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Assessment of College Students' Understanding of the Equals Relation: Development and Validation of an Instrument

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ASSESSMENT OF COLLEGE STUDENTS' UNDERSTANDING OF THE EQUALS

RELATION: DEVELOPMENT AND VALIDATION OF AN INSTRUMENT

by

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ABSTRACT

Assessment of College Students' Understanding of the Equals Relation:

A Development and Validation of an Instrument

by

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Research indicates that many elementary students do not comprehend that the equal sign is an indication that an equality relation exists between two structures. Instead, they perceive the equal sign as an indication that a particular procedure is to be performed. As students mature, and as their exposure to the equal sign and equality relations in multiple contexts increases, most obtain the ability to interpret the equal sign as an indicator of an equivalence relation. Incorrect usages of the equal sign, however, by post-algebra students indicate a tendency for students to regress back to a comprehension of the equal sign as an operator symbol or to ignore the equal sign altogether.

The purpose of this project was to develop an instrument that is relevant to objectives associated with the interpretation of the equals relation, and to perform a test reliability analysis to assess measurement reliability and construct validity for the instrument. The model that was utilized to develop items for the instrument followed a general item development and validity assessment model proposed by Cangelosi. This model requires an iterative process that includes a peer review of objectives and instrument items by a panel of experts and a revision of the items based upon recommendations from the panel. A pilot test was synthesized from the revised items and administered to a group of subjects, and an instrument reliability analysis and an item efficiency analysis were performed. The quantitative and qualitative data obtained from this process were used to create the 18-item instrument entitled, *Wheeler Test for Comprehension of Equals*. The researcher recommends further validity assessments for the instrument across multiple settings and subject groups.

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Gregory D. Wheeler

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CHAPTER I

INTRODUCTION

Background and Rationale

Increased high school graduation standards and college admissions criteria are requiring students to take more mathematics classes than ever before (Planty, Provansik, & Daniel, 2007). Schoenfeld (1995) believed that algebra is a new literacy requirement for citizenship: “If one does not have algebra, one cannot understand much of science, statistics, business, or today’s technology” (p. 11). Algebra courses now function as a gatekeeper for students to take the higher-level math and science classes necessary to participate and succeed in higher education (Bass, 2006; Viadero, 2005). But studies suggest that students in the United States generally struggle to obtain mathematical understanding beyond basic arithmetic, algebraic, and geometric skills and processes (Brown et al., 1988).

Most students are not able to apply basic skills and procedures in problem solving situations because they lack understanding of the structures that define and explain these skills and processes (Brown et al., 1988). Students in the United States generally believe that learning mathematics is an exercise in memorizing rules and procedures and using those rules and procedures to derive correct answers to numerical problems (Brown et al., 1988). This misconception about what constitutes mathematics is prohibitive to the study of algebra and other subjects dependent upon algebraic understanding. When students believe that a mathematical expression represents a string of operations that are to be performed, they encounter a conflict with implicit objectives of algebra that require a view of the expression as an object that can be manipulated. For students to learn and understand algebra they must have the ability to see a mathematical expression as a structure (Kieran, 1992; Sfard, 1991).

Sfard (1991) offered a theory about how mathematical concepts and relations are conceived by learners and the connection between procedural and structural comprehension of mathematical ideas. The theory suggests that development of a structural understanding of a mathematical notion must evolve in stages starting with a procedural understanding. Students first make sense of a mathematical notion by interpreting it as a process or "as a potential rather than an actual entity," and the capacity for a student to transition to a structural understanding where the notion is perceived as a "static structure" is done only with great difficulty, if it is done at all (p. 4). Although there is an ontological difference in classifying a mathematical notion as a thing or as a potential, the two interpretations are not mutually exclusive. Procedural and structural understandings are "complementary" and the ability of seeing a mathematical concept as both a process and an object "is indispensable for a deep understanding of mathematics" (p. 5).

Research has shown that procedural emphasis on arithmetic computations dominates the elementary math curriculum in the United States (Valverde & Schmidt, 1997/1998). Little or no attention to structural understanding can lead to misconceptions about the fundamental structure of arithmetic and impede a students' ability to understand algebraic concepts (Baroudi, 2006; McNeil & Alibali, 2005a). Because of the arithmetic dominated curriculum of most elementary schools, the notion of “equals” and the meaning of the equal sign are misunderstood by most elementary students. Most elementary students do not comprehend that the equal sign is an indication that an equality relation exists between two structures. Instead, they perceive the equal sign as an indication that a particular procedure is to be performed (Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Knuth, Stephens, McNeil, & Alibali, 2006; Molina & Ambrose, 2006; McNeil & Alibali, 2005b). This misunderstanding can be attributed to the emphasis of arithmetic learning in elementary math education and use of the equals sign in expressing arithmetic procedures. Students’ misunderstanding of the equals relation is also reinforced and perpetuated by problem sets and instructional materials found in most elementary and middle school textbooks in the United States (Capraro, Ding, Matteson, Capraro, & Xiaobao, 2007; McNeil et al., 2006).

The problems that misconception of the equals relation pose for learning mathematical notions cannot be overstated. In algebra, students are introduced to properties and structures of arithmetic operations primarily through use of equations. Equations are used in algebra to indicate which mathematical objects, written in different forms, are the same. If students misinterpret the intent and meaning of the equals sign, then an equation has no value as a means of helping students learn the relation proposed by the equation. In order for students to have a structural understanding of many mathematical notions, they need to have an appropriate understanding of the equals relation.

As students mature and their exposure to the equal sign in multiple contexts increases, most obtain the ability to interpret the equal sign as an indicator of an equivalence relation. This is evidenced by the fact that most high school students are able to accept equality statements containing multiple operations on each side (Herscovics & Kieran, 1980; McNeil & Alibali, 2005b). It is not clear, however, whether this ability to interpret the equals sign in terms of an equivalence relation develops into an understanding of equivalent equations in algebra or calculus (Kieran, 1981).

Problem Statement

Incorrect usage of the equal sign by post-algebra students as they solve equations or calculate derivatives indicates a tendency to regress back to a comprehension of the equal sign as an operator symbol. It is possible that such misuses of the equal sign are simply careless mistakes made by students because they lack knowledge of an appropriate notation. Is it appropriate, however, to assume that college students correctly interpret the equals sign in all contexts? Should we assume that because a college student has completed many years of math, including an algebra curriculum, that their understanding of the equals relationship is sufficient to allow them to succeed in their continued pursuit of mathematical learning?

The studies that have been done thus far have focused on student understanding of the equals relation in very specific contexts. The instruments used in these studies presented participants with various equations and then asked the participants to interpret the meaning of the equal sign in those equations. The responses were then classified as appropriate if the participants gave a relational interpretation and inappropriate if the interpretation was procedural. There were no studies found that measured understanding of the equals relation beyond interpretation of the equal sign, or instrument validation studies on measurement of the equals relation.

Because the process of transitioning from a procedural to a structural understanding in mathematics is difficult, and because research suggests misinterpretation of the equal sign by most pre-algebra students, it is important for researchers to find out what post-algebra students understand about the equals relation in contexts relevant to algebraic learning. In order to assess understanding about the equals relation we must determine what constitutes an understanding of the relation. This is done by systematically identifying a set of specific learning objectives that are relevant to the overall learning goal of comprehending the equals relation (Cangelosi, 2000). Once these objectives are derived, an instrument can be developed which is relevant to these objectives and which measures student achievement relative to the objectives.

Purpose of the Study

The purpose of this research was to: (a) identify and classify specific learning objectives that define what behaviors will be expressed by a student who correctly comprehends the equals relation and has the ability to correctly interpret the relation in different contexts; (b) develop an instrument that is relevant to the identified objectives; and (c) perform a test reliability analysis to assess measurement reliability, construct validity, intraobserver consistency, and interobserver consistency.

CHAPTER II

LITERATURE REVIEW

Introduction

This chapter reports published findings related to student understanding of the equals relation and current efforts to develop instruments to measure student understanding of equals. The purpose of this review was to determine: (a) what is known about student understanding of the equals relation, (b) what is known about the reasons for student misconceptions of the equals relation, and (c) research-based procedures that can be applied to develop and analyze a relevant instrument to measure student understanding of the equals relation.

Student Understanding of Equals

The equals relationship is typically introduced during students' elementary education. However, little instructional time is spent on the relation during later years even though the notion of equals is fundamental in the study of mathematics at all levels (Knuth et al., 2006). Students are left to interpret the meaning of equals and the use of the equal sign based upon their personal experiences (McNeil & Alibabli, 2005a). Many studies suggest that the equals relation is difficult for students to understand and the procedural misinterpretation of the equal sign is prevalent.

Preschool children generally display an intuitive understanding of the equals relation that is based on their ability to count the elements in a set and compare the cardinalities of two sets. Gelman and Gallistel (1978) proposed two strategies that can be used to determine that two finite sets are equivalent. One strategy is to count the number of elements in each set and show the sets yield the same cardinality. The other strategy involves a demonstration that a one-to-one correspondence exists between the elements of a set. Gellman and Gallistel used their "magic experiments" to show that preschool children require that decisions about equivalence be based upon the cardinalities of sets rather than the establishment of a one-to-one correspondence between them. "The practical decision about whether [the equivalence relation] holds or not rests on counting. If counting yields identical representations for the numerosities of the two sets, the sets are judged to satisfy the equivalence relation" (Gelman & Gallistel, 1978, p. 164).

Gellman and Gallistel (1978) also showed that the ability to count and compare the number of elements in sets is usually followed by the ability to count the total number of elements that belong to two sets and report the cardinality of the union of the sets. At first, preschool children can only evaluate the outcome of addition by combining the elements from two disjoint sets and counting the number of elements in the union. As children continue to receive training at home and at school, however, they progress through different strategies until they can perform addition operations from memory. Kieran (1981) believed that the ability to count and report the number of elements in a union of sets leads preschool students to an operator notion of equality that emphasizes the *result* of the arithmetic operation. Kieran suggested that preschoolers have two interpretations of equality: (1) children interpret two sets as equal if they can count elements in the sets and determine that they have the same cardinality; (2) children interpret equals as an operator that is an indication that numbers need to be combined to produce a result. Many studies show that it is the second interpretation—the interpretation of the equal sign as a "do something signal"—that is resorted to when the symbol is used in different mathematical contexts" (p. 318).

Behr and colleagues (1980) investigated children's understanding of equals through nonstructured individual interviews with children from 6 to 12 years of age. In these interviews the researchers presented children with number sentences and asked the children to interpret the meaningfulness of the number sentences and the meaning of the "+" and "=" symbols within those sentences. They found that almost all of the children that were interviewed viewed equality as an operator and interpreted the equals sign as an indication to perform an operation. During these interviews the children demonstrated "an extreme rigidity" in their insistence that number sentences need to be written in a particular form (p. 15). The children viewed equations such as *2=2* or □ = 3+2 as incomplete, wrong, or meaningless and insisted that such equations be changed by reordering or including more numbers or addition operations in order for the statements to be meaningful. The researchers determined that the children were entirely focused on what actions were to be performed when presented with the number sentences and they gave no attempt to "reflect, make judgments, or infer meanings" when considering the number sentences (p. 15).

Rittle-Johnson and Alibali (1999) conducted a study with fourth- and fifth-grade students to assess the causal relations between children's structural and procedural knowledge of equivalence. The children were asked to solve standard equivalence problems of the form $a+b+c=a+$. to assess their ability to solve such problems prior to receiving specific instruction related solving such problems. After the students received the instruction—in the form of procedural or conceptual instruction—the students were asked to solve standard equivalence problems that differed from the pretest problems. The differences reflected changes made in the operation used in the equation or the position of the blank in the equation. Along with solving equivalence problems, they were also asked to evaluate and rate three correct and three incorrect proposed procedures for solving such problems. This study suggests that most children of this age group understand what it means for quantities to be equal, but there is still an incomplete understanding of the equals relation and the structure of equations.

Even when students encounter the equals relation in multiple contexts as they progress through their formal education, they do not fully grasp the complete, relational meaning of the equal sign (Rittle-Johnson & Alibali, 1999). The lack of an appropriate relational understanding of the equal sign can become a handicap to students as they transition from arithmetic to algebra. A study by Knuth and colleagues (2006) found that almost half (141 out of 300) of the sixth-, seventh-, and eighth-grade students proposed an operational definition for the meaning of the equal sign in an algebraic expression. Much fewer than half of those same students (106 out of 300) proposed an appropriate relational definition. The study also showed that students who did have an appropriate understanding of the equal sign were more likely to utilize an appropriate strategy to solve a basic linear equation. This association between understanding of the equal sign and the ability to solve algebraic equations was even manifested after controlling for mathematical ability as measured by standardized achievement scores.

As students progress through middle school and high school they begin to encounter the equality relation in nonarithmetic contexts. These contexts include algebraic equations and scientific relations that require a relational interpretation of the equal sign. These encounters often contradict understanding of the equals sign and force students to change their interpretation of equals (Kieran, 1981). McNeil and Alibali (2005a) found that most junior high school students, when exposed to an equal sign in a "typical addition context," held onto an operational interpretation of the symbol (pp. 290-291). When the context changed, where a relational interpretation of the equal sign was required, the students were able to interpret the equal sign appropriately. The study also found that college students who had completed at least one semester of calculus were able to give a relational interpretation of the equal sign in all three of the contexts that were studied. The authors concluded that with increased exposure to the equal sign in contexts that require a relational interpretation, students eventually supplant the operational interpretation of equals with an appropriate relational interpretation. This study, however, only assessed student's interpretation of the equal sign in the three basic contexts and did not provide evidence that these same students correctly interpreted the equals relation in all contexts they may be exposed to within even a basic algebra course.

