claim is true because we can measure only relative magnitudes. When the size of a table is given in meters it is being compared to a certain number of wavelengths of light; the mass of a chair is ultimately in comparison to the masses of particles or nuclei. The situation is no different at the subatomic level, where we maintain our use of standard units.

Nor is it necessary for our standard units to be chosen in the same way everywhere. The choice is made in accordance with the purpose at hand. A physicist generally chooses a uniform scale because with such a scale motion has a simple description. But an artist represents objects using a scale having distant objects smaller than the same types of object nearby because this is consistent with our visual process. It is certainly possible and consistent to translate our physical descriptions into such units.

While the Poincaré group does not generate scale changes, the conformal group respects this freedom of choosing scale by preserving only relative lengths. The preservation of one scale choice by the Poincaré group is not physical – scale need not be preserved. Thus, we reach the first conclusion: the local spacetime symmetry which does not claim more than we can know is the conformal group, not the Poincaré group.
RESIDUAL SYMMETRY OF CONFORMAL GRAVITY

To understand the second conclusion requires some understanding of the conformal group. We have, of course, the usual Poincaré symmetries—Lorentz transformations and translations. Lorentz transformations \( \mathfrak{o}(3, 1) \) or \( \mathfrak{sl}(2, \mathbb{C}) \) include both rotations and boosts, and are generated by \( 4 \times 4 \) matrices, \( M_{ab} \). The translations, with generators \( P_a \), give simple displacements. In addition to these are two new transformations:

1. Scale changes, generated by the dilation operator, \( D \) directly produce the allowed choices of units by rescaling lengths locally.

2. Conformal translations. The four generators, \( K_a \), translate infinitesimal vectors in inverse coordinates. Such a vector is inverted through a unit sphere, translated, then re-inverted. The transformation clearly involves an arbitrary displacement vector. The thing to notice—or rather not to notice—is that the vector is also rotated and rescaled. Since both rotating and stretching are already independent transformations, the group closes.

It is central to what follows to understand that with the scale symmetry available, one mode of translation is just as good as the other. Since we only compare relative magnitude, translating with \( K_a \) or \( P_a \) always gives the same physical result: things are moved from here to there and have the same relative magnitude. We now return to the second motivational observation. Seeking a conformal theory of gravity, we introduce gauge fields for each of the generators of the algebra:

\[
M_{ab}, P_a, K_a, D \rightarrow \omega^{ab}, u^a, v^a, W_a
\]

When we gauge the Poincaré group, the first of these, \( \omega^{ab} \), is the spin connection, while the second, the gauge field of translations, \( u^a \), is identified with the vierbein, \( e^a_\alpha \). The vierbein (the soldering form of the fibre bundle) is related to the metric tensor, \( g_{\alpha\beta} \), via the relation

\[
g_{\alpha\beta} = e^a_\alpha e^b_\beta \eta_{ab}
\]

where \( \eta_{ab} \) is the Minkowski metric.

If we compute the conformal curvature tensor we find that it has certain components corresponding to a generalization of the usual Riemannian curvature, and additional components related to the \( u^a_\alpha \), \( v^a_\alpha \), and \( W_\alpha \) fields. A problem arises as soon as we write down a scale-invariant action from this curvature. It has been shown [2,3] that for any such action we write, the \( v^a_\alpha \) field is auxiliary. What happens is that, when the action (which contains 9 independent terms) is varied with respect to \( v^a_\alpha \), the resulting field equation for \( v^a_\alpha \) can be solved and substituted back into the action. The resulting effective action is independent of \( v^a_\alpha \). A second problem arises as soon as we consider the quantization of a scale-invariant theory. The action of a dilation on the generator of translations is

\[
e^{-\lambda D} P_a e^{\lambda D} = e^{\lambda} P_a
\]
which simply says that the action of a dilation on an infinitesimal translation is to change the magnitude, but not the direction, of the displacement. But since $P^2$ is identified with the square of the mass, this implies the relation

$$e^{-\lambda D} P^2 e^{\lambda D} = e^{2\lambda} P^2.$$ 

As a consequence, masses may be changed by an arbitrary factor, $e^{2\lambda}$. The mass spectrum becomes continuous instead of discrete, in contradiction with experiment. For this reason, it is generally assumed that scale-invariance is not a good symmetry at the quantum level [4,5].

