Utah State University

DigitalCommons@USU

5-2010

Classical Foundations for a Quantum Theory of Time in a Two-Dimensional Spacetime

Nathan Thomas Carruth
Utah State University

Recommended Citation

UtahStateUniversity
MERRILL-CAZIER LIBRARY

CLASSICAL FOUNDATIONS FOR A QUANTUM THEORY OF TIME

IN A TWO-DIMENSIONAL SPACETIME

by

Nathan Thomas Carruth

A thesis submitted in partial fulfillment
of the requirements for the degree

of

MASTER OF SCIENCE

in

Physics

Approved:

_____   _____
Dr. Charles G. Torre   Dr. David Peak
Major Professor   Committee Member

_____   _____
Dr. Mark E. Fels   Dr. Byron R. Burnham
Committee Member   Dean of Graduate Studies

UTAH STATE UNIVERSITY
Logan, Utah

2010

ABSTRACT

Classical Foundations for a Quantum Theory of Time

in a Two-Dimensional Spacetime

by

Nathan Thomas Carruth, Master of Science

Utah State University, 2010

Major Professor: Dr. Charles G. Torre
Department: Physics

We consider the set of all spacelike embeddings of the circle S^1 into a spacetime $\mathbf{R}^1 \times S^1$ with a metric globally conformal to the Minkowski metric. We identify this set and the group of conformal isometries of this spacetime as quotients of semidirect products involving diffeomorphism groups and give a transitive action of the conformal group on the set of spacelike embeddings. We provide results showing that the group of conformal isometries is a topological group and that its action on the set of spacelike embeddings is continuous. Finally, we point out some directions for future research.

(76 pages)

ACKNOWLEDGMENTS

First, a disclaimer: all statements on this page are the responsibility of myself alone, and are not intended to reflect the opinions of the other signatories to this thesis.

I would like to thank my thesis advisor, Dr. Charles Torre, for thinking of this project in the first place – the central idea behind what we do, that the conformal group acts transitively on the set of spacelike embeddings, is due to him. I also wish to thank him for guidance throughout the project and for helping me to get down to the work of writing everything up. I would like to thank Dr. Mark Fels for his assistance at various points throughout the project, and specifically for bringing to my attention the existence of the distinct topologies mentioned in Chapter 2. My thanks are also due to Dr. David Peak for his help and guidance throughout my studies of physics – without what he has done I would be far from where I am now.

I would also like to thank my parents and family for their support throughout.

Finally, in the words of Isaiah (6:3): "Holy, holy, holy, is the LORD of hosts: the whole earth is full of his glory."–*Soli Deo Gloria.*

Nathan Thomas Carruth

Replace this page with the generated CONTENTS page

CHAPTER 1

INTRODUCTION

Our goal in this thesis is to provide mathematical foundations needed to implement Isham's quantization program [1] for the system whose configuration space is the set of spacelike embeddings in a two-dimensional spacetime with compact spatial dimension and metric globally conformal to the Minkowski metric.[1]

We present three main results in this direction. The third, whose importance is the most immediately obvious, is the proof that the group of conformal isometries of the spacetime has a natural continuous transitive action on this collection of spacelike embeddings. As we indicate in Chapter 5, we believe that this group may be extended to a group of symplectic transformations acting transitively on the corresponding phase space. In Isham's program the unitary representations of this latter group then give information about the quantum system (more specifically, the self-adjoint generators of the one-parameter unitary groups coming from such a representation are the observables of the quantum system).

Our other two main results, also presented in Chapter 4, are the identification of the group of conformal isometries and the set of spacelike embeddings in terms of

---

[1]We note that every Lorentzian metric in a two-dimensional spacetime is locally conformal to the Minkowski metric; this global condition serves to eliminate certain pathologies, such as closed timelike curves, which could otherwise occur. We note also that there is another obvious choice of topology for a two-dimensional spacetime, namely $\mathbf{R}^1 \times \mathbf{R}^1$, in which the spatial dimension is noncompact. As far as the results which we establish go, the compact case appears the more difficult of the two; it appears, though, that the infinite-dimensional differential geometry discussed in Chapter 5 is easier in the compact case.

diffeomorphism groups. In our opinion it is the methods used in proving these two results which constitute the main technical innovation of the present work, especially the openness arguments and their use in providing homeomorphism arguments. The continuity arguments given below have received very similar treatments in the literature [5], [6], but we are not presently aware of any previous appearance of results similar to the openness results presented in Chapter 3, particularly those in Proposition 3.6 and Lemmas 3.1, 3.2, and 3.3; nor are we aware of prior use of the methods used in those proofs, the concept of openness of a map onto its image, and the related result presented in Proposition 3.2. Continuity and transitivity of the group action presented in Theorem 4.3 could possibly be proved more directly, but the identification in Theorem 4.1 of the conformal group in terms of diffeomorphism groups (which are in fact covering groups of $\mathrm{Diff}(S^1)$) will presumably be helpful in studying the representation theory necessary to complete Isham's program.

We proceed as follows. In Chapter 2 we set notation and some basic conventions, show that two certain groups of $\mathbf{R}^1$-diffeomorphisms are in fact topological groups in the C^∞ topology (defined below), and establish that a certain semidirect product constructed from one of these groups is also a topological group. This semidirect product group is that used in our constructions in Chapter 4. In Chapter 3 we prove certain openness and topological results which are needed in Chapter 4; we also establish that the group of diffeomorphisms used in the semidirect product is a covering group of $\mathrm{Diff}(S^1)$, among other things. In Chapter 4 we present homeomorphisms between the conformal group and the set of spacelike embeddings

on the one hand and quotients of this semidirect product group on the other, and demonstrate the transitive action of the conformal group on the set of spacelike embeddings in terms of these homeomorphisms. In Chapter 5 we give a brief, informal account of the next steps necessary to implement Isham's program and also provide an indication of the type of questions which can be studied therein.

CHAPTER 2

TOPOLOGICAL GROUP STRUCTURES

First we would like to set some notation which will be used in the following. The symbol $\mathbf{N}$ refers to nonnegative integers, i.e., $0 \in \mathbf{N}$; $\mathbf{N}^* = \mathbf{N}\backslash\{0\}$ is the set of positive integers. $\mathbf{R}$ will denote the set of real numbers and $\mathbf{C}$ the set of complex numbers. The letters V and W will be used to denote finite-dimensional vector spaces, either real or complex, with the usual topology. Typically we will use the standard Euclidean metric on $\mathbf{R}^n$ and $\mathbf{C}^n$, i.e., $|(x_1, x_2, \cdots, x_n)| = \sqrt{|x_1|^2 + |x_2|^2 + \cdots + |x_n|^2}$; however, we will also have occasion to consider sets of the form $\prod_{i=1}^{n} X_i$, where X_i is a compact subset of $\mathbf{R}$ or $\mathbf{C}$, and unless stated otherwise we will use the supremum metric on such spaces, i.e.,

$$d((x_1, x_2, \cdots, x_n), (y_1, y_2, \cdots, y_n)) = \max_{1 \leq i \leq n} |x_i - y_i|. \tag{2.1}$$

(This metric of course gives the same topology as does the Euclidean metric.) We consider the circle S^1 as the group of complex numbers of unit modulus. We let id denote the identity map; its domain space will be determined by the context.

If V and W, as above, are finite-dimensional vector spaces, we let $C^\infty(V, W)$ denote the set of all smooth (i.e., infinitely differentiable) maps from V to W. We shall always topologize these spaces and their subsets using the family of seminorms given as follows, following Yosida [3], pp. 26-27. Let $v = \dim(V)$, and let $\{x_1, x_2, \cdots, x_v\}$ be a basis for V; then, for $n \in \mathbf{N}$, $K \subset V$ compact, $(\alpha_i) = (\alpha_1, \cdots, \alpha_v) \in \mathbf{N}^v$, $|(\alpha_i)| = \sum \alpha_i$, and (for $f \in C^\infty(V, W)$)

$$\partial^{(\alpha_i)} f = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}} \frac{\partial^{\alpha_2}}{\partial x_2^{\alpha_2}} \cdots \frac{\partial^{\alpha_v}}{\partial x_v^{\alpha_v}} f, \tag{2.2}$$

where we make the convention that $\frac{\partial^0}{\partial x_i^0} = \mathrm{id}$, we define

$$p_{n,K}(f) = \sup_{|(\alpha_i)| \leq n, x \in K} \left| \left(\partial^{(\alpha_i)} f \right)(x) \right|. \tag{2.3}$$

When clarity is desired we may occasionally emphasize V or W by writing $p_{n,K}^V$, $p_{n,K}^{V,W}$, or $p_{n,K}^W$. We shall, however, never write $\mathbf{R}^1$ as a superscript. We note that, for any $n \in \mathbf{N}$, $p_{n,K}(f - g) = 0$ for all K implies that $f(x) = g(x)$ for all x; thus (see Yosida [3], p. 26) the topology on $C^\infty(V, W)$ defined by this family of semi-norms gives $C^\infty(V, W)$ the structure of a locally convex space. We shall always give these spaces this topology.[2] Thus sets of the form $\{ f \in C^\infty(V, W) | p_{n,K}(f - f_0) < \epsilon \}$, $n \in \mathbf{N}$, $K \subset V$ compact, $f_0 \in C^\infty(V, W)$, and $\epsilon > 0$ form a basis for the topology of $C^\infty(V, W)$. (Though not necessary for this definition of the topology, we shall usually tacitly assume that all compact sets K are nonempty.)

---

[2]This topology corresponds to the so-called weak topology introduced in [15], Section 2.1. Other topologies for these spaces which we have seen include the so-called strong topology introduced also in [15], Section 2.1, and the so-called very-strong topology introduced in [16]. For spaces of C^∞ functions on compact manifolds all three definitions appear to coincide (see [15], p. 35, [16], p. 413). As we will see in Chapter 4, the group of conformal transformations of interest to us behaves in some ways like a set of C^∞ functions on a compact manifold; thus the weak topology is a reasonable choice. Further, the strong and very strong topologies are not in general first countable ([15], p. 35, [16], p. 413); also, as we point out after Theorem 4.1 below, the identity is the only element of the conformal group which has compact support, and thus the results in [17] (see especially p. 19 and Proposition 4.1) appear to imply that the conformal group is totally disconnected in the very strong topology. While this would imply that our group in the very strong topology is at least first countable, it is still (as is lack of first countability) very undesirable from the perspective of the infinite-dimensional differential geometry mentioned in Chapter 5. The use of fundamentally different function spaces, such as the Sobolev spaces used in [18], is beyond the scope of our present work.

We shall denote these sets by $U(n, K, \epsilon, f_0)$; as with the $p_{n,K}$ we may occasionally emphasize the vector spaces and write $U_{V,W}(n, K, \epsilon, f_0)$, etc. We note that $U(n, K_1, \epsilon, f_0) \cap U(n, K_2, \epsilon, f_0) = U(n, K_1 \cup K_2, \epsilon, f_0)$. We note also that $p_{m,K}(f) \le p_{n,K}(f)$ for any compact K and any $m, n \in \mathbf{N}$ for which $m \le n$ (in particular,

$$p_{0,K}(f) = \sup_{x \in K} |f(x)| \le p_{n,K}(f) \text{ for any } n \in \mathbf{N}); \text{ thus } U(n, K, \epsilon, f_0) \subset U(m, K, \epsilon, f_0)$$

for $n \ge m$ and we may restrict to $n \ge n_0$ for any fixed $n_0 \in \mathbf{N}$ and obtain an equivalent topology. If $X \subset W$ is a submanifold then we topologize $C^\infty(V, X)$ as a subspace of $C^\infty(V, W)$. (The term subspace will always be used to mean simply a topological subspace, i.e., a subset equipped with the subspace topology, unless noted otherwise.) The only spaces of C^∞ maps which we shall consider below in which the domain space is not a vector space are spaces of the form $C^\infty(S^1, X)$ or $C^\infty(\mathbf{R}^1 \times S^1, X)$; we shall topologize these roughly as in Milnor [2] (i.e., roughly with the weak topology) – we give the details in Chapter 3 below.

For any finite-dimensional vector space V, we set

$$C_p^\infty(\mathbf{R}^1, V) = \{f \in C^\infty(\mathbf{R}^1, V) | f(x + 2\pi) = f(x) \text{ for all } x \in \mathbf{R}\}, \qquad (2.4)$$

and topologize it as a subset of $C^\infty(\mathbf{R}^1, V)$. We let $\mathrm{Diff}(\mathbf{R}^1)$ denote the set of all C^∞ diffeomorphisms of the real line, i.e., invertible smooth maps with smooth inverses. We let

$$\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) = \{f \in \mathrm{Diff}(\mathbf{R}^1) | f(x + 2\pi) = f(x) + 2\pi \text{ for all } x \in \mathbf{R}\} \qquad (2.5)$$

and note that it is a subgroup of $\mathrm{Diff}(\mathbf{R}^1)$. We topologize these last two spaces as subspaces of $C^\infty(\mathbf{R}^1, \mathbf{R}^1)$.

We shall often denote the nth derivative of a function $f \in C^\infty(\mathbf{R}^1, V)$ by $f^{(n)} \in C^\infty(\mathbf{R}^1, V)$. We make the convention that $f^{(0)} = f$.

We recall (see Bredon [4], pp. 4-5, especially Proposition I.2.6) that a function $f : X \to Y$ between two topological spaces is continuous if and only if for every $x \in X$ and any basis of open sets $B_{f(x)}$ at $f(x) \in Y$, the set $f^{-1}(U)$ is a neighborhood of x for every $U \in B_{f(x)}$. In our case, this gives rise to the following result.

Proposition 2.1. Let V_1, V_2, W_1, and W_2 be finite-dimensional vector spaces. Then a function $F : C^\infty(V_1, W_1) \to C^\infty(V_2, W_2)$ is continuous if and only if for every $g_0 \in C^\infty(V_1, W_1)$, $n \in \mathbf{N}$, $K \subset V_2$ compact, and $\epsilon > 0$, there exist $n' \in \mathbf{N}$, $K' \subset V_1$ compact, and $\delta > 0$ so that $p_{n',K'}^{V_1,W_1}(g - g_0) < \delta$ implies that $p_{n,K}^{V_2,W_2}(F(g) - F(g_0)) < \epsilon$.

Proof. This criterion is equivalent to

$$F^{-1}(U_{V_2,W_2}(n, K, \epsilon, F(g_0))) \supset U_{V_1,W_1}(n', K', \delta, g_0), \tag{2.6}$$

and the result is then clear since by definition the collections of all sets of the form $U_{V_i,W_i}(n, K, \epsilon_i, f_i)$ form neighborhood bases about $f_i \in C^\infty(V_i, W_i)$. QED.

Similar results clearly hold in the case of maps between products of these spaces.

With these preliminaries out of the way we now proceed to our first main results, namely that $\mathrm{Diff}(\mathbf{R}^1)$ and $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ are topological groups with respect to the C^∞ topology specified above. We refer the reader to the papers by Glöckner [5], [6] for a somewhat clearer proof of this result along the same lines. We present the proofs below for the sake of completeness, and also because the form of the polynomials in Proposition 2.2 and an estimate in the proof of Lemma 2.1 are both

needed for the proof of Lemma 2.2 presented in Chapter 3 below.

Our first proposition, giving an explicit formula for the polynomials mentioned in [6], Appendix, in the case $d = n = 1$, is fairly easy, even if it at first sight appears somewhat cumbersome. The reader is strongly encouraged to work out a few cases such as $n = 1, 2, 3, \ldots$ by hand, collecting terms to bring the result into the form given below, so as to get a feel for what is going on. For a similar but much more general result, see Keller [7], p. 111.

Proposition 2.2. Let $n \in \mathbf{N}^*$. There is a finite collection $I_n = \{\{(j_i, m_i, l_i)\}_i\}$ of finite, ordered sequences of triples of positive natural numbers and a collection of positive natural numbers $\{a_k^n\}_{k \in I_n}$ such that, if V is any finite-dimensional vector space, $f \in C^\infty(\mathbf{R}^1, V)$, and $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, then

$$(f \circ g)^{(n)} = (f' \circ g)(x)g^{(n)}(x)$$
$$+ \sum_{k=\{(j_i, m_i, l_i)\}_1^N \in I_n} a_k^n \left[\left(f^{(j_1)} \circ g \right)(x) \prod_{(j_i, m_i, l_i) \in k} \left(g^{(l_i)}(x) \right)^{m_i} \right].(2.7)$$

Moreover, if $k = \{(j_i, m_i, l_i)\}_{i=1}^N \in I_n$, then $j_1 = j_2 = \cdots = j_N$, $\sum_{i=1}^N m_i l_i = n$, $\sum_{i=1}^N m_i = j_1$, and $l_i < n$ for all i. (Since j_i is independent of i, we shall simply write $k = \{(j, m_i, l_i)\}_{i=1}^N$ below.)

Proof. The proof proceeds by induction. For $n = 1$ we have by the chain rule $(f \circ g)'(x) = (f' \circ g)(x)g'(x)$, which is simply the leading term above; we may thus take $I_1 = \emptyset$ and the result follows vacuously. Now suppose that the result holds for $n \le q$, $q \ge 1$. We shall differentiate the expansion above with $n = q$ and show that each resulting term is of the correct form. We see that

$$(f \circ g)^{(q+1)}(x) = (f' \circ g)(x)g^{(q+1)}(x) + (f'' \circ g)(x)g^{(q)}(x)g'(x)$$

$$+ \sum_{k=\{(j,m_i,l_i)\} \in I_q} a_k^q \frac{d}{dx}\left[\left(f^{(j)} \circ g\right)(x) \prod_{(j,m_i,l_i) \in k} \left(g^{(l_i)}(x)\right)^{m_i} \right]. \quad (2.8)$$

The first term is the expected leading term in the expansion with $n = q + 1$. The second term is a term of the correct form with sequence $\{(2,1,q),(2,1,1)\}$; since for this sequence $j_1 = j_2 = 2$, $\sum m_i l_i = q + 1$, $\sum m_i = 2 = j_1$, and $l_i < q + 1$ for each i, this sequence is of the desired form, and hence so is this term.

Now each term in the sum in equation (2.8) may be expanded as (for $k = \{(j,m_i,l_i)\} \in I_q$; we set $z^0 = 1$ for all $z \in \mathbf{R}^1$ and note also that $m_i \geq 1$ for all i)

$$a_k^q\left[\left(f^{(j+1)} \circ g\right)(x)\, g'(x) \prod_{(j,m_i,l_i) \in k} \left(g^{(l_i)}(x)\right)^{m_i} \right.$$

$$+ \sum_{(j,m_i,l_i) \in k} \left(f^{(j)} \circ g\right)(x)\, m_i \left(g^{(l_i)}(x)\right)^{m_i-1} g^{(l_i+1)}(x) \prod_{\substack{(j,m,l) \in k \\ (m,l) \neq (m_i,l_i)}} \left(g^{(l)}(x)\right)^{m} \left. \right]. \quad (2.9)$$

Now the first of these terms is a term of the stated form whose sequence is k with each j replaced by $j + 1$ and the term $(j + 1, 1, 1)$ added at the end (recall that each k is considered as an *ordered* sequence!); since this new sequence satisfies $j_1 = \cdots = j_{N+1} = j + 1$, $\sum ml = q + 1$, $\sum m = j + 1 = j_1$, and $l < q < q + 1$ for each l, it is of the desired form, and thus so is this term. Similarly, fix some $(j,m_i,l_i) \in k$, and consider the corresponding term in the sum in (2.9). Consider the sequence obtained from k by adjoining the element $(j, 1, l_i + 1)$ at the end and replacing (j,m_i,l_i) by $(j,m_i - 1,l_i)$ if $m_i > 1$, or deleting it if $m_i = 1$. This new sequence is easily seen to correspond to the term under consideration. Now for this

sequence we have $j_1 = j_2 = \cdots = j$, $\sum ml = q + (l_i + 1) - m_i l_i + (m_i - 1)l_i = q + 1$, $\sum m = j - m_i + (m_i - 1) + 1 = j$, and $l < q + 1$ for all l; thus this sequence is of the desired form. Hence we see that each term in (2.9) is of the desired form; since (2.9) is a generic term of the sum in equation (2.8), the entire sum is of the desired form, and hence the result holds for $n = q + 1$ and thus for all n by induction. (Each a_k^n will be a positive natural number since $\mathbf{N}^*$ is closed under the operations of addition and multiplication.)   QED.

We note that we could dispense with the first term above by including the singleton sequence $\{(1, 1, n)\}$ in I_n and writing $a_{\{(1,1,n)\}}^n = 1$ for all n, and we shall frequently do this below. In this case we shall write J_n for the collection of sequences rather than I_n. (The only condition in the above proposition not satisfied by J_n is $l < n$, which must be replaced with $l \leq n$.) The explicit form of the leading term is important in the proof of Lemma 2.2 given in Chapter 3 below, but we shall not need it until then.

We now proceed to prove that composition of C^∞ functions is continuous, using the above proposition. First we make one more definition. Fix $n \in \mathbf{N}^*$. For each $k \in J_n$ we define a multilinear form $\beta_k : \mathbf{R}^n \to \mathbf{R}$ by $\beta_k((x_i)_{i=1}^n) = \prod_{(j,m,l) \in k} (x_l)^m$; thus by the proposition we may write

$$(f \circ g)^{(n)}(x) = \sum_{k \in J_n} a_k^n \left(f^{(j)} \circ g \right)(x)\, \beta_k \left((g^{(i)}(x))_{i=1}^n \right). \qquad (2.10)$$

We note that if $k \in I_n$ then β_k only depends on the first $n - 1$ elements of (x_i).

Let V denote a finite-dimensional vector space.

Lemma 2.1. Composition of C^∞ functions,

$$\alpha : C^\infty(\mathbf{R}^1, V) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, V)$$

$$(f, g) \mapsto f \circ g,$$

(2.11)

is continuous.

Proof. Basically, this follows from continuity of the multilinear functions β_k along with certain compactness arguments. Some of the estimates are somewhat involved, though, so for the sake of completeness, and also because one of the estimates below is necessary for the proof of Lemma 2.2, we give a detailed proof of this result, as follows.

Continuity of α is equivalent to openness of $\alpha^{-1}(U)$ in $C^\infty(\mathbf{R}^1, V) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ for each basic open set $U \subset C^\infty(\mathbf{R}^1, V)$. Fix $(f_0, g_0) \in C^\infty(\mathbf{R}^1, V) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, $n \in \mathbf{N}^*$, $K \subset \mathbf{R}^1$ compact, and $\epsilon > 0$, and let $U = U_V(n, K, \epsilon, f_0 \circ g_0)$ be the corresponding basic neighborhood of $f_0 \circ g_0 = \alpha(f_0, g_0)$. By our comments about restricting n on pp. 5-6 the set of all such U is a basis for the topology of $C^\infty(\mathbf{R}^1, V)$ since α is clearly surjective (as $\alpha(f, \mathrm{id}) = f$ for all $f \in C^\infty(\mathbf{R}^1, V)$). We must then show that there exist $n_1, n_2 \in \mathbf{N}$, $K_1, K_2 \subset \mathbf{R}^1$ compact, and $\delta_1, \delta_2 > 0$ so that $U_V(n_1, K_1, \delta_1, f_0) \times U(n_2, K_2, \delta_2, g_0) \subset \alpha^{-1}(U)$; i.e., so that for all $f \in C^\infty(\mathbf{R}^1, V)$, $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ $p^V_{n_1, K_1}(f - f_0) < \delta_1$ and $p_{n_2, K_2}(g - g_0) < \delta_2$ implies that $p^V_{n, K}(f \circ g - f_0 \circ g_0) < \epsilon$.

