Quaternions, Octonions, and Electromagnetism

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Review: Real and Imaginary numbers

Real numbers ($\mathbb{R}$):

- $1, 2, 3...$
- $\frac{1}{2}, \frac{7}{3}...$
- $\sqrt{2}, e \approx 2.718...$

Imaginary (Complex) numbers ($\mathbb{C}$):

- $i = \sqrt{-1}$
- General Form: $a + bi$, where $a, b \in \mathbb{R}$

Quaternions: $\mathbb{H}$

• General Form: $a + bi + cj + dk$. Multiplication:
$i^2 = j^2 = k^2 = ijk = -1$

• Noncommutative: $q_1 q_2 \neq q_2 q_1$ for $q_1, q_2 \in \mathbb{H}$

• Pure Quaternion: $bi + cj + dk$, called the imaginary part. Looks like $b\hat{x} + c\hat{y} + d\hat{k}$.

• $q_1 \cdot q_2 = \text{Re}(q_1 \cdot q_2) + \text{Im}(q_1 \cdot q_2)$.
$\implies$ Dot product: $-\text{Re}(q_1 \cdot q_2)$
$\implies$ Cross product: $\text{Im}(q_1 \cdot q_2)$

• E & M: Given by Maxwell equations, originally expressed in terms of $\mathbb{H}$. Vectors eventually became the convention.

Octonions: $\mathbb{O}$

- General form: $a_0e_0 + a_1e_1 + a_2e_2 + a_3e_3 + a_4e_4 + a_5e_5 + a_6e_6 + a_7e_7$, where each $a_i \in \mathbb{R}$, and each $e_i^2 = -1$ except for e_0, which is 1 (placeholder). Many multiplication tables exist.

- Not associative! i.e. $a(bc) \neq (ab)c$. Not commutative.

- Alternative: $a(bc) = (ab)c$ iff either $b = a$ or $b = c$.
$a(ac) = (aa)c$, $(ac)c = a(cc)$.

- Can we use octonions instead of 7+1 dimensional vectors in Electromagnetism? If so, why even bother?

Source-Free Octonionic Maxwell Equations

• Conjecture: ME's look the same.

• Some key vector identities hold true for Octonions. In particular, one that results in wave equations for both $\mathbf{E}$ and $\mathbf{B}$. Solutions: $\mathbf{E} = \mathbf{E_0} \cos(\mathbf{k} \cdot \mathbf{x} - \omega t)$. Similar for $\mathbf{B}$.

• Maxwell Equations impose extra conditions: $\mathbf{B} = \dfrac{1}{\omega}\mathbf{k} \times \mathbf{E}$. Additionally, $\mathbf{k}$ is perpendicular to both $\mathbf{E}$ and $\mathbf{B}$.

• In order to compare to the Conventional theory, we will need to construct an Octonionic Faraday Tensor (Matrix). We use the same formulas as in 3+1 dimensional theory.

• We can express this Tensor purely in terms of components of $\mathbf{E}$.

Source-Free Conventional Maxwell Equations

• Conventionally, there is no vector field notion of $\mathbf{B}$: no cross product.

• $\mathbf{B}$ is a Tensor (Matrix), part of $F_{\mu\nu}$: F_{ij}. Electric field: F_{i0}.

• Solutions to the conventional equations involve derivatives of $\mathbf{A}$, the 8-dimensional vector potential. $F_{\mu\nu}$ is then written in terms of $\mathbf{A}$.

• "Gauge freedom" allows us to choose $\mathbf{A}$ such that the divergence of $\mathbf{A}$ is zero. Then, we get wave equation solutions.

Comparison of Conventional and Octonionic theories

- Equal Components = Equal Tensors = Equal Theories

- Octonionic Faraday Tensor in terms of components of $\mathbf{E}$, Conventional Faraday Tensor in terms of $\mathbf{A}$.

- Well-defined function with a well-defined inverse using $E^j = \omega A^j - k_j A^0$.

- Note: only seven independent components of $\mathbf{A}$ due to choosing ${A^\nu}_{,\nu} = 0$.

- Reminiscent of $\mathbf{E} = -\frac{\partial \mathbf{A}}{\partial t} - \nabla V$.

Summary

• One can approach source-free, 7+1 dimensional electrodynamics conventionally or by way of Octonions.

• What about the general Maxwell Equations?

• What about Quantum Mechanics? Is GR + QM a matter of QM in various algebraic environments?

• Or are Octonions the solution to GR + QM? Dr. Cohl Furey

Thank You

Questions?