**Introduction**

- Objectives:
  - Recovering sparse signal $X$ from a small set of linear noisy measurements using either single- or multiple-measurement vectors (SMV or MMV).

- Assumptions:
  - Sparse Clustered Pattern: Non-zero elements of the underlying signal may cluster in columns with an unknown structure on each column of $X$.
  - Joint-Sparsity: Columns of $X$ have the same non-zero locations $Y$.

**Proposed algorithm:**
- **C-SBL:** Sparse Bayesian learning model for sparse signals with unknown clustered pattern.

**Graphical Representation of the Proposed Bayesian Model**

**Proposed Bayesian Model and Defining Priors**

- **Basic MMV Model:** Solve for $X_{0m}$ in $Y = AX_{0m} + E$, where $Y \in \mathbb{R}^{M \times N}, A \in \mathbb{R}^{M \times P}, X_{0m} \in \mathbb{R}^{P \times N}$, and $(M < P)$.

- **Promoting sparsity:** Gaussian-Bernoulli prior by defining $X_{0m} \sim \mathcal{B}(s \equiv X, \text{where } s \in \{0, 1\}^{(P \times N)})$.

- **Promoting clustering pattern:** Incorporating total variation prior on the support learning vector $s$ using a measure of sparsity $(\|\Delta_{1\chi}\|_{2,1} \equiv \sum_{n \in \chi} |s_n| - |\chi|)$.

- **There exist two links for the clustered pattern supports:**

**Examples:**

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- **Prior on the solution-value matrix $X$:**

  $X - (x_1, \ldots, x_N)$ are drawn i.i.d. from the normal-gamma distribution $x_\mu \sim N(0, \tau^{-1} \mu), \gamma \sim \text{Gamma}(a_0, b_0), n = 1, \ldots, N$.

- **Prior on the support-learning component $s$:**

  Model the elements of $s$ as Bernoulli random variables $(s_1, s_2, \ldots, s_P) \sim \text{Bernoulli}(\frac{\gamma}{\tau})$, $\gamma = 1, 2, \ldots, P$.

  $w_{s,j} \sim \text{Beta}(\gamma - 1, \tau - 1, \gamma - 1)$.

  $\text{Beta}(\gamma)$.

  $\gamma = 1, \ldots, P$.

**Samples for support learning (MCMC-Gibbs Sampler)**

**Proposed Bayesian Model and Defining Priors (Contd)**

- **Prior on the noise component $E$:**

  $e_{\nu} \sim N(0, \tau^{-1} \nu), m = 1, \ldots, M, n = 1, \ldots, N$.

- **Prior on the sparsity index $n$:**

  $n \sim \text{Gamma}(a_0, b_0)$.

**Update Rule for $n$ (EM-based)**

$$Q(\tilde{\nu}) = \mathbb{E}_{\nu | Y, X, \tilde{\nu}} \left[ \log p(Y | X, \tilde{\nu}) \right]$$

- **Maximization step of EM:**

  $\mathbb{E}_{\nu | Y, X, \tilde{\nu}}$ arg max $E_{\nu | Y, X, \tilde{\nu}} [Q(\tilde{\nu})]$. 

- **Update rule for $n$:**

  $n \sim \text{Gamma}(a_0, b_0)$.

**C-SBL Algorithm**

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**Simulation Results on Synthetic Data**

- **Aspects of performance of C-SBL for the SMV problem:**

- **Aspects of performance of C-SBL for MMV (with $N = 2$):**

- **Comparisons of various algorithms for the SMV case:**

- **Comparisons of various algorithms for the MMV ($N = 2$) case:**

**Performance Comparison on Hand-Written Digits**

- **Original images (borrowed from MNIST) are scaled up to be of size 100 x 100 pixels:**

- **The pixel values were normalized to be within [0, 1], then they were subtracted from 1, and those with value of less than 0.3 were deemed to zero:**

- **For SMV we solve each column of $X$ (for each digit) one at the time:**

- **The number of measurements for each column of $X$ is set to 50:**

- **The measurements for each chunk of digit are computed by $y_{cd} = Ax_{cd} + e_{cd}$, where $x_{cd} \in \mathbb{R}^{100}$:**

- **We set the estimated pixels values lower value than 0.3 to 0:**

- **The true $(\|\Delta\|_{1})$ for digits 0, 1, 4, 5, 6, 7, 8, 9, 208, 160, 161, 316, and 220, respectively:**

- **C-SBL learning the clustering pattern via our measure of sparsity $(\|\Delta\|_{2,1})$:**

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