There is also evidence that high school and college students misunderstand the equals relation as they solve equations or evaluate expressions in algebra and calculus (Byers & Herscovics, 1977; Clement, 1982). Students erroneously use the equal sign as they write out procedures to solve story problems or calculate the derivative of a function. Even after students have acquired a basic relational interpretation of the equal sign, there are still tendencies to view the equal sign as an operator symbol that indicates the result of an operation (Kieran, 1981).

Reasons for Student Misconceptions of the Concept of Equals

There is some contention as to why students hold so strongly to the operator interpretation of equals instead of a relational interpretation. Baroody and Ginsburg (1983) identify two views regarding the foundation of the understanding of equals. One view is that the operator interpretation is a result of student's early arithmetic training. Throughout their experiences in elementary school, students repeatedly encounter the term "equals" and the equal sign in the context of arithmetic operation problems such as addition, subtraction, or multiplication where there is no need to interpret the equals sign with a relational perspective suggesting equivalence. This encourages student interpretation of equals as "the answer" and the equals sign as a proclamation of the result of an arithmetic operation (Knuth et al., 2006; McNeil & Alibali, 2005b; Weaver, 1973). According to this view, students will be able to obtain a relational understanding of equals if the nature of the instruction they receive emphasizes this understanding.

A second view is that preference for the operator interpretation of equals is the result of cognitive limitations of children. Collis (1974) has observed that children between the ages of 6 and 10 years require "closure" when dealing with unevaluated operations, i.e. they need to see a result before the operations on the numbers are meaningful. Some children are simply unable to comprehend equations such as $2+2=5-1$

and are, therefore, unable to incorporate a relational understanding of equals in all contexts. According to this view, the cognitive limitations of children prohibit them from being able to obtain a relational interpretation of equals. Changing instructional experiences related to the concept will do little to change their interpretation of equals as an operator (Kieran, 1981).

Studies have shown that both views hold some truth as to why children hold so strongly to an operational interpretation of equals. Baroody and Ginsburg (1983) studied a group of students who had participated in the Wynroth (1975) curriculum. This curriculum defined equals as “the same as” and provided students with instruction where they experienced a variety of equation sentence forms. Researchers presented the children with many equality sentences, one at a time, and asked them a series of questions to determine their understanding of equals in different contexts. The results suggest that the Wynroth program was successful in developing a relational understanding of equals. The majority of the participating first grade children did consider equations such as $2+2 = 5-1$ as sensible thus contradicting the assertions of Collis (1974) and Kieran (1981), that cognitive limitations prevent children of this age group from obtaining a relational understanding of equals. The study also showed that while the Wynroth curriculum did promote a basic relational understanding of equals, this understanding was often in conflict with an operator understanding and a relational understanding would succumb to an operator understanding in some contexts. This suggests that a cognitive barrier to a relational interpretation of equals does exist and that “both cognitive and instructional factors contribute to a child’s view of equals as an operator symbol” (Baroody &

Ginsburg, 1983, p. 209).

A study by Seo and Ginsburg (2003) suggested that use of curriculum intent on exposing children to the equals relation and use of the equals sign in different contexts does not always encourage a relational interpretation in all contexts. Second-grade students, who were exposed to the equal sign in a variety of contexts in their mathematics class, were tested on their understanding of the equal sign. The results showed that students did obtain a relational understanding of equals, but they also retained an operator view of equals in certain contexts. The students would rely upon one interpretation or the other in a given context and did not have the ability to make a connection between the two views.

Even with some debate concerning the cognitive ability for children at different ages to obtain a relational interpretation of the equal sign, these studies all suggested that there remains a tendency for students to hold to an operator interpretation within at least some contexts. McNeil and Alibali (2005a) contend that this is due in large part to early mathematical experiences dominated by an emphasis on arithmetic operations. Their study showed a negative correlation between adherence to operational patterns prevalent in arithmetic and their ability to learn procedures for solving algebraic equations. The study also tested a group of college students who were randomly selected to receive a computer mediated stimulus that activated their knowledge of arithmetic operational patterns. The study found that students who received the stimulus were less likely to utilize appropriate strategies when solving a set of equations than the students who did not receive the stimulus.

A recent study by Capraro and colleagues (2007) examined methods books used to prepare U.S. elementary preservice teachers and student mathematics textbooks for first- through sixth-grade students. The authors found very little background information regarding the equal sign or the definition for equal. The authors also studied a set of first- through sixth-grade textbooks from China. All of these books introduce the equal sign in conjunction with greater than and less than signs before introducing the concepts of addition and subtraction. The textbooks also encouraged teachers to teach the equals sign within relational contexts. The Chinese textbooks also provide students with operations to be performed without the inclusion of an equal sign and continuous operations where arrows are used instead of equal signs to indicate the transition from one completed operation to the next. Capraro and colleagues also supported the findings of other studies (Knuth et al., 2006; McNeil & Alibali, 2005a) in that a sample of U.S. sixth-grade students had many misconceptions about the equal sign. Their study examined a sample of sixth-grade students from China and found that students were able to correctly interpret the equal sign as a relational symbol, which may be an indication of pedagogical differences in the two countries regarding the approach to teaching the equals relation.

Validation Methods

Measurement validation studies typically utilize research methods from classical test theory, correlation analysis, and classification of learning objectives. These methods arise from theories that enable quantitative and qualitative judgments concerning the validity of an instrument. There are also models proposed by researchers to guide the development of relevant measurement items (Cangelosi, 2000; Ebel, 1965; Gronlund & Linn, 1990; Popham, 1981).

Classical Test Theory

Classical test theory (CTT) was initially developed in the early 1900s as a way to explain why some tests gave more consistent results than others (National Research Council, 2001). Classical test theory is based on the model that individual ability (true score) is the sum of the measured ability on an instrument (observed score) and a measurement error (reliability error). The assumptions of CTT have been used to develop many measurement tools such as reliability statistics, standard error estimation formulas, and test equating practices that are used to link scores from one test to scores on another. In measurement validation studies, this simple theory provides the justification and framework for researchers to approximate the reliability error by computing reliability coefficients using a variety of different formulas.

Correlation Analysis

In statistics, the correlation between two random variables is a theoretical parameter with a value between -1 and 1 that indicates the strength and direction of the linear relationship that exists between variables (Cohen, 2001). Major contributions to correlation statistics were made by Sir Francis Galton (England 1822-1911) and Karl Pearson (England 1857-1936) as they tried to quantify hereditary influences by comparing physical attributes of children to their parents (Freedman, Pisani, & Purves, 1998). A result of their studies was the formulation of a statistic—the Pearson product-moment correlation coefficient—that could be used to approximate the correlation that describes the linear relationship between separate but paired random variables. Today, the Pearson product-moment correlation coefficient is still the prevailing statistic used to approximate the linear relationship between two random variables. The calculation of a correlation coefficient is an essential part of any validation study where a coefficient is calculated as a measure of an instrument's reliability (Cangelosi, 2000).

Classification of Learning Objectives

When assessing the relevance of an instrument, it is essential that evidence be collected to describe the congruity between what an instrument measures and what the instrument is intended to measure. According to Cangelosi (2003), an appropriate statement for a learning objective should be such that the content and the learning level—"the manner in which students will mentally interact with the objective's mathematical content once the objective is achieved"—is inherently understood by the reader of the objective (p. 166). Bloom's taxonomy of educational objectives is a famous example of a model that can clarify learning levels by classifying different behavioral constructs. Bloom's taxonomy is a categorization of educational objectives that serves to improve "the exchange of ideas and materials among test workers, as well as other persons concerned with educational research and curriculum development" (Bloom, Engelhart, Furst, Hill, & Krathwohl, 1956, p. 264). This taxonomy provides the researcher with a tool to categorize an educational objective into a domain (i.e., cognitive, affective, or psychomotor) and a learning level within each domain. This taxonomy can be a useful organizer for judgments about how well items on an instrument pertain to the content, domain, and learning level of the objectives they are intended to measure.

Cangelosi (2003) has proposed a taxonomy of learning levels specified by objectives similar to Bloom's but which "takes into account the need for inquiry instruction from the constructivist perspective, as well as for direct instruction for skill building" (p. 166). This particular classification scheme may be useful for the development of instruments used to measure mathematical understanding. Table 1 is an outline of Cangelosi's taxonomy.

Table 1

Cangelosi's Scheme for Categorizing Learning Levels Specified by Objectives

Domain	Objectives
1. Cognitive	A. **Construct a concept**—Students achieve an objective at the construct-a-concept level by using inductive reasoning to distinguish examples of a particular concept from nonexamples of that concept.
	B. **Discover a relation**—Students achieve an objective at the discover-a-relation learning level by using inductive reasoning to discover that a particular relationship exists or why the relationship exists.
	C. **Simple knowledge**—Students achieve an objective at the simple-knowledge learning level by remembering a specified response (but not multi-step process) to a specified stimulus.
	D. **Comprehension and communication**—Students achieve an objective at the comprehension-and-communication level by (i) extracting and interpreting meaning from an expression, (ii) using the language of mathematics, and (iii) communicating with and about mathematics.
	E. **Algorithmic skill—**Students achieve an objective at the algorithmic-skill level by remembering and executing a sequence of steps in a specific procedure.
	F. **Application—**Students achieve an objective at the application level by using deductive reasoning to decide how to utilize, if at all, a particular mathematical content to solve problems.
	G. **Creative thinking—**Students achieve an objective at the creative-thinking level by using divergent reasoning to view mathematical content in unusual and novel ways.
2. Affective	A. **Appreciation**-Students achieve an objective at the appreciation level by believing the mathematical content specified in the objective has value.
	B. **Willingness to try**-Students achieve an objective at the willingness to try level by choosing to attempt a mathematical task specified by the objective.

Models for Developing Relevant Measures

Ebel (1965) suggested that relevance of an instrument cannot be measured statistically, but is a matter of logical analysis and expert judgment. “Relevance must be built into the test. What a test actually does measure is determined by the test constructor as he works, step by step, to build the test” (p. 390). All decisions made by the constructor of the instrument at each step of development will determine the relevance of the instrument.

The models used by Fodor-Davis (1993) and Rowley (1996) to develop relevant test items included procedures that utilized the input from subject matter experts in a peer review process. These models were inspired by Cangelosi’s (1990), Ebel’s (1965), and Popham’s (1981) proposals for a string of steps that promote the opportunity for relevance to be built into the test items. These models initially require the identification of specific behaviors to be examined. The objectives are then catagorized by specific learning level as defined by a learning level catogorization scheme. Next, the objectives are submitted to experts within a related field and revised according to recommendations. After the objectives had been determined and weighted, the researchers develop prompts and scoring rubrics relevent to those objectives. The development of test items is followed by a peer review and revision based upon recommendations from the subject experts. The test items are then compiled into an instrument which is administered and scored. Further revisions of the test items are then made based upon quantitative and qualitative data obtained through.reliability analysis, item efficiency analysis, and participant review of items.

Summary

Research suggested that the equals relation is difficult for students to understand. Many studies have demonstrated the inability of elementary students to correctly interpret the equal sign as an indication of an equivalence relation. Studies do indicate that as students get older they begin to interpret the equal sign correctly within certain contexts, but the contexts that have been studied are far fewer than those experienced by even a beginning algebra student.

Research provided little information regarding college student's interpretation of the equal sign and their understanding of the equals relation in contexts of algebraic identities, graphs of equations, function notation, set theory, and the difference between equals and equivalent. An exhaustive search through literature provided no evidence that an instrument exists for measuring understanding of the equals relation in multiple contexts relevant to algebraic understanding exists. The development of such an instrument is an important step in evaluating mathematical understanding. Using methods and procedures prescribed by researchers in the field of psychometrics, it is possible to develop a relevant measure of student understanding of the equals relation.

CHAPTER III

PROCEDURES

Overview of Instrument Development Model

The purpose of this project was to develop and validate an instrument that assesses student understanding of the equals relation. The model that was utilized to develop items for the instrument followed a general item development and validity assessment model proposed by Cangelosi (2000). This model provided “a practical system for developing valid and usable measurements” (p. 254). The first step was to identify objectives that define understanding and ability related to the behavior being measured, and to classify the objectives according to a specified taxonomy. The second step was to develop test items that were relevant to the objectives. Each test item consisted of a prompt and a rubric for scoring the item. Step three was to submit instructional objectives and test items for expert review. The review process was utilized to evaluate how well the test items matched with their respective objectives. The fourth step was to revise the test items according to input from the expert review process. Step five was to compile the measurement and administer the measurement to a group of subjects. The final steps of this process included a test reliability analysis that measured the internal consistency of the measurement, and an item efficiency analysis to find weak items that were revised or eliminated.

Development of the Instrument

The researcher identified written objectives and classified the behavioral construct of each objective according to the scheme developed by Cangelosi (2003). Because the purpose of the instrument was to measure student comprehension of a mathematical relationship, all of the objectives were classified as comprehension level objectives. The researcher also determined objectives through examination of literature, interviews with peers, reflection of personal experiences in teaching algebra, and analysis of existing studies concerning student understanding of the equals relation.

An initial set of objectives was proposed to a panel of three subject-content experts who are experienced in teaching algebra at multiple levels. One member of the panel was an associate professor of mathematics education who has helped develop assessment items for secondary teachers in the state of Georgia. The second member of the panel was currently a mathematics lecturer who oversaw a university intermediate algebra course. The third member of the panel was a professor of math education at Utah State University and has written extensively about assessment and instrument development.

