

Quaternions, Octonions, and Electromagnetism

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Review: Real and Imaginary numbers

Real numbers (\mathbb{R}):

- 1, 2, 3...
- $\frac{1}{2}, \frac{7}{3} \dots$
- $\sqrt{2}, e \approx 2.718 \dots$

Imaginary (Complex) numbers (\mathbb{C}):

- $i = \sqrt{-1}$
- General Form: $a + bi$, where $a, b \in \mathbb{R}$

Quaternions: \mathbb{H}

- **General Form:** $a + bi + cj + dk$. **Multiplication:**
 $i^2 = j^2 = k^2 = ijk = -1$
- **Noncommutative:** $q_1q_2 \neq q_2q_1$ for $q_1, q_2 \in \mathbb{H}$
- **Pure Quaternion:** $bi + cj + dk$, called the imaginary part. Looks like $b\hat{x} + c\hat{y} + d\hat{k}$.
- $q_1 \cdot q_2 = \text{Re}(q_1 \cdot q_2) + \text{Im}(q_1 \cdot q_2)$.
 \implies Dot product: $-\text{Re}(q_1 \cdot q_2)$
 \implies Cross product: $\text{Im}(q_1 \cdot q_2)$
- **E & M:** Given by Maxwell equations, originally expressed in terms of \mathbb{H} . Vectors eventually became the convention.

Octonions: \mathbb{O}

- General form: $a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$, where each $a_i \in \mathbb{R}$, and each $e_i^2 = -1$ except for e_0 , which is 1 (placeholder). Many multiplication tables exist.
- Not associative! i.e. $a(bc) \neq (ab)c$. Not commutative.
- Alternative: $a(bc) = (ab)c$ iff either $b = a$ or $b = c$.
 $a(ac) = (aa)c$, $(ac)c = a(cc)$.
- Can we use octonions instead of 7+1 dimensional vectors in Electromagnetism? If so, why even bother?

Source-Free Octonionic Maxwell Equations

- Conjecture: ME's look the same.
- Some key vector identities hold true for Octonions. In particular, one that results in wave equations for both \mathbf{E} and \mathbf{B} . Solutions: $\mathbf{E} = \mathbf{E}_0 \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Similar for \mathbf{B} .
- Maxwell Equations impose extra conditions: $\mathbf{B} = \frac{1}{\omega} \mathbf{k} \times \mathbf{E}$. Additionally, \mathbf{k} is perpendicular to both \mathbf{E} and \mathbf{B} .
- In order to compare to the Conventional theory, we will need to construct an Octonionic Faraday Tensor (Matrix). We use the same formulas as in 3+1 dimensional theory.
- We can express this Tensor purely in terms of components of \mathbf{E} .

Source-Free Conventional Maxwell Equations

- Conventionally, there is no vector field notion of \mathbf{B} : no cross product.
- \mathbf{B} is a Tensor (Matrix), part of $F_{\mu\nu} : F_{ij}$. Electric field: F_{i0} .
- Solutions to the conventional equations involve derivatives of \mathbf{A} , the 8-dimensional vector potential. $F_{\mu\nu}$ is then written in terms of \mathbf{A} .
- “Gauge freedom” allows us to choose \mathbf{A} such that the divergence of \mathbf{A} is zero. Then, we get wave equation solutions.

Comparison of Conventional and Octonionic theories

- Equal Components = Equal Tensors = Equal Theories
- Octonionic Faraday Tensor in terms of components of \mathbf{E} ,
Conventional Faraday Tensor in terms of \mathbf{A} .
- Well-defined function with a well-defined inverse using
 $E^j = \omega A^j - k_j A^0$.
- Note: only seven independent components of \mathbf{A} due to choosing
 $A^\nu_{,\nu} = 0$.
- Reminiscent of $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V$.

Summary

- One can approach source-free, $7+1$ dimensional electrodynamics conventionally or by way of Octonions.
- What about the general Maxwell Equations?
- What about Quantum Mechanics? Is GR + QM a matter of QM in various algebraic environments?
- Or are Octonions the solution to GR + QM? Dr. Cohl Furey

Thank You

Questions?