First, define $K' = g_0(K) + [-1, 1]$; then K' is compact. Similarly, for each $n' \in \mathbf{N}^*$, $n' \le n$ let $K''_{n'} = \prod_{i=1}^{n'} \left(g_0^{(i)}(K) + [-1, 1] \right) \subset \mathbf{R}^{n'}$; the $K''_{n'}$ are also compact since $g_0 \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$. Note that K' and $K''_{n'}$ are *fixed* – they are determined solely

by the initial data fixed in the paragraph above. Now suppose that $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ satisfies $p_{n,K}(g - g_0) < 1$; then for each $i \in \mathbf{N}^*$, $i \leq n$, we have $g^{(i)}(K) \subset g_0^{(i)}(K) + [-1, 1]$. Thus $\prod_{i=1}^{n'} g^{(i)}(K) \subset K''_{n'}$ for each $n' \leq n$. Similarly, $g(K) \subset K'$. Now for each $q \in \mathbf{N}$ $f_0^{(q)}|_{K'} : K' \to V$ is uniformly continuous (see, e.g., Munkres [8], Theorem 27.6), as are the $\beta_k|_{K''_{n'}} : K''_{n'} \to \mathbf{R}^1$ for each $k \in J_{n'}$, where $1 \leq n' \leq n$. Let $s = \max_{1 \leq n' \leq n} \left(\sum_{k \in J_{n'}} |a_k^{n'}| \right)$; then $s > 0$ since $n \geq 1$. Let $\epsilon' = \dfrac{\epsilon}{9(1 + s)\left(1 + p_{n,K}(g_0)\right)^n}$, $\epsilon'' = \dfrac{\epsilon}{3(1 + s)(1 + p_{n,K'}^V(f_0))}$; then $0 < \epsilon' < \infty$, $0 < \epsilon'' < \infty$ and ϵ' and ϵ'' are both fixed by our choices in the previous paragraph. By the uniform continuity of the functions noted above we may then choose $\delta', \delta'' > 0$ so that (i) $x, y \in K'$ and $|x - y| < \delta'$ implies that $|f_0^{(q)}(x) - f_0^{(q)}(y)| < \epsilon'$ for all $q \in \mathbf{N}$, $q \leq n$ and (ii) $x, y \in K''_n$ and $\max_{1 \leq i \leq n} |x_i - y_i| < \delta''$ (recall our convention about the metric used on products of compact subsets of $\mathbf{R}^1$!) implies that $|\beta_k((x_i)_1^{n'}) - \beta_k((y_i)_1^{n'})| < \epsilon''$ for all $k \in J_{n'}$, $1 \leq n' \leq n$. (δ'' exists since the collection of functions $\{\beta_k | k \in J_{n'}$ for some $1 \leq n' \leq n\}$ is finite and each member is uniformly continuous, and since $x, y \in K''_n$ and $\max_{1 \leq i \leq n} |x_i - y_i| < \delta''$ if and only if $(x_i)_1^{n'}, (y_i)_1^{n'} \in K''_{n'}$ and $\max_{1 \leq i \leq n'} |x_i - y_i| < \delta''$ for each $1 \leq n' \leq n$.)

Now let $\delta = \min\{1, \delta', \delta'', \epsilon'\}$, and let $f \in C^\infty(\mathbf{R}^1, V)$ and $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ satisfy $p_{n,K'}^V(f - f_0) < \delta$ and $p_{n,K}(g - g_0) < \delta$. We note that $p_{q,K'}^V(f) < 1 + p_{q,K'}^V(f_0)$ and $p_{q,K}(g) < 1 + p_{q,K}(g_0)$ for all $q \in \mathbf{N}$, $q \leq n$. Thus in particular $g(K) \subset K'$ and $\prod_{i=1}^{q} g^{(i)}(K) \subset K''_q$ for $1 \leq q \leq n$. Further, for all $q \in \mathbf{N}$, $q \leq n$ we see that $x, y \in K'$ and $|x - y| < \delta$ implies

$$|f^{(q)}(x) - f^{(q)}(y)| \leq |f^{(q)}(x) - f_0^{(q)}(x)| + |f^{(q)}(y) - f_0^{(q)}(y)| + |f_0^{(q)}(x) - f_0^{(q)}(y)|$$

$$< \delta + \delta + \epsilon' \leq 3\epsilon', \tag{2.12}$$

since $p_{n,K'}^V(f - f_0) < \delta \leq \epsilon'$, and $\delta \leq \delta'$ implies that $|f_0^{(q)}(x) - f_0^{(q)}(y)| < \epsilon'$ by uniform continuity of f_0 on K'. Thus for all $x \in K$,

$$|(f \circ g)(x) - (f_0 \circ g_0)(x)| \leq |(f \circ g)(x) - (f \circ g_0)(x)| + |(f \circ g_0)(x) - (f_0 \circ g_0)(x)|$$

$$< 3\epsilon' + \delta \leq 4\epsilon' < \epsilon, \tag{2.13}$$

since $|g(x) - g_0(x)| < \delta$ and $g(x), g_0(x) \in K'$, and $|f(y) - f_0(y)| < \delta \leq \epsilon'$ for all $y \in K'$.

Now fix $n' \in \mathbf{N}^*$, $n' \leq n$, and $x \in K$. We see that

$$\left|(f \circ g)^{(n')}(x) - (f_0 \circ g_0)^{(n')}(x)\right| \leq \sum_{k \in J_{n'}} \left|a_k^{n'}\right| \cdot$$

$$\left|\left(f^{(j)} \circ g\right)(x)\,\beta_k\left((g^{(i)}(x))_1^{n'}\right) - \left(f_0^{(j)} \circ g_0\right)(x)\,\beta_k\left((g_0^{(i)}(x))_1^{n'}\right)\right|. \tag{2.14}$$

Now fixing $k = \{(j, m_\iota, l_\iota)\} \in J_{n'}$, we see that

$$\left|\left(f^{(j)} \circ g\right)(x)\,\beta_k\left((g^{(i)}(x))_1^{n'}\right) - \left(f_0^{(j)} \circ g_0\right)(x)\,\beta_k\left((g_0^{(i)}(x))_1^{n'}\right)\right|$$

$$\leq \left|\left(f^{(j)} \circ g\right)(x)\,\beta_k\left((g^{(i)}(x))\right) - \left(f^{(j)} \circ g_0\right)(x)\,\beta_k\left((g^{(i)}(x))\right)\right|$$

$$+ \left|\left(f^{(j)} \circ g_0\right)(x)\,\beta_k\left((g^{(i)}(x))\right) - \left(f^{(j)} \circ g_0\right)(x)\,\beta_k\left((g_0^{(i)}(x))\right)\right|$$

$$+ \left|\left(f^{(j)} \circ g_0\right)(x)\,\beta_k\left((g_0^{(i)}(x))\right) - \left(f_0^{(j)} \circ g_0\right)(x)\,\beta_k\left((g_0^{(i)}(x))\right)\right|$$

$$= \left|\left(f^{(j)} \circ g\right)(x) - \left(f^{(j)} \circ g_0\right)(x)\right| \prod_{(j, m_\iota, l_\iota) \in k} \left|g^{(l_\iota)}(x)\right|^{m_\iota}$$

$$+ \left|\left(f^{(j)} \circ g_0\right)(x)\right| \left|\beta_k\left((g^{(i)}(x))\right) - \beta_k\left((g_0^{(i)}(x))\right)\right|$$

$$+ \left|\left(f^{(j)} \circ g_0\right)(x) - \left(f_0^{(j)} \circ g_0\right)(x)\right| \prod_{(j, m_\iota, l_\iota) \in k} \left|g_0^{(l_\iota)}(x)\right|^{m_\iota}. \tag{2.15}$$

Recalling that $1 \leq j, l \leq n' \leq n$ and $\sum m_\iota = j$ for $(j, m_\iota, l_\iota) \in k$, that $x \in K$, and hence that $g(x), g_0(x) \in K'$, that $p_{n,K}(g - g_0) < \delta \leq 1$, and that $p_{n,K'}^V(f - f_0) < \delta \leq \epsilon'$, we see that

$$\left. \begin{aligned} |(f^{(j)} \circ g)(x) - (f^{(j)} \circ g_0)(x)| &\leq 3\epsilon' \\ \prod_{(j,m_\iota,l_\iota) \in k} \left| g^{(l_\iota)}(x) \right|^{m_\iota} \leq (1 + p_{n',K}(g_0))^j &\leq (1 + p_{n,K}(g_0))^n \\ |(f^{(j)} \circ g_0)(x)| &\leq 1 + p_{n,K'}^V(f_0) \\ |(f^{(j)} \circ g_0)(x) - (f_0^{(j)} \circ g_0)(x)| &< \delta \leq \epsilon' \\ \prod_{(j,m_\iota,l_\iota) \in k} \left| g_0^{(l_\iota)}(x) \right|^{m_\iota} \leq (1 + p_{n',K}(g_0))^j &\leq (1 + p_{n,K}(g_0))^n; \end{aligned} \right\} \tag{2.16}$$

moreover, we see that $\displaystyle\max_{1 \leq i \leq n} |g^{(i)}(x) - g_0^{(i)}(x)| \leq p_{n,K}(g - g_0) < \delta \leq \delta''$; since $(g^{(i)}(x))_1^n, (g_0^{(i)}(x))_1^n \in K_n''$, uniform continuity of β_k gives, for each $k \in J_{n'}$,

$$\left| \beta_k((g^{(i)}(x))_1^{n'}) - \beta_k((g_0^{(i)}(x))_1^{n'}) \right| < \epsilon''. \tag{2.17}$$

(In passing note our (admittedly buried) usage of the uniform continuity of $f^{(j)}$ and β_k: we need the estimates given above to hold uniformly for all g and all $x \in K$ and hence need uniform continuity, not just pointwise continuity.) We therefore obtain

$$\begin{aligned} \left| \left(f^{(j)} \circ g \right)(x) \, \beta_k \left((g^{(i)}(x)) \right) - \left(f_0^{(j)} \circ g_0 \right)(x) \, \beta_k \left((g_0^{(i)}(x)) \right) \right| & \\ \leq 3\epsilon' \left(1 + p_{n,K}(g_0) \right)^n + \left(1 + p_{n,K'}^V(f_0) \right) \epsilon'' + \epsilon' \left(1 + p_{n,K}(g_0) \right)^n & \\ = \frac{4\epsilon}{9(1+s)} + \frac{\epsilon}{3(1+s)} < \frac{\epsilon}{1+s}; \end{aligned} \tag{2.18}$$

and substituting this back in to our original expression in equation (2.14) above gives finally

$$\left| (f \circ g)^{(n')}(x) - (f_0 \circ g_0)^{(n')}(x) \right| < \sum_{k \in J_{n'}} \left| a_k^{n'} \right| \frac{\epsilon}{1+s}$$

$$\leq \epsilon \frac{s}{1+s} < \epsilon, \tag{2.19}$$

so $f \circ g \in U$. We may thus take $\delta_1 = \delta_2 = \delta$, $K_1 = K'$, $K_2 = K$, and $n_1 = n_2 = n$.

QED.

This establishes, in particular, that composition in $\mathrm{Diff}(\mathbf{R}^1)$ is continuous. See [16], Proposition 2.3, [19], Proposition 1, for similar results in other topologies.

The last step in proving that $\mathrm{Diff}(\mathbf{R}^1)$ is a topological group is to prove that inversion $\iota : \mathrm{Diff}(\mathbf{R}^1) \to \mathrm{Diff}(\mathbf{R}^1)$, $f \mapsto f^{-1}$ is continuous. A large part of the technical work necessary for this is applicable also to proving one of the openness results in Chapter 3, and so we give it as a separate result, as follows. As usual, let V be some fixed finite-dimensional real vector space. Now let

$$C_0^\infty(\mathbf{R}^1, V) = \{ f \in C^\infty(\mathbf{R}^1, V) | f'(x) \neq 0 \text{ for all } x \in \mathbf{R}^1 \}. \tag{2.20}$$

It is to be topologized, of course, as a subspace of $C^\infty(\mathbf{R}^1, V)$. We then have the following result.

Lemma 2.2. Let $f_0 \in C_0^\infty(\mathbf{R}^1, V)$, $g_0 \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$. For every $\epsilon > 0$, $n \in \mathbf{N}$, and $K \subset \mathbf{R}^1$ compact, there exist $\delta > 0$ and $K_1, K_2, K_3 \subset \mathbf{R}^1$ compact so that, for all $f \in C^\infty(\mathbf{R}^1, V)$ and $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, $p_{n,K_1}^V(f - f_0) < \delta$, $p_{n,K_2}^V(f \circ g - f_0 \circ g_0) < \delta$, and $p_{0,K_3}(g - g_0) < \delta$ implies that $p_{n,K}(g - g_0) < \epsilon$.

We defer the proof to Chapter 3 and instead show first how this result may be applied to show continuity of inversion.

Corollary 2.1. Inversion

$$\iota : \text{Diff}(\mathbf{R}^1) \to \text{Diff}(\mathbf{R}^1)$$

$$f \mapsto f^{-1}$$

(2.21)

is continuous.

Proof. We proceed by showing (essentially) that ι is continuous in the compact-open topology on $\text{Diff}(\mathbf{R}^1)$ (i.e., the C^∞ topology restricted to $n = 0$) and then applying Lemma 2.2. The technical details of the following argument basically involve a lot of juggling of inequalities. There is, nevertheless, a fairly clear intuitive picture to most of what we are doing: we may think of a neighborhood of a function g_0 in $\text{Diff}(\mathbf{R}^1)$ as consisting of certain functions having their graphs in a strip in the plane around the graph of g_0; recalling that the graph of the inverse of a function is simply the graph of the original function reflected in the line $\{(x, x)|x \in \mathbf{R}^1\}$, we then see that the inverses of all such functions must have their graphs in the reflection in this line of the original strip around g_0, which is a strip around g_0^{-1}. This underlies the inequality-juggling below.

Fix $f_0 \in \text{Diff}(\mathbf{R}^1)$. We show that, if $K \subset \mathbf{R}^1$ is compact and $\epsilon > 0$, then there exist $K' \subset \mathbf{R}^1$ compact and $\delta > 0$ so that $p_{0,K'}(f - f_0) < \delta$ implies $p_{0,K}(f^{-1} - f_0^{-1}) < \epsilon$. Fix K and ϵ. Set $K^* = [\inf K - 1, \sup K + 1]$; then $K \subset K^*$ and K^* is compact. Now since $f_0 \in \text{Diff}(\mathbf{R}^1)$ we see that either $f_0' > 0$ on $\mathbf{R}^1$ or $f_0' < 0$ on $\mathbf{R}^1$. First suppose that $f_0' > 0$. Let $K'' \subset \mathbf{R}^1$ be any compact set with $\sup K'' > \inf K''$. Then we see that $f_0(\sup K'') - f_0(\inf K'') > 0$; if $f \in \text{Diff}(\mathbf{R}^1)$ is such that $p_{0,K''}(f - f_0) < \frac{1}{2}(f_0(\sup K'') - f_0(\inf K''))$ then $f_0(x) - \frac{1}{2}(f_0(\sup K'') - $

$f_0(\inf K'')) < f(x) < f_0(x) + \frac{1}{2}(f_0(\sup K'') - f_0(\inf K''))$ for all $x \in K''$; thus

$$\begin{aligned}
f(\sup K'') - f(\inf K'') &> \left[f_0(\sup K'') - \frac{1}{2}(f_0(\sup K'') - f_0(\inf K'')) \right] \\
&\quad - \left[f_0(\inf K'') + \frac{1}{2}(f_0(\sup K'') - f_0(\inf K'')) \right] \\
&= \frac{1}{2}(f_0(\sup K'') + f_0(\inf K'')) \\
&\quad - \frac{1}{2}(f_0(\sup K'') + f_0(\inf K'')) \\
&= 0, \hspace{5cm} (2.22)
\end{aligned}$$

and thus $f'(x) > 0$ for some $x \in K''$, so $f' > 0$ on $\mathbf{R}^1$, since f is a diffeomorphism of $\mathbf{R}^1$.

Now $K^* + [-1, 1] \subset \mathbf{R}^1$ is compact; thus there is a $\delta'' > 0$ so that $x, y \in K^* + [-1, 1]$ and $|x - y| < \delta''$ imply that $|f_0^{-1}(x) - f_0^{-1}(y)| < \epsilon$. Let now $K' = f_0^{-1}(K^*)$, which is compact since $f_0 \in \mathrm{Diff}(\mathbf{R}^1)$; we note that $f_0(\sup K') = \sup K^*$ and $f_0(\inf K') = \inf K^*$, so $f_0(\sup K') > f_0(\inf K')$ (and hence $\sup K' > \inf K'$) by our definition of K^*. Now set $\delta = \min\{\frac{1}{2}\delta'', \frac{1}{2}(\sup K^* - \inf K^*), \frac{1}{2}\}$, and let $f \in \mathrm{Diff}(\mathbf{R}^1)$ satisfy $p_{0,K'}(f - f_0) < \delta$. We note that $f' > 0$ on $\mathbf{R}^1$.

Now we begin our juggling of inequalities. First, we note that for all $x, a \in \mathbf{R}^1$, $f_0(f_0^{-1}(x - a)) + a = x$, so $(f_0 + a)^{-1}(x) = f_0^{-1}(x - a)$, and similarly $(f_0 - a)^{-1}(x) = f_0^{-1}(x + a)$. Further, if $g_1, g_2 \in \mathrm{Diff}(\mathbf{R}^1)$, $g_1', g_2' > 0$, and $g_1(x) > g_2(x)$ for all x in some set A, then for all $y \in g_1(A)$ we have $g_2(g_2^{-1}(y)) = y = g_1(g_1^{-1}(y)) > g_2(g_1^{-1}(y))$, so $g_2^{-1}(y) > g_1^{-1}(y)$ since $g_2' > 0$; similarly, for $y \in g_2(A)$ we have $g_1(g_1^{-1}(y)) = y = g_2(g_2^{-1}(y)) < g_1(g_2^{-1}(y))$, so again $g_2^{-1}(y) > g_1^{-1}(y)$. Now $f(f_0^{-1}(K^*)) = f(K') \subset K^* + [-1, 1]$; also, $f_0(x) + \delta > f(x) > f_0(x) - \delta$ for all

$x \in f_0^{-1}(K^*) = K'$, so for all $y \in f_0(K') + \{\delta\} \cap f_0(K') - \{\delta\} \subset K^* + [-\frac{1}{2}, \frac{1}{2}]$ we have

$f_0^{-1}(y + \delta) > f^{-1}(y) > f_0^{-1}(y - \delta)$. But $f_0^{-1}(y + \delta) < f_0^{-1}(y) + \epsilon$ and $f_0^{-1}(y - \delta) >$

$f_0^{-1}(y) - \epsilon$, by our choice of δ'' and since $\delta \leq \frac{1}{2}\delta''$ and $y, y \pm \delta \in K^* + [-1, 1]$; thus

$|f^{-1}(y) - f_0^{-1}(y)| < \epsilon$ for $y \in K^* + \{\delta\} \cap K^* - \{\delta\}$. But $\delta < 1$, so since K^* is

connected $K^* + \{\delta\} \cap K^* - \{\delta\} \supset K^* + \{1\} \cap K^* - \{1\} \supset [\inf K, \sup K] \supset K$. Thus

$p_{0,K}(f^{-1} - f_0^{-1}) < \epsilon$, as desired.

If $f_0' < 0$, then $(-f_0)' > 0$; thus there exist $K' \subset \mathbf{R}^1$ compact and $\delta > 0$

so that $f \in \text{Diff}(\mathbf{R}^1)$ and $p_{0,K'}((-f) - (-f_0)) = p_{0,K'}(f - f_0) < \delta$ implies that

$p_{0,-K}((-f)^{-1} - (-f_0)^{-1}) = p_{0,K}(f^{-1} - f_0^{-1}) < \epsilon$. Thus the result holds for all f_0.

We now apply the lemma. We first note that $C_0^\infty(\mathbf{R}^1, \mathbf{R}^1) \supset \text{Diff}(\mathbf{R}^1)$. Fix

$f_0 \in \text{Diff}(\mathbf{R}^1)$, let $g_0 = f_0^{-1}$, and fix $K \subset \mathbf{R}^1$ compact, $\epsilon > 0$, and $n \in \mathbf{N}$. By the

lemma there exist $\delta > 0$ and $K_1, K_2, K_3 \subset \mathbf{R}^1$ compact so that $f \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$,

$g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, $p_{n,K_1}(f - f_0) < \delta$, $p_{n,K_2}(f \circ g - f_0 \circ g_0) < \delta$, and $p_{0,K_3}(g - g_0) < \delta$

implies that $p_{n,K}(g - g_0) < \epsilon$. Now by the preceding there exist $K' \subset \mathbf{R}^1$ and $\delta' > 0$

so that $p_{0,K'}(f - f_0) < \delta'$ implies that $p_{0,K_3}(f^{-1} - f_0^{-1}) < \delta$. Let $f \in \text{Diff}(\mathbf{R}^1)$ satisfy

$p_{n,K' \cup K_1}(f - f_0) < \min\{\delta', \delta\}$. Then $p_{0,K_3}(f^{-1} - g_0) < \delta$, $p_{n,K_1}(f - f_0) < \delta$, and

$p_{n,K_2}(f \circ f^{-1} - f_0 \circ g_0) = 0 < \delta$, and therefore $p_{n,K}(f^{-1} - g_0) = p_{n,K}(f^{-1} - f_0^{-1}) < \epsilon$.

ι is therefore continuous, as desired. QED.

We thus have the following theorem.

Theorem 2.1. $\text{Diff}(\mathbf{R}^1)$ and $\text{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ are topological groups.

Proof. Lemma 2.1 and Corollary 2.1 together show that $\text{Diff}(\mathbf{R}^1)$ is a topologi-

cal group. That $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ is also a topological group follows from consideration of the restriction of the map $\mathrm{Diff}(\mathbf{R}^1) \times \mathrm{Diff}(\mathbf{R}^1) \to \mathrm{Diff}(\mathbf{R}^1)$, $(f, g) \mapsto f \circ g$ to $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$, which maps into $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ since $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ is a subgroup of $\mathrm{Diff}(\mathbf{R}^1)$, and of the restriction of $\iota : \mathrm{Diff}(\mathbf{R}^1) \to \mathrm{Diff}(\mathbf{R}^1)$ to $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$, which similarly maps into $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$. (For a more detailed argument, see e.g. Pontrjagin [9], p. 58.)   QED.