Thus, naive implementation of the conformal group leads to the conclusion that the gauge field of the conformal translations may be eliminated and the dilation symmetry must be broken. We are left with the Poincaré symmetry from which we started.

## 2 The metric structure of conformal gauge theory

The conclusions above concerning the elimination of the gauge field, $v^a_\alpha$, hold because we chose to identify the gauge field of translation, $u^a_\alpha$, with the vierbein, and hence with the metric. Had we chosen $v^a_\alpha$ as the vierbein instead, it would have been $u^a_\alpha$ that was auxiliary. The only reason for picking $u^a_\alpha$ is that that is what is done when gauging the Poincaré group. With the conformal group, there is an option. This observation is central. It means that there is an additional symmetry, implicit in the conformal group, which did not need to be broken. The only thing that breaks the symmetry between the two translations, $P_a$ and $K_a$, is our arbitrary choice. If we can rewrite or alter the conformal group in such a way as to make this symmetry explicit, then we may find some new content to conformal gauge theory after all.

This reworking of the group is dependent on the introduction of a metric on the underlying manifold. What we’re going to do is to find all metrics which can be constructed from the gauge fields $u^a_\alpha$ and $v^a_\alpha$, and rebuild the group in a way that guarantees that we can independently pick any of the possibilities as a gauge choice.

It is easy to establish that there are precisely three rank-2, symmetric tensor fields constructible from the gauge fields $u^a_\alpha$ and $v^a_\alpha$. They are:

$$g^1_{\alpha\beta} = u^a_\alpha u^b_\beta \eta_{ab}$$

$$g^2_{\alpha\beta} = u^a_{(\alpha} v^b_{\beta)} \eta_{ab}$$

$$g^3_{\alpha\beta} = v^a_\alpha v^b_\beta \eta_{ab}$$

While there is no guarantee that any of these metrics is invertible or torsion-free, the same is true of the gauge theory of the Poincaré group. Invertibility
and vanishing torsion are assumptions which must be made to reproduce general relativity from the gauge theory, and we make the same assumptions here. The status of these assumptions is a subject of debate. Certainly, invertibility holds generically. As for the vanishing of the torsion, it is still an open question whether torsion does vanish. While the macroscopic limits are quite stringent, there is always the possibility of consistently interpreting some physical field as torsion.

The next step is to write the symmetry so that $g^{\beta\alpha}_{\alpha\beta}$, $g^{2}_{\alpha\beta}$, and $g^{3}_{\alpha\beta}$ are possible gauge choices. This is achieved by introducing a vierbein, $e^{i\alpha}_{\alpha}(i = 1, 2, 3)$, for each possible metric, and introducing a translation generator, $T_{a}^{\alpha}$, as the operator for which $e^{i\alpha}_{\alpha}$ is the gauge field. Clearly, we can let $T_{1}^{\alpha} = P_{a}$ and $T_{3}^{\alpha} = K_{a}$, but $T_{2}^{\alpha}$ is new. One might think that $e^{2\alpha}_{\alpha}$ could be gotten by taking some combination $A_{1}P_{a} + A_{2}K_{a}$, but such combinations always introduce some measure of $g^{1}_{\alpha\beta}$ and $g^{3}_{\alpha\beta}$ in addition, so the middle metric would not be independent.

The presence of the new generator $T_{2}^{\alpha}$ changes the conformal algebra, and we are faced with a choice of several ways to close the new algebra. We could simply let the new translation commute with all of the other generators, but this means that it rotates as four scalars instead of as a vector. We can extend the group until it can contain the generator we want, or we can contract the group. Both of these latter ways work. For example, $O(5, 3)$ contains two translations and five scalar generators in addition to the conformal group, and would therefore work as an extension.

