Multi-resolution Analysis Using Wavelet Basis Conditioned on Homogenization

Student Research Symposium
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We consider elliptic problems of the form:

$$\frac{d}{dx} K(x) \frac{dh}{dx} = f(x)$$

where $K$ is the permeability or conductivity tensor, $h$ is the pressure variable, and $f$ is the forcing function.

This DE models, for example, fluid flow in porous media.
Homogenization

- Deals with derivation of equations for averaging of solutions of equations with rapidly varying coefficients.
- Begins with a problem that includes structural variations.
- Derives a simpler problem that serves as first-term approximation to the original problem.
- The homogenized solution converges to a weak solution of the original problem.
The average homogenization formula for computing the harmonic average:

\[ K^\# = \int_0^1 K \left( 1 + \frac{dw_0}{dy_1} \right) dy_1 \]

\[ = \int_0^1 K dy_1 + \int_0^1 K \left( \frac{dw_0}{dy_1} \right) dy_1 \]

where \( \int_0^1 K dy_1 \) is the arithmetic average, and

\[ \int_0^1 K \left( \frac{dw_0}{dy_1} \right) dy_1 \]

is the perturbation needed to produce the harmonic average.
$w_0$ satisfies

$$\frac{d}{dy_1} K(y_1) \frac{d}{dy_1} w_0 = - \frac{d}{dy_1} K(y_1)$$

For example, given $\{K_{1,0}, K_{1,1}\}$. The harmonic average is:

$$K^\# = \frac{2K_0 K_1}{K_0 + K_1}$$
The correct averaged value of $K(y)$ in one dimension is the harmonic average. If $K(y)$ is continuous:

$$K^\# = \left( \int \frac{1}{K(y)} d(y) \right)^{-1}$$

Or

Given a sequence of values, $\{K_{i,0}, K_{i,1}, K_{i,2}, \ldots, K_{i,2^m-1}\}$

$$K^\# = \left( \sum_{j=0}^{2^m-1} \frac{1}{K_{i,j}} \right)^{-1}$$
Consider a sequence of measurements of the conductivity that has $2^m$ samples.

$\{ K_{i,0}, K_{i,1}, K_{i,2}, \ldots, K_{i,2^m-1} \}$

Algorithm

$$a_{i,j} = \frac{(K_{i,j+1} + K_{i,j})}{2}$$

$$d_{i,j} = -\frac{(K_{i,j+1} - K_{i,j})^2}{2(K_{i,j+1} + K_{i,j})}$$

$$K_{i-1,j}^\# = a_{i,j} + d_{i,j}$$

$$= \frac{2K_{1,j}K_{1,j+1}}{K_{1,j} + K_{1,j+1}}$$
Assume that the permeability tensor $K_{i,j}$ represents samples of $K(x)$ at $2^m$ equally spaced points in each spatial dimension.

Compute $a_{i,j}$, the arithmetic average of the two neighboring values of permeability tensor.

Compute $d_{i,j}$, the detail involving the difference of the two neighboring values of permeability tensor.

Then, add the arithmetic average and detail to obtain $K^\#$, the harmonic average of the original signals.
Visual Illustration of the Procedure

1 \( K_{3,0} \) 2 \( K_{3,1} \) 3 \( K_{3,2} \) 4 \( K_{3,3} \) 5 \( K_{3,4} \) 6 \( K_{3,5} \) 7 \( K_{3,6} \) 8 \( K_{3,7} \) 9 \( K_{3,8} \) 10 \( K_{3,9} \) 11 \( K_{3,10} \) 12 \( K_{3,11} \) 13 \( K_{3,12} \) 14 \( K_{3,13} \) 15 \( K_{3,14} \) 16 \( K_{3,15} \)

1 \( K_{2,0} \) 2 \( K_{2,1} \) 3 \( K_{2,2} \) 4 \( K_{2,3} \) 5 \( K_{2,4} \) 6 \( K_{2,5} \) 7 \( K_{2,6} \) 8 \( K_{2,7} \)

1 \( K_{1,0} \) 2 \( K_{1,1} \) 3 \( K_{1,2} \) 4 \( K_{1,3} \)