We let S_2 denote the symmetric group on two letters, i.e., $S_2 = \{(1), (12)\}$, where (1) is the identity and $(12) : \{1, 2\} \to \{1, 2\}$, $1 \mapsto 2$, $2 \mapsto 1$. We note that $S_2 \cong \mathbf{Z}_2$. Now we may view $\mathbf{Z}_2$ as the multiplicative group of real numbers of unit modulus, and we shall often do this below. By virtue of the above isomorphism we may then write loosely $-\chi = (12)\chi$ for $\chi \in S_2$, and we shall often use this notation below. We give both of these groups the discrete topology.

We let $\mathrm{Diff}_{2\pi\mathbf{Z}}^+(\mathbf{R}^1)$, $\mathrm{Diff}_{2\pi\mathbf{Z}}^-(\mathbf{R}^1)$ represent, respectively, orientation-preserving and orientation-reversing elements of $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$, and set $\Delta = \mathrm{Diff}_{2\pi\mathbf{Z}}^+(\mathbf{R}^1) \times \mathrm{Diff}_{2\pi\mathbf{Z}}^+(\mathbf{R}^1) \cup \mathrm{Diff}_{2\pi\mathbf{Z}}^-(\mathbf{R}^1) \times \mathrm{Diff}_{2\pi\mathbf{Z}}^-(\mathbf{R}^1)$. Δ is clearly a topological group (see Hewitt and Ross [11], Theorem 6.2). We denote elements of Δ by either (f_1, f_2) or $f_1 \times f_2$. We shall also view elements $f_1 \times f_2$ as functions $\mathbf{R}^2 \to \mathbf{R}^2$, $(x, y) \mapsto (f_1(x), f_2(y))$. We define an action of S_2 on Δ by setting $\chi(f_1, f_2) = (f_{\chi(1)}, f_{\chi(2)})$ and note the following easy result.

Proposition 2.3. The map $\Sigma : \Delta \times S_2 \to \Delta$, $(f_1, f_2, \chi) \mapsto \chi(f_1, f_2)$, is continuous.

Proof. Let $U = U(n_1, K_1, \epsilon_1, f_0) \times U(n_2, K_2, \epsilon_2, g_0) \cap \Delta$ be open in Δ. Then

$$\Sigma^{-1}(U) = \{(f_1, f_2, \chi) \in \Delta \times S_2 | \chi(f_1, f_2) \in U\}$$

$$= \{(f_1, f_2, 1) \in \Delta \times S_2 | (f_1, f_2) \in U\}$$

$$\cup \{(f_1, f_2, (12)) \in \Delta \times S_2 | (f_2, f_1) \in U\}$$

$$= [U \times \{1\}] \cup [(U(n_2, K_2, \epsilon_2, g_0) \times U(n_1, K_1, \epsilon_1, f_0) \cap \Delta) \times \{(12)\}]; \quad (2.23)$$

this last set is open in $\Delta \times S_2$ since we give S_2 the discrete topology and since

the set $U(n_2, K_2, \epsilon_2, g_0) \times U(n_1, K_1, \epsilon_1, f_0) \cap \Delta$ is open in Δ (since the reflection

$(f_1, f_2) \mapsto (f_2, f_1)$ fixes Δ). $\hspace{4cm}$ QED.

We now define the semidirect product $\Delta \rtimes S_2$ by taking the group operation to be

$(f_1, f_2, \chi)(g_1, g_2, \xi) = (f_1 \circ g_{\chi(1)}, f_2 \circ g_{\chi(2)}, \chi\xi) = ((f_1 \times f_2) \circ (\chi(g_1 \times g_2)), \chi\xi)$; inver-

sion is given by (see Hungerford [10], p. 99) $(f_1, f_2, \chi)^{-1} = ((\chi^{-1})(f_1^{-1}, f_2^{-1}), \chi^{-1})$.

These operations are both continuous by the foregoing proposition, since Δ is a

topological group and S_2 has the discrete topology, and thus this semidirect prod-

uct is a toplogical group. This group is basic to our constructions in Chapter 4

below. As a topological space it equals $\Delta \times S_2$, and we shall occasionally write it

as such when we are not concerned with its group structure.

We shall also have occasion to use the action of S_2 on $\mathbf{R}^2$ given by $\chi(x_1, x_2) = $

$(x_{\chi(1)}, x_{\chi(2)})$, and when we write an element χ of S_2 standing alone as a function

on $\mathbf{R}^2$ it will always have this meaning. We note that $(\chi(f_1, f_2))(\chi(x_1, x_2)) = $

$(f_{\chi(1)}(x_{\chi(1)}), f_{\chi(2)}(x_{\chi(2)})) = \chi(f_1(x_1), f_2(x_2)) = (\chi \circ (f_1 \times f_2))(x_1, x_2)$; thus in

particular we take most careful note that $\chi \circ (f_1 \times f_2) \neq \chi(f_1 \times f_2)$ – in fact, this

is crucial to a correct understanding of our proof of Theorem 4.1 below!

CHAPTER 3

OPENNESS AND TOPOLOGICAL RESULTS

We present first the proof of Lemma 2.2. We recall that $\{(1,1,n)\} \notin I_n$, so that if $k = \{(j, m_i, l_i)\} \in I_n$ then $l_i < n$ for each i.

Proof of Lemma 2.2. Fix $\epsilon > 0$ and $K \subset \mathbf{R}^1$ compact and nonempty. We proceed by induction on n. Our basic idea is to take inequality (2.14) in the proof of Lemma 2.1 above and solve it for $g^{(n')}$ after writing out the leading term explicitly as in the statement of Proposition 2.2 above.

For $n = 0$ we may clearly take $K_1 = K_2 = K_3 = K$, $\delta = \epsilon$.

Suppose now that the result holds for $n \leq q$, $q \geq 0$. (We would like to here insert a word of caution. The case $q = 0$ is of course a key case, as without it our induction could never get started. However, it is a very special case in what follows as many of the conditions and terms below become vacuous and zero, respectively, when $q = 0$, and we invite the reader to pay special attention to the logic below for this particular case. The other cases ($q > 0$) are more straightforward. We make the convention that a sum taken over an empty range and a function taken on an empty set are both zero.) We see that (for all $f \in C^\infty(\mathbf{R}^1, V)$, $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$)

$$\left|(f \circ g)^{(q+1)}(x) - (f_0 \circ g_0)^{(q+1)}(x)\right| = \left|(f' \circ g)(x)g^{(q+1)}(x) - (f_0' \circ g_0)(x)g_0^{(q+1)}(x)\right.$$

$$+ \sum_{k \in I_{q+1}} a_k^{q+1} \left[\left(f^{(j)} \circ g\right)(x) \prod_{(j,m,l) \in k} \left(g^{(l)}(x)\right)^m\right.$$

$$\left.\left.- \left(f_0^{(j)} \circ g_0\right)(x) \prod_{(j,m,l) \in k} \left(g_0^{(l)}(x)\right)^m\right]\right|; \quad (3.1)$$

for $q = 0$ the range over which the sum is taken is empty (since $I_1 = \emptyset$) and we therefore take the sum to be zero. The condition $l < q + 1$ on the elements of the sequences in I_{q+1} implies that the terms in the sum above depend on the derivatives of g only up to the qth order. As in the proof of Lemma 2.1, if $p_{q,K}(g - g_0) < 1$ then $g^{(l)}(K) \subset g_0^{(l)}(K) + [-1, 1]$ for all $l \leq q$. As before, let $K' = g_0(K) + [-1, 1]$ and $K'' = \prod_{l=1}^q \left(g_0^{(l)}(K) + [-1, 1] \right)$ for $q > 0$; for $q = 0$ set $K'' = \emptyset$. As before, K' and K'' are fixed and compact. Let $m = \frac{1}{2} \inf_{x \in K'} |f_0'(x)|$, which is positive since f_0' is continuous and never zero and K' is compact and nonempty, and let $s = \sum_{k \in I_{q+1}} \left| a_k^{q+1} \right|$; we see that $s \geq 0$. Let $\epsilon' = \dfrac{m\epsilon}{27(1 + s)(1 + p_{q+1,K}(g_0))^{q+1}}$, $\epsilon'' = \dfrac{m\epsilon}{27(1 + s)(1 + p_{q+1,K'}^V(f_0))}$; as before, ϵ' and ϵ'' are positive, finite, and fixed. As in Lemma 2.1 above, choose $\delta' > 0$, $\delta'' > 0$ so that (i) $x, y \in K'$ and $|x - y| < \delta'$ implies that $|f_0^{(q')}(x) - f_0^{(q')}(y)| < \epsilon'$ for all $q' \in \mathbf{N}$, $q' \leq q + 1$ and (ii) $(x^i)_{i=1}^q, (y^i)_{i=1}^q \in K''$ and $\max_{1 \leq i \leq q} |x^i - y^i| < \delta''$ implies that $|\beta_k((x^i)^q) - \beta_k((y^i)^q)| < \epsilon''$ for all $k \in I_{q+1}$. (We recall that β_k only depends on the first q of its arguments if $k \in I_{q+1}$.) For $q = 0$ condition (ii) is entirely vacuous since K'' and I_{q+1} are empty; in this case we set $\delta'' = 1$. Now by our induction hypothesis the result is true for $n = q$ and $\epsilon = \min\{\delta'', 1\}$; let δ''', K_1', K_2', and K_3' be the positive number and compact sets resulting in this case. Set $K_1 = K' \cup K_1'$, $K_2 = K \cup K_2'$, and $K_3 = K \cup K_3'$, set $\delta = \min\{1, \frac{1}{2}m, \delta', \delta'', \delta''', \epsilon'\}$, and let $f \in C^\infty(\mathbf{R}^1, V)$, $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ satisfy $p_{q+1,K_1}^V(f - f_0) < \delta$, $p_{q+1,K_2}^V(f \circ g - f_0 \circ g_0) < \delta$, $p_{0,K_3}(g - g_0) < \delta$. Then by our induction hypothesis $p_{q,K}(g - g_0) < \min\{\delta'', 1\} \leq 1$; thus $\prod_{i=1}^q g^{(i)}(K) \subset K''$ if $q > 0$. Moreover, as before, for $0 \leq q' \leq q + 1$, $x, y \in K'$, $|x - y| < \delta$ we have

$$\left| f^{(q')}(x) - f^{(q')}(y) \right| \leq \left| f^{(q')}(x) - f_0^{(q')}(x) \right| + \left| f_0^{(q')}(x) - f_0^{(q')}(y) \right|$$

$$+ \left| f_0^{(q')}(y) - f^{(q')}(y) \right|$$

$$< 2\delta + \epsilon' \leq 3\epsilon'. \tag{3.2}$$

Similarly, if $q > 0$, then for all $x \in K$ we see that $(g^{(i)}(x))_1^q$, $(g_0^{(i)}(x))_1^q \in K''$, and similarly (by the induction hypothesis) $\max_{1 \leq i \leq q} \left| g^{(i)}(x) - g_0^{(i)}(x) \right| < \delta''$; thus $\left| \beta_k((g^{(i)}(x))_1^q) - \beta_k((g_0^{(i)}(x))_1^q) \right| < \epsilon''$ for all $k \in I_{q+1}$. Finally, $p_{1,K'}^V(f - f_0) \leq p_{q+1,K_1}^V(f - f_0) < \delta \leq \frac{1}{2}m$ (note that $q \geq 0$ implies $q + 1 \geq 1$!), so for all $x \in K'$ we have $|f'(x)| > |f_0'(x)| - \frac{1}{2}m$ (since $|f_0'(x)| - |f'(x)| \leq |f'(x) - f_0'(x)| < \frac{1}{2}m$), and thus $\inf_{x \in K'} |f'(x)| > 2m - \frac{1}{2}m = \frac{3}{2}m$. Further, $p_{q+1,K'}^V(f) \leq 1 + p_{q+1,K'}^V(f_0)$. Thus we see that, for all $x \in K$

$$\left| g^{(q+1)}(x) - g_0^{(q+1)}(x) \right|$$

$$\leq \frac{1}{|(f' \circ g)(x)|} \Big[\left| (f \circ g)^{(q+1)}(x) - (f_0 \circ g_0)^{(q+1)}(x) \right|$$

$$+ \Big(\left| (f' \circ g)(x) - (f' \circ g_0)(x) \right| + \left| (f' \circ g_0)(x) - (f_0' \circ g_0)(x) \right| \Big) \Big(\left| g_0^{(q+1)}(x) \right| \Big)$$

$$+ \sum_{k \in I_{q+1}} \left| a_k^{q+1} \right| \left| \left(f^{(j)} \circ g \right)(x) \prod_{(j,m,l) \in k} \left(g^{(l)}(x) \right)^m \right.$$

$$\left. - \left(f_0^{(j)} \circ g_0 \right)(x) \prod_{(j,m,l) \in k} \left(g_0^{(l)}(x) \right)^m \right| \Big]$$

$$< \frac{2}{3m} \Big[\delta + (3\epsilon' + \delta)(1 + p_{q+1,K}(g_0)) + \sum_{k \in I_{q+1}} \left| a_k^{q+1} \right| \Big[$$

$$|(f^{(j)} \circ g)(x) - (f^{(j)} \circ g_0)(x)||\beta_k((g^{(i)}(x))_1^q)|$$

$$+ |(f^{(j)} \circ g_0)(x)||\beta_k((g^{(i)}(x))_1^q) - \beta_k((g_0^{(i)}(x))_1^q)|$$

$$+ |(f^{(j)} \circ g_0)(x) - (f_0^{(j)} \circ g_0)(x)||\beta_k((g_0^{(i)}(x))_1^q)| \Big] \Big]. \tag{3.3}$$

Noting that estimates analogous to those in equations (2.16) and (2.17) in the proof of Lemma 2.1 above hold here, and recalling that all l values in the sum are no greater than q (and that the entire sum is taken to be zero when $q = 0$) we thus obtain

$$\left| g^{(q+1)}(x) - g_0^{(q+1)}(x) \right| < \frac{2}{3m} \left(\delta + (3\epsilon' + \delta)(1 + p_{q+1,K}(g_0)) \right.$$

$$+ \sum_{k \in I_{q+1}} \left| a_k^{q+1} \right| \left[3\epsilon'(1 + p_{q,K}(g_0))^{q+1} + (1 + p_{q+1,K'}^V(f_0))\epsilon'' + \epsilon'(1 + p_{q,K}(g_0))^{q+1} \right] \right)$$

$$\leq \frac{2}{3m} \left(\frac{m\epsilon}{27} + \frac{m\epsilon}{9} + \frac{m\epsilon}{27} + \frac{m\epsilon}{9} + \frac{m\epsilon}{27} + \frac{m\epsilon}{27} \right)$$

$$= \frac{20}{81} \epsilon < \epsilon. \tag{3.4}$$

Thus the result is true for $q + 1$ as well with the above choices of δ, K_1, K_2, and K_3. We note in passing that we may take $K_2 = K_3 = K$ for all n. QED.

We say that a map $F : X \to Y$ between topological spaces is open onto its image if the restriction $F : X \to F(X) \subset Y$ is open; such a function is open as a map into Y if and only if its image $F(X)$ is open in Y. We have the following criterion, related to Proposition 2.1 above, which links this notion with Lemma 2.2. Compare Pontrjagin [9], §18, C). (We note that if $B \subset P(Y)$ is a basis for the topology of Y, then $\{U \cap F(X) | U \in B\}$ is for $F(X)$, and similarly if B is instead a basis of neighborhoods at some point x_0.)

Proposition 3.1. Let X and Y be two topological spaces, and let $F : X \to Y$. F is open onto its image if and only if for every $x_0 \in X$ the following condition is satisfied: if B_{x_0} is a basis of neighborhoods at x_0 and $B_{F(x_0)}$ is a basis of neigh-

borhoods at $F(x_0)$, then for every $U \in B_{x_0}$ there is a $V \in B_{F(x_0)}$ so that for every $x \in X$ satisfying $F(x) \in V$ there is an $x^* \in U$ such that $F(x) = F(x^*)$.

Proof. Suppose that $F : X \to Y$ is open onto its image, and let $x_0 \in X$, B_{x_0} be some basis of neighborhoods at x_0, and $U \in B_{x_0}$. Then $F(U)$ is open in $F(X)$ as a subspace of Y, and it contains $F(x_0)$; thus there is a $V \in B_{F(x_0)}$ so that $V \cap F(X) \subset F(U)$. Let $x \in X$ be such that $F(x) \in V$; then $F(x) \in F(U)$ so there is an $x^* \in U$ such that $F(x) = F(x^*)$, as desired.

Now suppose that the above criterion holds, let $W \subset X$ be open, and consider $y_0 \in F(W)$. Then $y_0 = F(x_0)$ for some $x_0 \in W$; since W is open there is a $U \in B_{x_0}$ so that $U \subset W$. Let $V \in B_{F(x_0)}$ be as given by the criterion; thus $y_0 = F(x_0) \in V$. Let $y = F(x) \in V \cap F(X)$; then there is an $x^* \in U \subset W$ so that $y = F(x) = F(x^*) \in F(U) \subset F(W)$. Thus $V \cap F(X) \subset F(W)$ and $y_0 \in V \cap F(X)$. Since $y_0 \in F(W)$ is arbitrary this shows that $F(W)$ is open in $F(X)$. Thus F is open onto its image, as desired. QED.

Specializing to the case where X and Y are both spaces of C^∞ functions, we have the following result.

Corollary 3.1. Let V_1, V_2, W_1, and W_2 be finite-dimensional vector spaces. A map $F : C^\infty(V_1, W_1) \to C^\infty(V_2, W_2)$ is open onto its image if and only if for every $n \in \mathbf{N}$, $K \subset V_1$ compact, $g_0 \in C^\infty(V_1, W_1)$, and $\epsilon > 0$ there exist $n' \in \mathbf{N}$, $K' \subset V_2$ compact, and $\delta > 0$ so that for every $g \in C^\infty(V_1, W_1)$ satisfying $p_{n',K'}^{V_2,W_2}(F(g) - F(g_0)) < \delta$ there is a $g^* \in C^\infty(V_1, W_1)$ such that $F(g^*) = F(g)$ and

$$p_{n,K}^{V_1,W_1}(g^* - g_0) < \epsilon.$$

Proof. This follows immediately from the proposition since sets of the form $U(n, K, \epsilon, g_0)$ and $U(n', K', \delta, F(g_0))$ for the stated ranges of $n, n', K, K', \epsilon, \delta$ form neighborhood bases at g_0 and $F(g_0)$, respectively. QED.

Proposition 3.2. Let X and Y be topological spaces, and let $F : X \to Y$ be a map open onto its image. Let $S \subset X$ be such that $S = F^{-1}(F(S))$. Then $F|_S : S \to Y$ is also open onto its image. This also holds if S is open in X.

Compare Munkres [8], Theorem 22.1.

Proof. Let $U \subset S$ be open. Then there is an open $V \subset X$ such that $U = V \cap S$. Since F is open onto its image $F(V)$ is open in $F(X)$. Consider $F(U)$. Clearly $F(U) \subset F(V) \cap F(S)$. Let $y_0 \in F(V) \cap F(S)$; then there exist $v_0 \in V$, $s_0 \in S$ such that $y_0 = F(v_0) = F(s_0)$. But this implies that $v_0 \in F^{-1}(F(S)) = S$, so $v_0 \in S \cap V$ and $y_0 \in F(V \cap S)$. Thus $F(U) = F(V) \cap F(S)$ is open in $F(S)$, since $F(S) \subset F(X)$ and $F(V)$ is open in $F(X)$. Thus $F|_S$ is open onto its image, as desired.

If S is open and $U \subset S$ is open in S, then U is also open in X so $F(U) \subset F(S)$ is open in $F(X)$ and hence in $F(S)$. QED.

We note that the first condition is clearly satisfied if F is a homeomorphism onto its image; thus the restriction of a homeomorphism is a homeomorphism onto its image. (This can also be seen more directly, of course.)

Corollary 3.2. Let $F : X \to Y$ and $G : Y \to Z$ be two maps between topological spaces, each open onto its image. Then $G \circ F : X \to Z$ is open onto its image if $G^{-1}(G(F(X))) = F(X)$, or if $F(X)$ is open in Y.

Proof. By Propostion 3.2, $G^{-1}(G(F(X))) = F(X)$ or $F(X)$ open in Y implies that $G|_{F(X)} : F(X) \to G(F(X))$ is open. But $F : X \to F(X)$ is open, and thus $G \circ F = (G|_{F(X)}) \circ F : X \to G(F(X))$ is open; i.e., $G \circ F : X \to Z$ is open onto its image, as desired. QED.

We desire now to prove certain results concerning the topology of spaces of the form $C^\infty(S^1, V)$, where V is some finite-dimensional vector space, and also concerning openness of certain maps between such spaces. First we set some more notation. We let $q : \mathbf{R}^1 \to S^1$, $x \mapsto e^{ix}$ denote the standard covering map, and $p : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1 \times S^1$, $(t, \theta) \mapsto (t, e^{i\theta})$ the universal covering map of the cylinder; thus $p = \mathrm{id} \times q$. We note that q and p are local diffeomorphisms; specifically, if $U \subset S^1$ is evenly covered by q, so that $q^{-1}(U) = \cup_i V_i$, $V_i \subset \mathbf{R}^1$ open and disjoint and such that $q|_{V_i} : V_i \to U$ is a homeomorphism for each i, then $q|_{V_i} : V_i \to U$ is in fact a diffeomorphism, since $q|_{V_i}$ is smooth and bijective with nowhere vanishing derivative. A similar statement holds for p. Before proceeding we note a property of such maps.

Proposition 3.3. Let M, N, and L be finite-dimensional C^∞ manifolds, and let $\pi : M \to N$ be a covering map which is also a local diffeomorphism in the above sense. Suppose that maps $f : M \to L$ and $\phi : N \to L$ are such that $f = \phi \circ \pi$.

Then f is C^∞ if and only if ϕ is. The same holds for $f : L \to M$ and $\phi : L \to N$ satisfying $\pi \circ f = \phi$ if f is continuous.

Proof. We start with the first statement. The if direction is obvious. Suppose now that f is C^∞ and let $n \in N$. Let $U \subset N$ be a neighborhood of n evenly covered by π, so that $\pi^{-1}(U) = \cup_i V_i$, where the $V_i \subset M$ are disjoint and open and $\pi|_{V_i} : V_i \to U$ is a diffeomorphism for each i. Then we see that $f|_{V_i} = (\phi \circ \pi)|_{V_i} = \phi|_U \circ \pi|_{V_i}$; now $\pi|_{V_i}$ is a diffeomorphism, and thus $\phi|_U = f|_{V_i} \circ (\pi|_{V_i})^{-1} : U \to L$ is C^∞ on U, and in particular at n. But $n \in N$ was arbitrary and thus $\phi : N \to L$ is C^∞, as desired.