However, since we know already that the conformal group effectively reduces to the Poincaré group, and don’t want to artificially introduce extraneous symmetries, it is most natural to look at contractions. It is possible to rescale the $P_{a}$ and $K_{a}$ generators in such a way that their commutator vanishes. That is, the commutator

$$[P_{a}, K_{b}] = 2\eta_{ab}D - 2M_{ab}$$

is contracted to

$$[P_{a}, K_{b}] = 0$$

with all other commutators remaining the same. Then it is natural to take the third translation to have the same commutation relations as $P_{a}$ and $K_{a}$. The algebra is simply

$$[M_{ab}, M_{cd}] = \eta_{bc}M_{ad} - \eta_{ac}M_{bd} - \eta_{bd}M_{ac} + \eta_{ad}M_{bc}$$

$$[M_{ab}, T_{c}^{i}] = \eta_{bc}T_{a}^{i} - \eta_{ac}T_{b}^{i}$$

$$[T_{a}^{i}, T_{b}^{j}] = 0$$

$$[T_{a}^{i}, D] = (i - 2)T_{a}^{i}$$

$$[M_{ab}, D] = 0$$

$$[D, D] = 0$$
Here the factor \((i - 2)\) in the \(DT\)-commutator gives the scaling weight of each translation. A direct check shows that all Jacobi identities are satisfied.

This Lie algebra contains as much of the conformal group as survives with the standard gauging – i.e., the Poincaré group – and includes in addition the desired symmetry among the translational generators. The gauge field of any of these translations may be identified with the vierbein.

The next question is: What symmetry applies to the class of metrics generated by this algebra? Given the three component vierbein fields as a starting point, any inner product on the \(i\) index produces a possible metric. The general inner product is given in component form by

\[
g_{\alpha\beta} = g_{ij} e^i_\alpha e^j_\beta \eta_{ab}
\]

where \(g_{ij}\) is a symmetric array of numbers. Notice that there is no need for \(g_{ij}\) to be invertible. For example, with the usual identification of \(u^a\) as the vierbein, \(g_{11} = 1\) and all of the other components vanish. Thus, \(g_{ij}\) may be any nonzero symmetric matrix.

The matrix \(g_{ij}\) determines the symmetry group of the class of spacetime metrics, \(g_{\alpha\beta}\). This symmetry group is taken as the covering group of the compact part of the smallest transitive group giving all elements of the class: \(SU(4)\). This specification follows for several reasons:

1. The smallest transitive group is used in order to give all possible metrics while not overcounting.
2. The covering group is chosen to give the maximal expression of the symmetry, and because we require spinor representations.
3. The compact part is taken to insure a discrete mass spectrum [4].

The space of metrics, \(R^6 - \{0\}\), is the same set as the group manifold, \(R \times O(6)\), so the only noncompact part to be neglected is an overall scale factor. The compact part, \(O(6)\), has covering group \(SU(4)\). Note that the scale factor is not the scale generated by \(D\), but does give an overall scaling of the metric. It is therefore this factor of \(R\) which, according to Wess, must be broken to preserve the mass spectrum.

3 The standard model

The emergence of the standard \(SU(3) \times SU(2) \times U(1)\) model depends on two further points. First, we show that picking a particular metric from the \(SU(4)\) class breaks the symmetry to \(SU(3) \times U(1) \times Z\), where the integers, \(Z\), provide a discrete symmetry. Second, we note the presence of an additional \(SU(2)\) Yang-Mills field arising from the antisymmetric product of the vierbeins.
To understand the symmetry breaking that results from choosing a metric, observe that the matrices \( g_{ij} \) form a 6-dimensional vector space. The vector

\[
g^A = (g_{11}, g_{22}, g_{33}, g_{12}, g_{13}, g_{23}) \quad (A = 1, 2, \ldots, 6)
\]

may be written as a bispinor under \( SU(4) \):

\[
g^A \rightarrow [g]_{\alpha \beta} = \text{Re}(\psi_\alpha \psi_\beta) = \begin{pmatrix}
0 & g_{11} & g_{12} & g_{13} \\
-g_{11} & 0 & g_{22} & g_{23} \\
g_{12} & -g_{22} & 0 & g_{33} \\
g_{13} & -g_{23} & -g_{33} & 0
\end{pmatrix}
\]