The panel members were first contacted by a letter that contained a brief description of the project along with a request to serve on the panel. After panel members agreed to participate as reviewers, they were provided with: (a) a set of guidelines for reviewing the proposed objectives and preliminary instrument items for assessment of the objectives, (b) a list of proposed objectives, (c) a set of preliminary instrument items used to assess the objectives, and (d) a list of questions that the reviewers were asked to respond to as they reviewed the objectives and preliminary items. The original contact letter, the guidelines for review, the list of proposed objectives, questions for guiding the review, and the preliminary set of items are presented in Appendix A.

During this initial review, the panel concluded that there was no way to determine for sure that the set of objectives represented a comprehensive description of understanding the equals relation. However, only one other objective was suggested for inclusion by the panel: *Given an equation, students interpret the equal sign as an indication that both sides are the same structure and not as a symbol that separates two structures.* The panel had no suggestions for removing or rewording any of the originally included objectives.

The panel members were intrigued by the preliminary items that they reviewed. The consensus was that the items showed promise as a means of measuring student understanding of the equals relation. They alleged that the items matched the objectives with regards to content but they were worried that there was an uncertain potential for construct invalidity. Their suggestion was to field-test these items to determine how students would respond, and determine student thought processes as they responded to the prompts during interviews. The suggestion was to start with the items that were initially presented and revise items as appropriate based on feedback from those field tests. After these revisions the panel would consider the data from the interviews and again review the items.

The panel also reported that all of the preliminary items would be rated as medium or hard items on an item-difficulty scale. They suggested that additional items should be written that would rate as easy on an item-difficulty scale. They recommended that new, easy-level items should be included with the field tests.

After additional items were written, the researcher obtained a sample of college students who were enrolled in a university mathematics or statistics course. Four students were selected based on mathematical experiences in higher education. One student had just completed an introductory statistics course and rated herself as a poor mathematics student. Another student was enrolled in an intermediate algebra course taught by the researcher and was doing B grade-level work in the course. The third student was enrolled in a science based calculus course taught by the instructor and was doing C grade-level work at the time of the interview. The fourth student was enrolled in the same calculus course and was doing A grade-level work at the time of the interview.

Because it is difficult to quantitatively assess the learning-level relevance and construct validity of the test items, the researcher attempted to assess these by meeting one-on-one with the students in the sample. The students were presented with the preliminary items and additional lower difficulty-level items that were written as suggested by the review panel. All items were presented one at a time and participants were instructed to respond to the prompts and to think out loud as they responded. After the participants responded to all of the prompts, the researcher pointed out any mistakes, identified appropriate responses, and requested further insight as to why participants responded the way they did. The participants were also encouraged to express their opinions concerning the instructions provided with each prompt and their perceived effectiveness of the individual items.

During the interviews, it was apparent that all of the students understood that equals is a relation that defines two entities as “the same.” It was also apparent that when working with equations, students often ignored the equals relations that were suggested and focused on computing; even when there was no indication in the prompt that there was anything to compute. The participant’s ability to correctly interpret an equals relation was contingent on their perceived notions of what they were “supposed” to be doing as they responded to each item. When directly asked during interviews, “What does the equals sign mean?” all four students responded that it meant things were “the same.” But when responding to some of the prompts they would ignore this understanding and proceed in a way consistent with how they thought they were supposed to respond.

This information was a justification for the changes made to some of the preliminary items on the survey. Some of the open-ended response items were changed to closed forms, such as multiple choice or true or false. When students responded incorrectly to open-form response items it was difficult to determine if those mistakes were due to a misunderstanding of the equals relation or if they simply ignored the equals relation as they responded in a way they thought they should. The closed forms encouraged participants to consider different ways that an equation can be interpreted. This increased the likelihood their responses would reflect their interpretations of the equals relations. These prompts made it less likely that the participant’s conditioned responses to dealing with mathematics would suppress their understanding of the equals relation.

One of the students who were interviewed showed an ability to recognize the equals relations in some of the equations that the other students did not. She mentioned the existence of the equals sign as she thought through her response to some of the prompts and her response was predicated on her understanding of the equals relation. The other students responded to the same prompts by computing and said nothing about the equals signs. When the researcher demonstrated errors in their responses to those prompts, they admitted that they would have responded differently if they had paid attention to the equals signs. In general, it is difficult to determine if equals signs encourage students to erroneously compute or if they are conditioned to compute and they simply ignore the equals sign. Students that were interviewed felt that they had simply ignored the equals signs. There is also evidence of this when comparing student's responses to fill-in-the-blank items where the equation is terminated by a constant to fill-in-the-blank items where the equation is terminated by a blank. Three of the students that were interviewed would erroneously compute all binomial expressions in an equation terminated by a blank (e.g. 2 + 3 = ____ - 1 = _____), but would correctly identify an equals relation for the equations that were terminated by a constant (e.g. 2 + 3 = ____ - 1 = _____ + 3). The students explained that they correctly responded to the equations ending in a constant because the constant at the end created a scenario that was obviously flawed if they attempted to compute binomial expressions. One student explained: "It didn't make sense if I did all of the math and then had it ended in plus 3, so I decided that I needed to make each part equal and then it made sense."

These observations demonstrated that for various students who have some understanding of the equals relation, their understanding contributes little to their interpretation of equations unless the equations are such that the suggestion of an equals relation cannot be ignored. Therefore, the researcher added another objective to the group: *Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.*

These interviews also influenced the researcher to adjust the number and format of the easy difficulty-level items so that they could serve two important functions in the instrument: (a) measure basic understanding of the equals relation in multiple contexts, and (b) discourage conditioned responses to the medium and hard difficulty-level items. These items were written with the intent to encourage interpretations of equations as relations and suppress the inclination to calculate. To measure the impact of these items on encouraging relational interpretations, an alternative instrument was created using the same medium and hard difficulty-level items but utilized different easy difficulty-level items. The easy difficulty-level items on the alternative instrument could be correctly addressed through computations, thus encouraging the conditioned inclination of students to calculate. A *t* test assuming unequal variances was then performed using two groups of students where the scores of one group taking the primary instrument were compared to the scores of a similar group who took the alternative instrument to see if the format of the easy difficulty-level items significantly increased scores on the more difficult items.

The revised items and objectives were synthesized into a preliminary instrument to measure student understanding of the equals relation and student response to stimuli associated with the equals sign. These objectives and the instrument are presented in Appendix B. The revised objectives and the preliminary instrument were again presented to the review committee. Each member of the committee was asked to judge the overall quality of the instrument items and determine if the items appeared to be an effective measure of the associated objectives. In this follow-up review, committee members provided information on: (a) clarity of instructions, (b) content relevance to corresponding objective, (c) construct relevance to corresponding objective, and (d) additional comments or suggestions concerning individual items or the measurement as a whole.

One member of the review panel expressed concerns about the wording of the prompt for item #10. Their opinion was that many students don't understand that in mathematics the word "expression" is used to signify a specific structure that can have many different forms. The item therefore becomes a measure of a student's understanding of the word "expression" rather than a measure of a student's interpretation of the equals relation. The item was revised, and the phrase "mathematical entity" was used in the place of "expression" in response (a).

All of the panel members expressed concern for the scoring rubrics for items #10 and #11. They believed that item #10 was flawed because responses (a) and (b) were both correct interpretations of the equation and that the implication of a "best" interpretation was inappropriate. Likewise, they felt that both responses (b) and (c) were correct interpretations of the equation given in the prompt for item #11. One member of the panel provided the following review of items 10 and 11.

> Although I like questions 10 and 11, there is something a bit disconcerting about them. I think it is that all three answers have truth and I'm not sure that students will interpret the word "best." How are students going to distinguish between two clearly true prompts? I'm not convinced that responses to these questions are

> going to provide much information…. Personally, I would select "A" on number eleven, not B. I am more confident in the wording of that response than part "B." In that choice I am less comfortable with my interpretation of "the same mathematical entity," worried that "the same" might mean "identical" when it comes to "entities."

I expressed my intention to use these prompts to determine how a student will likely interpret an arithmetic and algebraic equation; will then interpret the equation as a guideline for a specific calculation or will they interpret the equation as an expression of a relation?.To determine student interpretations of these equations, the panel suggested a distinction be made between a computational interpretation and a relational interpretation without discrediting a correct interpretation. One panel member explained: "I like [the prompt] as is and it can be used to discriminate between A-type correct responses [i.e. relational interpretation] and B-type correct responses [i.e. computational interpretation]." In an attempt to address this point, the researcher changed the scoring rubric so that a point was given for either of the correct interpretations. A secondary rubric was also created where the scoring of these items, along with two others, would be used to measure a student's inclination to interpret an equation as a relation compared to their inclination to interpret an equation as a guideline for a specific calculation.

The panel also suggested word changes in many of the prompts that improved the coherence of the prompts without changing the outlines or premises of the prompts. These suggestions included: (a) changes to all of the true/false items and some of the other items in order to emphasize literal language and correct semantics, (b) inclusion of qualifiers in some of the prompts so that a literal interpretation of all prompts leads to a correct response, and (c) inclusion of words or phrases that highlight essential aspects of a suggested relation. A second revision of the instrument is shown in Appendix C.

After the second revision, the researcher administered the resulting instrument to a group of students. Every student enrolled in an intermediate algebra class during spring 2010 semester at Utah State University had the opportunity to participate in this study. Each student was randomly placed into one of two groups. One group of students, hereafter referred to as group A, would take an online form of the primary instrument. Two hundred forty-two out of 696 students from group A volunteered to participate, and there were 222 students from group A who completed the quiz. The students voluntarily participated by responding to the prompts on the instrument as they were presented one at a time in an online format. The format did allow for participants to go backward and forward and revisit any of the items before they submitted the quiz. Before deciding to participate, the group was informed that they would be taking an online quiz that was part of a research project. They were instructed that the quiz consisted of items aimed at measuring student understanding of basic algebra concepts. They did not know before participating that the quiz was an attempt to measure their understanding of the equals relation. Students were not given any instructions relative to the use of notes, textbooks, or calculators. They were neither prohibited nor encouraged to use any supplementary materials. There was a time limit of 30 minutes allowed for the completion of the quizzes. Students were only allowed one attempt at the quiz. The online quiz was generated and managed using internet classroom software.

Validation Study

Test Reliability Analysis

After the students had completed the quizzes, the researcher scored each of the instruments taken by group A. The data that was collected consisted of a total score for each student as well as a score for each item for each student. The responses and scoring protocol for each item are presented in Appendix C.

The total score for student *i* will be referred to as T_i and the score for item *j* and student *i* will be referred to as t_{ij}. This data was then used to calculate Cronbach's reliability coefficient a. The formula for Cronbach's reliability coefficient is given by:

$$a = \frac{k}{k-1}\left(1-\frac{\sum_{j=1}^{k}\sigma_j^2}{\sigma^2}\right)$$

where *k* is the number of test items, s^2 is the variance of the total scores and s_j^2 is the variance of the scores from item *j* (Cangelosi, 2000). The descriptive statistics for the primary instrument and Cronbach's reliability coefficient a are shown in Table 2.

There are three aspects of an instrument that may contribute to a low reliability coefficient, (a) individual item scores that are skewed or have restricted variability often do not correlate well with other items on an instrument, (b) ineffective items may not correlate well with other items, and (c) the instrument is not one-dimensional but contains items measuring several different constructs.

Table 2

Descriptive Statistics and Cronbach's Alpha for Instrument Used During Pilot Test

Variable	k	N	μ	σ^2	α
Primary instrument scores	18	222	12.91	3.361	0.3374

If individual item scores are skewed or have restricted variability then the assumption of normality required by the regression model cannot be satisfied. This implies that the nonrobust Pearson-product correlation coefficient, which is a basis for Cronbach's alpha, may be a misleading indicator of association between items. A review of the variability of the scores for items used during the pilot test shows that restricted variability and skewness is probably a contributing factor to the low reliability coefficient. It also calls into question the ability of alpha to disclose pertinent information about the reliability of the instrument used during the pilot test. Table 3 shows the variance for each of the 18 items that were used during the pilot test along with the variability of the total score. Because the items were all dichotomously scored, a variance close to 0.5 represents a symmetric distribution whereas a variance close to "0" indicates a skewed distribution.

To determine if there were ineffective items that may have contributed to the low reliability coefficient, data obtained from an item analysis was reviewed. As described in the next section, the item analysis provided evidence that 14 out of the 18 items used on the pilot test were effective. The other four items had efficiency indices between 0.5 and 0.42. The items that scored the lowest item efficiencies were scrutinized and refined in an

Table 3

Variance of Items and Total Score for Instrument Used During Pilot Test

Item	Variance	Item	Variance
1	0.000	10	0.233
2	0.035	11	0.162
3	0.043	12	0.189
4	0.248	13	0.237
5	0.026	14	0.086
6	0.000	15	0.227
7	0.142	16	0.245
8	0.142	17	0.071
9	0.005	18	0.201
TOTAL	3.361		

attempt to increase the efficiency indices and an increase in item efficiency should result in a higher internal consistency.

The low reliability coefficient (alpha = 0.3374) may be an indication that the instrument contains items that measure many different constructs associated with understanding the equals relation and that the total score may not be a measure of any specific construct. To determine which items may form a subset for which a common construct can be attributed and which items act in isolation, a correlation analysis between pairs of items was conducted. Table 4 shows the correlation matrix obtained during the correlation analysis. Correlations for items #1 and #2 were not calculated because the variance for those items was 0.

