RÉNYI ORDERING OF TOURNAMENTS

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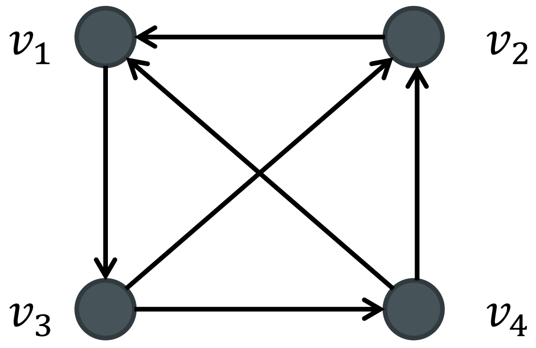
TOURNAMENTS

Definition: A *tournament* $\mathrm{T} = (V, A)$ consists of a set V of vertices and a set A containing exactly one directed arc (x, y) between each pair of vertices x and y. If (x, y) is an arc, we say x *beats* y, written $x \to y$.

Definition: Let $V = \{v_1, \ldots, v_n\}$. Then the *score* s_i of vertex v_i is the number of vertices in T beaten by v_i, and the *score sequence* s of T is the ordered n-tuple $(s_1, \ldots, s_n)$.

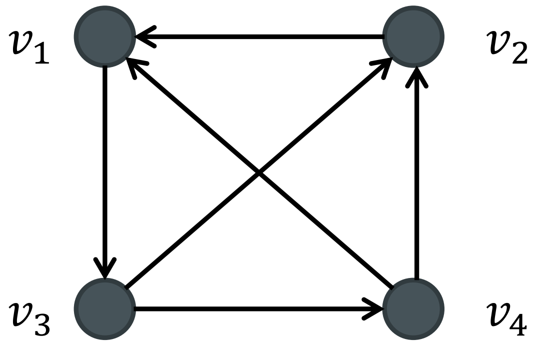


has score sequence $(1, 1, 2, 2)$.

ADJACENCY AND LAPLACIAN MATRICES

Definition: The *adjacency matrix* $A = (a)_{ij}$ of T is defined by and $a_{ij} = 1$ if $v_i \to v_j$ and $a_{ij} = 0$ otherwise.

Definition: The *Laplacian matrix* L is given by $D - A$, where $D = (d)_{ij}$ is the diagonal matrix with $d_{ii} = s_i$ for $i = 1, \dots, n$.



$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & 0 & -1 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 2 & -1 \\ -1 & -1 & 0 & 2 \end{bmatrix}.$$

LANDAU'S h

Definition: A tournament is called *transitive* if it has the property that if $x \to y$ and $y \to z$, then $x \to z$.

- H. G. Landau, a pioneer in tournament theory, defined a measure of how closely a tournament T resembles the transitive tournament on a scale from 0 to 1, given by

$$h(T) = \frac{12}{n^3 - n} \sum_{i=1}^{n} \left(s_i - \frac{n-1}{2} \right)^2,$$

where $n = |V|$ and $s = (s_1, \ldots, s_n)$ is the score sequence of T.

- The different values of h partition the set of tournaments on n vertices into roughly $\frac{1}{4}\binom{n+1}{3}$ equivalence classes.

ENTROPY MEASURES

- Let $p = (p_1, \dots, p_n)$ be a discrete probability distribution, that is, each p_i is nonnegative and $\sum p_i = 1$. The Shannon entropy $S(p)$ and Rényi α-entropy $H_\alpha(p)$ are defined by

$$S(p) = \sum_{i=1, p_i \neq 0}^{n} p_i \log_2 \frac{1}{p_i};$$

$$H_\alpha(p) = \frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{n} p^\alpha \right), \alpha > 1.$$

GRAPH ENTROPY

- In quantum mechanics, the *von Neumann entropy* is of interest, which is obtained by treating the *spectrum,* or multiset of eigenvalues, of the positive semidefinite density matrix as a discrete probability distribution, and applying the formula for the Shannon entropy.
- A simple graph, unlike a tournament, has a positive semidefinite Laplacian matrix.
- We ask if the entropy formulas can tell us something about tournaments even though the eigenvalues are often complex.

TOURNAMENT ENTROPY

- Call $\bar{L} = \frac{1}{\operatorname{tr} L} L$ the *normalized Laplacian matrix* of T. Because $\bar{L}$ has trace 1, its spectrum $\{\lambda_i\}_{i=1}^{n}$ satisfies $\sum \lambda_i = 1$, so we define the von Neumann entropy and Rényi α-entropy of a tournament T by

$$S(T) \coloneqq \frac{1}{\ln 2} \sum_{i=1, \lambda_i \neq 0}^{n} \lambda_i \operatorname{Log} \frac{1}{\lambda_i};$$

$$H_\alpha(T) := \frac{1}{1-\alpha} \log_2 \left(\sum_{i=1}^{n} \lambda^\alpha \right), \alpha > 1,$$

where Log is the principal value of the complex logarithm.

- It's often convenient and necessary to consider the simpler Rényi α-entropy*:

$$H_\alpha^*(T) := -\sum_{i=1}^{n} \lambda_i^\alpha, \alpha > 1.$$

COMBINATORIAL APPROACH TO RÉNYI ENTROPY

- If α is an integer, then the matrix $\overline{L}^\alpha$ has spectrum $\{\lambda_i^\alpha\}_{i=1}^n$.
- Also, for any tournament on n vertices, $\operatorname{tr} L = \binom{n}{2}$.
- Therefore, the Rényi α-entropy* can be expressed simply as

$$\begin{aligned} H_\alpha^*(T) &= -\operatorname{tr} \overline{L}^\alpha \\ &= -\frac{\operatorname{tr} L^\alpha}{\binom{n}{2}^\alpha}. \end{aligned}$$

- Thus for fixed α and n, all of the information about the Rényi α-entropy is contained in the integer $\operatorname{tr} L^\alpha$.