Here, \( \psi_\alpha (\alpha = 1, 2, 3, 4) \), are the complex, anticommuting components of a spinor, \( \psi \). This antisymmetric form is preserved by \( SU(4) \) rotations, as is the Euclidean norm

\[
|g|^2 = \sum_{i<j} (g_{ij})^2 = \frac{1}{2} \text{tr}([g]^2) = \sum_\alpha (\psi_\alpha^* \psi_\alpha)^2
\]

Since \( SU(4) \) acts transitively on the spinors, and the form of \([g]_{\alpha \beta}\) includes arbitrary real, antisymmetric matrices as \( \psi_\alpha \) ranges over all 4-spinors, the possible metrics are fully characterized in terms of a single of 4-spinor. Because the norm is preserved, transformations of \( \psi \) produce rotations of \( g^A \).

When a particular metric is singled out, the \( SU(4) \) gauge is partially fixed. There remains an \( SU(3) \) subgroup which leaves \( \psi \) invariant. In addition, there additional symmetries present. These symmetries may depend on the particular \( \psi \) chosen. We prove the following theorem:

**Thm. 1** Fixing the metric reduces the \( SU(4) \) symmetry to \( SU(3) \times C(1) \times K \), where \( C(1) \) is a bounded, one-parameter group and \( K \) is a discrete group. When the spacetime metric has definite scaling weight, \( C(1) = U(1) \) and \( K \) includes the integers.

**Proof:** We first demonstrate that the subgroup which leaves a fixed spinor invariant is \( SU(3) \). Let \( \psi \) be fixed, let \( U \) be a unitary transformation and let \( \tilde{U} \) be that transformation that maps \( \psi \) to

\[
\tilde{U} \psi = \psi_0 = \begin{pmatrix}
\alpha \\
\alpha^* \\
0 \\
0
\end{pmatrix}
\]

Then for every transformation \( U \) that leaves \( \psi \) invariant,

\[
U \psi = \psi,
\]

we can construct another one,

\[
U_0 = \tilde{U} U \tilde{U}^\dagger,
\]
that leaves $\psi_0$ invariant, and vice-versa:

$$U_0\psi_0 = (\bar{U}U\bar{U}^\dagger)(\bar{U}\psi) = \bar{U}U\psi = \bar{U}\psi = \psi_0$$

$$U\psi = (\bar{U}^\dagger U_0\bar{U})(\bar{U}^\dagger\psi_0) = \bar{U}^\dagger U_0\psi_0 = \bar{U}^\dagger\psi_0 = \psi$$

Therefore, the group that leaves $\psi$ invariant is the same as the group that leaves $\psi_0$ invariant. But this group was shown by Wheeler [6] to be $SU(3)$, by a direct construction of the infinitesimal Hermitian generators. These generators are required to satisfy $H\psi = 0$.

The existence of a one-parameter, bounded symmetry, $C(1)$, follows immediately from the expression for $\psi$ in terms of $\bar{g}$. Each component of $\psi$ is bounded by the norm of $\bar{g}$. Furthermore, $\psi$ has four complex components, constrained by the constancy of the norm, $\psi^\dagger\psi$. Therefore, seven degrees of freedom in $\psi$ parameterize the six independent components of $\bar{g}$, leaving a one-parameter family of solutions to the algebraic equations for $\psi_\alpha(\bar{g})$. Since the equations to be solved for $\psi(\bar{g})$ are of quadratic or higher order, there will be more than a single root, providing a discrete symmetry, $K$. Additional discrete symmetry may also be provided by the phase transformations of the components.