The correlation matrix demonstrates a low correlation between all of the individual items on the instrument. The highest correlation existed between item #12 and item #18 (r = 0.200) and only 12 out of 153 of the calculated correlations were even

Table 4

Correlation Matrix for Items on Instrument Used During the Pilot Test

Item	#2	#3	#4	#5	#7	#8	#9	#10	#11	#12	#13	#14	#15	#16	#17
#3	0.19														
#4	-0.03	0.19													
#5	-0.03	-0.04	-0.10												
#7	-0.09	-0.04	0.07	-0.08											
#8	-0.04	-0.02	0.05	0.08	0.02										
#9	-0.01	-0.01	0.07	-0.01	-0.03	0.03									
#10	0.00	-0.12	0.01	-0.01	-0.05	0.15	-0.05								
#11	0.02	0.11	0.03	-0.08	-0.05	0.05	-0.03	-0.01							
#12	0.00	0.17	0.03	0.03	-0.02	0.02	-0.04	0.05	-0.06						
#13	-0.04	0.04	-0.09	0.02	-0.01	-0.02	0.05	0.03	0.08	0.27					
#14	-0.04	-0.19	-0.04	-0.04	0.07	0.02	-0.01	0.03	-0.03	-0.02	0.14				
#15	0.04	0.11	0.06	0.06	0.03	-0.03	0.05	-0.05	0.06	0.14	0.02	0.00			
#16	0.07	0.14	0.17	-0.08	0.05	0.09	0.06	0.04	0.07	0.10	0.02	0.12	0.19		
#17	-0.06	0.10	-0.02	0.06	0.09	-0.09	-0.02	-0.01	0.11	-0.05	0.09	0.00	0.17	0.01	
#18	0.01	-0.01	0.13	0.04	0.04	-0.02	-0.11	-0.15	0.06	0.20	0.16	-0.04	0.05	0.01	0.03

statistically significant. These small correlations suggest that student performance on a given item is not a good indication of performance on another item. As was mentioned in the preceding paragraph, the fact that the item scores deviate considerably from a normal distribution diminishes the ability of Pearson's product-moment correlation coefficient to represent an association that exists between item scores. Nevertheless, it does provide some evidence that instrument items in the pilot test could be measuring distinct constructs that are independent from each other.

Item Analysis

The individual item scores were used with the total scores to calculate the index of discrimination D_j, index of difficulty P_j , and the index of item efficiency E_j which are used to assess the effectiveness of item j (Cangelosi, 2000). In calculating D_j.the total scores were divided into subgroups. The scores that make up the highest 25% were assigned to group H (the high score group) and the lowest 25% of scores were designated group L (the low score group). The formula for index of discrimination is given by:

$D_j = PH_j - PL_j$

and PH_j and PL_j are given by: $PH_j = \dfrac{\sum_{i=1}^{N_H} j_i}{N_H w_j}$ $\quad PL_j = \dfrac{\sum_{i=1}^{N_L} j_i}{N_L w_j}$

where N_H is the number of students in group H, N_L is the number of students in group L, w_j is the maximum number of points possible for item j, and j_i is the score for item j obtained by student i from the appropriate group H or L. The conditions wherein D_j can be judged sufficiently large to reflect effectiveness are not straightforward and depend upon many factors. One of the factors that directly influences the value of D_j is the difficulty of item j. Items that are very easy or very hard will necessarily have indices of discrimination near zero (Cangelosi). In order to better assess the effectiveness of item j.the index of difficulty $P_j = \frac{PH_j + PL_j}{2}$, the maximum value of the absolute value of the item of discrimination $Max|D_j| = \begin{Bmatrix} 2P_j \text{ when } P_j \leq 0.5 \\ 2(1 - P_j) \text{when } P_j > 0.5 \end{Bmatrix}$, and the index of item efficiency $E_j = \frac{D_j}{Max|D_j|}$.were also calculated. The index of item discrimination, the index of difficulty, and the index of item efficiency for all items on the primary instrument are presented in Table 5.

This analysis utilized guidelines suggested by Hoffman (1975) for interpreting the individual indices of item efficiency. An item with an efficiency index of less than 0.5 was identified as a potentially weak item and was scrutinized by the researcher in an attempt to discover the reasons for the low index of item efficiency. Items #1, #6, and #9

Table 5

Index of Item Discrimination and Index of Item Efficiency for the Primary Instrument

Item	D_i	P_i	E_i	Item	D_i	P_i	E_i
1	0.000	1.000	*	10	0.339	0.634	0.463
2	0.036	0.964	0.500	11	0.286	0.786	0.615
3	0.125	0.938	1.000	12	0.536	0.696	0.882
4	0.446	0.501	0.455	13	0.482	0.402	0.600
5	0.036	0.964	0.500	14	0.143	0.929	1.000
6	0.000	1.000	*	15	0.536	0.393	0.682
7	0.214	0.804	0.546	16	0.571	0.482	0.593
8	0.232	0.205	0.565	17	0.161	0.902	0.818
9	0.000	1.000	*	18	0.464	0.321	0.722

had no item efficiency index due to the fact that no students in the upper and lower 25% responded incorrectly to those items. The low item efficiencies for items #2 and #5 (both 0.5) are attributed to the fact that they were very easy items. With so few students responding incorrectly, the item efficiency over emphasizes the observed differences in the scores between the top 25% and the bottom 25% of the students in those cases.

In order to elicit reasons why the other items (#4, and #10) had low efficiency indices, the researcher interviewed four students in a think-out-loud administration of the instrument (these items and the other items used on the instrument that was administered to four students can be seen in Appendix C). The students were all enrolled in an intermediate algebra course and had completed 10 weeks of a 15-week course. The students were presented with the same prompts as the large sample of students from group A, who responded to the prompts in an online format. The students were asked to respond to the prompts out-loud and were encouraged to ask questions or express concerns over the instructions provided with each prompt as they responded to each prompt. After the students had responded to all of the prompts, the researcher pointed out any mistakes, identified an appropriate response, and asked the students for further insight as to why they responded correctly or incorrectly. The students were also encouraged to provide their opinions about the effectiveness of individual prompts and the instructions provided with each prompt.

For items #10 and #11, it was initially intended that students would select one of two correct responses (b or c) or one incorrect response (a) to each of the prompts. Response (c) suggests a correct and preferable relational interpretation of the equation while response (b) suggests a correct but less preferable procedural interpretation of the equation. Response (a) is an incorrect interpretation that also indicates procedural interpretation of the equation. While conducting the think-out-loud sessions, the researcher observed that one of the students who selected the correct relational interpretation did so by default. That student chose the correct relational response (c) because he felt responses (a) and (b) were so similar that one could not be correct unless they were both correct, and by elimination chose the correct relational response. The researcher determined that eliminating the correct procedural responses for each item will make the two items more relevant and reliable and should increase the item efficiency index for these two items.

Item #4 had an efficiency index of 0.455 and interviews with students showed that the item appeared to measure an ability to recognize the importance of the equals sign in the equation. Two of the students ignored the equals sign and computed binary expressions between the equals signs. When the researcher reviewed responses and asked the students what the equals signs meant in that prompt, they quickly recognized their mistake and attributed their error to working too quickly and not looking carefully enough at the prompt. Another student had computed each binary expression and then realized the mistake before the instrument was completed. He stated, "I forgot to make all of the parts equal and I just started to do all of the math."

Based on these interviews, the researcher determined that the item was an effective measure. The efficiency of the item is decreased because many students who scored high on the instrument were distracted by their preconceived notion of what they were supposed to do when working with an expression of that form. Consequently, their understanding of the equals relation was never conjured when they responded to the item. Evidence from field tests and from the pilot test itself showed that students who had high or low total scores on the instrument often failed to recognize the equals relation defined by the equation. In all of these instances, students attributed these misunderstandings to inattention to the details of the equation. Because this evidence validates the purpose of the item, no changes were made to this item even though the efficiency index for this item was relatively low.

The think-out-loud sessions also revealed some problems with construct validity for item #8. The item was intended to assess two behaviors: (1) student ability to interpret an equation as an indication of a relation as opposed to an indication to calculate, and (2) student understanding that there are infinitely many ways to write an expression that is equal to a given expression. One student who was interviewed suggested that the only way to make the equation true is to rewrite expression $\sqrt{13} + 2$.in the blank on the right side of the equals sign. When the researcher suggested that there are many ways to write $\sqrt{13} + 2$, such as $\sqrt{13} + 1 + 1$ or $\sqrt{12 + 1} + 2$ the student expressed the opinion that it was a trick question because he considered all of those the same thing. This indicated a good understanding that equal expressions are the same expression, while at the same time suggesting a lack of understanding that there are many forms for a given expression. This response indicated that students may correctly determine that there is only one expression that is equal to $\sqrt{13} + 2$ and erroneously conclude that there is only one form of an expression that is equal to$\sqrt{13} + 2$. Since the item was intended to assess two objectives, an understanding of one of the objectives but not the other was not acknowledged in the scoring of the item. This item was changed into a two part prompt, so that an actual expression that is equal to $\sqrt{13} + 2$ is requested and then a student is asked to determine how many forms could have been given to represent an equal expression. The wording of the multiple choices was also changed to better insinuate the possibility of multiple forms rather than multiple expressions.

Significance Test

One objective that was added to the original set of objectives was: *Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.* Items #4, #10, and #11 from the primary instrument were written to measure this objective. Measuring this objective posed a problem because it was difficult to determine if a student misinterpreted the equations in these items because he or she misinterpreted the equals sign or if he or she simply ignored the equals sign.

To address this concern, a set of items (#1, #2, #3, #5, #6, #7, #9) were written where a computational interpretation of the expression is awkward and a basic relational understanding of equals allows for a simple and correct response to the prompt. The purpose of these items is twofold: (a) to measure a student's ability to correctly interpret the equals sign as an indication that two expressions are the same, and (b) to discourage students from ignoring the equals relation in contexts where it may seem logical that a calculation is required. The relevance of these seven items as a measure of student's ability to correctly interpret the equals relation was established by the expert review panel. If a student has a basic understanding that equals means “the same” then they should be able to respond correctly to the majority of these prompts.

To measure the ability of these items to discourage students from ignoring the equals relation in many contexts, an alternative instrument was written that included items that were identical to items #4, #10, and #11, but also contained different easy level items. All of the easy level items on the alternative instrument could be correctly addressed by computing. This was done to encourage the conditioned inclination of students to calculate a mathematical expression. The alternative instrument is shown in Appendix D. Each of the items, which are also included on the primary instrument, are marked with an asterisk.

A test of significance was then performed using two groups of students to determine if there was a difference in the scores on three of the items that are identical on both instruments. The scores for items #4, #10, and #11 on the primary instrument were compared with the scores for the identical items #4, #15, and #10 on the alternative instrument respectively. The average and total scores on the three items from the primary instrument were calculated from the 222 students from group A, and were compared to the average and total scores of the three items from a similar group who completed the alternative instrument. The group who completed the alternative instrument will hereafter be referred to as group B.

There were 667 students in group B that were selected randomly from all students enrolled in an intermediate algebra course at Utah State University during the 2010 spring semester. A total of 204 students from group B volunteered to participate, and there were a total of 191 students from group B who completed the alternative quiz. The students responded to the prompts on the instruments as they were presented one at a time in an online format. The format did allow for participants to go backward and forward and revisit any of the items before they submitted the quiz.

Before deciding to participate, both groups were told that they would be taking an online quiz that was part of a research project. They were instructed that the quiz consisted of items aimed at measuring student understanding of basic algebra concepts. They did not know before participating that the quiz was an attempt to measure their understanding of the equals relation. Students were not given any instructions relative to the use of notes, textbooks, or calculators. They were neither prohibited nor encouraged to use any supplementary materials. There was a time limit of 30 minutes allowed for the completion of the quizzes. Students were only allowed one attempt at the quiz. The online quiz was generated and managed using internet classroom software.

A t-test, assuming unequal variances, was conducted to determine if there is a significant difference in the average score for the three identical items from group A (the primary instrument) and group B (the alternative instrument). Because the purpose of the test was to determine if the easy level prompts on the instrument have an effect on a student's ability to recognize and attend to the equals relationship expressed in a given equation, items #10 and #15 from the primary and alternative instruments respectively were scored using an alternative rubric. Although both responses b and c were correct interpretations of the equations given in those prompts, only response c suggests a relational interpretation. Therefore, 1 point was given for those items only if response c was selected. Likewise, only response b was awarded a point for items #10 and #11 from the primary and alternative instrument respectively. Item #4 was scored the same as it was on the item analysis portion of the study. The test found that there was a statistical difference in the average scores for all three items and the total of the scores for the three identical items. The summary statistics for this test of significance are presented in Table 6.

Final Refinement of Items

Based on the data from the item analysis, the significance test, and the interviews conducted with students, the researcher refined the items a final time. The resulting items were used to create the final instrument that is shown in Appendix E. Test reliability analysis and item analysis have yet to be performed on this final version of the

Table 6

Summary Statistics for Test of Significance

Item	*M*	*σ*	*df*	*t*	*P*
Item #4	0.5450	0.4991			
Item #4A	0.0733	0.2613	344	12.264	0.000
Item #8	0.1712	0.3775			
Item #7A	0.0890	0.2855	405	2.513	0.006
Item #10	0.0856	0.2804			
Item #15A	0.0314	0.1749	377	2.389	0.009
Item #11	0.5360	0.4998			
Item #10A	0.466	0.5002	402	1.42	0.078
Total of 4 Items	1.3378	0.9116			
Total of 4 Items A	0.6597	0.6916	405	8.580	0.000

instrument. The process utilized in the development of this instrument does provide evidence suggesting that the items on the final version of the instrument are relevant to the stated objectives. The final version of the instrument shall be referred to hereafter as the Wheeler Test for Comprehension of Equals (WTCE).