1 \( K_{0,0} \) 2 \( K_{1,0} \)
Visual Illustration of the Computation of Harmonic Average

\[ K^\# = K_{0,0} \]
Homogenization methods give the harmonic average of the permeability field and the wavelets characterization let us compute the harmonic average using wavelets transform.

$$K^\# = \int_0^1 K \left( 1 + \frac{dw_0}{dy_1} \right) dy_1$$

$$= \int_0^1 K dy_1 + \int_0^1 K \left( \frac{dw_0}{dy_1} \right) dy_1$$
Analogy: Homogenization and the Fast Transform Method

For

\[
K = \begin{cases} 
K_0 & y_1 \in \left[0, \frac{1}{2}\right) \\
K_1 & y_1 \in \left[\frac{1}{2}, 1\right] 
\end{cases}
\]

\[
\frac{dw_0}{dy_1} = \frac{K_1 - K_0}{K_0 + K_1} \begin{cases} 
1, & y_1 \in \left[0, \frac{1}{2}\right) \\
-1, & y_1 \in \left[\frac{1}{2}, 1\right] 
\end{cases}
\]

\[
\frac{dw_0}{dy_1} = w_{0,0}\varphi(y_1)
\]
\[ \int_0^1 K \left( \frac{dw_0}{dy_1} \right) dy_1 = d_{i,j} \]
\[ = \frac{(K_{i,j+1} - K_{i,j})^2}{2(K_{i,j+1} + K_{i,j})} \]

\[ \int_0^1 K dy_1 = a_{i,j} \]
\[ = \frac{(K_{i,j+1} + K_{i,j})}{2} \]
Reconstruction Algorithm Using MRA

Given the finest scale permeability samples $K_{i,j}$, apply the wavelet transform to compute the homogenized values on the entire domain, and compute the details at all scales.

Compute the solution of the coarsest level homogenized problem on the entire domain.

Compute the piecewise defined function $\frac{dh_n}{dy_n}$ using the form:

$$h = h_0 + \epsilon h_1 + \epsilon^2 h_2 + \ldots$$

and use the details to compute the next term in the perturbation series.
Computing the Pressure using Wavelet Decomposition

Brute Force Verification of the HWR Method

For one scale problem

\[ h_0(y_0) = 2(B_R - B_L)y_0 + B_L \]

For two-scale problem

\[ h_1(y_1) = \begin{cases} 
2\beta_{0,0}(B_R - B_L)(y_1) + B_L & y_1 \in \left[0, \frac{1}{2}\right) \\
2\alpha_{0,0}(B_R - B_L)(y_1 - 1) + B_R & y_1 \in \left[\frac{1}{2}, 1\right] 
\end{cases} \]
Computing the Pressure using Wavelet Decomposition

For three-scale problem

\[ h_2(y_2) = \begin{cases} 
2\beta_{0,0}\beta_{1,0}(B_R - B_L)(y_2) + B_L \\
2\beta_{0,0}\alpha_{1,0}(B_R - B_L)(y_2 - 1) + \beta_{0,0}B_R + (1 - \beta_{0,0})B_L \\
2\alpha_{0,0}\beta_{1,1}(B_R - B_L)(y_2 - 1) + (1 - \alpha_{0,0})B_R + \alpha_{0,0}B_L \\
2\alpha_{0,0}\alpha_{1,1}(B_R - B_L)(y_2 - 2) + B_R 
\end{cases} \]
Computing the Differences between the Solutions

We derived close form formulae for the solutions and compute the differences $h_n(y_n) - h_{n-1}(y_{n-1})$

$$h_1(y_1) - h_0(y_0) = (B_R - B_L) \begin{cases} 
(\beta_{0,0} - \alpha_{0,0})y_1 \\
(\beta_{0,0} - \alpha_{0,0})(1 - y_1)
\end{cases}$$
Computing the Differences between the Solutions

\[ h_2(y_2) - h_1(y_1) = (B_R - B_L) \begin{cases} 
\beta_{0,0}(\beta_{1,0} - \alpha_{1,0})y_2 \\
\beta_{0,0}(\beta_{1,0} - \alpha_{1,0})(1 - y_2) \\
\alpha_{0,0}(\beta_{1,1} - \alpha_{1,1})(y_2 - 1) \\
\alpha_{0,0}(\beta_{1,1} - \alpha_{1,1})(2 - y_2) 
\end{cases} \]
Computing the Differences between the Solutions