For the second part, we note first that if f is C^∞ then clearly so is ϕ. Suppose now that ϕ is C^∞ and that f is continuous, let $l \in L$, and as before let U be an evenly covered neighborhood of $\phi(l) \in N$, with $\pi^{-1}(U) = \cup_i V_i$ where the V_i are disjoint and mapped diffeomorphically onto U by π. Now $\pi(f(l)) = \phi(l) \in U$, so $f(l) \in V_i$ for some i. Now $f^{-1}(V_i)$ is open in L since f is continuous; moreover, we see that on $f^{-1}(V_i)$ $\pi|_{V_i} \circ f|_{f^{-1}(V_i)} = \phi|_{f^{-1}(V_i)}$, so $f|_{f^{-1}(V_i)} = (\pi|_{V_i})^{-1} \circ \phi|_{f^{-1}(V_i)}$ and hence f is C^∞ at $l \in f^{-1}(V_i)$ since $\pi|_{V_i}$ is a diffeomorphism. Thus f is C^∞ on L, as desired.   QED.

Continuity of f in the second part is necessary: consider the case $M = L = \mathbf{R}^1$, $N = S^1$, $\pi = \phi = q$, and $f(x) = x, x < 0, x + 2\pi, x \geq 0$. Then clearly $\pi \circ f = \phi$ and ϕ is C^∞, but f is not C^∞ at 0.

Let now V be some finite-dimensional vector space, and consider $C^\infty(S^1, V)$. We shall topologize this space as done in Milnor [2] (in other words, with essentially

the weak topology; see our note on p. 5). Specifically, for every $\sigma = e^{i\theta_0} \in S^1$ we have a coordinate system $\theta_\sigma : S^1\backslash\{-\sigma\} \to (-\pi, \pi)$ given by inverting $e^{i(\cdot + \theta_0)}|_{(-\pi,\pi)} :$ $(-\pi, \pi) \to S^1\backslash\{-\sigma\}$; we then take as a basis for the topology of $C^\infty(S^1, V)$ the set of all finite intersections of sets of the form

$$\{\phi \in C^\infty(S^1, V) | p^V_{n, \theta_\sigma(K)}(\phi \circ \theta_\sigma^{-1} - \phi_0 \circ \theta_\sigma^{-1}) < \epsilon\}, \tag{3.5}$$

where $n \in \mathbf{N}$, $\sigma \in S^1$, $K \subset S^1\backslash\{-\sigma\}$ is compact (and hence $\theta_\sigma(K) \subset (-\pi, \pi)$ is compact), $\epsilon > 0$, and $\phi_0 \in C^\infty(S^1, V)$. Consider now the two compact sets $K_1 = q([\frac{\pi}{4}, \frac{7\pi}{4}])$ and $K_2 = q([-\frac{3\pi}{4}, \frac{3\pi}{4}])$. We see that $K_1 \subset S^1\backslash\{1\}$ and $K_2 \subset S^1\backslash\{-1\}$. Now $\theta_{-1} : S^1\backslash\{1\} \to (-\pi, \pi)$ is the inverse of $e^{i(\cdot + \pi)}|_{(-\pi,\pi)} = q(\cdot + \pi)|_{(-\pi,\pi)}$, and $\theta_1 : S^1\backslash\{-1\} \to (-\pi, \pi)$ is the inverse of $q|_{(-\pi,\pi)}$. Thus for all $n \in \mathbf{N}$ and all $\phi \in C^\infty(S^1, V)$ we have

$$p^V_{n, \theta_{-1}(K_1)}(\phi \circ \theta_{-1}^{-1}) = p^V_{n, [-\frac{3\pi}{4}, \frac{3\pi}{4}]}(\phi \circ q(\cdot + \pi))$$

$$= p^V_{n, [\frac{\pi}{4}, \frac{7\pi}{4}]}(\phi \circ q), \tag{3.6}$$

and similarly $p^V_{n, \theta_1(K_2)}(\phi \circ \theta_1^{-1}) = p^V_{n, [-\frac{3\pi}{4}, \frac{3\pi}{4}]}(\phi \circ q)$. Thus the set

$$\{\phi \in C^\infty(S^1, V) | p^V_{n, [-\frac{3\pi}{4}, \frac{7\pi}{4}]}(\phi \circ q - \phi_0 \circ q) < \epsilon\} \tag{3.7}$$

is a basic open set in $C^\infty(S^1, V)$ for all $n \in \mathbf{N}$, $\epsilon > 0$, and $\phi_0 \in C^\infty(S^1, V)$; we see that the collection of all such sets forms a basis for the topology of $C^\infty(S^1, V)$ as defined above. By the periodicity of q this means that $\{\phi | p^V_{n,K}(\phi \circ q - \phi_0 \circ q) < \epsilon\}$ is open in $C^\infty(S^1, V)$ for all $n \in \mathbf{N}$, $K \subset \mathbf{R}^1$ compact, $\epsilon > 0$, and $\phi_0 \in C^\infty(S^1, V)$. We similarly topologize $C^\infty(\mathbf{R}^1 \times S^1, V)$ by taking as a basis all finite intersections

of sets of the form $\{\Psi \in C^\infty(\mathbf{R}^1 \times S^1, V)|p^V_{n,K \times [-\frac{3\pi}{4}, \frac{7\pi}{4}]}(\Psi \circ p - \Psi_0 \circ p) < \epsilon\}$, $n \in \mathbf{N}$,

$K \subset \mathbf{R}^1$ compact, $\Psi_0 \in C^\infty(\mathbf{R}^1 \times S^1, V)$, $\epsilon > 0$. We then see that any set of

this form with $[-\frac{3\pi}{4}, \frac{7\pi}{4}]$ replaced by an arbitrary $K' \subset \mathbf{R}^1$ compact is also open

in $C^\infty(\mathbf{R}^1 \times S^1, V)$. We let $C^\infty_p(\mathbf{R}^1 \times \mathbf{R}^1, V)$ for any vector space V denote the

set of all C^∞ functions $f : \mathbf{R}^1 \times \mathbf{R}^1 \to V$ satisfying the periodicity condition

$f(x, y + 2\pi) = f(x, y)$ for all $(x, y) \in \mathbf{R}^1 \times \mathbf{R}^1$, and we topologize this space as a

subspace of $C^\infty(\mathbf{R}^2, V)$.

We now have the following result.

Proposition 3.4. The map

$$\Theta : C^\infty(S^1, V) \to C^\infty_p(\mathbf{R}^1, V)$$

$$\phi \mapsto \phi \circ q$$

(3.8)

is a homeomorphism.

Proof. Clearly Θ maps into $C^\infty_p(\mathbf{R}^1, V)$, by periodicity and smoothness of q. It

is clearly injective: $\Theta(\phi_1) = \Theta(\phi_2)$ implies that $\phi_1(q(x)) = \phi_2(q(x))$ for all $x \in \mathbf{R}^1$,

which implies that $\phi_1 = \phi_2$ by surjectivity of q. To see that Θ is surjective, let

$f \in C^\infty_p(\mathbf{R}^1, V)$. Then f is constant on the fibers of q, so $f(q^{-1}(\sigma))$ is well-defined

for each $\sigma \in S^1$. Denote this value by $\phi(\sigma) \in V$. Thus $\phi : S^1 \to V$ satisfies

$\phi \circ q = f$; by Proposition 3.3 and our observations about q above, this implies that

$\phi \in C^\infty(S^1, V)$. Now by definition $\Theta(\phi) = f$, and thus Θ is indeed surjective.

To see that Θ is continuous, let $U_1 = U_V(n, K, \epsilon, f_0) \cap C^\infty_p(\mathbf{R}^1, V)$ be a basic

open set in $C^\infty_p(\mathbf{R}^1, V)$, where $n \in \mathbf{N}$, $K \subset \mathbf{R}^1$ is compact, $\epsilon > 0$, and $f_0 \in$

$C_p^\infty(\mathbf{R}^1, V)$. Choose $\phi_0 \in C^\infty(S^1, V)$ such that $\Theta(\phi_0) = f_0$. Then $\Theta^{-1}(U_1) = \{\phi \in C^\infty(S^1, V)|p_{n,K}^V(\phi \circ q - \phi_0 \circ q) < \epsilon\}$, which is open in $C^\infty(S^1, V)$ by our comments above. To see that Θ is open, let $U_2 = \{\phi|p_{n,[-\frac{3\pi}{4}, \frac{7\pi}{4}]}^V(\phi \circ q - \phi_0 \circ q) < \epsilon\}$ be a basic open set in $C^\infty(S^1, V)$; then $\Theta(U_2) = \{\phi \circ q|p_{n,[-\frac{3\pi}{4}, \frac{7\pi}{4}]}^V(\phi \circ q - \phi_0 \circ q) < \epsilon\}$. Since Θ is surjective, we see that $\Theta(U_2) = U_V(n, [-\frac{3\pi}{4}, \frac{7\pi}{4}], \phi_0 \circ q, \epsilon) \cap C_p^\infty(\mathbf{R}^1, V)$, which is open. Thus Θ is open. Hence Θ is a homeomorphism, as desired. QED.

Restricting to $C^\infty(S^1, S^1) \subset C^\infty(S^1, \mathbf{C})$ and $C_p^\infty(\mathbf{R}^1, S^1) \subset \mathbf{C}_p^\infty(\mathbf{R}^1, \mathbf{C})$, we have the following result.

Corollary 3.3. The map

$$\hat{\Theta} : C^\infty(S^1, S^1) \to C_p^\infty(\mathbf{R}^1, S^1)$$

$$\phi \mapsto \phi \circ q$$

(3.9)

is a homeomorphism.

Proof. Again, $\hat{\Theta}$ maps into $C_p^\infty(\mathbf{R}^1, S^1)$ by smoothness and periodicity of q. It is injective as a restriction of the injective Θ. To see that it is surjective, note that $C_p^\infty(\mathbf{R}^1, S^1) \subset C_p^\infty(\mathbf{R}^1, \mathbf{C})$; if $f \in C_p^\infty(\mathbf{R}^1, S^1)$ and $\phi = \Theta^{-1}(f) \in C^\infty(S^1, \mathbf{C})$, then $\phi \circ q = f$ so clearly $\phi : S^1 \to S^1$. ϕ is smooth by Proposition 3.3, as before, so $\phi \in C^\infty(S^1, S^1)$; thus $\hat{\Theta}(\phi) = f$ and $\hat{\Theta}$ is indeed surjective, as desired. Now $\hat{\Theta}$ is clearly continuous as the restriction of the continuous Θ; it is open by Proposition 3.2 (or alternatively since $\hat{\Theta}^{-1} = (\Theta^{-1})|_{C_p^\infty(\mathbf{R}^1, S^1)}$). Thus $\hat{\Theta}$ is a homeomorphism, as desired. QED.

Proposition 3.5. Let $X \subset C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times S^1)$, topologized as a subspace

of $C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times \mathbf{C})$, be any subset. Then the map

$$Z_X : X \to C_p^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$$

$$f \mapsto f \circ p$$

(3.10)

is a homeomorphism onto its image.

Proof. Z_X is clearly injective, since p is surjective. It clearly maps into $C_p^\infty(\mathbf{R}^1 \times$

$\mathbf{R}^1, \mathbf{R}^1 \times S^1)$ by periodicity and smoothness of p. To see that it is continuous, let

$U = U_{\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times \mathbf{C}}(n, K, \epsilon, f_0) \cap C_p^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$, $n \in \mathbf{N}$, $K \subset \mathbf{R}^1 \times \mathbf{R}^1$

compact, $f_0 \in C_p^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$, and $\epsilon > 0$ be open in $C_p^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$.

We may assume that $K = K_1 \times K_2$, $K_1, K_2 \subset \mathbf{R}^1$ compact. Then $Z_X^{-1}(U) =$

$\{\Psi \in X | p_{n,K}^{\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times \mathbf{C}}(\Psi \circ p - f_0) < \epsilon\}$; defining $\Psi_0 : \mathbf{R}^1 \times S^1 \to \mathbf{R}^1 \times S^1$ by

$\Psi_0 \circ p = f_0$, which exists since $f_0 \in C_p^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$ and is C^∞ by Proposition

3.3, we see that this is a basic open set in X with respect to the topology on

$C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times \mathbf{C})$ given above since $K = K_1 \times K_2$.

To see that Z_X is open onto its image, let $V = \{\Psi \in X | p_{n,K_1 \times K_2}^{\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times \mathbf{C}}(\Psi \circ p - \Psi_0 \circ$

$p) < \epsilon\}$ be a basic open set in X. Then $Z_X(V) = \{\Psi \circ p | \Psi \in X, p_{n,K_1 \times K_2}^{\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times \mathbf{C}}(\Psi \circ$

$p - \Psi_0 \circ p) < \epsilon\} = Z_X(X) \cap U_{\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times \mathbf{C}}(n, K_1 \times K_2, \epsilon, \Psi_0 \circ p)$, which is open in

$Z_X(X)$, as desired.   QED.

Proposition 3.6. The map

$$I : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, S^1)$$

$$f \mapsto q \circ f$$

(3.11)

is open.

Proof. We note first of all that I maps into $C^\infty(\mathbf{R}^1, S^1)$ by smoothness of q. Moreover, I is surjective, since we may lift any smooth curve $\mathbf{R}^1 \to S^1$ to a continuous curve $\mathbf{R}^1 \to \mathbf{R}^1$ by the covering map $q : \mathbf{R}^1 \to S^1$ since $\pi_1(\mathbf{R}^1)$ is trivial (see Bredon [4], Theorem III.4.1), and such a lift will be C^∞ by Proposition 3.3 above. Now fix $K \subset \mathbf{R}^1$ compact and connected (as in Corollary 2.1 above), $\epsilon > 0$, $n \in \mathbf{N}$, and $g_0 \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, and let $\delta > 0$, $K_1 \subset \mathbf{R}^1$ compact, $K_2 = K_3 = K$ be as given by Lemma 2.2 above with $f_0 = q \in C_0^\infty(\mathbf{R}^1, \mathbf{C})$. Note that K, ϵ, n, g_0, δ, and K_1 are all fixed throughout the rest of the proof. Let $\delta' = \min\{\delta, \frac{\pi}{2}\}$. Since $q : \mathbf{R}^1 \to S^1$ is open (since it is a covering map), there is a $\delta'' > 0$, $\delta'' < \delta'$, so that $x \in \mathbf{R}^1$ and $|e^{ix} - 1| < \delta''$ implies that $|x - 2\pi m| < \delta'$ for some $m \in \mathbf{Z}$. (Alternatively, this follows by considering an evenly covered neighborhood of $1 \in S^1$.) Let $h \in C^\infty(\mathbf{R}^1, S^1)$ satisfy $p_{n,K}^{\mathbf{C}}(h - q \circ g_0) < \delta''$. Choose $g \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ so that $q \circ g = h$; then g satisfies $p_{0,K}^{\mathbf{C}}(q \circ g - q \circ g_0) = p_{0,K}^{\mathbf{C}}(e^{i(g-g_0)} - 1) \leq p_{n,K}^{\mathbf{C}}(q \circ g - q \circ g_0) < \delta'' < \delta$. Thus for each $x \in K$ there is an $m(x) \in \mathbf{Z}$ such that $|g(x) - g_0(x) - 2\pi m(x)| < \delta'$. We claim that $m : K \to \mathbf{Z}$ is continuous. Choose $\delta''' > 0$ so that $x, y \in K$ and $|x - y| < \delta'''$ implies that $|g(x) - g(y)| < \delta'$, $|g_0(x) - g_0(y)| < \delta'$. ($\delta'''$ exists since g and g_0 are continuous on K and K is compact.) Then for such x, y,

$$|2\pi m(x) - 2\pi m(y)| \leq |g(x) - g_0(x) - 2\pi m(x)| + |g(x) - g_0(x) - (g(y) - g_0(y))|$$

$$+ |g(y) - g_0(y) - 2\pi m(y)|$$

$$< 2\delta' + |g(x) - g(y)| + |g_0(x) - g_0(y)|$$

$$< 4\delta' \leq 2\pi; \tag{3.12}$$

thus $|m(x) - m(y)| < 1$ so $m(x) = m(y)$ and $m : K \to \mathbf{Z}$ is necessarily continuous. Since K is connected and $\mathbf{Z}$ is discrete this means that m is constant. We denote its constant value simply by m as well. (Note that the equation $m(x) = m(y)$ only holds for $|x - y| < \delta'''$ and thus does not immediately imply that m must be constant.) Thus $g - 2\pi m \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$. Moreover, $p_{0,K_3}(g_0 - g - 2\pi m) < \delta' \le \delta$ by construction, and $p_{n,K_2}^{\mathbf{C}}(q \circ g - q \circ g_0) = p_{n,K_2}^{\mathbf{C}}(q \circ (g + 2\pi m) - q \circ g_0) < \delta$ also; finally, taking $f = f_0 = q$, we see that $p_{n,K_1}^{\mathbf{C}}(f - f_0) = 0 < \delta$. The lemma then shows that $p_{n,K}(g_0 - g - 2\pi m) < \epsilon$; since $q \circ (g + 2\pi m) = q \circ g = h$, an application of Proposition 3.1 to the map $I : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, \mathbf{C})$ shows that this map is open onto its image, which means that $I : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, S^1)$ is open, as desired.   QED.

We note that $I(f_1) = I(f_2)$ for $f_1, f_2 \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ implies that for each x there is an $m(x) \in \mathbf{Z}$ such that $f_1(x) = f_2(x) + 2\pi m(x)$. Then $m = \frac{1}{2\pi}(f_1 - f_2)$ is C^∞, in particular continuous, and therefore constant. Clearly $I(f + 2\pi m) = I(f)$ for all $f \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ and all $m \in \mathbf{Z}$, and thus $I^{-1}(I(S)) = \{f + 2\pi m | f \in S, m \in \mathbf{Z}\}$ for any $S \subset C^\infty(\mathbf{R}^1, \mathbf{R}^1)$. In particular, $I^{-1}(I(\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)) = \mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$, and thus $I|_{\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)} : \mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \to C^\infty(\mathbf{R}^1, S^1)$ is open onto its image.

Suppose now that G is any Lie group and $H \subset G$ is any subgroup which is discrete in the induced topology. We will now show that the canonical projection $\pi : G \to G/H$, $g \mapsto Hg$ is a covering map. We need the following elementary result from topological group theory (cf. Bredon [4], Proposition I.15.9). If G is a topological group, a neighborhood $V \subset G$ of the identity e is said to be symmetric

if $V = V^{-1}$.

Proposition 3.7. Let G be a topological group and $U \subset G$ be any neighborhood of the identity e. Then there is a symmetric neighborhood V of e such that $V^2 \subset U$.

Proof. By Hewitt and Ross [11], Theorem 4.5, there is a neighborhood W of e contained in U such that $W^2 \subset U$; by ibid., Theorem 4.6, there is a symmetric neighborhood V of e contained in W. We note that $V^2 \subset W^2 \subset U$, as desired.

QED.

We now have the following result about covering spaces. Compare Spanier [12], p. 62.

Proposition 3.8. Let G be a topological group, and let $H \subset G$ be a discrete subgroup, i.e., a subgroup which is discrete in the induced topology. Then the canonical map $\Xi : G \to G/H$, $g \mapsto Hg$, is a covering map. ($G/H = \{Hg | g \in G\}$ is of course given the quotient topology.)

Proof. Since H is discrete there is a neighborhood U of e satisfying $U \cap H = \{e\}$. Let $V \subset U$ be a symmetric neighborhood of e satisfying $V^2 \subset U$. Now let $g \in G$, and consider $gV \cap H$. If $h_1, h_2 \in gV \cap H$, then $h_1 = gv_1$, $h_2 = gv_2$, $v_1, v_2 \in V$, so $h_2^{-1}h_1 = (v_2^{-1}g^{-1})(gv_1) = v_2^{-1}v_1 \in V^2 \subset U$; thus $h_2^{-1}h_1 \in U \cap H$ so $h_2^{-1}h_1 = e$ and $h_2 = h_1$. Thus $gV \cap H$ has at most one point. Since G is T_1, this implies that H is closed ($gV \backslash (gV \cap H)$ is a neighborhood of g disjoint from H for any $g \in G \backslash H$, since $gV \cap H$ is closed in G as it is either empty or a singleton). Similarly, if $g \in G$, $h_1, h_2 \in H$, and $h_1 V g \cap h_2 V g \neq \emptyset$, then there must be $v_1, v_2 \in V$

such that $h_1 v_1 g = h_2 v_2 g$; thus $h_2^{-1} h_1 = v_2 v_1^{-1} \in U \cap H$, so $h_1 = h_2$. Thus the collection $\{hVg | h \in H\}$ is a disjoint collection of open sets for each $g \in G$. Now fix $x_0 = H g_0 \in G/H$. We see that $\Xi(hVg_0) = \Xi(Vg_0)$ for each $h \in H$; moreover, $W = \Xi(Vg_0)$ is open in G/H, since $\Xi^{-1}(W) = \cup_{h \in H} hVg_0$ is open in G, and W is therefore a neighborhood of $x_0 = \Xi(g_0)$. We claim that it is evenly covered by Ξ. Since $\Xi^{-1}(W) = \cup_{h \in H} hVg_0$, it suffices to show that $\Xi|_{hVg_0} : hVg_0 \to W$ is a homeomorphism for each $h \in H$. Since hVg_0 is open in G and $\Xi(U') = \Xi(\cup_{h \in H} hU')$ is therefore open for all open $U' \subset hVg_0$ by Munkres [8], p. 137, this restriction is open; it is also continuous and surjective. Thus it suffices to show that it is one-to-one. Let $hv_1 g_0, hv_2 g_0 \in hVg_0$ satisfy $\Xi(hv_1 g_0) = \Xi(hv_2 g_0)$; then there exist $h_1, h_2 \in H$ so that $h_1 h v_1 g_0 = h_2 h v_2 g_0$. But then $h^{-1}(h_2^{-1} h_1)h = v_2 v_1^{-1} \in U$, so $h^{-1}(h_2^{-1} h_1)h = e = v_2 v_1^{-1}$, so $hv_1 g_0 = hv_2 g_0$ and Ξ is injective. This completes the proof. QED.

We now set

$$\mathrm{Diff}_H(G) = \{f \in \mathrm{Diff}(G) | f(hg) = h^{\pm 1} f(g) \text{ for all } g \in G \text{ and all } h \in H\}. \quad (3.13)$$

We then have the following result.

Proposition 3.9. Let G and H be as above, let now $\pi : G \to G/H$ denote the canonical projection, and suppose that G is pathwise connected and locally pathwise connected and that H is infinite cyclic. Then the map $\Pi_\pi : \mathrm{Diff}_H(G) \to \mathrm{Diff}(G/H)$ given by $\Pi_\pi(f) \circ \pi = \pi \circ f$ is surjective.

Note that we have put no topology on $\mathrm{Diff}(G)$ nor $\mathrm{Diff}(G/H)$ in general and

thus make no claim as to the continuity or openness of this map.