Observer Consistency Analysis

It was anticipated that scoring some of the items on the instrument would require rubrics that left some room for judgment so the initial proposal for this study called for an analysis of intraobserver and interobserver consistency. However, all of the items on the instruments that were given to the students in the validation analysis portion of this study were multiple choice, true/false, or fill in the blank. The researcher determined that an analysis of intraobserver and interobserver consistency was not warranted for an instrument of this nature.

CHAPTER IV

CONCLUSIONS AND DISCUSSION

Introduction

The purpose of this research was to (a) identify and classify specific learning objectives that define what behaviors will be expressed by a student who correctly comprehends the equals relation and has the ability to correctly interpret the relation in different contexts; (b) develop an instrument that is relevant to the identified objectives; and (c) perform a test reliability analysis to assess measurement reliability, construct validity, intraobserver consistency, and interobserver consistency. This chapter will address each of these objectives.

Instrument Development

Methodology

The researcher used both quantitative and qualitative methodologies to develop and validate the items on the instrument. The researcher utilized an iterative process while developing the instrument items as recommended by Cangelosi (2000). Using this process, the learning objectives and instrument items were developed, analyzed, and repeatedly revised by the researcher. Process activities included: (a) collaboration with content experts where opinions regarding the objectives and relevance of the instrument items was sought and recorded, (b) field trials of potential instrument items using students and colleagues, and (c) recorded think-out-load interviews with individual students.

The quantitative portion of the study involved administration and scoring of potential instrument items. Data relevant to the instrument and the individual items that formed the instrument were collected and analyzed. Item analysis indices were used to assess item effectiveness. A reliability coefficient was calculated to access the ability of the instrument to provide a measure of a specific construct: a student's general understanding of the equals relation. Results of this analysis were presented in Chapter III.

Both quantitative and qualitative methods were necessary in developing quality items for the WTCE. The data obtained from the item analysis showed that there were some potentially ineffective items on the instrument that was administered during the pilot test. These statistics did not, however, identify the reason that items were ineffective or how they should be modified. It was through the interviews and think-out-loud sessions that the researcher was able to pinpoint weaknesses and revise or eliminate those items.

Relevance of the Instrument Items

The content relevance of each item on the instrument was determined through a purely qualitative process. The researcher identified a set of seven behavioral objectives that correspond to an understanding of the equals relation in different contexts along with a set of preliminary items corresponding to those objectives. The preliminary items were developed by the researcher and were adjusted based on information gleaned from field tests of the items administered to students and peers.

The objectives and preliminary items were then reviewed by a panel of content experts. This panel provided insight into appropriateness of the objectives and relevance of each instrument item as a means of assessing student adherence to the corresponding objectives. Based on feedback from the panel, items were removed, revised, and added in an effort to synthesize an instrument that measures student understanding of the equals.

The construct relevance of potential items was determined by a combination of qualitative and quantitative methods. An item efficiency analysis showed that sixteen of the eighteen items that made up the instrument used during the pilot test were effective. Qualitative data collected from field tests and think-out-loud interview sessions was used to refine the potentially ineffective items and increase the construct relevance of the instrument.

A *t* test was used to determine if easy level items that were included for the purpose of encouraging recognition of the equals relation had an effect on a student's ability to appropriately respond to items where a computational interpretation of a given equation is possible. The test showed that there was a significant difference in the scores between students who were given the instrument containing easy level items that encouraged recognition of the equals relation in an equation, and students who were given the alternative instrument containing easy level items where computations could lead to an appropriate response to the corresponding equations.

Based on the quantitative and qualitative data gleaned from experts, students, and peers, the researcher has proposed a set of 18 items that are relevant as a means of measuring nine behavioral objectives associated with understanding and interpreting the equals relation. Those items, along with their corresponding behavioral objectives, are presented in Appendix D.

Validation Study

A validation study involves two components: (a) an assessment of the relevance of the instrument, and (b) an assessment of the reliability of the instrument. As discussed in the previous section, the proposed items can be considered sufficiently relevant due to the nature of the process that was utilized in the development of the items. The assessment of item reliability is discussed in the following section.

Reliability

To assess the internal consistency of the instrument that was used during the pilot test, Cronbach's reliability coefficient α was calculated. The low value of α (0.3374) suggests that the instrument administered during the pilot test lacks internal consistency. To better understand the reasons for the lack of internal consistency, the researcher examined quantitative data from the item analyses and a correlation analysis. The item analysis showed that 14 of the 18 items used on the pilot test were effective. The other four items had efficiency indices between 0.5 and 0.42. The items that scored the lowest item efficiencies were scrutinized and refined in an attempt to increase the efficiency indices. An increase in item efficiency should result in higher internal consistency, but because none of the items had negative or extremely low efficiencies, these refinements will not substantially increase the internal consistency.

Correlation analyses of the data obtained during this study indicated that a correct interpretation of the equals relation in one context does not correlate well with a correct interpretation of equals in the other contexts. However, qualitative data obtained during student interviews and the quantitative data obtained from instrument scores suggest high construct relevance of the items. This finding supports the claim that instrument items can be used to effectively measure appropriate interpretation of equations in specific contexts.

Item Analysis

Fourteen of the 18 items used during the pilot study were found to have item efficiency indices $E_i \geq 0.5$. Two of the test items had efficiency indices $0.5 < E_i < 0.4$. Three of the items had no item efficiency indices due to the fact that all of the students in the upper 25% and lower 25% of total score responded correctly to those items. Refinements were made to one of the items with an item efficiency of less than 0.5 and the refined item was included on the final version of the instrument. Data obtained from field tests and think-out-loud interviews suggested that the other low efficiency item was measuring the desired objective. Because the item efficiency index was not alarmingly low, the item was included on the WTCE without refinement.

The WTCE consists of fifteen items (#1, #2, #3, #4, #5, #6, #7, #9, #12, #13, #14, #15, #16, #17, #18) that are identical to items that were included during the pilot test. All but one of these items (#4) were credited with an item efficiency index of 0.5 or above during an item analysis of the pilot test data. Three of those items (#1, #6, #9) had no item efficiency indices because they were so easy. The *t* test showed that inclusion of those items increased discrimination in how the equals relation is interpreted in different contexts. These three items also served as an indication that a student can correctly interpret an equals relation in contexts where the need for a relational interpretation is exaggerated. Three of the items that are included on the WTCE (#8, #10, #11) are similar to items that were included on the pilot-test instrument. Refinements were made on the pilot-test instrument based on qualitative data obtained using field tests and think-out-loud interviews in an attempt to increase the construct validity of the items. Accordingly, the WTCE is comprised of items that have been refined to improve item effectiveness, as well as, a high percentage of items that were already shown to be effective and appropriate.

Summary

The purpose of this study was to: (a) identify and classify specific learning objectives that define what behaviors will be expressed by a student who correctly comprehends the equals relation and has the ability to correctly interpret the relation in different contexts; (b) develop an instrument that is relevant to the identified objectives; and (c) perform a test reliability analysis to assess measurement reliability, construct validity, intraobserver consistency, and interobserver consistency. A review of the instrument development process and the results of the instrument validation study indicate that these purposes were accomplished during this study.

A set of eight objectives were identified by the researcher and confirmed by a panel of experts to be appropriate indications of correct understanding of the equals relation in different contexts. Each of the objectives was classified as a comprehensive learning level objective appropriate for assessing student comprehension of the equals relation in different contexts and forms. Responding to these learning objectives, the researcher synthesized an instrument composed of eighteen items using a refinement process involving field testing of items, expert review of items, and think-out-loud interviews with students. The nature of the process used to develop these items affords validation to the claim that the items are sufficiently relevant to the corresponding objectives.

The results of the test reliability analysis provide evidence that the WTCE consists of a high number of items deemed effective by an item analysis. The instrument also consists of items that were refined according to qualitative data obtained through field tests and think-out-loud interviews conducted with students. Although the items that make up the instrument have been shown to be effective measures of their associated objectives, a low reliability coefficient suggests that the items do not form a reliable measure of a construct common to all objectives.

The WTCE is comprised of dichotomously scored items of the form: multiple choice, true/false, and fill in the blank. Therefore, a high level of intraobserver and interobserver consistency is expected. The researcher determined that an analysis of these consistencies was not warranted for the proposed instrument.

Discussion

What Students Understand About Equals

It was the original intent of the investigator to create an instrument that measures distinct constructs that, taken together, completely define an appropriate understanding of the equals relation. The vast majority of college students interviewed did understand equals as an indication that two representations are the same structure. But they often failed to implement this understanding when confronted with equations in different contexts. This study illuminates two significant reasons for this finding. First, students fail to recognize the extent of the sameness suggested by an equation. Second, when students focus on solving, evaluating, or coming up with "the answer" they fail to recognize the contribution of the equals sign or other indications of the equals relation in a given context. This could be a conditioned response from their previous experience with situations featuring math problems.

These findings are consistent with the theory offered by Sfard (1991), and suggest that students do not instinctively offer a structural understanding of equations. While most college students have the ability to transition from a procedural understanding of equations to a structural understanding, they do not bring that knowledge to bear without prompting or encouragement. In a context where a procedural interpretation of an equation is consistent with a student's perception of how they should interact with the equation, they will most likely fail to interpret the equals relation. If an equation is presented such that any procedural interpretation contradicts student perceptions of the equation's purpose, the students are more likely to interpret the equation correctly as a relation. The study also showed that exposing college students to multiple contexts where a structural understanding seems more natural than a procedural understanding increases the likelihood that students correctly interpret equations *of all kinds* as relations rather than as expressions to be evaluated.

These findings indicate that student mistakes on prompts involving the equals relation are often a result of the students' failure to pay sufficient attention to the equality designation, especially in specific contexts. Analysis of qualitative data obtained in this study suggests that when college students are confronted with an equation, they proceed according to what they think they are "supposed to do," and the equals sign or other equals designations do little to discourage their response patterns. This indicates students' misinterpretation is rooted in their conditioning that the purpose of math is to evaluate expressions; and has less to do with misunderstanding the equals relation itself. In other words, rather than misunderstanding the meaning of an equivalence designation in an equation, students simply ignore it in order to proceed as they feel they are supposed to.

Practical Applications

The research-based procedures used to construct objectives and measurement items have led to the development of an instrument featuring relevant and valid items that measure student understanding of the equals relation in many contexts. The WTCE may provide benefits to teachers and researchers.

1. Researchers and teachers can use the instrument to determine the contexts in which individual students fail to correctly comprehend an expressed equals relation.

2. Researchers and teachers can use the instrument to determine the contexts in which groups of students are most likely to misinterpret an equals relation.

3. Researches and teachers can use the instrument to assess the need for curriculum development that addresses the comprehension of the equals relation.

4. Researchers and teachers can use the instrument to assess the effectiveness of interventions designed to increase the correct comprehension of an equals relation.

5. The instrument can be easily scored by teachers and researchers and can be administered to large samples of students in many formats.

Existing studies indicate that ability to correctly interpret the equals sign improves as a student matures and is exposed to a larger number of non-arithmetic contexts. The existing studies indicate that most college students will correctly interpret an equals relation when directed to express the meaning of an equals sign in a given equation. But these studies provide little understanding relative to a student's ability to utilize their understanding of an equals relation when solving problems, simplifying expressions, answering questions, modeling real world scenarios, or other mathematical tasks that require interpretations of equations. The WTCE, that is the product of this study, represents an original, significant, and practical contribution to the assessment of student understanding of the equals relation which is an integral component of mathematical learning.

Limitations and Implications for Future Research

Although the primary purposes of this development study have been realized, there are some limitations that should be mentioned. One limitation is that the instrument provides an assessment of student tendencies associated with interpreting equals, but does not provide a valid assessment of why these tendencies occur nor does it suggest interventions that are appropriate to address these tendencies. Future studies may build upon knowledge of these tendencies provided by the instrument to determine the reasons that students correctly or incorrectly interpret an equals relation within certain contexts, and prescribe interventions that adequately address these concerns.

The students that participated in the pilot testing of the instrument were all students at Utah State University who were enrolled in a remedial intermediate algebra math course. This is limiting factor prohibits generalization of the results to groups with more or less mathematical backgrounds, groups with different maturity levels, and groups in other regions of the country. Therefore, studies involving groups that are older, younger, have more mathematical experience, less mathematical experience, and reside in other regions of the country should be performed to determine if the instrument is valid to divergent groups of students.

The pilot testing of the instrument was done in an online format. Students responded to the prompts as they were presented on their computer screens. There were no limitations as to the environment in which students participated. It is likely that the lack of a structured environment decreased the likelihood that a student put forth their best effort to respond to the prompts. A study should be conducted where the instrument is administered to students in a structured environment that encourages a student's best effort, in order to determine the environmental effect on internal consistency and item efficiency.

The WTCE is composed of items that are refined versions of some of the items that were administered to students during the pilot test. The reliability analysis and item analysis were performed on the items from the pilot test in order to justify appropriate refinements that led to the product instrument—no item analysis or reliability analysis has been performed on the WTCE. Because the refinements to the pilot instrument were done in accordance to data obtained from the reliability and item analyses, it is assumed that the reliability and item effectiveness have been improved. Therefore, a follow-up reliability analysis and item analysis should be performed to determine the validity of the WTCE.