The spacetime metric has definite Weyl weight if and only if $\bar{g}$ has only one nonvanishing component. In these cases $\psi$ has exactly two nonvanishing components. For example, we may have

$$\psi = \begin{pmatrix} \alpha \\ \beta \\ 0 \\ 0 \end{pmatrix}$$

If the phase of $\alpha$ is shifted by $\delta$ and the phase of $\beta$ by $-\delta$, $[\bar{g}]_{12} = \text{Re}(\alpha\beta)$ remains invariant. This is a $U(1)$ symmetry. Note that it does not commute with the $SU(3)$ symmetry, but for any change of phase, $\delta$, it is trivial to write down the new generators of $SU(3)$.

Finally, when the metric is of definite Weyl weight, there is an Hermitian transformation, $A$, which commutes with $SU(3)$ satisfying $A\psi = \psi$. When exponentiated, the net effect is a phase change

$$U\psi = e^{i\delta A}\psi = e^{i\delta}\psi$$

In general this does not leave $\bar{g}$ invariant. However, when $\delta = n\pi$, $[\bar{g}]$ is unchanged. The transformation remains nontrivial when acting on spinors other than $\psi$, and is distinct for different values of $n$. The symmetry group, $K$, therefore contains the integers.

4 The $SU(2)$ symmetry

There is a remaining symmetry of the metric choice, which is most naturally thought of as arising because the metric is symmetric in the gauge fields $e_\alpha^a$. 
It is natural to ask about the character of the antisymmetric combination
\[ F^i_{\alpha\beta} = \epsilon^i_{jk} e^j_\alpha e^k_\beta \eta_{ab} = -F^i_{\beta\alpha} \]
where \( \epsilon_{ijk} \) is the Levi-Civita symbol. \( F^i_{\alpha\beta} \) has the index structure of a Yang-Mills field. Applying the same criteria used to arrive at \( SU(4) \), we have the space \( R^3 - \{0\} \) covered by \( R \times O(3) \), with compact part \( O(3) \) and covering group \( SU(2) \). This is precisely the additional group required to give the standard model.

For \( F^\alpha_{\alpha\beta} \) to be a gauge field, it must arise from a gauge potential. Interestingly, the necessary condition depends on the vanishing of part of the torsion. Normally, in the gauging of Poincaré symmetry, we impose the condition
\[ D[\alpha e^a_{\beta}] = T^a_{\alpha\beta} = 0 \]
on the vierbein. It is sufficient to demand that each metric in our class be torsion-free. We therefore require the same condition of each of the three vierbein components,
\[ D[\alpha e^k_{\beta}] = 0 \]
Contracting with \( \eta_{ab} \epsilon^j_{ik} e^j_\mu \) and antisymmetrizing yields the Bianchi identity for \( F^i_{\alpha\beta} \):
\[ D[\alpha F^i_{\mu\beta}] = 0 \]
\( F^i_{\alpha\beta} \) therefore arises from an \( SU(2) \) gauge potential, providing the final symmetry required for the standard model.

5 Gauging

We now have shown the existence of \( SU(3) \times SU(2) \times U(1) \times Z \) symmetry as the residual gauge group following any choice of metric of definite scaling weight. The full symmetry is therefore the standard model, together with the Poincaré group and a discrete symmetry. We assumed that the dilation symmetry is broken even though it does not directly give a scaling of the mass as assumed by Wess [4]. Even if dilations were allowed, the standard model symmetries would still remain.

It is important to note that the new unitary symmetries are independent of the Poincaré symmetry. The Poincaré symmetry is the remnant of the original conformal symmetry. The unitary symmetry was introduced to classify the metrics allowed by the conformal gauge fields, but has no direct relationship to the translation, rotation or boost symmetries. The gauging of the group may proceed along the usual lines, with the exception that the product of the electromagnetic and strong symmetries is semi-direct and not direct.

Still more interesting is the possibility of investigating what happens if the entire \( SU(4) \) symmetry is maintained. The spacetime metric may be regarded as an \( SU(4) \)-valued tensor field, \( g^A_{\alpha\beta} \), and the curvature for the full symmetry
derived. It remains to be seen whether an appropriate action emerges naturally in this approach.

Finally, we conjecture that the correct way to introduce matter fields is through supersymmetrization of the model.

References


