Rényi $\alpha$-Entropy of Tournament Digraphs

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What Is Entropy?

Definition

Given a discrete probability distribution \( p = (p_1, \ldots, p_n) \), *Shannon entropy* is defined to be

\[
S(p) := \sum_{i=1}^{n} p_i \log_2 \left( \frac{1}{p_i} \right)
\]

- A measure of uncertainty or disorder.
- Expected information content.
A tournament is a complete directed graph, or a model of pairwise comparisons.
Calculating Tournament Entropy
The Adjacency Matrix

\[ A(T) \text{ where } a_{ij} = 1 \text{ if } i \rightarrow j \]

\[
\begin{bmatrix}
0 & 1 & 1 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 \\
0 & 0 & 0 & 1 & 1 \\
1 & 0 & 0 & 0 & 1 \\
1 & 1 & 0 & 0 & 0 \\
\end{bmatrix}
\]
Calculating Tournament Entropy

The Diagonal Matrix

\[
D(T) = \begin{bmatrix}
2 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 \\
0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 2
\end{bmatrix}
\]
Calculating Tournament Entropy

The Laplacian

Definition

The Laplacian of a tournament $T$ is defined as $L(T) = D(T) - A(T)$

Definition

The normalized Laplacian is defined as

$$\bar{L}(T) := \frac{1}{\text{tr}L(T)}L(T)$$
Typically, use the spectrum of $\bar{L}(T)$ as the discrete probability distribution.

- With tournaments, the spectrum of $\bar{L}(T)$ will contain complex eigenvalues.
- How do we handle complex eigenvalues, and what do they mean?
Alfréd Rényi generalized the definition of entropy to what is known as Rényi $\alpha$-entropy, for $\alpha > 1$.

**Definition**

For $\alpha > 1$, given a discrete probability distribution $p = (p_1, ..., p_n)$, the Rényi $\alpha$-entropy is

$$H_\alpha(p) := \frac{1}{1 - \alpha} \log_2 \left( \sum_{i=1}^{n} p_i^\alpha \right)$$
We thus define the entropy of a tournament $T$ as the Rényi $\alpha$-entropy of the spectrum of $\overline{L}(T)$.

$$H_\alpha(T) := -\frac{1}{1 - \alpha} \log_2 \left( \sum_{i=1}^{n} \lambda_i^\alpha \right)$$
Definition

The score sequence of a tournament is the sequence consisting of the out-degrees of all vertices of the tournament in nondecreasing order.

- The out-degree of vertex $i$, denoted $d_i$, is the number of vertices which vertex $i$ beats.
Theorem (Brown, et al. 2017)

Rényi 2-Entropy depends only upon the score sequence of a tournament and is equivalent to

\[ H_2(T) = - \log_2 \left( \frac{1}{\binom{n}{2}^2} \sum_{i=1}^{n} d_i^2 \right) \]

- Follows from the fact that \( \sum_{i=1}^{n} \lambda_i^2 = Tr(\overline{L}(T))^2 \)
Theorem (Brown, et al. 2017)

Rényi 3-Entropy depends only upon the score sequence of a tournament and is equivalent to

$$H_3(T) = -\frac{1}{2} \log_2 \left[ \frac{1}{(n/2)^3} \left( \sum_{i=1}^{n} d_i^3 - 3 \binom{n}{3} - 3 \sum_{i=1}^{n} \binom{d_i}{2} \right) \right]$$

- Follows from the fact that $\sum_{i=1}^{n} \lambda_i^3 = Tr \left( \bar{L}(T)^3 \right)$
- Depends on the number of directed 3-cycles, which can be written as $3$-cycles $= \binom{n}{3} - \sum_{i=1}^{n} \binom{d_i}{2}$
Tournaments with Maximal Entropy

**Definition**

A tournament is *regular* if each vertex has an out-degree of \( \frac{n-1}{2} \).

**Definition**

A tournament is *semi-regular* if exactly half of the vertices have an out-degree of \( \frac{n}{2} \) and half have an out-degree of \( \frac{n-2}{2} \).
Tournaments with Maximal Entropy

Theorem (Brown, et al. 2017)

For all tournaments on \( n \) vertices:

- If \( n \) is odd, then the Rényi 2- and 3-entropy will be maximized by all regular tournaments on \( n \) vertices.
- If \( n \) is even, then the Rényi 2- and 3-entropy will be maximized by all semi-regular tournaments on \( n \) vertices.
Tournaments with Maximal Entropy

\[ \text{Rényi } \alpha \text{-Entropy of Tournament Digraphs} \]
Tournaments with Maximal Entropy

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Definition

The *transitive tournament* on $n$ vertices is a tournament such that there exists exactly one vertex with out-degree $n-1$, $n-2$, ..., 0.

Theorem (Brown, et al. 2017)

For all tournaments on $n$ vertices, the Rényi 2- and 3-entropy is minimized by the *transitive tournament* on $n$ vertices.
Tournaments with Minimal Entropy

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Further Directions

- Can we find a closed formula (in terms of graph invariants) for Rényi 4-Entropy?
- Is there another form of entropy which can further distinguish tournaments?
Thank You