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APPENDICES

Appendix A

Letter for Request of Service on Review Panel

Guidelines for the Panel Review of Objectives and Items

Objectives Associated with Understanding Equals Relation

Preliminary Set of Items Presented to Review Panel

Letter for Request of Service on Review Panel

Greg Wheeler
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Dear Colleague:

My name is Greg Wheeler and I am a doctoral candidate in Mathematics Education at Utah State University. I am contacting you because I know of your expertise in mathematics instruction. I am writing this letter to request your services as a participant on a panel that will review the items on an instrument that I am constructing for my dissertation. I intend to assess the appropriateness and validity of the instrument items. The purpose of the instrument is to measure student understanding of the equals relation and the degree to which their interpretation of the equals relation is influenced by their experiences in school mathematics and arithmetic.

Research suggests that most 6^{th} grade students interpret the equals sign as an indication to calculate a result. The ability to correctly interpret the equal sign increases as a student matures and has more experience with the equals relation in algebraic settings. Research also indicates, however, that students continue to misuse the equals sign in work done in higher level science and mathematics classes.

I am constructing this instrument to assess a student's interpretation of the equal sign and the equals relation in many contexts not considered in previous research. Researchers in most studies have assessed the understanding of the equals relation in contexts that are limited to equality of arithmetic expressions. I intend to also assess understanding of the equals relation in contexts such as equality of algebraic expressions, equality of numbers, equality of geometric entities, and equality of physical, real world entities.

Additionally, using this instrument, I intend to measure whether or not a student's ability to correctly interpret the equals relation is influenced by conditioned responses to what they believe they are "supposed" to do. My thought is that most students do understand that equals means that two structures are the same structure. However, if a student believes they are "supposed" to be calculating then they are conditioned to interpret equals as an indication to evaluate a result. I think this is a significant issue because most students in most problems think they are supposed to be calculating "correct answers."

Although the instrument is not yet ready for review, I have attached two documents associated with the instrument so that you can get a glimpse into what this request will entail. The first document consists of 8 items used to measure understanding of the equals relation and they are embedded with other items asking students to compute and evaluate expressions. It is anticipated that this document will influence student's conditioned responses to interpreting equals computationally. The third document will consist of the same 8 items embedded with other items where they are asked to discover relationships in expressions. It is anticipated that this second document will influence students to correctly interpret the equals relation on those common eight items.

I would seek your participation as a reviewer during the months of December and January as I would like to administer the instrument to groups of students by the end of January 2010. It is anticipated that you would spend from one to three hours reviewing the prompts and scoring rubric of the initial instrument and less time reviewing subsequent updates of the instrument based on changes recommended by you and the other panel participants.

I would very much appreciate your help as outlined and if you are willing I will send you a more detailed description of the feedback I request. If this is something that you are not able to do then I would appreciate any suggestions of peers who might be able to assist me in this work.

Thank you very much for considering this request. If you have questions that I can answer that will help you determine if this is something that you are able to help me with then please do not hesitate to email me or call me.

Sincerely,

Greg Wheeler

Guidelines for the Panel Review of Objectives and Items

The purpose of this instrument is to measure a student's ability to correctly comprehend the equals relation in contexts not considered in previous research. Researchers in most studies have assessed the understanding of the equals relation in contexts that are limited to equality of arithmetic expressions. This instrument is intended to measure the ability to comprehend the equals relation in contexts such as equality of algebraic expressions, equality of numbers, equality of geometric entities, and equality of physical, real world entities. My personal experiences in the classroom, research done by others, and information gleaned from field-testing possible items for this instrument have shown the following misinterpretations or misunderstandings of the equals relation and the equals sign:

1) Students interpreting the equals relation as an indication that an expression is to be evaluated.
2) Students interpreting an algebraic equation as a formula for generating a result rather than expressing a relation.
3) Students interpreting two things as equal when they have similar properties but are not the same entity.
4) Students interpreting the equals sign as a separation of expressions rather than a very specific designation of a relation.
5) Students using the equals sign to communicate the end of one step and the beginning of another step during a problem-solving process.

The eight blue items on the instrument are the real focus of the instrument as they address the student misinterpretations listed above. The items in black are included for two reasons, 1) to encourage students to consider relational interpretations while they respond to all of the prompts and 2) to measure a significant lack of understanding of the equals relation if students are not able to correctly respond to those prompts. It is intended that students who have an understanding of equals as a relation will correctly respond to the prompts in black and will possibly respond incorrectly to some of the blue prompts because of a misinterpretation of the equals relation rather than a lack of understanding. The items will not be distinguished as black and blue when the instrument is administered to the sample of students.

There is separate instrument that will be administered to a similar sample of students that consists of many prompts where students are asked to calculate and evaluate expressions along with the same eight blue prompts found on this instrument. That instrument will be administered as a way measuring the effect that a student's belief that they are "supposed" to be calculating has on their ability to interpret equals as relation. Please evaluate that instrument after evaluating this instrument.

Before listing the items on the instrument I have listed the objectives that are to be measured by the items. Please read through the objectives, the prompts, and scoring rubrics on the instrument. Then please respond to the following:

1) Do the listed objectives serve as a comprehensive description of what it means to correctly comprehend the equals relation?.Are there any of these objectives that you would leave out?.Are there other objectives that should be included?

2) Are there errors or oversights on the prompts or on the scoring rubrics for each prompt?

3) Would you reword the prompts in any way that you feel would make the item a more reliable measure of the corresponding objective.

4) Please identify any items that you feel are questionable in their content or construct reliability as a measure of the corresponding objective.

5) Do you have any suggestions on improving the reliability of any of the items on the instrument.

Original Objectives Associated with Understanding the Equals Relation

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

This is the trap that many students fall into. Even when students know that equals means that two structures are the same, they are still conditioned to interpret equals as an indication to calculate or evaluate in some contexts. A misinterpretation of the equals relation in a specific context prohibits students from comprehending the solution to a problem, the steps of an algorithm, the result obtained from applying a formula, or the meaning of an algebraic property. This is placed as the objective of greatest relevance because research and experience suggest that this is why students most often make mistakes and misinterpretations related to the equals relation.

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

If students do not comprehend that equal structures are the same structure then they will not be able to solve equations, model with equations, or comprehend equations at any level higher than the simple knowledge/memorization level.

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

To learn and to apply algebra, students must understand that the equals relation is used to express different forms of the same structure. Recognizing equal structures in different forms is more relevant to understanding algebra than an ability to calculate or evaluate expressions. Field trials suggest that these contexts are where students most consistently show an understanding of equals as a relation. Interviews suggest that in these contexts they are aware that they are not supposed to calculate but are comparing two quantities that are expressed differently.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Students should understand that equals implies something very specific: two structures are the same structure. Saying that two things are equal means more than they have properties in common or that one structure can be used in place of another structure in certain circumstances. This understanding is essential if students are going to comprehend equations such as: $\frac{x-1}{x^2-1} = \frac{1}{x+1}\ for\ x \neq 1$, or to understand why some triangles are congruent and not equal.

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)

Students will often use the equal sign or the equals relation to organize information when modeling or problem solving without considering the relational implications of the equations they form. When students write an equation to represent the fact that there are twice as many boys as girls, they will often model the scenario with the equation $2g = b$ because that is the way they read the problem. They can easily see the mistake of such an equation when they are reminded of the meaning of the equation. Students also use the equal sign erroneously to indicate the end of one step and the beginning of another step as they solve a problem.

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application and Comprehension)

Students should be able to identify unknowns in an equation by interpreting an equation as a relation.

Preliminary Set of Items Presented to Review Panel

Preliminary Item 1:

Fill in the blanks so that the statement below is a true proposition. If it is impossible to make the statement below true, then please explain why.

$2 + 5 = \_\_\_ - 3 = \_\_\_ + 9 = \_\_\_ - 5 = \_\_\_\_\_\_$

Scoring Rubric: +1 for 10, -2 , 12 , 7

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Preliminary Item 2:

Fill in the blank so that the statement below is a true proposition. If it is impossible to make the statement below true, then please explain why.

$2 + \sqrt{13}$ is equal to ___________.

Scoring Rubric: + 1 for any entity equal to the number that is two greater than the square root of 13.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate.

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Preliminary Item 3:

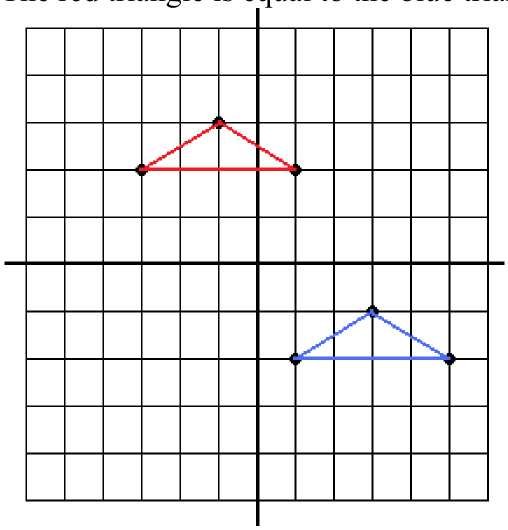
The following is a true proposition:
For all real numbers x, $(x + 1)(x + 5)$ is equal to $x^2 + 5x + x + 5$.
What is the meaning of the phrase "**is equal to**" as used in the proposition above?

Scoring Rubric: +1 for a statement that suggests or attempts to suggest that the expressions are the same structure as opposed to a statement suggesting that one expression is the result of an evaluative process.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Preliminary Item 4:

Determine if the proposition below is true or false and then explain why you believe the proposition is true or false:
The red triangle is equal to the blue triangle.



Scoring Rubric: +1 for false and an explanation suggesting that they are not the same triangle.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Preliminary Item 5:

Determine if the proposition below is true or false and then choose the response below that **best** explains why the proposition is true or false:
The number. $\frac{2}{7}$.is equal to the number. $\frac{6}{21}$.

a) True, they are the same number.
b) False, the result of dividing 2 by 7 is different than the result of dividing 6 by 21.
c) True, if you multiply both the top and bottom of the fraction $\frac{2}{7}$ by 3 you get the fraction $\frac{6}{21}$
d) False, they are not the same number.
e) True, when you cross multiply you get 2x21=42 and 6x7=42.

Scoring Rubric:.+1 for selecting a)

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Preliminary Item 6:

Let ***g*** represent the number of girls attending a specific party and let ***b*** represent the number of boys attending that same party. Assume that the equation below is true:

$\boldsymbol{g + 4 = b}$.

If there were 10 boys who attended the party, then how many girls attended the party?

Scoring Rubric: +1 for stating that there are 6 girls that attended the party.

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application and Comprehension)

Preliminary Item 7:

Let **A** be the number of apples in a basket and let **B** be the number of bananas in a basket. If there are three times more apples in the basket than bananas, then which of the equations below expresses the relationship between the number of bananas and the number of apples?

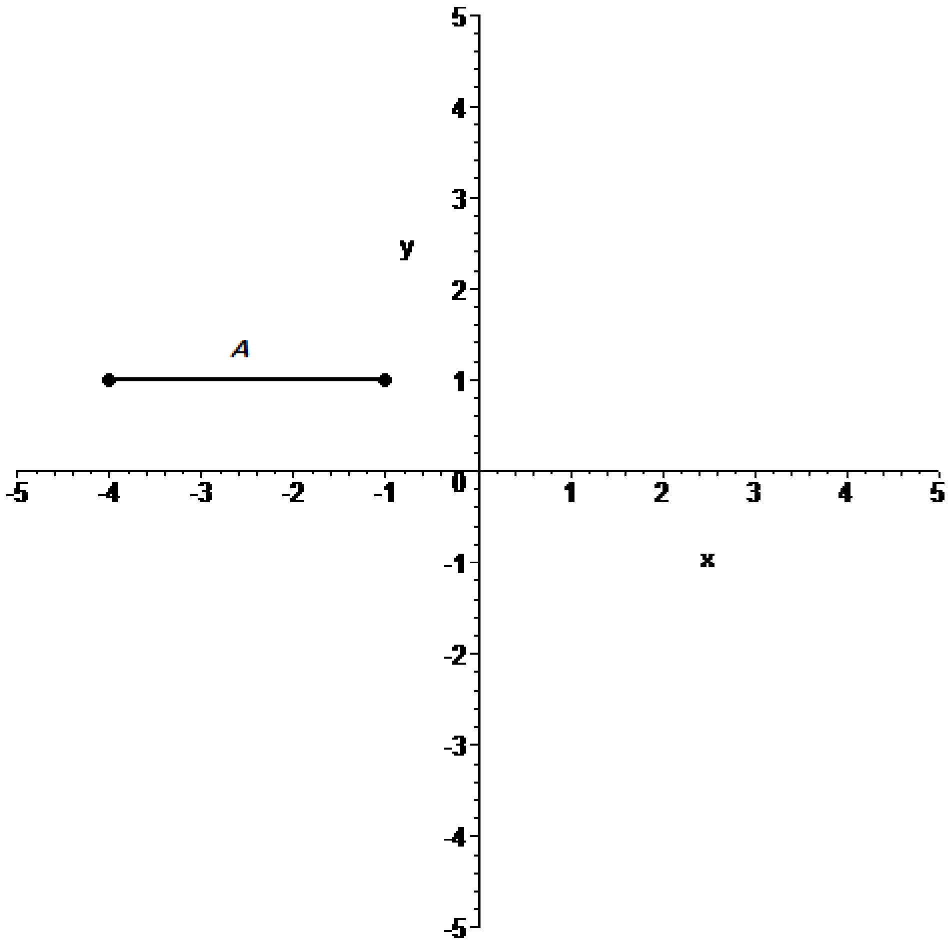
a) $A + 3 = B$
b) $3A = B$
c) $A = 3 + B$
d) $A = 3B$
e) *All of the above*

Scoring Rubric: +1 for selecting d)

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)

Preliminary Item 8:

Below is the line segment *A* drawn in a plane. In that same plane, draw a line segment that is equal to the line segment *A*.



