In 1834 a scientist named John Scott Russell observed the first “soliton,” floating in a barge. Attempt to Recreate John Russell’s Soliton.

Background:

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Then in 1895 Korteweg and de Vries (often abbreviated KdV) described their famous integrable system. Since then, dozens of systems besides the KdV have been discovered, such as the Kaup Kupershmidt and Nonlinear Schrodinger equations.

Attempts have been made to create a database of integrable systems and their properties, but in the past all these projects were abandoned partway through.*

* Wang, Jing Ping. "A list of 1+1 dimensional integrable equations and their properties." Journal of Nonlinear Mathematical Physics 16.1(1975): 1


My work:

• Executing and tweaking the following algorithm presented by Hickman, et. al.** for discovering a property called a Lax pair for an integrable system:
  • Find a scaling symmetry for the integrable system
  • Using this, generate terms consistent with a Lax pair
  • I have made a program to do this automatically
  • With these terms, build generic Lax pair with constant coefficients
  • Derive equations for these constants using the definition of a Lax pair
  • Solve these equations, then substitute to form concrete Lax pairs

• Adding Lax pairs, new and old, to an integrable systems database started by Thomas Hill in Maple

Future:

• Expanding the database from its current 30 systems
• Adding Lax pairs, Zero curvatures, and others to it
• Write software to automatically derive Lax pairs

Lax Pair Derivation of Korteweg de Vries Equation

1. derive the following scaling for the KdV equation:
   \[ u_t = u_{xxx} \]
   \[ \lambda \text{ is a scaling of } u \text{ and } x \text{ by } \lambda \]
   \[ \lambda s = \lambda^3 x, \lambda x \]

2. Consider the Lax pair
   \[ L = D_x^2 - u \]
   \[ M = 4D_x^3 - 6uD_x - 3u \]
   where
   \[ \lambda \text{ is a scaling of } u \text{ and } x \text{ by } \lambda \]

3. The scaling of \( u_x = u^2 \) can be found from the generic scaling \( x \to \lambda x, u \to \lambda u \)
   • Note that since \( \lambda x, u \)
   • Substitute: \( \lambda^2 \) \( u_x = \lambda^2 u \]
   • For this equation to be unchanged we need \( \lambda^2 a = \lambda^2 b \), and therefore \( -2a = 2b \).
   • Fix \( a = -1 \) and \( b = 2 \). Then, \( x \to \lambda x, u \to \lambda u \)
   • Substitute to obtain: \( \lambda^4 u_x = \lambda^4 u_0 \) or \( \lambda^4 (u_x - u^2) = 0 \)

4. Finally, note that the weight of this equation is 4

Definitions:

• A Lax Pair is a pair of differential operators \( L \) and \( M \) that satisfy
  \[ [L, M] = L_t \]
  where \([,]\) is the commutator giving \([L, M] = L M - M L \)

• A differential operator is of the form
  \[ a_0 + a_1 D_x + \ldots + a_k D_x^k \]
  for some \( k \), where
  \[ a_i = a_{i,1} x + a_{i,2} x^2 + \ldots \]
  and where \( u = u(x) \)

• A Scaling Symmetry for a differential equation is a scaling of the dependent and independent variables that preserves the equation.

• Given a scaling, we define the weight of a variable to be the power of \( \lambda \) under the scaling.

• We say the weight of a differential operator is \( y \) if the weight of each term is \( y \)

Examples:

• The Korteweg de Vries equation is a differential operator given by
  \[ u_t = 6 u u_x - u_{xxx} \]
  with Lax pair \( L = D_x^2 - u \) and \( M = 4D_x^3 - 6uD_x - 3u \)

• The scaling of \( u_x = u^2 \) can be found from the generic scaling \( x \to \lambda x, u \to \lambda u \)
  • Note that since \( \lambda x, u \)
  • Substitute: \( \lambda^2 \lambda u_x = \lambda^2 \lambda u \]
  • For this equation to be unchanged we need \( \lambda^2 a = \lambda^2 b \), and therefore \( -2a = 2b \).
  • Fix \( a = -1 \) and \( b = 2 \). Then, \( x \to \lambda x, u \to \lambda u \)
  • Substitute to obtain: \( \lambda^4 u_x = \lambda^4 u_0 \) or \( \lambda^4 (u_x - u^2) = 0 \)

• Finally, note that the weight of this equation is 4