\[ h_3(y_3) - h_2(y_2) = (B_R - B_L) \]

\[
\begin{align*}
\beta_{0,0}\beta_{1,0}(\beta_{2,0} - \alpha_{2,0})y_3 \\
\beta_{0,0}\beta_{1,0}(\beta_{2,0} - \alpha_{2,0})(1 - y_3) \\
\beta_{0,0}\alpha_{1,0}(\beta_{2,1} - \alpha_{2,1})(y_3 - 1) \\
\beta_{0,0}\alpha_{1,0}(\beta_{2,1} - \alpha_{2,1})(2 - y_3) \\
\alpha_{0,0}\beta_{1,1}(\beta_{2,2} - \alpha_{2,2})(y_3 - 2) \\
\alpha_{0,0}\beta_{1,1}(\beta_{2,2} - \alpha_{2,2})(3 - y_3) \\
\alpha_{0,0}\alpha_{1,1}(\beta_{2,3} - \alpha_{2,3})(y_3 - 3) \\
\alpha_{0,0}\alpha_{1,1}(\beta_{2,3} - \alpha_{2,3})(4 - y_3)
\end{align*}
\]
Let $\psi = (\beta - \alpha)$ and $\phi = (B_R - B_L)$

$$h_1(y_1) - h_0(y_0) = \phi \begin{cases} 
\psi_{0,0} y_1 \\
\psi_{0,0}(1 - y_1)
\end{cases}$$
Let $\psi = (\beta - \alpha)$ and $\phi = (B_R - B_L)$

$$h_2(y_2) - h_1(y_1) = \phi \begin{cases} 
\beta_{0,0} \psi_{1,0} y_2 \\
\beta_{0,0} \psi_{1,0} (1 - y_2) \\
\alpha_{0,0} \psi_{1,1} (y_2 - 1) \\
\alpha_{0,0} \psi_{1,1} (2 - y_2) 
\end{cases}$$
Let $\psi = (\beta - \alpha)$ and $\phi = (B_R - B_L)$

$$h_3(y_3) - h_2(y_2) = \phi$$

\[
\begin{align*}
\beta_{0,0}\beta_{1,0}\psi_{2,0}y_3 \\
\beta_{0,0}\beta_{1,0}\psi_{2,0}(1 - y_3) \\
\beta_{0,0}\alpha_{1,0}\psi_{2,1}(y_3 - 1) \\
\beta_{0,0}\alpha_{1,0}\psi_{2,1}(2 - y_3) \\
\alpha_{0,0}\beta_{1,1}\psi_{2,2}(y_3 - 2) \\
\alpha_{0,0}\beta_{1,1}\psi_{2,2}(3 - y_3) \\
\alpha_{0,0}\alpha_{1,1}\psi_{2,3}(y_3 - 3) \\
\alpha_{0,0}\alpha_{1,1}\psi_{2,3}(4 - y_3)
\end{align*}
\]
### Numerical Results

**Table:** Results of the Approximate Pressure Variable, $h$

<table>
<thead>
<tr>
<th>$y$</th>
<th>$h_1(y_1)$</th>
<th>$h_2(y_2)$</th>
<th>$h_3(y_3)$</th>
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</table>
The graph of $h_1(y_1)$ against $y$
The graph of $h_2(y_2)$ against $y$
The graph of $h_3(y_3)$ against $y$
The graph of $h_1(y_1)$, $h_2(y_2)$, and $h_3(y_3)$ against $y$
Conclusion

- We developed a fast transform algorithm in one dimension for computing harmonic average of functions representing the fine scale parameter values that uses a dyadic mesh in the spatial domain.
- The fast transform algorithm introduced is built on the idea of wavelet multi-resolution.
- The method preserves the harmonic average of the permeability.
- We provide a process that computes the solution of the pressure variable using wavelet multi-resolution analysis.
- We implement Java codes that compute the pressure variables and came up with a close form generalization of the formula in our solution.
Thank you!