Scoring Rubric: +1 for retracing the line A or indicating that a line segment equal to A must be the same line segment (the same set of points).

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Preliminary Item 9:

Fill in the blank with a mathematical expression so that the proposition is true.
For all real numbers r,.$r^2 + r \; is \; equal \; to$ ____________________.

Scoring Rubric: +1 for an expression that is equal to $r^2 + r$

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate.

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Preliminary Item 10:

Describe a scenario in which the proposition below is true:
Sven is equal to Ole.

Scoring Rubric: +1 for a scenario in which Sven and Ole are defined to be the name of the same structure.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Appendix B

Revised Objectives Associated with Understanding the Equals Relation

First Revised Items Presented to Review Panel

Revised Objectives Associated with Understanding the Equals Relation

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

This is the trap that many students fall into. Even when students know that equals means that two structures are the same, they are still conditioned to interpret equals as an indication to calculate or evaluate in some contexts. A misinterpretation of the equals relation in a specific context prohibits students from comprehending the solution to a problem, the steps of an algorithm, the result obtained from applying a formula, or the meaning of an algebraic property. This is placed as the objective of greatest relevance because research and experience suggest that this is why students most often make mistakes and misinterpretations related to the equals relation.

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

If students do not comprehend that equal structures are the same structure then they will not be able to solve equations, model with equations, or comprehend equations at any level higher than the simple knowledge/memorization level.

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

To learn and to apply algebra, students must understand that the equals relation is used to express different forms of the same structure. Recognizing equal structures in different forms is more relevant to understanding algebra than an ability to calculate or evaluate expressions. Field trials suggest that these contexts are where students most consistently show an understanding of equals as a relation. Interviews suggest that in these contexts they are aware that they are not supposed to calculate but are comparing two quantities that are expressed differently.

Objective: Given an equation, students interpret the equal sign as an indication that both sides are the same structure and not as a symbol that separates two structures. (Comprehension)

Students should interpret equations as relations suggesting that the two sides are the same and not as two separate expressions that can take on infinitely many values determined by evaluating the expressions for different values of the variables.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)
Students should understand that equals implies something very specific: two structures are the same structure. Saying that two things are equal means more than they have properties in common or that one structure can be used in place of another structure in certain circumstances. This understanding is essential if students are going to comprehend equations such as: $\frac{x-1}{x^2-1} = \frac{1}{x+1}\ for\ x \neq 1$, or to understand why some triangles are congruent and not equal.

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)
Students will often use the equal sign or the equals relation to organize information when modeling or problem solving without considering the relational implications of the equations they form. When students write an equation to represent the fact that there are twice as many boys as girls, they will often model the scenario with the equation 2g = b because that is the way they read the problem. They can easily see the mistake of such an equation when they are reminded of the meaning of the equation. Students also use the equal sign erroneously to indicate the end of one step and the beginning of another step as they solve a problem.

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application and Comprehension)
Students should be able to identify unknowns in an equation by interpreting an equation as a relation.

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

First Revised Items Presented to Review Panel

1).Fill in the blank so that the equation below is true.

12 +.______. =. 13

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 1

2).Fill in the blank so that the equation below is true.

8.=._____ -.5

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 13

3) Fill in the blank so that the equation below is true.

8 + 4. =. _____ + 2

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for 10

4) Fill in the blank so that the equation below is true.

3 + 7. =.._____.+ 2. =.._____.- 2. =. _____.+ 1. =. _____

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for 8,12,9,10 or an indication that they were attempting to make each quantity 10.

5) Is the equation below true or false?

$5 \times 2 \ = \ 4 \times 2 + 2$

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

6) Is the equation below true or false?
$8 - 2 \ = \ 6. -3$

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for False

7) Is the equation below true or false?
7^{10}.=.$(2+5)^{10}$

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

8) Fill in the blank so that the equation below is true.
$\sqrt{13} + 2. =. \_\_\_\_$

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for any quantity that is equal to the number that is two greater than the square root of 13.

9).Is the equation below true or false?.Explain your answer to the question using complete sentences.
4 + 3.=.14.÷ 2

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for true and an explanation that suggests that both quantities are the same.

10).Which of the following <u>best</u> describes the meaning of the equation.$15 \div 3. = .5$?

a) (15 ÷ 3) and the number 5 are the same number.
b) When the number 15 is divided by the number 3 then the result is the number 5.
c) Given the expression 15 divided by 3, the solution to the equation is the number 5.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for A

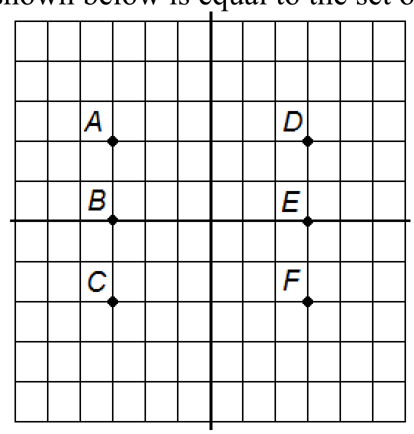
11) The distributive property states: *a (b + c) = ab + ac.*
Which of the following statements <u>best</u> describes the meaning of the distributive property?

a) The value of the expression *a (b + c)* can be calculated by adding the product *ab* to the product *ac*.
b) *a (b+c)* and *ab + ac* are the same the same mathematical entity.
c) When solving a problem related to the expression *a (b + c)* , the correct solution is *ab + ac*.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for B

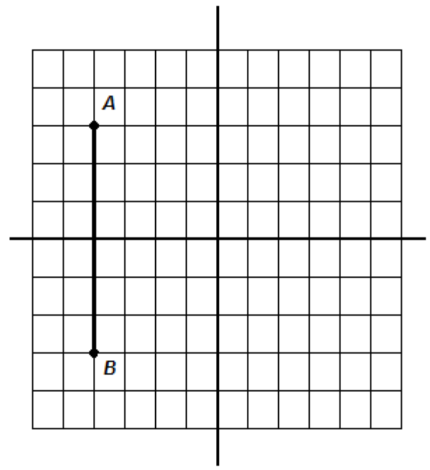
12) Is the statement below true or false?
The set of points {A, B, C} shown below is equal to the set of points {D, E, F}.



Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

13).*A line segment.*$\overline{AB}$*.is defined to be the set of points on a line that include A and B and all points between A and B.*
Draw a line segment on the grid below that is equal to the line segment $\overline{AB}$ that is shown.



Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for drawing or indicating the exact same set of points.

14).Let *A* be the number of apples in the basket and let *P* be the number of peaches in the basket. If there are 10 apples in the basket and if.$P + 4 = A$.then how many peaches are in the basket?

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application)

Scoring Rubric: +1 for 6

15).Given the equation:.$x + 10 = 10x$,.which side of the equal sign is larger?
a) The left side
b) The right side
c) They are the same
d) You cannot determine which side is larger unless you know what *x* is.

Objective: Given an equation, students interpret the equal sign as an indication that both sides are the same structure and not as a symbol that separates two structures. (Comprehension)

Scoring Rubric: +1 for C

16) Let B be the number of boys at a graduation party and let G be the number of girls at the same party. If there are twice as many boys at the party than there are girls at the party, write an equation that describes the relationship between B and G.

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)

Scoring Rubric: +1 for any form of the equation $B=2G$

17) Is the statement below true or false?
If Timmy is 5 feet tall and Danny is 60 inches tall then Timmy's height is equal to Danny's height?

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

18) Is the statement below true or false?
If Abby was born on April 4, 2001 and Sarah was born on April 4, 2001 then Abby is equal to Sarah.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

Appendix C

Instrument Used During Pilot Test

Pilot Test Instrument

1).Fill in the blank so that the equation below is true.
12 +.______. =. 13

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 1

2).Fill in the blank so that the equation below is true.
8.=._____ -.5

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 13

3) Fill in the blank so that the equation below is true.
8 + 4. =. _____ + 2

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for 10

4) Fill in the blank so that the equation below is true.
3 + 7. =.._____.+ 2. =.._____.- 2. =. _____.+ 1. =. _____

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Scoring Rubric:.+1 for 8,12,9,10 or an indication that they were attempting to make every quantity 10.

Secondary Scoring Rubric: +1 for 8,12,9,10 or an indication that they were attempting to make each quantity 10.

5).Is the equation below true or is the equation below false?
$5 \times 2 = 4 \times 2 + 2$
True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

6) Is the equation below true or is the equation below false?
$8 - 2 = 6. -3$
True ☐ False ☐

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for False

7) Is the equation below true or is the equation below false?
$7^{10}.=.(2+5)^{10}$
True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

8) How many different ways are there to fill in the blank below so that the resulting equation is true?
$\sqrt{13} + 2. =.$_______

a) There is no response that can be provided to make the resulting equation true.
b) There is exactly one response that can be provided to make the resulting equation true.
c) There are exactly two responses that can be provided to make the resulting equation true.
d) There are many responses that can be provided so that the resulting equation is true.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Scoring Rubric: +1 for d. ;.Secondary Rubric: +1 for d.

9).Is the equation below true or is the equation below false?

4 + 3.=.14.÷ 2

True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for true.

10).Which of the following best describes the meaning of the equation. $15 \div 3. = .5$?

a) Given the expression 15 divided by 3, the solution to the equation is the number 5.
b) (15 ÷ 3) and the number 5 are the same number.
c) When the number 15 is divided by the number 3 then the result is the number 5.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Scoring Rubric: +1 for (B or C).;..+1 for B on Secondary Rubric

11) The distributive property states:.*a (b + c) = ab + ac.*

Which of the following statements best describes the meaning of the distributive property?

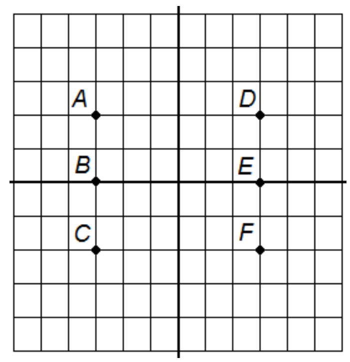
a) When solving a problem related to the expression *a (b + c)* , the correct solution is.*ab + ac*.
b) The value of the expression *a (b + c)* is calculated by adding the product *ab* to the product *ac*.
c) *a (b+c)*.and.*ab + ac*.are the same mathematical entity.

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for (B or C).;..+1 for C on Secondary Rubric

12) Is the statement below true or is the statement below false?
The set of points {A, B, C} shown below is equal to the set of points {D, E, F}.

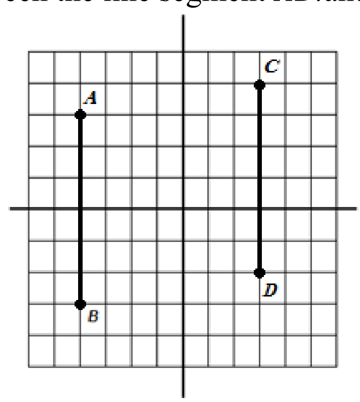


True ☐ False ☐

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

13).*A line segment.*$\overline{AB}$*.is defined to be the set of points on a line that include A and B and all points between A and B.*
Look at the diagram below and then choose the statement below the diagram that best describes the relationship between the line segment $\overline{AB}$.and the line segment $\overline{CD}$.



a) The line segments are equal because they have the same length.
b) The line segments are not equal because they have different lengths.
c) The line segments are not equal because they are not the same line segment.
d) You can't determine the length of the line segments so you can't determine if they are equal.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)
Scoring Rubric: +1 for C.

14).Let A be the number of apples in a basket and let P be the number of peaches in the same basket. If there are 10 apples in the basket and if $P + 4 = A$ then how many peaches are in the basket?

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application)

Scoring Rubric: +1 for 6

15).Which of the following is suggested by the equation $x + 10 = 10 x$?

a) The left side is larger than the right side.
b) The right side is larger than the left side.
c) They are the same.
d) You cannot determine which side is larger unless you know what x is.

Objective: Given an equation, students interpret the equal sign as an indication that both sides are the same structure and not as a symbol that separates two structures. (Comprehension)

Scoring Rubric: +1 for C

16) Let B be the number of boys at Joe's party and let G be the number of girls at Joe's party. If there are twice as many boys at Joe's party than there are girls at Joe's party, then which of the equations below describe the relationship between B and G.

a) 2B=G
b) 2+B=G
c) B=2G

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)

Scoring Rubric: +1 for c).

17) Is the statement below true or is the statement below false?
If Timmy is 5 feet tall and Danny is 60 inches tall then Timmy's height is equal to Danny's height.
True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True.

18) Is the statement below true or is the statement below false?
If Abby was born on April 4, 2001 at exactly the same time that Sarah was born on April 4, 2001 then Abby is equal to Sarah.
True ☐ False ☐

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

Appendix D

Alternative Instrument Used During Pilot Test for Significance Test

Alternative Instrument Used During Pilot Test

1).Fill in the blank so that the resulting equation is true.
$3 + 7 =.$_____

2).Fill in the blank so that the resulting equation is true.
$24 \div 8 =$ _____

3) Fill in the blank so that the resulting equation is true.
$10 \times 2.1 =.$_____

4) Fill in the blank so that the equation below is true.
3 + 7. =.._____.+ 2. =.._____.- 2. =. _____.+ 1. =. _____

Scoring Rubric: +1 for 8,12,9,10 or an indication that they were attempting to make each quantity 10.