Proof. Let $f \in \mathrm{Diff}_H(G)$. We first show that $\Pi_\pi(f) \in \mathrm{Diff}(G/H)$. Clearly $\pi \circ f$ maps into G/H. We note that $\Pi_\pi(f)$ is defined on G/H: if $Hg \in G/H$ then $(\Pi_\pi(f))(Hg) = (\Pi_\pi(f))(\pi(g)) = \pi(f(g))$. It is also well-defined: if $g_1, g_2 \in G$ are such that $Hg_1 = Hg_2$, then there is an $h \in H$ so that $g_1 = hg_2$; thus $\Pi_\pi(f)(Hg_1) = \pi(f(g_1)) = \pi(f(hg_2)) = \pi(h^{\pm 1}f(g_2)) = \pi(f(g_2)) = \Pi_\pi(f)(Hg_2)$. Now we note that $[\Pi_\pi(f) \circ \Pi_\pi(f^{-1})] \circ \pi = (\Pi_\pi(f)) \circ (\pi \circ f^{-1}) = \pi \circ (f \circ f^{-1}) = \pi$; since π is surjective, this means that $\Pi_\pi(f) \circ \Pi_\pi(f^{-1}) = \mathrm{id}$. Thus clearly (since $(f^{-1})^{-1} = f$) $\Pi_\pi(f^{-1}) \circ \Pi_\pi(f) = \mathrm{id}$, so $\Pi_\pi(f) : G/H \to G/H$ is bijective with inverse $\Pi_\pi(f^{-1})$. $\Pi_\pi(f)$ and $\Pi_\pi(f^{-1})$ are both C^∞ by Proposition 3.3. Thus $\Pi_\pi(f) \in \mathrm{Diff}(G/H)$, as desired.

We now show that Π_π is surjective. Let $\phi \in \mathrm{Diff}(G/H)$. Then $\phi \circ \pi \in C^\infty(G, G/H)$; thus there is a continuous lift $\widetilde{\phi} : G \to G$ satisfying $\pi \circ \widetilde{\phi} = \phi \circ \pi$. $\widetilde{\phi}$ exists by Bredon [4], Theorem III.4.1, since clearly $(\phi \circ \pi)_\#(\pi_1(G/H)) \subset \pi_\#(\pi_1(G/H))$ (see also ibid., p. 132). $\widetilde{\phi}$ is then C^∞ by Proposition 3.3. We claim that $\widetilde{\phi} \in \mathrm{Diff}_H(G)$. Let $\widetilde{\phi^{-1}} : G \to G$ be the lift of $\phi^{-1} \circ \pi$ satisfying $\widetilde{\phi^{-1}}(\widetilde{\phi}(e)) = e$, where $e \in G$ is the identity. $\widetilde{\phi^{-1}}$ exists since $(\phi^{-1} \circ \pi)(\widetilde{\phi}(e)) = (\phi^{-1} \circ \phi)(\pi(e)) = \pi(e)$ (see Bredon [4], Theorem III.4.1). Thus $\widetilde{\phi^{-1}}$ is also C^∞. Then $\pi \circ (\widetilde{\phi^{-1}} \circ \widetilde{\phi}) = \phi^{-1} \circ (\pi \circ \widetilde{\phi}) = (\phi^{-1} \circ \phi) \circ \pi = \pi$, so $(\widetilde{\phi^{-1}} \circ \widetilde{\phi})(g) \in Hg$ for each $g \in G$. Thus $(\widetilde{\phi^{-1}} \circ \widetilde{\phi})(g) \cdot (g^{-1}) \in H$ for each $g \in G$. But $(\widetilde{\phi^{-1}} \circ \widetilde{\phi})(\cdot) \cdot (\cdot^{-1}) : G \to G$ is C^∞ and hence continuous; since it maps into H, and H is discrete, it must be constant. Thus there is some $h_0 \in H$ such that $(\widetilde{\phi^{-1}} \circ \widetilde{\phi})(g) = h_0 g$ for all $g \in G$.

Since $\widetilde{\phi^{-1}}(\widetilde{\phi}(e)) = e = h_0 e$, we see that $h_0 = e$ and $\widetilde{\phi^{-1}} \circ \widetilde{\phi} = \mathrm{id}$. Similarly, $\pi \circ (\widetilde{\phi} \circ \widetilde{\phi^{-1}}) = \pi$, so as before there is some $h_0' \in H$ so that $(\widetilde{\phi} \circ \widetilde{\phi^{-1}})(g) = h_0' g$ for all $g \in G$. But taking $g = \widetilde{\phi}(e)$ gives $\widetilde{\phi}(\widetilde{\phi^{-1}}(\widetilde{\phi}(e))) = \widetilde{\phi}(e) = h_0' \widetilde{\phi}(e)$, so again $h_0' = e$ and $\widetilde{\phi} \circ \widetilde{\phi^{-1}} = \mathrm{id}$. Thus $\widetilde{\phi}$ is a diffeomorphism and $\widetilde{\phi}^{-1} = \widetilde{\phi^{-1}}$.

To see that $\widetilde{\phi} \in \mathrm{Diff}_H(G)$, let $h \in H$ be a generator for H. Then all elements of H can be written as nh (h or h^{-1} applied to itself n times as n is positive or negative, respectively, or e if $n = 0$), where $n \in \mathbf{Z}$. Fix $g \in G$ and consider $\widetilde{\phi}(hg)$ and $\widetilde{\phi}^{-1}(h\widetilde{\phi}(g))$. We see that

$$(\phi \circ \pi)(\widetilde{\phi}^{-1}(h\widetilde{\phi}(g))) = (\pi \circ \widetilde{\phi})(\widetilde{\phi}^{-1}(h\widetilde{\phi}(g)))$$

$$= (\pi \circ \widetilde{\phi})(g)$$

$$= (\phi \circ \pi)(g), \tag{3.14}$$

so $\widetilde{\phi}^{-1}(h\widetilde{\phi}(g)) = n'hg$ for some $n' \in \mathbf{Z}\setminus\{0\}$; further,

$$(\pi \circ \widetilde{\phi})(hg) = (\phi \circ \pi)(hg)$$

$$= (\phi \circ \pi)(g)$$

$$= (\pi \circ \widetilde{\phi})(g), \tag{3.15}$$

so there is some $m \in \mathbf{Z}\setminus\{0\}$ so that $\widetilde{\phi}(hg) = mh\widetilde{\phi}(g)$. ($m$ and n' are nonzero since $\widetilde{\phi}$ is injective and $h \neq e$.) Thus $\widetilde{\phi}(n'hg) = n'mh\widetilde{\phi}(g)$; but also $\widetilde{\phi}(n'hg) = h\widetilde{\phi}(g)$, so $n'm = 1$ since H is infinite cyclic and thus $m = \pm 1$. Thus $\widetilde{\phi}(hg) = h^{\pm 1}\widetilde{\phi}(g)$; since H is cyclic and generated by h and $g \in G$ is arbitrary, we see that $\widetilde{\phi} \in \mathrm{Diff}_H(G)$, as desired. We note that $\widetilde{\phi}$ was an arbitrary lift of $\phi \circ \pi$, and thus we have actually shown that all lifts via π of $\phi \circ \pi$ are in $\mathrm{Diff}_H(G)$. QED.

Corollary 3.4. The map $\Pi : \mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \to \mathrm{Diff}(S^1)$ given by $\Pi(f) \circ q = q \circ f$ is surjective, continuous, and open. It is also a group homomorphism with kernel $\{f \in \mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) | f(x) = x + 2\pi n$ for some $n \in \mathbf{Z}$, for all $x \in \mathbf{R}^1\}$.

Proof. Identifying S^1 with $\mathbf{R}^1/\mathbf{Z}^1$, surjectivity of Π follows immediately from Proposition 3.9. Now we note that $\Pi(f) = \hat{\Theta}^{-1}(q \circ f)$; thus Π is continuous by Lemma 2.1 and Corollary 3.3, and open onto its image by Proposition 3.6 and our following comment, Corollary 3.3 and Corollary 3.2.

Π is easily seen to be a group homomorphism:

$$\Pi(f_1 \circ f_2) \circ q = q \circ (f_1 \circ f_2)$$

$$= (\Pi(f_1) \circ q) \circ f_2$$

$$= (\Pi(f_1) \circ \Pi(f_2)) \circ q. \tag{3.16}$$

Its kernel may be computed as follows. $f \in \mathrm{Ker}\,\Pi$ implies that $\Pi(f) \circ q = q = q \circ f$; thus for each $x \in \mathbf{R}^1$ there is an $n(x) \in \mathbf{Z}$ such that $f(x) = x + 2\pi n(x)$. As usual n is constant, so $f(x) = x + 2\pi n$ for some $n \in Z$. Conversely, any f of this form is clearly in $\mathrm{Ker}\,\Pi$, as desired. QED.

Since $\mathrm{Ker}\,\Pi$ is discrete, Proposition 3.8 above together with Pontrjagin [9], Theorem 12 shows that Π is in fact a covering map. The work which has gone into this corollary will be used again in Theorems 4.1 and 4.2 below.

We now prove a few special results about openness of maps between C^∞ spaces which will be of great importance in Chapter 4 below.

Lemma 3.1. The map

$$\omega : C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1) \to C^\infty(\mathbf{R}^2, S^1)$$

$$(f, g) \mapsto ((x, y) \mapsto f(x)g(y)) \tag{3.17}$$

is continuous, and open onto its image.

Proof. We show first that ω is open onto its image. Choose $n \in \mathbf{N}$, $\epsilon > 0$, $K \subset \mathbf{R}^1$ compact and nonempty, and $(f_0, g_0) \in C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$, and consider the basic open set $U = U_{\mathbf{C}}(n, K, \epsilon, f_0) \times U_{\mathbf{C}}(n, K, \epsilon, g_0) \cap C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$. Sets of this form clearly form a neighborhood basis at (f_0, g_0) in $C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$. Let $\delta = \min \left\{ \frac{1}{2}\epsilon, \frac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(g_0))} \right\}$, let $V = U_{\mathbf{C}}(n, K \times K, \delta, \omega(f_0, g_0)) \cap C^\infty(\mathbf{R}^2, S^1)$, and let $(f, g) \in C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$ be such that $\omega(f, g) \in V$. Pick $x_0 \in K$, and define $f^*, g^* : \mathbf{R}^1 \to S^1$ by $f^*(x) = f(x)g(x_0)g_0(x_0)^{-1}$, $g^*(x) = g(x)g_0(x_0)g(x_0)^{-1}$. Then clearly $f^*, g^* \in C^\infty(\mathbf{R}^1, S^1)$ and $\omega(f^*, g^*) = \omega(f, g)$. Moreover, for all $x \in K$ and all $n' \in \mathbf{N}$, $n' \leq n$,

$$\left| f^{*(n')}(x) - f_0^{(n')}(x) \right| = \left| f^{(n')}(x)g(x_0) - f_0^{(n')}(x)g_0(x_0) \right|$$

$$\leq p_{n,K \times K}^{\mathbf{C}}(\omega(f, g) - \omega(f_0, g_0)) < \delta < \epsilon \tag{3.18}$$

$$\left| g^{*(n')}(x) - g_0^{(n')}(x) \right| = \left| g^{(n')}(x)g_0(x_0)g(x_0)^{-1} - g_0^{(n')}(x) \right|$$

$$= \left| g^{(n')}(x)f(x_0) - g_0^{(n')}(x)f^*(x_0) \right|$$

$$\leq \left| g^{(n')}(x)f(x_0) - g_0^{(n')}(x)f_0(x_0) \right|$$

$$+ \left| g_0^{(n')}(x) \right| |f_0(x_0) - f^*(x_0)|$$

$$< \frac{1}{2}\epsilon + \frac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(g_0))} p_{n,K}^{\mathbf{C}}(g_0) < \epsilon, \tag{3.19}$$

so $(f^*, g^*) \in U$ and ω is open by Proposition 3.1.

To see that ω is continuous, let $(f_0, g_0) \in C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$, $n \in \mathbf{N}$, $K \subset \mathbf{R}^1$ compact, and $\epsilon > 0$, and let $U = U_{\mathbf{R}^2, \mathbf{C}}(n, K \times K, \epsilon, \omega(f_0, g_0)) \cap C^\infty(\mathbf{R}^2, S^1)$ be a basic open set in $C^\infty(\mathbf{R}^2, S^1)$. As before, sets of this form clearly form a neighborhood basis at $\omega(f_0, g_0)$. Now let $\delta = \min\left\{ \dfrac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(f_0))}, \dfrac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(g_0))}, 1 \right\}$, let $V = U_{\mathbf{C}}(n, K, \delta, f_0) \times U_{\mathbf{C}}(n, K, \delta, g_0) \cap C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$, and let $(f, g) \in V$. Then for all $x, y \in K$ and all $n_1, n_2 \in \mathbf{N}$, $n_1 + n_2 \leq n$,

$$
\left| \frac{\partial^{n_1 + n_2}}{\partial x^{n_1} \partial y^{n_2}} (f(x)g(y) - f_0(x)g_0(y)) \right|
$$
$$
\leq \left| g^{(n_2)}(y) \right| \left| f^{(n_1)}(x) - f_0^{(n_1)}(x) \right| + \left| f_0^{(n_1)}(x) \right| \left| g^{(n_2)}(y) - g_0^{(n_2)}(y) \right|
$$
$$
< (1 + p_{n,K}^{\mathbf{C}}(g_0)) \left(\frac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(g_0))} \right) + p_{n,K}^{\mathbf{C}}(f_0) \left(\frac{\epsilon}{2(1 + p_{n,K}^{\mathbf{C}}(f_0))} \right)
$$
$$
< \epsilon. \tag{3.20}
$$

Thus $\omega^{-1}(U) \supset V$, and ω is continuous by Bredon [4], Proposition I.2.6 (cf. Proposition 2.1 above). QED.

Our next result is used in Chapter 4 below to allow us to work in null coordinates on a Minkowski spacetime. As usual, let V denote an arbitrary finite-dimensional real or complex vector space.

Lemma 3.2. For each $\chi \in S_2$, the map

$$
\eta_\chi : C^\infty(\mathbf{R}^2, V) \to C^\infty(\mathbf{R}^2, V)
$$
$$
f \mapsto \left((x, y) \mapsto (f \circ \chi) \left(\frac{1}{\sqrt{2}}(x + y), \frac{1}{\sqrt{2}}(x - y) \right) \right) \tag{3.21}
$$

is continuous and open.

Proof. First we note that, using the isomorphism $S_2 \cong \mathbf{Z}_2$, we may write $\eta_\chi(f)(x,y)$ more explicitly as $f\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)$. Now fix $\chi \in S_2$. We note that $\eta_\chi^2 = \mathrm{id}$; thus continuity and openness are equivalent. (Also, η_χ is bijective.) We prove continuity. Fix $K \times K \subset \mathbf{R}^2$ compact, $\epsilon > 0$, $n \in \mathbf{N}$, $f_0 \in C^\infty(\mathbf{R}^2, V)$, let $U = U_V(n, K \times K, \epsilon, f_0)$, and consider $\eta_\chi^{-1}(U)$. Let $\delta = \frac{\epsilon}{(n+1)^2 n!}$, and choose $f \in C^\infty(\mathbf{R}^2, V)$ satisfying $p^V_{n, \frac{1}{\sqrt{2}}(K+\chi K) \times \frac{1}{\sqrt{2}}(K-\chi K)}(f - f_0) < \delta$. Now

$$\frac{\partial}{\partial y}\left(f\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)\right) = \frac{\chi}{\sqrt{2}}(-\partial_2 f + \partial_1 f)|_{\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)}$$

$$(3.22)$$

and

$$\frac{\partial}{\partial x}\left(f\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)\right) = \frac{1}{\sqrt{2}}(\partial_2 f + \partial_1 f)|_{\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)}, \quad (3.23)$$

where by ∂_1, ∂_2 we mean differentiation with respect to the first or second variable, respectively. Let $n_1, n_2 \in \mathbf{N}$, $n_1 + n_2 \leq n$, and let $(x, y) \in K \times K$; then $(\frac{1}{\sqrt{2}}(x + \chi y), \frac{1}{\sqrt{2}}(x - \chi y)) \in \frac{1}{\sqrt{2}}(K + \chi K) \times \frac{1}{\sqrt{2}}(K - \chi K)$, so

$$\left|\frac{\partial^{n_1+n_2}}{\partial x^{n_1} \partial y^{n_2}}\left(f\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right) - f_0\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)\right)\right|$$

$$= \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2}\left|[(-\partial_2 + \partial_1)^{n_2}(\partial_2 + \partial_1)^{n_1}(f - f_0)]|_{\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)}\right|$$

$$\leq \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2}\left[\left|\sum_{k_1=0}^{n_1}\binom{n_1}{k_1}\right|\partial_2^{n_1-k_1}\partial_1^{k_1}\right.$$

$$\left.\left(\sum_{k_2=0}^{n_2}\binom{n_2}{k_2}(-\partial_2)^{n_2-k_2}\partial_1^{k_2}(f - f_0)\right)\right|_{\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)}\right]$$

$$\leq \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} n_1! n_2! \cdot$$

$$\left|\partial_2^{(n_1+n_2)-(k_1+k_2)} \partial_1^{k_1+k_2}(f-f_0)\right|_{\left(\frac{1}{\sqrt{2}}(x+\chi y), \frac{1}{\sqrt{2}}(x-\chi y)\right)}\right|$$

$$< \left(\frac{1}{\sqrt{2}}\right)^{n_1+n_2} \sum_{k_1=0}^{n_1} \sum_{k_2=0}^{n_2} n! \delta \leq \epsilon. \tag{3.24}$$

Thus η_χ is continuous. We note that $\eta_\chi^2 = \mathrm{id}$ implies that η_χ is then a homeomorphism. QED.

We end this chapter with a lemma on the continuity and openness of a variety of maps. As usual, W and V denote finite-dimensional real or complex vector spaces.

Lemma 3.3. (i) The maps below are open onto their images:

(a) $\alpha_\pm : C^\infty(\mathbf{R}^1, V) \to C^\infty(\mathbf{R}^2, V)$, $f \mapsto \left((x,y) \mapsto f\left(\frac{1}{\sqrt{2}}(x \pm y)\right)\right)$.

(b) $\beta_1 : C^\infty(W, V) \times C^\infty(W, V) \to C^\infty(W, V)$, $(f,g) \mapsto f + g$; $\beta_2 : C^\infty(W, V) \times C^\infty(W, V) \to C^\infty(W \times W, V)$, $(f,g) \mapsto ((x,y) \mapsto f(x) + g(y))$.

(c) $q_1 \times q_2 : X_1 \times X_2 \to Y_1 \times Y_2$, $(x_1, x_2) \mapsto (q_1(x_1), q_2(x_2))$, X_1, X_2, Y_1, Y_2 arbitrary topological spaces, $q_1 : X_1 \to Y_1$, $q_2 : X_2 \to Y_2$ maps open onto their images.

(d) $C^\infty(W, V) \to C^\infty(W, V)$, $f \mapsto cf$, $c \neq 0$.

(e) $C^\infty(W, V) \to C^\infty(W, V)$, $f \mapsto (x \mapsto f(cx))$, $c \in \mathbf{R}$ or $c \in \mathbf{C}$, $c \neq 0$.

The map β_1 in (b) is open into $C^\infty(W, V)$, and that in (c) is open into $Y_1 \times Y_2$ if q_1 and q_2 are open into Y_1 and Y_2, respectively. The maps in (a), (b), (d), and (e) are also continuous, as is that in (c) if the maps q_1 and q_2 are continuous.

(ii) If V_1 and V_2 are finite-dimensional real or complex vector spaces, then the

map $\iota_{V_1,V_2}^W : C^\infty(W,V_1) \times C^\infty(W,V_2) \to C^\infty(W,V_1 \times V_2)$, $(f,g) \mapsto (x \mapsto (f(x),g(x)))$

is a homeomorphism. (Compare [16], Proposition 3.1.)

Proof. (i)(a) We see that

$$\partial^{(\alpha_i)}\left(f(\frac{1}{\sqrt{2}}(x \pm y))\right) = \left(\frac{1}{\sqrt{2}}\right)^{|\alpha|} (-1)^k f^{(|\alpha|)}\left(\frac{1}{\sqrt{2}}(x \pm y)\right), \qquad (3.25)$$

where $k \in \mathbf{N}$ depends only on $\alpha = (\alpha_i)$. Thus $p_{n,K\sqrt{2}\times\{0\}}^{\mathbf{R}^2,V}(\alpha_\pm(f) - \alpha_\pm(f_0)) <$ $\left(\frac{1}{\sqrt{2}}\right)^{n+1} \epsilon$ implies $p_{n,K}^V(f - f_0) < \epsilon$, and $p_{n,\frac{1}{\sqrt{2}}(K\pm K)}^V(f - f_0) < \epsilon$ implies that $p_{n,K\times K}^{\mathbf{R}^2,V}(\alpha_\pm(f) - \alpha_\pm(f_0)) < \epsilon$, where $n \in \mathbf{N}$, $K \subset \mathbf{R}^1$ compact, $f, f_0 \in C^\infty(\mathbf{R}^1,V)$, and $\epsilon > 0$ are arbitrary. Thus $\alpha_\pm$ is continuous and open onto its image, as desired.

(b) $p_{n,K}^{W,V}((f+g) - (f_0+g_0)) < \epsilon$ implies that $p_{n,K}^{W,V}((f+g-g_0) - f_0) < \epsilon$; since $\beta_1(f+g-g_0,g_0) = \beta_1(f,g)$, we see that β_1 is open onto its image by Proposition 3.1 above. Clearly β_1 is surjective ($\beta_1(f,0) = f$), and thus it is open into $C^\infty(W,V)$. Now clearly $f \in U_{W,V}(n,K,\frac{1}{2}\epsilon,f_0)$, $g \in U_{W,V}(n,K,\frac{1}{2}\epsilon,g_0)$ implies that $\beta_1(f,g) \in U_{W,V}(n,K,\epsilon,f_0+g_0)$ for all $n \in \mathbf{N}$, $K \subset W$ compact, $\epsilon > 0$, and $f_0, g_0 \in C^\infty(W,V)$, and therefore β_1 is continuous.

To see that β_2 is open onto its image, let $n \in \mathbf{N}$, $K \subset W$ compact and nonempty, $\epsilon > 0$, and $(f_0,g_0) \in C^\infty(W,V) \times C^\infty(W,V)$, and let $(f,g) \in C^\infty(W,V) \times C^\infty(W,V)$ satisfy $\beta_2(f,g) \in U_{W\times W,V}(n,K\times K,\frac{1}{2}\epsilon,\beta_2(f_0,g_0))$. Pick $x_0 \in K$, and define $\hat{f},\hat{g} \in C^\infty(W,V)$ by $\hat{f}(x) = f(x) + g(x_0) - g_0(x_0)$, $\hat{g}(x) = g(x) + g_0(x_0) - g(x_0)$. Then clearly $\beta_2(\hat{f},\hat{g}) = \beta_2(f,g)$; moreover,

$$p_{n,K}^{W,V}(\hat{f} - f_0) \leq p_{n,K\times K}^{W\times W,V}(\beta_2(f,g) - \beta_2(f_0,g_0)) < \frac{1}{2}\epsilon$$

$$p_{n,K}^{W,V}(\hat{g} - g_0) \leq p_{n,K\times K}^{W\times W,V}(\beta_2(\hat{f},\hat{g}) - \beta_2(f_0,g_0)) + p_{n,K}^{W,V}(\hat{f} - f_0) < \epsilon. \qquad (3.26)$$

Thus β_2 is open onto its image by Proposition 3.1, as desired. (We note, incidentally, that β_2 is very far from being surjective – in fact its image is very far from being an open set.) Continuity follows as with β_1: if $n \in \mathbf{N}$, $K \subset W$ is compact, $\epsilon > 0$, and $f_0, g_0 \in C^\infty(W, V)$, then $f \in U_{W,V}(n, K, \frac{1}{2}\epsilon, f_0)$ and $g \in U_{W,V}(n, K, \frac{1}{2}\epsilon, g_0)$ implies that $\beta_2(f, g) \in U_{W \times W, V}(n, K \times K, \epsilon, \beta_2(f_0, g_0))$.