COMBINATORIAL RÉNYI 2-ENTROPY

- For $\alpha = 2$, we focus on the value of $\mathrm{tr}L^2$. By the linearity of the trace operator,

$$-H_\alpha^*(T) = -\frac{\mathrm{tr}L^2}{\binom{n}{2}^2} = -\frac{\mathrm{tr}(D - A)^2}{\binom{n}{2}^2}$$

$$= \binom{n}{2}^{-2} (\mathrm{tr}D^2 - 2\mathrm{tr}DA + \mathrm{tr}A^2)$$

$$= -\binom{n}{2}^{-2} \mathrm{tr}D^2$$

$$= -\binom{n}{2}^{-2} \sum_{i=1}^{n} s_i^2.$$

- Therefore, the Rényi 2-entropy* can be expressed entirely in terms of the score sequence. $\alpha = 3$ gives a similar result.

RÉNYI α-CLASSES

- The idea is to use Rényi α-entropy* to compare tournaments on the same number of vertices, using a refinement structure.
- We say T and T' are in the same *Rényi 2-class* if they have the same Rényi 2-entropy*.
- We say T and T' are in the same *Rényi 3-class* if they are in the same Rényi 2-class and have the same Rényi 3-entropy*.
- In general, we say T and T' are in the same *Rényi α-class* if they have the same Rényi k-entropy for $k = 2, 3, \ldots, \alpha$.

THE RÉNYI ORDER

- We can then order the tournaments lexicographically according to their Rényi α-entropy*, as α goes from 2 to ∞.
- In other words, T precedes T' in the Rényi order if
  1. $H_2^*(T) > H_2^*(T')$, or
  2. T and T' are in the same Rényi $(\alpha - 1)$-class and $H_\alpha^*(T) > H_\alpha^*(T')$.

RÉNYI VS. LANDAU

Observation: *T and T' are in the same Rényi 2-class iff they have the same h value.*

Proof: We simply write $h(T)$ as a function of $H_2^*(T)$.

$$h(T) = \frac{12}{n^3 - n} \sum_{i=1}^{n} \left(s_i - \frac{n-1}{2} \right)^2$$

$$= \frac{12}{n^3 - n} \left(\sum_{i=1}^{n} s_i^2 - (n-1) \sum_{i=1}^{n} s_i + \frac{n(n-1)^2}{4} \right)$$

$$= \frac{12}{n^3 - n} \left(-H_2^*(T) - (n-1) \binom{n}{2} + \frac{n(n-1)^2}{4} \right).$$

RÉNYI VS. LANDAU

	7 vertices	8 vertices
Tournaments*	456	6880
Score sequences**	59	167
Landau *h* equivalence classes	15	21
Rényi 2-classes	15	21
Rényi 3-classes	56	145
Rényi 4-classes	165	778
Rényi 5-classes	270	2152
Rényi 6-classes	334	4176
Rényi 7-classes	334	4664

*Up to isomorphism
**Up to permutation

RESULTS

Lemma: *For every ordered* n*-tuple* $(c_1, \ldots, c_n) \in \mathbb{C}^n$*, there exists a unique multiset* $S \subset \mathbb{C}$ *with* $|S| = n$ *such that for each* $k \in [n]$*, we have*

$$\sum_{\lambda \in S} \lambda^k = c_k.$$

Theorem: *If* T *and* T' *are in the same Rényi* $(n-1)$*-class, then their normalized Laplacian matrices have the same spectrum.*

- This means that there is no more refinement after $\alpha = n - 1$. If T and T' are in the same Rényi $(n-1)$-class, then they have the same Rényi α-entropy* for all α.

RESULTS

Definition: A tournament on $n = 2k + 1$ vertices is *regular* if $s_1 = s_2 = \cdots = s_n = k$.

Definition: A tournament on $n = 2k$ vertices is *semiregular* if $s = (k-1, \dots, k-1, k, \dots, k)$.

Theorem: *If n is odd, then all regular tournaments maximize Rényi 2-entropy* and Rényi 3-entropy*.*

If n is even, then all semiregular tournaments maximize Rényi 2-entropy and Rényi 3-entropy*.*

Theorem: *The last tournament in the Rényi order is the transitive tournament. Furthermore, the transitive tournament minimizes Rényi 2-entropy* and 3-entropy*.*

RESULTS

Definition: A regular tournament on $n = 4k + 3$ vertices is called *doubly regular* if every pair of vertices beats k common vertices.

Theorem: *If* $n \equiv 3$ *(mod 4), then all doubly regular tournaments are in the same Rényi* $(n-1)$*-class, and are first in the Rényi order.*

- Rényi 2- and 3-entropy* can't distinguish between regular tournaments.
- However, a regular tournament T achieves maximal Rényi 4-entropy among regular tournaments if and only if T is doubly regular.

FURTHER QUESTIONS

- Can Rényi $(n-1)$-entropy* distinguish between over half of all tournaments for any n?
- What proportion of score sequences can be distinguished by Rényi 3-entropy*?
- Are there tournaments with different score sequences in the same Rényi $(n-1)$-class?
- Does the transitive tournament minimize all Rényi entropies and the von Neumann entropy?
- Do (doubly) regular tournaments maximize the von Neumann entropy as well?

THANK YOU