Signal Recovery with Unknown Sparsity Pattern via Multiple Measurement Vectors

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Introduction

Objective:

- Recovering sparse signal X from a small set of linear noisy measurements using multiple measurement vectors (MMVs)

Assumption:

- Sparse Clustered Pattern: Non-zero elements of the underlying signal may appear in clusters with an unknown structure on each column of X.
- Joint Sparsity: Non-zero elements of X appear at the same rows (support set of the solution is the same for all columns of X).

Model:

\[ Y = A(X + E) + E \]

\[ Y \in \mathbb{R}^{m \times N}, \quad A \in \mathbb{R}^{m \times L}, \quad x \in \mathbb{R}^{L \times 1}, \quad E \in \mathbb{R}^{m \times N}, \quad (M < P). \]

\[ \text{s accounts for the supports of the solution and } \text{denotes Hadamard product} \]

Proposed algorithm:

- C-SBL: sparse Bayesian learning model for sparse signals with unknown clustered pattern

Proposed Statistical Model and Defining Priors

- Measure of sparseness: \( \|\Theta\|_0 = \sum x_i = 0 \), where \( \lambda \) is the support learning vector of the solution.
- There exist two transitions for the case where the supports of the solution have a block-sparse structure.
- For example, a constant vector (one all or all zeros) has a \( \|\Theta\|_0 \) of 0.
- More examples:

Prior on the support-learning component \( \alpha \):

- we model the elements of \( \alpha \) as Bernoulli random variables
  \[ \alpha \sim \text{Bernoulli}(\alpha) \]

Joint Probability Distribution of the Proposed Model

\[ P(Y, X) \propto P(Y | X) P(X) \]

Statistical Model and Defining Priors (Continued)

- Prior on the emphasizing factor on sparseness of supports \( \alpha \):
  \[ \alpha \sim \text{Gamma}(\alpha_0, \beta_0) \]

- Initial setting: \( \alpha_0 = 5 \) and \( \beta_0 = 1 \)
- The parameter \( \alpha_0 \) specifies the significance of \( (\|\Delta\|_0, \|\Delta\|_1) \)
- Large values of \( \alpha_0 \) encourage more contiguity in the support of \( x \), while small values of \( \alpha_0 \) cause \( x \) to have many transitions

Joint Probability Distribution of the Proposed Model

\[ P(Y, X) \propto P(Y | X) P(X) \]

Graphical Model of the Proposed Bayesian Model

C-SBL Algorithm

C-SBL Algorithm for sparse signal recovery of either SMV or MMVs.

For \( p = 1 \) to \( P \)
  \[ x_p = \text{Bernoulli}(\alpha) \]

Simulation Results (Performance of C-SBL)

Behavior of \( \alpha \) with respect to \( (\|\Delta\|_0, \|\Delta\|_1) \)

- For each element \( x_s \) we have:
  \[ x_s \sim \text{Bernoulli}(\theta) \]

- For example consider the case where forcing either \( x_s \) or \( x_s \) to 0 does not make any change in the evaluation of \( (\|\Delta\|_0, \|\Delta\|_1) \).

- In this case, \( \theta \) promotes the sparseness in the solution, it discourages the solution to be sparse.

- Therefore, \( \alpha \) needs to be decreased

Simulation Results

- Our MMV model is a set of linear equations where the supports of the true solution are binary and drawn from a Bernoulli distribution in such a way to have a random clustered-sparsity structure.
- The number of columns in \( X \) and \( Y \) is set to \( N = 2 \).
- The entries of \( \theta \) are drawn i.i.d. from \( N(0, 1) \), where \( i = 1 \).
- The true solution is constructed from \( X = A \cdot X \).
- The sensing matrix \( A \in \mathbb{R}^{m \times n} \), with \( a_{ki} \sim N(0, 1) \), where \( M \) varies and \( P = 100 \).
- The entries of the noise vector are drawn i.i.d. from \( N(0, 1) \), in such a way to have \( SNR = 25dB \).
- The measurement matrix \( Y \) is computed from \( Y = AX + E \).
- In all the simulations, the sparsity level is set to \( K = 25 \).
- In the simulations for C-SBL we set \( N_{low} = 500 \) and \( N_{high} = 500 \).
- We consider two case scenarios:
  - Case 1: \( N \) of true solution \( X \) are uncorrelated i.e., \( p = 0 \).
  - Case 2: \( N \) of true solution \( X \) have correlation factor of \( \rho = 0.85 \).

- In order to investigate the performance, we generate 200 random cases using the above settings and then averaging over all the obtained results.

- In the figures, \( \lambda \) is the sampling rate and is defined as \( \lambda = M/P \).

Conclusion

- A new algorithm for the recovery of sparse signals with unknown clustered pattern is proposed (C-SBL algorithm).
- The proposed algorithm can be used for either single-or multiple-measurement vectors in the compressive sensing (CS) applications.
- Based on the simulation results, C-SBL outperforms the famous M-SBL [2].
- T-SBL [3], and MFOCUSS [4] algorithms

Main References