(c) Let $U_1 \subset X_1$, $U_2 \subset X_2$ be open; then since $(q_1 \times q_2)(X_1 \times X_2) = q_1(X_1) \times q_2(X_2)$, we see that $(q_1 \times q_2)(U_1 \times U_2) = q_1(U_1) \times q_2(U_2)$ must be open in the image of $q_1 \times q_2$, and thus $q_1 \times q_2$ must be open onto its image. If q_1 and q_2 are open into Y_1 and Y_2, then $q_1(X_2) \times q_2(X_2)$ is open in $Y_1 \times Y_2$ and thus $q_1 \times q_2$ is open into $Y_1 \times Y_2$, as desired. For continuity, note that $(q_1 \times q_2)(x_1, x_2) = ((q_1 \circ \pi_1)(x_1, x_2), (q_2 \circ \pi_2)(x_1, x_2))$, where π_1 and π_2 are the projections on the first and second factors, respectively, and apply Munkres [8], Theorem 19.6.

(d) Trivial $(p_{n,K}^V(cf - cf_0) = |c| p_{n,K}^V(f - f_0))$.

(e) Let $n \in \mathbf{N}$, $K \subset W$ compact, and $f \in C^\infty(W, V)$. Then for all $x \in W$ and all sequences (α_i), $|(\alpha_i)| \le n$, we have

$$\left| \partial^{(\alpha_i)}(f(cx)) \right| = |c|^{|(\alpha_i)|} \left| \left(\partial^{(\alpha_i)}(f) \right)(cx) \right|. \tag{3.27}$$

Now let $c_< = \min\{1, |c|\} > 0$ and $c_> = \max\{1, |c|\} > 0$; then $c_<^n \le |c|^{|(\alpha_i)|} \le c_>^n$ and

$$c_<^n \left| \left(\partial^{(\alpha_i)}(f) \right)(cx) \right| \le \left| \partial^{(\alpha_i)}(f(cx)) \right| \le c_>^n \left| \left(\partial^{(\alpha_i)}(f) \right)(cx) \right|, \tag{3.28}$$

so $c_<^n p_{n,cK}^{W,V}(f) \le p_{n,K}^{W,V}(f(c \cdot)) \le c_>^n p_{n,cK}^{W,V}(f)$. Thus the map is continuous and open onto its image. Since the inverse of $f \mapsto f(c \cdot)$ is clearly $f \mapsto f(\frac{1}{c} \cdot)$, each of these

maps is in fact a homeomorphism.

(ii) First, ι_{V_1,V_2}^W is clearly surjective, as the components of a C^∞ map are also C^∞. It is likewise clearly injective. Now we note that for all $(x_1, x_2) \in V_1 \times V_2$, $\max\{|x_1|, |x_2|\} \leq |(x_1, x_2)| = \sqrt{|x_1|^2 + |x_2|^2}$; thus for all $n \in \mathbf{N}$, $K \subset W$ compact, and $(f_0, g_0), (f, g) \in C^\infty(W, V_1) \times C^\infty(W, V_2)$,

$$\max\{p_{n,K}^{W,V_1}(f - f_0), p_{n,K}^{W,V_2}(g - g_0)\} \leq p_{n,K}^{W,V_1 \times V_2}(\iota_{V_1,V_2}^W(f,g) - \iota_{V_1,V_2}^W(f_0,g_0))$$

$$\leq \sqrt{p_{n,K}^{W,V_1}(f - f_0)^2 + p_{n,K}^{W,V_2}(g - g_0)^2}; \quad (3.29)$$

thus ι_{V_1,V_2}^W is continuous and open onto its image. Since it is surjective and injective it is therefore a homeomorphism, as desired. QED.

CHAPTER 4

CONSTRUCTION OF THE TRANSITIVE ACTION

We are interested in the cylinder $\mathbf{R}^1 \times S^1$ and its universal cover $\mathbf{R}^1 \times \mathbf{R}^1$. We shall use (t, σ) to denote an arbitrary point in $\mathbf{R}^1 \times S^1$ and (t, θ) to denote an arbitrary point in $\mathbf{R}^1 \times \mathbf{R}^1$. (t, θ) are then the standard coordinates on $\mathbf{R}^1 \times \mathbf{R}^1$. Considering the covering map p restricted to sets of the form $\mathbf{R}^1 \times (\theta_0 - \pi, \theta_0 + \pi)$, we see that these coordinates on $\mathbf{R}^1 \times \mathbf{R}^1$ also give rise to coordinates $\mathrm{id} \times \theta_{\sigma_0}$: $\mathbf{R}^1 \times S^1 \to \mathbf{R}^1 \times (-\pi, \pi)$, where $\sigma_0 = e^{i\theta_0}$ and $\theta_{\sigma_0} : S^1 \backslash \{-\sigma_0\} \to (-\pi, \pi)$ is as defined in Chapter 3 above. Now for every $\sigma \in S^1$ the tangent space $T_\sigma S^1$ is one-dimensional with basis $f \mapsto \partial_\theta[(f \circ \theta_\sigma^{-1})(\theta)]|_{\theta=0}$, which for simplicity we denote by $\partial_{\theta_\sigma}|_\sigma$. We note that (letting $\theta_0 \in (-\pi, \pi)$, $\sigma_0 = e^{i\theta_0}\sigma \in S^1$) $\partial_\theta[(f \circ \theta_\sigma^{-1})(\theta)]|_{\theta=\theta_0} = \partial_{\theta_{\sigma_0}}|_{\sigma_0}(f)$; we shall also denote the tangent vector so defined by $\partial_{\theta_\sigma}|_{\sigma_0}$. For every $\sigma \in S^1$ the basis of the cotangent space $(T_\sigma S^1)^*$ dual to the basis $\{\partial_{\theta_\sigma}|_\sigma\}$ of $T_\sigma S^1$ is $d\theta_\sigma|_\sigma$. We define $\kappa : S^1 \to T^* S^1$ by $\kappa|_\sigma = d\theta_\sigma|_\sigma$ and claim that κ is a 1-form. $\kappa|_\sigma \in T_\sigma^* S^1$ by definition. To see that κ is C^∞, let $X : S^1 \to TS^1$ be a C^∞ vector field and pick $\sigma_0 \in S^1$. Then on $S^1 \backslash \{-\sigma_0\}$ we may write $X|_\sigma = f(\sigma) \partial_{\theta_{\sigma_0}}|_\sigma$, where $f : S^1 \backslash \{-\sigma_0\} \to \mathbf{R}^1$ is C^∞. Thus on $S^1 \backslash \{-\sigma_0\}$ we have $\kappa|_\sigma(X|_\sigma) = f(\sigma)$, which is C^∞ at σ_0. Thus κ is C^∞. Now we note that $p_*(\partial_\theta|_{\theta_0}) = \partial_{\theta_{\sigma_0}}|_{\sigma_0}$, where $\sigma_0 = e^{i\theta_0}$; thus $p^*\kappa = d\theta$. Clearly $p^* dt = dt$.

Let now $g = e^{w(\sigma,t)}(\kappa \otimes \kappa - dt \otimes dt)$ be a metric on $\mathbf{R}^1 \times S^1$ globally conformal to the Minkowski metric. g will be fixed throughout the remainder of our discussion.

Let $G(\mathbf{R}^1 \times S^1; g) \subset C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times \mathbf{C})$ denote the set of all conformal isometries of g, topologized of course as a subspace of $C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times \mathbf{C})$, and $K(S^1; \mathbf{R}^1 \times S^1; g) \subset C^\infty(S^1, \mathbf{R}^1 \times \mathbf{C})$ the set of all spacelike embeddings $X : S^1 \to \mathbf{R}^1 \times S^1$, i.e., embeddings such that X^*g is positive-definite, topologized as a subspace of $C^\infty(S^1, \mathbf{R}^1 \times \mathbf{C})$.[3] We note that $p^*g|_{(t,\theta)} = e^{w(t,e^{i\theta})}(d\theta \otimes d\theta - dt \otimes dt)$ is a metric on the covering space $\mathbf{R}^1 \times \mathbf{R}^1$ which is also globally conformal to the Minkowski metric. We now introduce null coordinates on $\mathbf{R}^1 \times \mathbf{R}^1$ with respect to this metric by defining $z^\pm : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1$, $z^\pm(t,\theta) = \frac{1}{\sqrt{2}}(t \pm \theta)$. Define $T : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^2$ by $T(t,\theta) = (z^+(t,\theta), z^-(t,\theta))$; then $T^2 = \text{id}$. Thus $t = \frac{1}{\sqrt{2}}(z^+ + z^-)$ and $\theta = \frac{1}{\sqrt{2}}(z^+ - z^-)$, so

$$
\begin{aligned}
d\theta \otimes d\theta - dt \otimes dt &= \frac{1}{2}(dz^+ \otimes dz^+ + dz^- \otimes dz^- - (dz^+ \otimes dz^- + dz^- \otimes dz^+)) \\
&\quad - \frac{1}{2}(dz^+ \otimes dz^+ + dz^- \otimes dz^- + (dz^+ \otimes dz^- + dz^- \otimes dz^+)) \\
&= -(dz^+ \otimes dz^- + dz^- \otimes dz^+)
\end{aligned}
\tag{4.1}
$$

and in null coordinates $p^*g = -e^w(dz^+ \otimes dz^- + dz^- \otimes dz^+)$.

We define $S : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^2$ by $S(t,\theta) = \left(\frac{1}{2}(t + \theta), \frac{1}{2}(t - \theta)\right)$ and note that $S^{-1}(t,\theta) = (t + \theta, t - \theta)$. We now prove two theorems allowing us to characterize $G(\mathbf{R}^1 \times S^1; g)$ and $K(S^1; \mathbf{R}^1 \times S^1; g)$; specifically, we shall demonstrate both of them as quotients of the semidirect product $\Delta \rtimes S_2$ defined at the end of Chapter 2 above.

Theorem 4.1. The map $\Phi : \Delta \rtimes S_2 \to G(\mathbf{R}^1 \times S^1; g)$ given by

---

[3]The letters 'G' and 'K' are from the Mandarin pronunciation of the Chinese words for 'conformal' and 'space', respectively.

$$\Phi(f_1, f_2, \chi) \circ p = p \circ \left(S \circ (f_1 \times f_2) \circ \chi \circ S^{-1} \right) \tag{4.2}$$

is a surjective, continuous, and open homomorphism with kernel

$$\mathrm{Ker}\,\Phi = \{(f_1, f_2, 1) \in \Delta \rtimes S_2 | f_1(x) = x + 2\pi n, f_2(x) = x - 2\pi n \text{ for some } n \in \mathbf{Z},$$

$$\text{for all } x \in \mathbf{R}\}. \tag{4.3}$$

Thus $G(\mathbf{R}^1 \times S^1; g) \cong (\Delta \rtimes S_2)/\mathrm{Ker}\,\Phi$ via a homeomorphic isomorphism, and in particular $G(\mathbf{R}^1 \times S^1; g)$ is a topological group.

More explicitly, using the isomorphism $S_2 \cong \mathbf{Z}_2$ and thinking of $\mathbf{Z}_2$ as the group of real numbers of unit modulus, we may write

$$[\Phi(f_1, f_2, \chi) \circ p]\,(t, \theta) = p\Big(\frac{1}{2}(f_1(t + \chi\theta) + f_2(t - \chi\theta)), \frac{1}{2}(f_1(t + \chi\theta) - f_2(t - \chi\theta))\Big).$$

$$\tag{4.4}$$

Proof. We note first of all that $\Phi(f_1, f_2, \chi)$ is defined since $f_1, f_2 \in \mathrm{Diff}_{2\pi\mathbf{Z}}(R^1)$. Further, Φ maps into $C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times S^1)$ by Proposition 3.3. We now show that its image is equal to $G(\mathbf{R}^1 \times S^1; g)$.

We show first that its image must contain $G(\mathbf{R}^1 \times S^1; g)$. Let $\Psi \in G(\mathbf{R}^1 \times S^1; g)$, and consider any p-lift $\tilde{\Psi} : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1 \times \mathbf{R}^1$ of $\Psi \circ p : \mathbf{R}^1 \times \mathbf{R}^1 \to \mathbf{R}^1 \times S^1$, which exists and is continuous by Bredon [4], Theorem III.4.1. Since $\Psi : \mathbf{R}^1 \times S^1 \to \mathbf{R}^1 \times S^1$ is a diffeomorphism and $\mathbf{R}^1 \times S^1$ may be viewed as $\mathbf{R}^1 \times \mathbf{R}^1/\{0\} \times \mathbf{Z}$, Proposition 3.9 applies to show that $\tilde{\Psi} \in \mathrm{Diff}(\mathbf{R}^1 \times \mathbf{R}^1)$ and that $\tilde{\Psi}$ must also satisfy $\tilde{\Psi}((t, \theta) + (0, 2\pi n)) = \tilde{\Psi}(t, \theta) \pm (0, 2\pi n)$ for all $n \in \mathbf{Z}$. Moreover, by definition $\tilde{\Psi}$ satisfies $p \circ \tilde{\Psi} = \Psi \circ p$; thus $\tilde{\Psi}^* \circ p^* = p^* \circ \Psi^*$, so $\tilde{\Psi}^*(p^*g) = p^*(\Psi^*g) = e^\gamma p^*g$, where γ

is some function, since Ψ is a conformal transformation of g. Thus $\tilde{\Psi}$ is a conformal isometry of p^*g. We consider its representation in null coordinates, which we write $(\tilde{\Psi}^+(z^+, z^-), \tilde{\Psi}^-(z^+, z^-)) = (T \circ \tilde{\Psi} \circ T^{-1})(z^+, z^-)$. Let $a_1 \partial_{z^+} + b_1 \partial_{z^-}$, $a_2 \partial_{z^+} + b_2 \partial_{z^-}$ be any two vectors at some point (z_0^+, z_0^-). Then since $\tilde{\Psi}$ is conformal we have

$$\tilde{\Psi}^*(p^*g)(a_1 \partial_{z^+} + b_1 \partial_{z^-}, a_2 \partial_{z^+} + b_2 \partial_{z^-}) = -e^w(2a_1 a_2 \tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,+}^- + 2b_1 b_2 \tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,-}^-$$
$$+ b_1 a_2 (\tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,+}^- + \tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,-}^-)$$
$$+ b_2 a_1 (\tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,-}^- + \tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,+}^-))$$
$$= -e^\gamma e^w (a_1 b_2 + b_1 a_2). \tag{4.5}$$

Thus $\tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,+}^- = \tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,-}^- = 0$ and $\tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,-}^- + \tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,+}^- > 0$; further, $\tilde{\Psi}_{,+}^+ \tilde{\Psi}_{,-}^- - \tilde{\Psi}_{,-}^+ \tilde{\Psi}_{,+}^- \neq 0$ since $\tilde{\Psi}^* = p^* \Psi^* p^{*-1}$ is nondegenerate (since p is a local diffeomorphism). Thus $\tilde{\Psi}_{,+}^+ \neq 0$ implies that $\tilde{\Psi}_{,+}^- = 0$, which implies that $\tilde{\Psi}_{,-}^- \neq 0$, so $\tilde{\Psi}_{,-}^+ = 0$; similarly, $\tilde{\Psi}_{,-}^+ \neq 0$ implies that $\tilde{\Psi}_{,-}^- = 0$, which implies that $\tilde{\Psi}_{,+}^- \neq 0$, so $\tilde{\Psi}_{,+}^+ = 0$. Thus $\tilde{\Psi}^+(x^+, x^-) = f(x^+)$, $\tilde{\Psi}^-(x^+, x^-) = h(x^-)$ or $\tilde{\Psi}^+(x^+, x^-) = f(x^-)$, $\tilde{\Psi}^-(x^+, x^-) = h(x^+)$, where in either case $f, h \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ and $f'h' > 0$. (This logic is not original to us. For an example in the literature, see [14], Section 2.5, especially Theorem 2.14.) This logic clearly works in reverse as well – thus any $\tilde{\Psi}$ so defined for $f, h \in C^\infty(\mathbf{R}^1, \mathbf{R}^1)$ with $f'h' > 0$ will preserve p^*g up to a conformal factor.

Now the periodicity condition satisfied by $\tilde{\Psi}$ becomes in null coordinates

$$\tilde{\Psi}^\pm(z^+ + n\pi\sqrt{2}, z^- - n\pi\sqrt{2}) = \tilde{\Psi}^\pm(z^+, z^-) \pm n\pi\sqrt{2}. \tag{4.6}$$

Thus if $\tilde{\Psi}^+(z^+, z^-) = f(x^+)$ and $\tilde{\Psi}^-(z^+, z^-) = h(x^-)$ then f and h satisfy $f(x +$

$n\pi\sqrt{2}) = f(x) + n\pi\sqrt{2}$, $h(x + n\pi\sqrt{2}) = h(x) + n\pi\sqrt{2}$, while if $\tilde{\Psi}^+(z^+, z^-) = f(x^-)$ and $\tilde{\Psi}^-(z^+, z^-) = h(x^+)$ then f and h satisfy $f(x + n\pi\sqrt{2}) = f(x) - n\pi\sqrt{2}$, $h(x + n\pi\sqrt{2}) = h(x) - n\pi\sqrt{2}$. Thus either $(\sqrt{2}f(\frac{1}{\sqrt{2}}x), \sqrt{2}h(\frac{1}{\sqrt{2}}x)) \in \mathrm{Diff}^+_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \mathrm{Diff}^+_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ or $(\sqrt{2}f(\frac{1}{\sqrt{2}}x), \sqrt{2}h(\frac{1}{\sqrt{2}}x)) \in \mathrm{Diff}^-_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times \mathrm{Diff}^-_{2\pi\mathbf{Z}}(\mathbf{R}^1)$. Now in either case

$$\tilde{\Psi}(t,\theta) = \left(\frac{1}{\sqrt{2}} \left(f\left(\frac{1}{\sqrt{2}}(t + \chi\theta) \right) + h\left(\frac{1}{\sqrt{2}}(t - \chi\theta) \right) \right), \right.$$
$$\left. \frac{1}{\sqrt{2}} \left(f\left(\frac{1}{\sqrt{2}}(t + \chi\theta) \right) - h\left(\frac{1}{\sqrt{2}}(t - \chi\theta) \right) \right) \right), \quad (4.7)$$

where $\chi = +1$ in the former case and $\chi = -1$ in the latter. Thus we see that $p \circ \tilde{\Psi} = \Phi(\sqrt{2}f(\frac{1}{\sqrt{2}}x), \sqrt{2}h(\frac{1}{\sqrt{2}}x), \chi) \circ p$, so $\Phi(\sqrt{2}f(\frac{1}{\sqrt{2}}x), \sqrt{2}h(\frac{1}{\sqrt{2}}x), \chi) = \Psi$ and $G(\mathbf{R}^1 \times S^1; g)$ is contained in the image of Φ, as desired.

To see that the image of Φ is in fact contained in $G(\mathbf{R}^1 \times S^1; g)$, we note first of all that by our comment at the end of the second-to-last paragraph and since Φ maps into $C^\infty(\mathbf{R}^1 \times S^1, \mathbf{R}^1 \times S^1)$ every element in the image of Φ will preserve g up to a conformal factor. That each element is in fact a diffeomorphism follows from the homomorphism property satisfied by Φ which we now demonstrate.

Let $(f_1, f_2, \chi), (g_1, g_2, \xi) \in \Delta \rtimes S_2$. Then we see that, recalling that (see the last paragraph of Chapter 2) $(\chi(g_1, g_2))(\chi(x_1, x_2)) = (\chi \circ (g_1 \times g_2))(x_1, x_2)$, i.e., $(\chi(g_1 \times g_2)) \circ \chi = \chi \circ (g_1 \times g_2)$,

$$\Phi((f_1, f_2, \chi)(g_1, g_2, \xi)) \circ p = \Phi(f_1 \circ g_{\chi(1)}, f_2 \circ g_{\chi(2)}, \chi\xi) \circ p$$
$$= p \circ \left(S \circ [(f_1 \circ g_{\chi(1)}) \times (f_2 \circ g_{\chi(2)})] \circ (\chi\xi) \circ S^{-1} \right)$$
$$= p \circ \left(S \circ (f_1 \times f_2) \circ \chi \circ [(g_1 \times g_2) \circ \xi \circ S^{-1}] \right)$$

$$= p \circ \left[\left(S \circ (f_1 \times f_2) \circ \chi \circ S^{-1} \right) \right.$$

$$\left. \circ \left(S \circ (g_1 \times g_2) \circ \xi \circ S^{-1} \right) \right]$$

$$= \Phi(f_1, f_2, \chi) \circ \Phi(g_1, g_2, \xi) \circ p. \tag{4.8}$$

Thus we see first of all, by consideration of

$$\Phi((f_1, f_2, \chi)(f_1, f_2, \chi)^{-1}) \tag{4.9}$$

and

$$\Phi((f_1, f_2, \chi)^{-1}(f_1, f_2, \chi)) \tag{4.10}$$

that every element in the image of Φ is in fact a diffeomorphism; thus the image of Φ is contained in and hence equal to $G(\mathbf{R}^1 \times S^1; g)$. Second, Φ is in fact a homomorphism.

