Quaternions, Octonions, and Electromagnetism

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Review: Real and Imaginary numbers

Real numbers ($\mathbb{R}$):

- $1, 2, 3...$
- $\frac{1}{2}, \frac{7}{3}...$
- $\sqrt{2}, e \approx 2.718...$

Imaginary (Complex) numbers ($\mathbb{C}$):

- $i = \sqrt{-1}$

- General Form: $a + bi$, where $a, b \in \mathbb{R}$
Quaternions: \( \mathbb{H} \)

- **General Form:** \( a + bi + cj + dk \). Multiplication:
  \[ i^2 = j^2 = k^2 = ijk = -1 \]

- **Noncommutative:** \( q_1 q_2 \neq q_2 q_1 \) for \( q_1, q_2 \in \mathbb{H} \)

- **Pure Quaternion:** \( bi + cj + dk \), called the imaginary part. Looks like \( b\hat{x} + c\hat{y} + d\hat{k} \).

- \( q_1 \cdot q_2 = \text{Re}(q_1 \cdot q_2) + \text{Im}(q_1 \cdot q_2) \).
  \[ \implies \text{Dot product:} -\text{Re}(q_1 \cdot q_2) \]
  \[ \implies \text{Cross product:} \text{Im}(q_1 \cdot q_2) \]

- **E & M:** Given by Maxwell equations, originally expressed in terms of \( \mathbb{H} \). Vectors eventually became the convention.
Octonions:

- General form: \( a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7 \), where each \( a_i \in \mathbb{R} \), and each \( e_i^2 = -1 \) except for \( e_0 \), which is 1 (placeholder). Many multiplication tables exist.

- Not associative! i.e. \( a(bc) \neq (ab)c \). Not commutative.

- Alternative: \( a(bc) = (ab)c \) iff either \( b = a \) or \( b = c \).
  \( a(ac) = (aa)c \), \( (ac)c = a(cc) \).

- Can we use octonions instead of 7+1 dimensional vectors in Electromagnetism? If so, why even bother?
Source-Free Octonionic Maxwell Equations

- Conjecture: ME’s look the same.

- Some key vector identities hold true for Octonions. In particular, one that results in wave equations for both $E$ and $B$. Solutions: $E = E_0 \cos(k \cdot x - \omega t)$. Similar for $B$.

- Maxwell Equations impose extra conditions: $B = \frac{1}{\omega} k \times E$. Additionally, $k$ is perpendicular to both $E$ and $B$.

- In order to compare to the Conventional theory, we will need to construct an Octonionic Faraday Tensor (Matrix). We use the same formulas as in 3+1 dimensional theory.

- We can express this Tensor purely in terms of components of $E$. 
Source-Free Conventional Maxwell Equations

- Conventionally, there is no vector field notion of $B$: no cross product.

- $B$ is a Tensor (Matrix), part of $F_{\mu\nu}: F_{ij}$. Electric field: $F_{i0}$.

- Solutions to the conventional equations involve derivatives of $A$, the 8-dimensional vector potential. $F_{\mu\nu}$ is then written in terms of $A$.

- “Gauge freedom” allows us to choose $A$ such that the divergence of $A$ is zero. Then, we get wave equation solutions.
Comparison of Conventional and Octonionic theories

• Equal Components = Equal Tensors = Equal Theories

• Octonionic Faraday Tensor in terms of components of $E$, Conventional Faraday Tensor in terms of $A$.

• Well-defined function with a well-defined inverse using
$E^j = \omega A^j - k_j A^0$.

• Note: only seven independent components of $A$ due to choosing $A^\nu, \nu = 0$.

• Reminiscent of $E = -\frac{\partial A}{\partial t} - \nabla V$. 
Summary

• One can approach source-free, 7+1 dimensional electrodynamics conventionally or by way of Octonions.

• What about the general Maxwell Equations?

• What about Quantum Mechanics? Is GR + QM a matter of QM in various algebraic environments?

• Or are Octonions the solution to GR + QM? Dr. Cohl Furey
Questions?