5) Fill in the blank so that the resulting equation is true.
$\sqrt{25}$ =._____

6) Fill in the blank so that the resulting equation is true.
$\sqrt{16} + 2 =$ ______

7) How many different ways are there to fill in the blank below so that the resulting equation is true?
$\sqrt{13} + 2. =.$________

a) There is no response that can be provided to make the resulting equation true.
b) There is exactly one response that can be provided to make the resulting equation true.
c) There are exactly two responses that can be provided to make the resulting equation true.
d) There are many responses that can be provided so that the resulting equation is true.

Scoring Rubric: +1 for d

8) Multiply and write the result on the blank space provided.
*4 (x + 3)…….*______________

9) Which of the following are equal to $5\ (x^2 + 2x - 3)$

a. $5x^2 + 2x - 3$
b. $5x^2 + 10x - 15$
c. 32
d. 17

10) The distributive property states:.$a\ (b + c) = ab + ac$.
Which of the following statements <u>best</u> describes the meaning of the distributive property?

a) The value of the expression $a\ (b + c)$ is calculated by adding the product ab to the product ac.
b) $a\ (b+c)$.and.$ab + ac$.are the same mathematical entity.
c) When solving a problem related to the expression $a\ (b + c)$, the correct solution is.$ab + ac$.

Scoring Rubric: +1 for B

11) True or False:.
2 + 3 = 5

12) True or False:
17 - 5 = 22

13) True or False:
5 x 10 = 500

14) True or False:
14 ÷ 4 = 10

15).Which of the following <u>best</u> describes the meaning of the equation.$15 \div 3. = .5$?

a) (15 ÷ 3) and the number 5 are the same number.
b) When the number 15 is divided by the number 3 then the result is the number 5.
c) Given the expression 15 divided by 3, the solution to the equation is the number 5.

Scoring Rubric: +1 for A

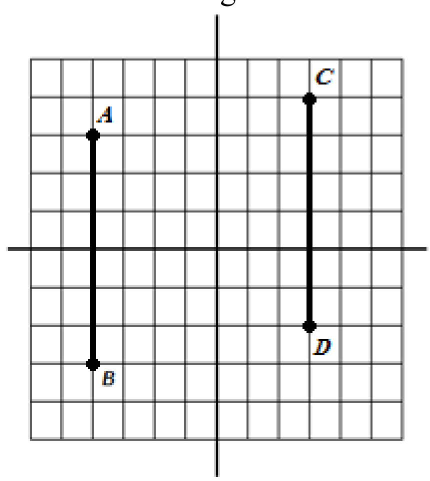
16) It took Tim 1 hour to complete an assignment, and it took Sam 70 minutes to complete the assignment. If they both started at the same time, who finished the assignment first?
a) Tim finished first
b) Sam finished first
c) They finished at the same time.

17) Stacey drove **20 miles** to get to school and Anne drove **20 kilometers** to get to school. Let *S* be the distance that Stacey drove to get to school and let *A* be the distance that Anne drove to get to school.
True or False: *A* is equal to *S*.

18) Is the statement below true or is the statement below false?
If Abby was born on April 4, 2001 and Sarah was born on April 4, 2001 then Abby is equal to Sarah.

Scoring Rubric: +1 for False

19).*A line segment.*$\overline{AB}$*.is defined to be the set of points on a line that include A and B and all points between A and B.*
Look at the diagram below and then choose the statement below the diagram that best describes the relationship between the line segment $\overline{AB}$.and the line segment $\overline{CD}$.



e) The line segments are equal because they have the same length.
f) The line segments are not equal because they have different lengths.
g) The line segments are not equal because they are not the same line segment.
h) You can't determine the length of the line segments so you can't determine if they are equal.

Scoring Rubric: +1 for drawing or indicating the exact same set of points.

20) If $t=5$ then which of the following expressions is larger: $(2 + t)$ or $(3 + t)$?
A) $(2+t)$ is larger.
B) $(3+t)$ is larger.
C) $(2+t)$ and $(3+t)$ are the same.
D) There is not enough information to determine which is larger.

21) If $a = 12.2$ and $b = 10.4$ then which of the following expressions is larger:.$(a - 1.2)$ or $(b + 0.6)$?
A) $(a – 1.2)$ is larger.
B) $(b + 0.6)$ is larger.
C) $(a – 1.2)$ and $(b + 0.6)$ are the same.
D) There is not enough information to determine.

22).Given the equation:.$x + 10.=.10 x$,.which side of the equal sign is larger?
A) The left side
B) The right side
C) They are the same
D) You cannot determine which side is larger unless you know what x is.

Scoring Rubric: +1 for C

23).If there are three apples in basket *A* and basket *B* contains three times more apples than basket *A* does, how many apples are in basket *B*?

24) If there are 10 fewer red marbles in a bag than there are blue marbles, and there are 18 blue marbles in the bag, how many red marbles are in the bag?

25) 16) Let B be the number of boys at Joe's party and let G be the number of girls at Joe's party. If there are twice as many boys at Joe's party than there are girls at Joe's party, then which of the equations below describe the relationship between B and G.

a) 2B=G
b) 2+B=G
c) B=2G

Scoring Rubric: +1 for c).

Appendix E

Wheeler Test for Comprehension of Equals

Wheeler Test for Comprehension of Equals

1).Fill in the blank so that the equation below is true.

12 +.______. =. 13

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 1

2).Fill in the blank so that the equation below is true.

8.=._____ -.5

Objective: Students will be able to interpret equals as a relation between two structures that are the same structure. (Comprehension)

Scoring Rubric: +1 for 13

3) Fill in the blank so that the equation below is true.

8 + 4. =. _____ + 2

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for 10

4) Fill in the blank so that the equation below is true.

3 + 7. =.._____.+ 2. =.._____.- 2. =. _____.+ 1. =. _____

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Scoring Rubric:.+1 for 8,12,9,10 or an indication that they were attempting to make every quantity 10.

Secondary Scoring Rubric: +1 for 8,12,9,10 or an indication that they were attempting to make each quantity 10.

5).Is the equation below true or is the equation below false?

$5 \times 2 = 4 \times 2 + 2$

True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

6) Is the equation below true or is the equation below false?
$8 - 2 \ = \ 6. -3$
True ☐ False ☐

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Scoring Rubric: +1 for False

7) Is the equation below true or is the equation below false?
$7^{10}.=.(2+5)^{10}$
True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True

8) Fill in the blank so that the resulting equation is true. Then answer the question that follows.
$\sqrt{13} \ + 2. =.$_______

How many different responses could you have provided in the blank above to make the resulting equation is true?
a) There is no response that can be provided to make the resulting equation true.
b) There is exactly one response that can be provided to make the resulting equation true.
c) There are exactly two responses that can be provided to make the resulting equation true.
d) There are many responses that can be provided so that the resulting equation is true.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Scoring Rubric: +1 for entering any expression that is equal to the number that is two bigger than the square root of 13. +1 for choosing d.

9).Is the equation below true or is the equation below false?

$4 + 3 = 14 \div 2$

True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for true.

10).Which of the following best describes the meaning of the equation $15 \div 3 = 5$?

a) Given the expression $15 \div 3$, the correct response is the number 5.
b) $(15 \div 3)$ and the number 5 are the same number.

Scoring Rubric: +1 for (b).;..+1 for (b) on Secondary Rubric

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

11) The distributive property states: $a(b + c) = ab + ac$.
Which of the following statements best describes the meaning of the equation above?

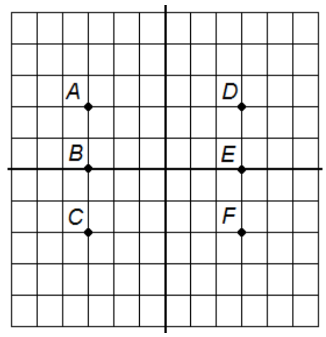
a) When solving a problem related to the expression $a(b + c)$, the correct response is $ab + ac$.
b) $a(b+c)$ and $ab + ac$ are the same mathematical entity.

Scoring Rubric: +1 for (b) ;..+1 for (b) on Secondary Rubric

Objective: Given an equation, students will consider the relevance of the equal sign as an essential part of interpreting the meaning of an equation.

Objective: When placed in an expression with numerals, variables, and operators, students will interpret equals as a relation and not as an indication to calculate. (Comprehension)

12) Is the statement below true or is the statement below false?
The set of points {A, B, C} shown below is equal to the set of points {D, E, F}.

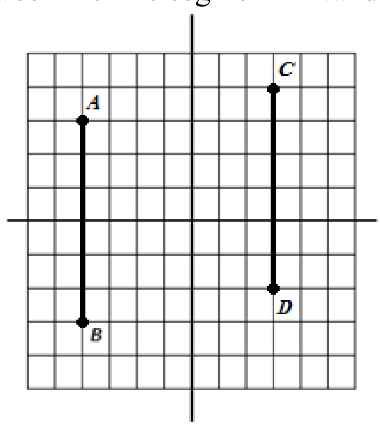


True ☐ False ☐

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

13).*A line segment.*$\overline{AB}$.*is defined to be the set of points on a line that include A and B and all points between A and B.*
Look at the diagram below and then choose the statement below the diagram that best describes the relationship between the line segment $\overline{AB}$.and the line segment $\overline{CD}$.



a) The line segments are equal because they have the same length.
b) The line segments are not equal because they have different lengths.
c) The line segments are not equal because they are not the same line segment.
d) You can't determine the length of the line segments so you can't determine if they are equal.

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for C.

14).Let A be the number of apples in a basket and let P be the number of peaches in the same basket. If there are 10 apples in the basket and if.$P + 4 = A$.then how many peaches are in the basket?

Objective: Given an equation, students can apply their understanding of equals to determine unknown quantities associated with the equation. (Application)

Scoring Rubric: +1 for 6

15).Which of the following is suggested by the equation.$x + 10.=.10x$?

a) The left side is larger than the right side.
b) The right side is larger than the left side.
c) They are the same.
d) You cannot determine which side is larger unless you know what x is.

Objective: Given an equation, students interpret the equal sign as an indication that both sides are the same structure and not as a symbol that separates two structures. (Comprehension)

Scoring Rubric: +1 for C

16) Let B be the number of boys at Joe's party and let G be the number of girls at Joe's party. If there are twice as many boys at Joe's party than there are girls at Joe's party, then which of the equations below describe the relationship between B and G.

a) 2B=G
b) 2+B=G
c) B=2G

Objective: Students are able to use their understanding of equals to correctly model a relation between equal structures. (Application and Comprehension)

Scoring Rubric: +1 for c).

17) Is the statement below true or is the statement below false?
If Timmy is 5 feet tall and Danny is 60 inches tall then Timmy's height is equal to Danny's height.
True ☐ False ☐

Objective: Students understand that two structures that are the same structure are equal even when different units, operators, or notations are used to express them. (Comprehension)

Scoring Rubric: +1 for True.

18) Is the statement below true or is the statement below false?
If Abby was born on April 4, 2001 at exactly the same time that Sarah was born on April 4, 2001 then Abby is equal to Sarah.
True ☐ False ☐

Objective: Students understand that two structures that have the same properties are not equal unless they are the same structure. (Comprehension)

Scoring Rubric: +1 for False

CURRICULUM VITAE

GREGORY D. WHEELER

EDUCATION

1988–1989, 1991-1992 Ricks College, Rexburg, ID

- Associate of Science General Education
- 1991-1992 Pre-Physical Therapy Student of the Year.

1992–1995 Utah State University, Logan, UT

- Bachelor of Science Mathematics Education.
- Graduated summa cum laude.

1995-1997 Utah State University, Logan, UT

- Masters of Mathematics
- Special Project: Computer aided manipulatives for mathematics teaching.
- 1996-1997 outstanding teaching assistant for the mathematics department.

2004-Present Utah State University, Logan, UT

- Doctorate of Education student with an emphasis in Curriculum and Development.
- June 2007 passed comprehensive examinations.
- October 2008 successfully defended dissertation proposal: Assessment of College Students' Understanding of the Equals Relation: A Development and Validation of an Instrument
- August 2010 successful final defense of dissertation: Assessment of College Students' Understanding of the Equals Relation: A Development and Validation of an Instrument

EXPERIENCE

April-June 1995 — *Student Teacher*, Willow Valley Middle School, Hyrum, UT

- Taught math and health classes to 6th and 7th graders.

August 1995-May 1997 — *Mathematics Teaching Assistant*, Utah State University, Logan, UT

- Taught math classes ranging from college algebra to calculus techniques.
- Teaching award for Outstanding Teaching Assistant. Awarded by Utah State University Department of Mathematics and Statistics, May 1997.

August 1997-Present — *Mathematics and Statistics Lecturer*, Utah State University Uintah Basin, Roosevelt, UT

- Taught math classes ranging from Intermediate Algebra to Introduction to Analysis.
- Taught mathematics-teaching methods course for prospective secondary education teachers
- Taught a wide range of concurrent enrollment mathematics classes to high school students.
- Served as Faculty Representative for USUUB from May 2001 to May 2003.
- Utah State University Regional Campus and Distance Education instructor of the year for 2003.
- Presentation of concurrent enrollment and EDNET practices at the 2004 Concurrent Enrollment Symposium held at UVSC for the state of Utah.
- Presentation on the use of internet conferencing technology to teach and tutor students at multiple locations. (2009) National Business Educators Association Annual Conference. Chicago, IL
- Award of recognition for innovated use of technology. Awarded by the Utah System of Higher Education, May 15, 2009.
- Aided in the development and implementation of the math 1050 online course along with Dr. Robert Heal. January 2010-Present.

- Presentation of paper: *The Scholarship of Teaching and the Standards for the assessment of Scholarship*. Regional Campus and Distance Education retreat in Vernal, UT. March 5, 2010

- Development and implementation of the USU math 1010 online course. August 2010-Present.