Finally, we note that (for $(f_1, f_2) \in \Delta$, $\chi \in S_2$)

$$\left(\eta_\chi \left(\beta_2 \left(\frac{1}{2} f_1(\sqrt{2} \cdot), \frac{1}{2} f_2(\sqrt{2} \cdot) \right) \right)(t, \theta), \eta_\chi \left(\omega \left((q \circ \frac{1}{2} f_1)(\sqrt{2} \cdot), (q \circ (-\frac{1}{2} f_2))(\sqrt{2} \cdot) \right) \right)(t, \theta) \right)$$

$$= p \left(\frac{1}{2} \left(f_1(t + \chi \theta) + f_2(t - \chi \theta) \right), \frac{1}{2} \left(f_1(t + \chi \theta) - f_2(t - \chi \theta) \right) \right)$$

$$= [\Phi(f_1, f_2, \chi) \circ p] (t, \theta). \tag{4.11}$$

We claim that this first map above, which maps Δ into $C^\infty(\mathbf{R}^1 \times \mathbf{R}^1, \mathbf{R}^1 \times S^1)$, is continuous and open onto its image. It is continuous by Lemmas 2.1, 3.1, 3.2, and 3.3. To see that it is open onto its image we will apply Corollary 3.2 and our work in Chapter 3. Since openness of Φ (together with openness of the map Ω in Theorem 4.2 below) is a key part of our results, and since a large part of the work done in the more than twenty pages of Chapter 3 is necessary to prove it,

we will present the argument in detail. (For the necessity of openness of Φ, see e.g. Pontrjagin [9], §19.) Fix $\chi \in S_2$ and consider the extension of this map to all of $C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$, i.e., the map

$$\tilde{\Phi} : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^2, \mathbf{R}^1) \times C^\infty(\mathbf{R}^2, S^1)$$

$$(f_1, f_2) \mapsto \left(\eta_\chi \left(\beta_2 \left(\frac{1}{2} f_1(\sqrt{2}\cdot), \frac{1}{2} f_2(\sqrt{2}\cdot) \right) \right), \eta_\chi \left(\omega \left((q \circ \frac{1}{2} f_1)(\sqrt{2}\cdot), (q \circ -\frac{1}{2} f_2)(\sqrt{2}\cdot) \right) \right) \right);$$
(4.12)

in other words,

$$\left(\tilde{\Phi}(f_1, f_2) \right)(t, \theta) = \left(\frac{1}{2} (f_1(t + \chi\theta) + f_2(t - \chi\theta)), q(\frac{1}{2} f_1(t + \chi\theta) - f_2(t - \chi\theta)) \right).$$
(4.13)

By Lemma 3.3(i)(c),(i)(d),(i)(e) the map

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$$

$$(f_1, f_2) \mapsto \left(\frac{1}{2} f_1(\sqrt{2}\cdot), \frac{1}{2} f_2(\sqrt{2}\cdot) \right)$$
(4.14)

is a homeomorphism; thus

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^2, \mathbf{R}^1)$$

$$(f_1, f_2) \mapsto \beta_2 \left(\frac{1}{2} f_1(\sqrt{2}\cdot), \frac{1}{2} f_2(\sqrt{2}\cdot) \right)$$
(4.15)

is open onto its image by Corollary 3.2. η_χ is a homeomorphism, and thus

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^2, \mathbf{R}^1)$$

$$(f_1, f_2) \mapsto \eta_\chi \left(\beta_2 \left(\frac{1}{2} f_1(\sqrt{2}\cdot), \frac{1}{2} f_2(\sqrt{2}\cdot) \right) \right)$$
(4.16)

is open onto its image by our comment after Proposition 3.2 above. Similarly, by

Lemma 3.3(i)(c),(i)(d),(i)(e), Proposition 3.6, and Corollary 3.2 the map

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, S^1) \times C^\infty(\mathbf{R}^1, S^1)$$

$$(f_1, f_2) \mapsto \left(\left(q \circ \frac{1}{2} f_1 \right)(\sqrt{2}\cdot), \left(q \circ -\frac{1}{2} f_2 \right)(\sqrt{2}\cdot) \right) \tag{4.17}$$

is open. Thus by Lemma 3.1 and Corollary 3.2 the map

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^2, S^1)$$

$$(f_1, f_2) \mapsto \omega \left(\left(q \circ \frac{1}{2} f_1 \right)(\sqrt{2}\cdot), \left(q \circ -\frac{1}{2} f_2 \right)(\sqrt{2}\cdot) \right) \tag{4.18}$$

is open onto its image. By Lemma 3.2 and Corollary 3.2 the map

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^2, S^1)$$

$$(f_1, f_2) \mapsto \eta_\chi \left(\omega \left(\left(q \circ \frac{1}{2} f_1 \right)(\sqrt{2}\cdot), \left(q \circ -\frac{1}{2} f_2 \right)(\sqrt{2}\cdot) \right) \right) \tag{4.19}$$

is open onto its image. Lemma 3.3(i)(c) then shows that $\tilde{\Phi}$ is open onto its image.

By Lemma 3.3(ii) (and since we give S_2 the discrete topology) the related map

$$\tilde{\tilde{\Phi}} : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2 \to C^\infty(\mathbf{R}^2, \mathbf{R}^1 \times S^1)$$

$$(f_1, f_2, \chi) \mapsto \iota_{\mathbf{R}^1, \mathbf{C}}^{\mathbf{R}^2} \left(\tilde{\Phi}(f_1, f_2) \right) \tag{4.20}$$

is open onto its image as well. We now investigate its restriction $\tilde{\tilde{\Phi}}|_{\Delta \times S_2}$, which is

the map on the left-hand side of equation (4.11) above. We compute the fibers of $\tilde{\tilde{\Phi}}$:

$\tilde{\tilde{\Phi}}(f_1, f_2, \chi) = \tilde{\tilde{\Phi}}(g_1, g_2, \xi)$ if and only if for all $(t, \theta) \in \mathbf{R}^2$ $\frac{1}{2}(f_1(t+\chi\theta)+f_2(t-\chi\theta)) =$

$\frac{1}{2}(g_1(t + \xi\theta) + g_2(t - \xi\theta))$ and $\frac{1}{2}(f_1(t + \chi\theta) - f_2(t - \chi\theta)) = \frac{1}{2}(g_1(t + \xi\theta) - g_2(t - $

$\xi\theta)) + 2\pi n$, where $n \in \mathbf{Z}$; but this is equivalent to $f_1(t + \chi\theta) = g_1(t + \xi\theta) + 2\pi n$,

$f_2(t - \chi\theta) = g_2(t - \xi\theta) - 2\pi n$. Differentiating with respect to t and θ at $\theta = 0$

gives, respectively, $f_1'(t) = g_1'(t)$, $\chi f_1'(t) = \xi g_1'(t)$, $f_2'(t) = g_2'(t)$, $\chi f_2'(t) = \xi g_2'(t)$; thus either $\chi = \xi$ or f_1, f_2, g_1, and g_2 are all constant. In either case we have, for all $x \in \mathbf{R}^1$, $f_1(x) = g_1(x) + 2\pi n$, $f_2(x) = g_2(x) - 2\pi n$; if f_1 and f_2 are not constant, this condition together with $\chi = \xi$ clearly implies that $\tilde{\tilde{\Phi}}(f_1, f_2, \chi) = \tilde{\tilde{\Phi}}(g_1, g_2, \xi)$. Thus

$$\tilde{\tilde{\Phi}}^{-1}\left(\tilde{\tilde{\Phi}}(\Delta \times S_2)\right)$$

$$= \left\{(f_1, f_2, \chi) \in C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2\right|$$

$$\tilde{\tilde{\Phi}}(f_1, f_2, \chi) = \tilde{\tilde{\Phi}}(g_1, g_2, \xi) \text{ for some } (g_1, g_2, \xi) \in \Delta \times S_2\Big\}$$

$$= \left\{(f_1, f_2, \chi) \in C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2\right|$$

$$f_1(x) = g_1(x) + 2\pi n, f_2(x) = g_2(x) - 2\pi n \text{ for some } (g_1, g_2) \in \Delta \text{ and } n \in \mathbf{Z},$$

$$\text{for all } x \in \mathbf{R}\Big\}$$

$$= \Delta \times S_2. \tag{4.21}$$

Thus the restriction $\tilde{\tilde{\Phi}}|_{\Delta \times S_2} : \Delta \times S_2 \to C^\infty(\mathbf{R}^2, \mathbf{R}^1 \times S^1)$ is open onto its image.

Now the image of Φ was shown above to be $G(\mathbf{R}^1 \times S^1; g)$; now $\tilde{\tilde{\Phi}}|_{\Delta \times S_2}$ is the map on the left-hand side of equation (4.11) above; thus for all $(f_1, f_2, \chi) \in \Delta \times S_2$ we have $\tilde{\tilde{\Phi}}(f_1, f_2, \chi) = \Phi(f_1, f_2, \chi) \circ p$, so $\tilde{\tilde{\Phi}}|_{\Delta \times S_2} = Z_{G(\mathbf{R}^1 \times S^1; g)} \circ \Phi$. Thus $\Phi = \left(Z_{G(\mathbf{R}^1 \times S^1; g)}\right)^{-1} \circ \tilde{\tilde{\Phi}}|_{\Delta \times S_2}$ is continuous and open onto its image, since $Z_{G(\mathbf{R}^1 \times S^1; g)} : G(\mathbf{R}^1 \times S^1; g) \to C^\infty(\mathbf{R}^2, \mathbf{R}^1 \times S^1)$ is a homeomorphism onto its image by Proposition 3.5 above. (We note that the image of $\tilde{\tilde{\Phi}}|_{\Delta \times S_2}$ equals that of $Z_{G(\mathbf{R}^1 \times S^1; g)}$ since $\tilde{\tilde{\Phi}}|_{\Delta \times S_2} = Z_{G(\mathbf{R}^1 \times S^1; g)} \circ \Phi$ and Φ is surjective from $\Delta \times S_2$ onto $G(\mathbf{R}^1 \times S^1; g)$.) Since the image of Φ is $G(\mathbf{R}^1 \times S^1; g)$ we see finally that

$\Phi : \Delta \rtimes S_2 \to G(\mathbf{R}^1 \times S^1; g)$ is open. Φ is therefore a continuous, open, surjective homomorphism, as desired.

To compute its kernel, we note that $\Phi(f_1, f_2, \chi) = \mathrm{id} = \Phi(\mathrm{id}, \mathrm{id}, 1)$ if and only if $\tilde{\tilde{\Phi}}(f_1, f_2, \chi) = \tilde{\tilde{\Phi}}(\mathrm{id}, \mathrm{id}, 1)$, since $Z_{G(\mathbf{R}^1 \times S^1; g)}$ is a homeomorphism; our previous work then shows that this is equivalent to (since id is not constant) $\chi = 1$, $f_1(x) = x + 2\pi n$, $f_2(x) = x - 2\pi n$ for some $n \in \mathbf{Z}$, for all $x \in \mathbf{R}$. Thus

$$\mathrm{Ker}\, \Phi = \{(f_1, f_2, \chi) \in \Delta \rtimes S_2 | \chi = 1, f_1(x) = x + 2\pi n, f_2(x) = x - 2\pi n$$

$$\text{for some } n \in \mathbf{Z}, \text{ for all } x \in \mathbf{R}\}, \quad (4.22)$$

as desired.

Thus we see that Φ descends to a map $\hat{\Phi} : (\Delta \rtimes S_2)/\mathrm{Ker}\, \Phi \to G(\mathbf{R}^1 \times S^1; g)$ which is an abstract group isomorphism and also a homeomorphism (see Munkres [8], Corollary 22.3; compare Pontrjagin [9], Theorem 12). Since $(\Delta \rtimes S_2)/\mathrm{Ker}\, \Phi$ is a topological group (Pontrjagin [9], p. 60), this shows, among other things, that $G(\mathbf{R}^1 \times S^1; g)$ is also a topological group, as desired. This completes our proof.

QED.

We note that for all $(f_1, f_2, \chi) \in \Delta \times S_2$, $n \in \mathbf{Z}$, $(t, \sigma) \in \mathbf{R}^1 \times S^1$ we have $\Phi(f_1, f_2, \chi)(t + 2\pi n, \sigma) = \Phi(f_1, f_2, \chi)(t, \sigma) \cdot (2\pi n, 1)$, i.e., $\Phi(f_1, f_2, \chi)$ commutes with time shifts of $2\pi n$, $n \in \mathbf{Z}$. Thus each conformal transformation is completely determined by its restriction to the compact subset $[0, 2\pi] \times S^1$, and it is in this sense that $G(\mathbf{R}^1 \times S^1; g)$ behaves like a group of diffeomorphisms of a compact manifold. Moreover, the identity is the only element in $G(\mathbf{R}^1 \times S^1; g)$ which has

compact support (see our comment on p. 5).

Theorem 4.2. The map $\Omega : \Delta \rtimes S_2 \to K(S^1; \mathbf{R}^1 \times S^1; g)$ given by

$$[\Omega(f_1, f_2, \chi) \circ q](x) = p\Big((S \circ (f_1 \times f_2))(\chi(x, -x))\Big) \tag{4.23}$$

is surjective, continuous, and open, and each of its fibers is a union of two cosets of Φ.

We note that this implies that Ω descends to a map $\hat{\Omega} : (\Delta \rtimes S_2)/\mathrm{Ker}\,\Phi \to K(S^1; \mathbf{R}^1 \times S^1; g)$; this map is however not injective. We note also that we may write more explicitly

$$[\Omega(f_1, f_2, \chi) \circ q](x) = p\left(\frac{1}{2}(f_1(\chi x) + f_2(-\chi x)), \frac{1}{2}(f_1(\chi x) - f_2(-\chi x))\right). \tag{4.24}$$

Proof. We note first that $\Omega(f_1, f_2, \chi)$ is defined since $(f_1, f_2) \in \Delta$; more explicitly, letting $\xi = \pm 1$ as $(f_1, f_2) \in \mathrm{Diff}_{2\pi\mathbf{Z}}^{\pm}(\mathbf{R}^1) \times \mathrm{Diff}_{2\pi\mathbf{Z}}^{\pm}(\mathbf{R}^1)$, we see that for all $n \in \mathbf{Z}$

$$p\Big((S \circ (f_1 \times f_2))(\chi(x + 2\pi n, -(x + 2\pi n)))\Big)$$

$$= p\Big(S(f_1(\chi x) + 2\pi\chi\xi n, f_2(-\chi x) - 2\pi\chi\xi n)\Big)$$

$$= p\left(\frac{1}{2}(f_1(\chi x) + f_2(-\chi x)), \frac{1}{2}(f_1(\chi x) - f_2(-\chi x)) + 2\pi\chi\xi n\right)$$

$$= p\left(\frac{1}{2}(f_1(\chi x) + f_2(-\chi x)), \frac{1}{2}(f_1(\chi x) - f_2(-\chi x))\right)$$

$$= p\Big((S \circ (f_1 \times f_2))(\chi(x, -x))\Big). \tag{4.25}$$

Further, by Proposition 3.3, $\Omega(f_1, f_2, \chi) : S^1 \to \mathbf{R}^1 \times S^1$ is C^∞.

To show that the image of Ω equals $K(S^1; \mathbf{R}^1 \times S^1; g)$, we proceed as in the proof of Theorem 4.1 above. Let $X \in K(S^1; \mathbf{R}^1 \times S^1; g)$ be some spacelike embedding, and consider any continuous p-lift $\tilde{X}$ of $X \circ q : \mathbf{R}^1 \to \mathbf{R}^1 \times S^1$; thus $\tilde{X} : \mathbf{R}^1 \to \mathbf{R}^1 \times \mathbf{R}^1$ satisfies $p \circ \tilde{X} = X \circ q$. As usual, we see that $\tilde{X}$ is C^∞. Now by definition X^*g is positive definite; since p^* and q^* are nondegenerate, this implies that $q^*X^*g = (X \circ q)^*g = (p \circ \tilde{X})^*g = \tilde{X}^*p^*g$ must also be positive definite. We now write $\tilde{X}$ in null coordinates as $(\tilde{X}^+(x), \tilde{X}^-(x)) = (T \circ \tilde{X})(x)$. Then we see that

$$\tilde{X}^*(p^*g) = \tilde{X}^*(-e^w(dz^+ \otimes dz^- + dz^- \otimes dz^+)) = -2e^w\tilde{X}^{+\prime}\tilde{X}^{-\prime}dx \otimes dx, \quad (4.26)$$

so $\tilde{X}$ must satisfy $\tilde{X}^{+\prime}\tilde{X}^{-\prime} < 0$ on $\mathbf{R}^1$. In particular, $\tilde{X}^+$ and $\tilde{X}^-$ must both be injective.

Now, for each x, $\tilde{X}$ must satisfy $\tilde{X}(x+2\pi) = \tilde{X}(x)+(0, 2\pi n(x))$ for some $n(x) \in \mathbf{Z}$. As usual (since the components of a C^∞ function are C^∞) this implies that n is a constant. Thus $\tilde{X}^\pm$ satisfy $\tilde{X}^\pm(x+2\pi) = \tilde{X}^\pm(x)\pm\pi n\sqrt{2}$; since $\tilde{X}^\pm$ are injective this implies that $n \neq 0$. Thus $\tilde{X}^\pm$ are surjective and hence are diffeomorphisms of $\mathbf{R}^1$, as they are smooth and have nowhere-vanishing derivatives. We note moreover that $\tilde{X}^{+\prime}\tilde{X}^{-\prime} < 0$ implies that $\tilde{X}^{\theta\prime} \neq 0$ on $\mathbf{R}^1$. Since $\tilde{X}^\theta(x + 2\pi) = \tilde{X}^\theta(x) + 2\pi n$ for all $x \in \mathbf{R}^1$, we see that $\tilde{X}^\theta$ is a diffeomorphism of $\mathbf{R}^1$. Now let $\chi = \operatorname{sgn}\tilde{X}^{\theta\prime}$ and define $\zeta : \mathbf{R}^1 \to \mathbf{R}^1$ by $\zeta(x) = (\tilde{X}^\theta)^{-1}(\tilde{X}^\theta(x) + 2\pi\chi)$; thus $\zeta(x) > x$ for all $x \in \mathbf{R}^1$. Now we note that (setting $\zeta^0 = \operatorname{id}$) $\zeta^m(x) = (\tilde{X}^\theta)^{-1}(\tilde{X}^\theta(x)+2\pi\chi m)$ for all $m \in \mathbf{Z}$ (since $\zeta^{-1}(x) = (\tilde{X}^\theta)^{-1}(\tilde{X}^\theta(x) - 2\pi\chi)$); thus in particular (since $\operatorname{sgn} n = \operatorname{sgn}\tilde{X}^{\theta\prime} = \chi$) $\zeta^{|n|}(x) = (\tilde{X}^\theta)^{-1}(\tilde{X}^\theta(x) + 2\pi n) = x + 2\pi$ and so $\tilde{X}^t(x) = \tilde{X}^t(\zeta^{|n|}(x))$ for all $x \in \mathbf{R}^1$. Now fix $x \in \mathbf{R}^1$ and consider the sequence $\{\tilde{X}^t(\zeta^m(x))-\tilde{X}^t(\zeta^{m-1}(x))|m =$

$1, 2, \cdots, |n|\}$. This sequence sums to zero since $\tilde{X}^t(x) = \tilde{X}^t(\zeta^{|n|}(x))$; thus there must

be an m in the given range, i.e., between 1 and $|n|$ (which is nonempty since $n \neq 0$),

so that $\tilde{X}^t(\zeta^{m+1}(x)) - \tilde{X}^t(\zeta^m(x)) = \tilde{X}^t(\zeta(\zeta^m(x))) - \tilde{X}^t(\zeta^m(x))$ is nonnegative and

$\tilde{X}^t(\zeta(\zeta^{m-1}(x))) - \tilde{X}^t(\zeta^{m-1}(x))$ is nonpositive or vice versa. In either case there

must be an $x^* \in [\zeta^{m-1}(x), \zeta^m(x)]$ so that $\tilde{X}^t(\zeta(x^*)) = \tilde{X}^t(x^*)$. But $\tilde{X}^\theta(\zeta(x^*)) = $

$\tilde{X}^\theta(x^*) + 2\pi\chi$; thus $p(\tilde{X}(\zeta(x^*))) = p(\tilde{X}(x^*))$, so $X(q(\zeta(x^*))) = X(q(x^*))$. By

injectivity of X this shows that $\zeta(x^*) = x^* + 2\pi n'$ for some $n' \in \mathbf{Z}$. But then

$$\tilde{X}^\theta(x^* + 2\pi n') = \tilde{X}^\theta(x^*) + 2\pi n' n$$

$$= \tilde{X}^\theta(x^*) + 2\pi\chi, \tag{4.27}$$

so $n'n = \pm 1$ and $n = \pm 1$. Thus $\sqrt{2}\tilde{X}^\pm \in \text{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$. Now we note that $\tilde{X}^-(-x)$

satisfies $\tilde{X}^-(-(x + 2\pi)) = \tilde{X}^-(-x - 2\pi) = \tilde{X}^-(-x) + n\pi\sqrt{2}$, while $\tilde{X}^+(x + 2\pi) = $

$\tilde{X}^+(x) + n\pi\sqrt{2}$; thus $(\sqrt{2}\tilde{X}^+, \sqrt{2}\tilde{X}^-(-\cdot)) \in \Delta$. (Recall that n could be either 1 or

-1.) Now we clearly have $\Omega(\sqrt{2}\tilde{X}^+, \sqrt{2}\tilde{X}^-(-\cdot), 1) \circ q = p \circ (\frac{1}{\sqrt{2}}(\tilde{X}^+ + \tilde{X}^-), \frac{1}{\sqrt{2}}(\tilde{X}^+ - $

$\tilde{X}^-)) = p \circ \tilde{X}$; thus $\Omega(\sqrt{2}\tilde{X}^+, \sqrt{2}\tilde{X}^-(-\cdot), 1) = X$ and $K(S^1; \mathbf{R}^1 \times S^1; g)$ is contained

in the image of Ω, as desired.

To see that $K(S^1; \mathbf{R}^1 \times S^1; g)$ contains the image of Ω, let $(f_1, f_2, \chi) \in \Delta \times S_2$ and

consider $X = \Omega(f_1, f_2, \chi) : S^1 \to \mathbf{R}^1 \times S^1$. X is C^∞ as usual. Then $\tilde{X} = (\frac{1}{2}(f_1(\chi x) + $

$f_2(-\chi x)), \frac{1}{2}(f_1(\chi x) - f_2(-\chi x))) : \mathbf{R}^1 \to \mathbf{R}^1 \times \mathbf{R}^1$ is a lift of $X \circ q$ by definition

of Ω. In null coordinates we have $\tilde{X}^+(x) = \frac{1}{\sqrt{2}}f_1(\chi x)$, $\tilde{X}^-(x) = \frac{1}{\sqrt{2}}f_2(-\chi x)$; thus

$\tilde{X}^{+\prime}\tilde{X}^{-\prime} = -\frac{1}{2}\chi^2 f_1'(\chi x)f_2'(-\chi x) < 0$ since $(f_1, f_2) \in \Delta$, so as above $\tilde{X}^* p^* g = q^* X^* g$

is positive definite. Hence so is $X^* g$ and X is therefore everywhere spacelike. X is

injective since $\tilde{X}(x_1) - \tilde{X}(x_2) = (0, 2\pi m)$ for some $m \in \mathbf{Z}$ implies that $f_1(\chi x_1) - $

$f_1(\chi x_2) = 2\pi m$; thus $x_1 - x_2 = \pm 2\pi m\chi$, so $q(x_1) = q(x_2)$. Now X is open onto its image since S^1 is compact and $X(S^1) \subset \mathbf{R}^1 \times S^1$ is Hausdorff (see Munkres [8], Theorem 26.6). Thus X is an embedding and $X \in K(S^1; \mathbf{R}^1 \times S^1; g)$, as desired. Thus the image of Ω equals $K(S^1; \mathbf{R}^1 \times S^1; g)$.

Now we note that we may write

$$\Omega(f_1, f_2, \chi) \circ q = \left(\beta_1 \left(\frac{1}{2} f_1(\chi \cdot), \frac{1}{2} f_2(-\chi \cdot) \right), I \left(\beta_1 \left(\frac{1}{2} f_1(\chi \cdot), -\frac{1}{2} f_2(-\chi \cdot) \right) \right) \right). \quad (4.28)$$

As before the map on the right-hand side is continuous by Lemmas 2.1 and 3.3(i). As in the proof of Theorem 4.1 above let

$$\tilde{\Omega}_\chi : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, S^1)$$

$$(f_1, f_2) \mapsto \left(\beta_1 \left(\frac{1}{2} f_1(\chi \cdot), \frac{1}{2} f_2(-\chi \cdot) \right), I \left(\beta_1 \left(\frac{1}{2} f_1(\chi \cdot), -\frac{1}{2} f_2(-\chi \cdot) \right) \right) \right) \quad (4.29)$$

denote the extension of this map to $C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$. As in the proof of Theorem 4.1, the maps

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1)$$

$$(f_1, f_2) \mapsto \left(\frac{1}{2} f_1(\chi \cdot), \pm \frac{1}{2} f_2(-\chi \cdot) \right) \quad (4.30)$$

are homeomorphisms by Lemma 3.3(i); thus the maps

$$C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \to C^\infty(\mathbf{R}^1, \mathbf{R}^1)$$

$$(f_1, f_2) \mapsto \beta_1 \left(\frac{1}{2} f_1(\chi \cdot), \pm \frac{1}{2} f_2(-\chi \cdot) \right) \quad (4.31)$$

are open by Lemma 3.3(i)(b). Thus the map $\tilde{\Omega}_\chi$ is open onto its image by Proposition 3.6, Corollary 3.2, and Lemma 3.3(i)(c). The associated map

$$\tilde{\tilde{\Omega}} : C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2 \to C^\infty(\mathbf{R}^1, \mathbf{R}^1 \times S^1)$$

$$(f_1, f_2, \chi) \mapsto \iota_{\mathbf{R}^1, \mathbf{C}}^{\mathbf{R}^1} \left(\tilde{\Omega}_\chi(f_1, f_2) \right) \tag{4.32}$$

is therefore open onto its image by Corollary 3.2 and Lemma 3.3(ii), since S_2 is given the discrete topology. We consider the restriction $\tilde{\tilde{\Omega}}|_{\Delta \times S_2}$. We compute the fibers of $\tilde{\tilde{\Omega}}$: $\tilde{\tilde{\Omega}}(f_1, f_2, \chi) = \tilde{\tilde{\Omega}}(g_1, g_2, \xi)$ implies that there is some $n \in \mathbf{Z}$ so that

$$\frac{1}{2}\left(f_1(\chi x) + f_2(-\chi x)\right) = \frac{1}{2}\left(g_1(\xi x) + g_2(-\xi x)\right)$$
$$\frac{1}{2}\left(f_1(\chi x) - f_2(-\chi x)\right) = \frac{1}{2}\left(g_1(\xi x) - g_2(-\xi x)\right) + 2\pi n, \tag{4.33}$$

so $f_1(\chi x) = g_1(\xi x) + 2\pi n$, $f_2(\chi x) = g_2(\xi x) - 2\pi n$ for all $x \in \mathbf{R}$. Thus $f_1(x) = g_1(\xi \chi x) + 2\pi n$, $f_2(x) = g_2(\xi \chi x) - 2\pi n$; this is clearly also sufficient, and so

$$\tilde{\tilde{\Omega}}^{-1}\left(\left\{\tilde{\tilde{\Omega}}(g_1, g_2, \xi)\right\}\right)$$
$$= \left\{(f_1, f_2, \chi) \in C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2 \right|$$
$$\left. f_1(x) = g_1(\xi \chi x) + 2\pi n, f_2(x) = g_2(\xi \chi x) - 2\pi n \text{ for some } n \in \mathbf{Z}, \text{ for all } x \in \mathbf{R}\right\}$$
$$= \left\{(f_1, f_2, \xi) \in C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2 \right|$$
$$\left. f_1(x) = g_1(x) + 2\pi n, f_2(x) = g_2(x) - 2\pi n\right\}$$
$$\cup \left\{(f_1, f_2, -\xi) \in C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C^\infty(\mathbf{R}^1, \mathbf{R}^1) \times S_2 \right|$$
$$\left. f_1(x) = g_1(-x) + 2\pi n, f_2(x) = g_2(-x) - 2\pi n\right\}. \tag{4.34}$$

Now $(f_1, f_2) \in \Delta$ implies that $(f_1(-\cdot), f_2(-\cdot)) \in \Delta$; thus $\tilde{\tilde{\Omega}}^{-1}\left(\tilde{\tilde{\Omega}}(\Delta \times S_2)\right) = \Delta \times S_2$ so $\Omega = \hat{\Theta}^{-1} \circ \tilde{\tilde{\Omega}}|_{\Delta \times S_2}$ is open onto its image by Corollaries 3.2 and 3.3, as desired.

Further, by our computation of the fibers of $\tilde{\tilde{\Omega}}$ above, we have clearly

$$\Omega^{-1}\left(\{\Omega(g_1, g_2, \xi)\}\right)$$

$$= \{(f_1, f_2, \xi) \in \Delta \rtimes S_2 | f_1(x) = g_1(x) + 2\pi n, f_2(x) = g_2(x) - 2\pi n\}$$

$$\cup \{(f_1, f_2, -\xi) \in \Delta \rtimes S_2 | f_1(x) = g_1(-x) + 2\pi n, f_2(x) = g_2(-x) - 2\pi n\}$$

$$= (\operatorname{Ker} \Phi)(g_1, g_2, \xi) \cup (\operatorname{Ker} \Phi)(g_1(-\cdot), g_2(-\cdot), -\xi), \tag{4.35}$$

so each fiber of Ω is a union of two disjoint cosets of Φ. This completes the proof.

QED.

We note for future use that

$$\Omega^{-1}(\{\Omega(g_1, g_2, \xi)\}) = (\operatorname{Ker} \Phi)(g_1, g_2, \xi) \cup (\operatorname{Ker} \Phi)(g_1, g_2, \xi)(-\mathrm{id}, -\mathrm{id}, -1). \tag{4.36}$$

This theorem implies (see Munkres [8], Theorem 22.2) that Ω descends to a continuous map $\hat{\Omega} : (\Delta \rtimes S_2)/\operatorname{Ker} \Phi \to K(S^1; \mathbf{R}^1 \times S^1; g)$. We have the following penultimate result.

Proposition 4.1. The map $\hat{\hat{\Omega}} = \hat{\Omega}|_{(\Delta \times \{1\})/\operatorname{Ker} \Phi} : (\Delta \times \{1\})/\operatorname{Ker} \Phi \to K(S^1; \mathbf{R}^1 \times S^1; g)$ is a homeomorphism.

Proof. We note that $\Delta \times \{1\}$ is an open subgroup of $\Delta \rtimes S_2$ containing $\operatorname{Ker} \Phi$, and $\Phi^{-1}(\Phi(\Delta \times \{1\})) = \Delta \times \{1\}$; thus $(\Delta \times \{1\})/\operatorname{Ker} \Phi$ is an open subgroup of $(\Delta \rtimes S_2)/\operatorname{Ker} \Phi$ (see also Bredon [4], Theorem I.15.11, and Pontrjagin [9], Theorem 11). Now $\hat{\Omega}$ is open: if $U \subset (\Delta \rtimes S_2)/\operatorname{Ker} \Phi$ is open, then by definition $U' = \{(f_1, f_2, \chi) \in \Delta \rtimes S_2 | (\operatorname{Ker} \Phi)(f_1, f_2, \chi) \in U\}$ is open in $\Delta \rtimes S_2$, so $\Omega(U')$ is open in $K(S^1; \mathbf{R}^1 \times S^1; g)$; but $\Omega(U') = \hat{\Omega}(U)$, and thus $\hat{\Omega}$ is open, as desired. Thus the map $\hat{\hat{\Omega}}$ is continuous and open as a restriction to an open set of an open and continuous map. To see that it is surjective, let $X = \Omega(f_1, f_2, \chi) \in K(S^1; \mathbf{R}^1 \times S^1; g)$,

$(f_1, f_2, \chi) \in \Delta \rtimes S_2$. Then $\Omega^{-1}\{X\} = (\mathrm{Ker}\,\Phi)(f_1, f_2, \chi) \cup (\mathrm{Ker}\,\Phi)(f_1(-\cdot), f_2(-\cdot), -\chi)$ – in other words, $\Omega(f_1(-\cdot), f_2(-\cdot), -\chi) = X$. Now if $\chi \neq 1$ then $-\chi = 1$, and thus $\Omega|_{\Delta \times \{1\}}$ is surjective. Hence so is the quotient $\hat{\Omega}|_{(\Delta \times \{1\})/\mathrm{Ker}\,\Phi} = \hat{\Omega}$. To see that $\hat{\Omega}$ is injective, choose $\omega_1 = (\mathrm{Ker}\,\Phi)(f_1, f_2, 1)$, $\omega_2 = (\mathrm{Ker}\,\Phi)(g_1, g_2, 1) \in (\Delta \times \{1\})/\mathrm{Ker}\,\Phi$, and suppose that $\hat{\Omega}(\omega_1) = \hat{\Omega}(\omega_2)$. Then $\Omega(f_1, f_2, 1) = \Omega(g_1, g_2, 1)$, so $(f_1, f_2, 1) \in (\mathrm{Ker}\,\Phi)(g_1, g_2, 1) \cup (\mathrm{Ker}\,\Phi)(g_1(-\cdot), g_2(-\cdot), -1)$; but every element in the latter has last component equal to -1 (since $\mathrm{Ker}\,\Phi \subset \Delta \times \{1\}$), and thus $(f_1, f_2, 1) \in (\mathrm{Ker}\,\Phi)(g_1, g_2, 1)$ and $\omega_1 = \omega_2$. Thus $\hat{\Omega}$ is a homeomorphism, as desired.

QED.

Since $(\Delta \times \{1\})/\mathrm{Ker}\,\Phi$ is an open subgroup of $(\Delta \rtimes S_2)/\mathrm{Ker}\,\Phi$, this last result shows that $K(S^1; \mathbf{R}^1 \times S^1; g)$ may be given the structure of a topological group via $\hat{\Omega}$; specifically, we may define a multiplication by $X_1 X_2 = \hat{\Omega}\left(\hat{\Omega}^{-1}(X_1)\hat{\Omega}^{-1}(X_2)\right)$ and inversion by $X^{-1} = \hat{\Omega}\left(\left[\hat{\Omega}^{-1}(X)\right]^{-1}\right)$, both of which are continuous since $(\Delta \times \{1\})/\mathrm{Ker}\,\Phi$ is a topological group and $\hat{\Omega}$ is a homeomorphism; the identity is given by $\hat{\Omega}(\mathrm{id}, \mathrm{id}, 1) = (\sigma \mapsto (0, \sigma))$. These operations are easily seen to satisfy the group axioms and therefore define a group structure on $K(S^1; \mathbf{R}^1 \times S^1; g)$. This group structure is not canonically defined, however, since our association of elements of $\Delta \times S_2$ with spacelike embeddings depends on our choice of coordinates (t, θ) on $\mathbf{R}^1 \times \mathbf{R}^1$.

We now have the following final result.

Theorem 4.3. Let ω_1, $\omega_2 \in (\Delta \rtimes S_2)/\mathrm{Ker}\,\Phi$. Then $\hat{\Phi}(\omega_1) \circ \hat{\Omega}(\omega_2) = \hat{\Omega}(\omega_1 \omega_2)$,

and the action of $G(\mathbf{R}^1 \times S^1; g)$ on $K(S^1; \mathbf{R}^1 \times S^1; g)$ defined by this equation is transitive. This action is also continuous, and the nontrivial element in its isotropy is conjugate to a conformal transformation representing time reversal.

Proof. Choose (f_1, f_2, χ), $(g_1, g_2, \xi) \in \Delta \rtimes S_2$ representing ω_1 and ω_2, respectively. Then by definition

$$
\begin{aligned}
\left(\hat{\Phi}(\omega_1) \circ \hat{\Omega}(\omega_2) \circ q\right)(x) &= (\Phi(f_1, f_2, \chi) \circ \Omega(g_1, g_2, \xi) \circ q)(x) \\
&= (\Phi(f_1, f_2, \chi) \circ p)\Big((S \circ (g_1 \times g_2))(\xi(x, -x))\Big) \\
&= p\Big((S \circ (f_1 \times f_2) \circ \chi)((g_1 \times g_2)\xi(x, -x))\Big) \\
&= p\Big((S \circ (f_1 \times f_2))((g_{\chi(1)} \times g_{\chi(2)})(\chi\xi(x, -x)))\Big) \\
&= \Big(\Omega((f_1, f_2, \chi)(g_1, g_2, \xi)) \circ q\Big)(x) \\
&= \Big(\hat{\Omega}(\omega_1\omega_2) \circ q\Big)(x), \tag{4.37}
\end{aligned}
$$

and thus $\hat{\Phi}(\omega_1) \circ \hat{\Omega}(\omega_2) = \hat{\Omega}(\omega_1\omega_2)$, as desired. Since $\hat{\Phi}$ and $\hat{\Omega}$ are surjective, the action of $G(\mathbf{R}^1 \times S^1; g)$ on $K(S^1; \mathbf{R}^1 \times S^1; g)$ given by $(\Psi, X) \mapsto \Psi \circ X$ is thus well-defined (i.e., it fixes $K(S^1; \mathbf{R}^1 \times S^1; g)$) and transitive.

To see that this action is actually continuous, we note that (for $\Psi \in G(\mathbf{R}^1 \times S^1; g)$ and $X \in K(S^1; \mathbf{R}^1 \times S^1; g)$)

$$
\Psi \circ X = \hat{\Phi}\left(\hat{\Phi}^{-1}(\Psi)\right) \circ \hat{\Omega}\left(\hat{\Omega}^{-1}(X)\right) = \hat{\Omega}\left(\hat{\Phi}^{-1}(\Psi)\hat{\Omega}^{-1}(X)\right), \tag{4.38}
$$

where the last multiplication is done in the topological group $(\Delta \rtimes S_2)/\mathrm{Ker}\,\Phi$. But $\hat{\Phi}$ and $\hat{\Omega}$ are homeomorphisms, and $\hat{\Omega}$ is continuous, so the action must be continuous, as desired. (Note our use of openness of Φ and Ω here.)

To compute the isotropy of this action, let $\iota = (\operatorname{Ker}\Phi)(-\mathrm{id}, -\mathrm{id}, -1) \in (\Delta \rtimes$

$S_2)/\operatorname{Ker}\Phi$. Then $\hat{\Omega}(\omega) = \hat{\Omega}(\omega')$ if and only if $\omega = \omega'$ or $\omega = \omega'\iota$. Thus $\hat{\Phi}(\omega_1) \circ$

$\hat{\Omega}(\omega_2) = \hat{\Omega}(\omega_1\omega_2) = \hat{\Omega}(\omega_2)$ if and only if $\omega_1 = \mathrm{id}$ or $\omega_1 = \omega_2\iota\omega_2^{-1}$, so that the

isotropy is conjugate to $\{\mathrm{id}, \hat{\Phi}(\iota)\}$. Now we see that

$$\left(\hat{\Phi}(\iota) \circ p\right)(t, \theta) = p\Big(\frac{1}{2}((-\mathrm{id})(t - \theta) + (-\mathrm{id})(t + \theta)),$$

$$\frac{1}{2}((-\mathrm{id})(t - \theta) - (-\mathrm{id})(t + \theta))\Big)$$

$$= p(-t, \theta), \tag{4.39}$$

i.e., $\hat{\Phi}(\iota)(t, \sigma) = (-t, \sigma)$, so that $\hat{\Phi}(\iota)$ corresponds to time reversal as claimed.QED.

CHAPTER 5

TOWARDS A QUANTUM THEORY

In this final section we would like to briefly and informally describe the steps still necessary to fully implement Isham's group-theoretic quantization program [1] and then indicate briefly some questions which can be studied within this framework.

Isham's program requires a group of symplectic transformations acting transitively on the phase space of the classical system. In attempting to use our results in Chapter 4 to construct such a group, we are immediately confronted with the problem of determining a manifold structure on the infinite-dimensional spaces introduced there, particularly $K(S^1; \mathbf{R}^1 \times S^1; g)$. This difficulty is compounded by the existence of many inequivalent notions of smoothness for maps between topological vector spaces such as $C_p^\infty(\mathbf{R}^1, \mathbf{R}^1)$ which are not Banach spaces. It can be shown that each connected component of $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ is homeomorphic to an open subset of $C_p^\infty(\mathbf{R}^1, \mathbf{R}^1)$; these homeomorphisms can then be taken to provide a differential structure on $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$. Now Proposition 4.1 and Proposition 3.8 show that $\Omega|_{\Delta \times \{1\}} : \Delta \times \{1\} \to K(S^1; \mathbf{R}^1 \times S^1; g)$ is a covering map, and in particular a local homeomorphism; thus it can presumably be used to transfer this differential structure to $K(S^1; \mathbf{R}^1 \times S^1; g)$. Letting $(C_p^\infty(\mathbf{R}^1, \mathbf{R}^1))'$ denote the strong topological dual of $C_p^\infty(\mathbf{R}^1, \mathbf{R}^1)$ (see Yosida [3], pp. 110-111), we may identify the cotangent bundle of $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1)$ with $\mathrm{Diff}_{2\pi\mathbf{Z}}(\mathbf{R}^1) \times (C_p^\infty(\mathbf{R}^1, \mathbf{R}^1))'$. This suggests that we may identify the cotangent bundle of $K(S^1; \mathbf{R}^1 \times S^1; g)$ with something like

$K(S^1; \mathbf{R}^1 \times S^1; g) \times (C_p^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C_p^\infty(\mathbf{R}^1, \mathbf{R}^1))'$. We may then take our symplectic form to be the canonical symplectic form on this bundle (see Chernoff and Marsden [13], p. 8) and attempt to extend the action of $G(\mathbf{R}^1 \times S^1; g)$ on $K(S^1; \mathbf{R}^1 \times S^1; g)$ to a transitive symplectic action on this symplectic manifold. The details and rigorous justification of the above would require a much more detailed understanding of the differential structure used and the precise definition of smoothness adopted. Beyond this, there are certain technical requirements which the group of symplectic transformations obtained must satisfy in order to finish applying Isham's program.

It is nevertheless possible to speculate concerning the nature of the resultant quantum theory, and we would like to describe briefly a preliminary result which appears to be valid regardless of the particulars of the construction indicated in the preceding paragraph. For each point in space $\sigma \in S^1$ let us define a classical observable $t_\sigma : K(S^1; \mathbf{R}^1 \times S^1; g) \times (C_p^\infty(\mathbf{R}^1, \mathbf{R}^1) \times C_p^\infty(\mathbf{R}^1, \mathbf{R}^1))' \to \mathbf{R}^1$ by $t_\sigma(X, P) = X^t(\sigma)$; thus roughly $t_\sigma(X, P)$ is the time coordinate of the spacetime point corresponding under X to σ. The quantum mechanical analogue of this observable will be a self-adjoint operator, say $\hat{t}_\sigma$, on some Hilbert space. After completing the procedure described above it is possible to ask, for example, whether the spectrum – which is of course the quantum-mechanically allowable values of the observable – of $\hat{t}_\sigma$ is continuous or discrete; in other words, whether, in our quantum system, time remains continuous in some sense or becomes discrete. Preliminary investigations of this operator appear to indicate that its spectrum must always be continuous (at least in a certain technical sense – see Yosida [3], p. 209), at least

given certain technical conditions such as continuity of the representation. There are other technical points which would need to be addressed before this result could be considered fully established (for example, we presently do not identify spacelike embeddings X and $X \circ \phi$ for $\phi \in \mathrm{Diff}(S^1)$, i.e., spacelike embeddings differing only by reparametrization – it is possible that identifying such spacelike embeddings may change the result). Regardless, any result which can be obtained about the spectrum of $\hat{t}_\sigma$ is of interest and indicates the use of our general procedure.

REFERENCES

1. Isham, C.J.: Topological and global aspects of quantum theory. In: DeWitt, B.S., Stora, R. (eds.) Relativity, Groups and Topology II, pp. 1059-1290. Elsevier Science Publishers B.V., Amsterdam (1984)

2. Milnor, J.: Remarks on infinite-dimensional Lie groups. In: DeWitt, B.S., Stora, R. (eds.) Relativity, Groups and Topology II, pp. 1007-1057. Elsevier Science Publishers B.V., Amsterdam (1984)

3. Yosida, K.: Functional Analysis, 6th ed. Springer-Verlag, Berlin-Heidelberg-New York (1980)

4. Bredon, G.E.: Topology and Geometry. Graduate Texts in Mathematics 139. Springer-Verlag, New York-Berlin-Heidelberg (1993)

5. Glöckner, H.: Diff($\mathbf{R}^n$) as a Milnor-Lie group. Math. Nachr. 278, 1025-1032 (2005)

6. Glöckner, H.: Discontinuous non-linear mappings on locally convex direct limits. arXiv:math/0503387v1 (2005)

7. Keller, H.H.: Differential Calculus in Locally Convex Spaces. Lecture Notes in Mathematics 417. Springer-Verlag, Berlin-Heidelberg-New York (1974)

8. Munkres, J.: Topology, 2nd ed. Prentice Hall, Upper Saddle River, NJ (2000)

9. Pontrjagin, L.S.: Topological Groups. Princeton University Press, Princeton (1939)

10. Hungerford, T.W.: Algebra. Graduate Texts in Mathematics 73. Springer-Verlag, New York-Heidelberg-Berlin (1974)

11. Hewitt, E., Ross, K.A.: Abstract Harmonic Analysis I, 2nd ed. Springer-Verlag, Berlin-Heidelberg-New York (1979)

12. Spanier, E.H.: Algebraic Topology. Springer-Verlag, New York (1966)

13. Chernoff, P.R., Marsden, J.E.: Properties of Infinite Dimensional Hamiltonian Systems. Lecture Notes in Mathematics 425. Springer-Verlag, Berlin-Heidelberg-New York (1974)

14. Schottenloher, M.: A Mathematical Introduction to Conformal Field Theory, 2nd ed. Lecture Notes in Physics 759. Springer-Verlag, Berlin-Heidelberg (2008)

15. Hirsch, M.W.: Differential Topology. Graduate Texts in Mathematics 33. Springer-Verlag, New York Berlin Heidelberg (1976)

16. Illman, S.: The very-strong C^∞ topology on $C^\infty(M, N)$ and K-equivariant maps. Osaka J. Math. 40, 409-428 (2003)

17. Banakh, T., Mine, K., Sakai, K., Yagasaki, T.: Homeomorphism and diffeomorphism groups of non-compact manifolds with the Whitney topology. arXiv:0802.0337v2 (2010)

18. Hájíček, P., Isham, C.J.: The symplectic geometry of a parametrized scalar field on a curved background. J. Math. Phys. 37, 3505-3521 (1996)

19. Mather, J.N.: Stability of C^∞ mappings: II. Infinitesimal stability implies stability. Ann. of Math., Second Series 89, 254-291 (1969)

CONTENTS