Gravitational Anomalies & Elliptic Curves

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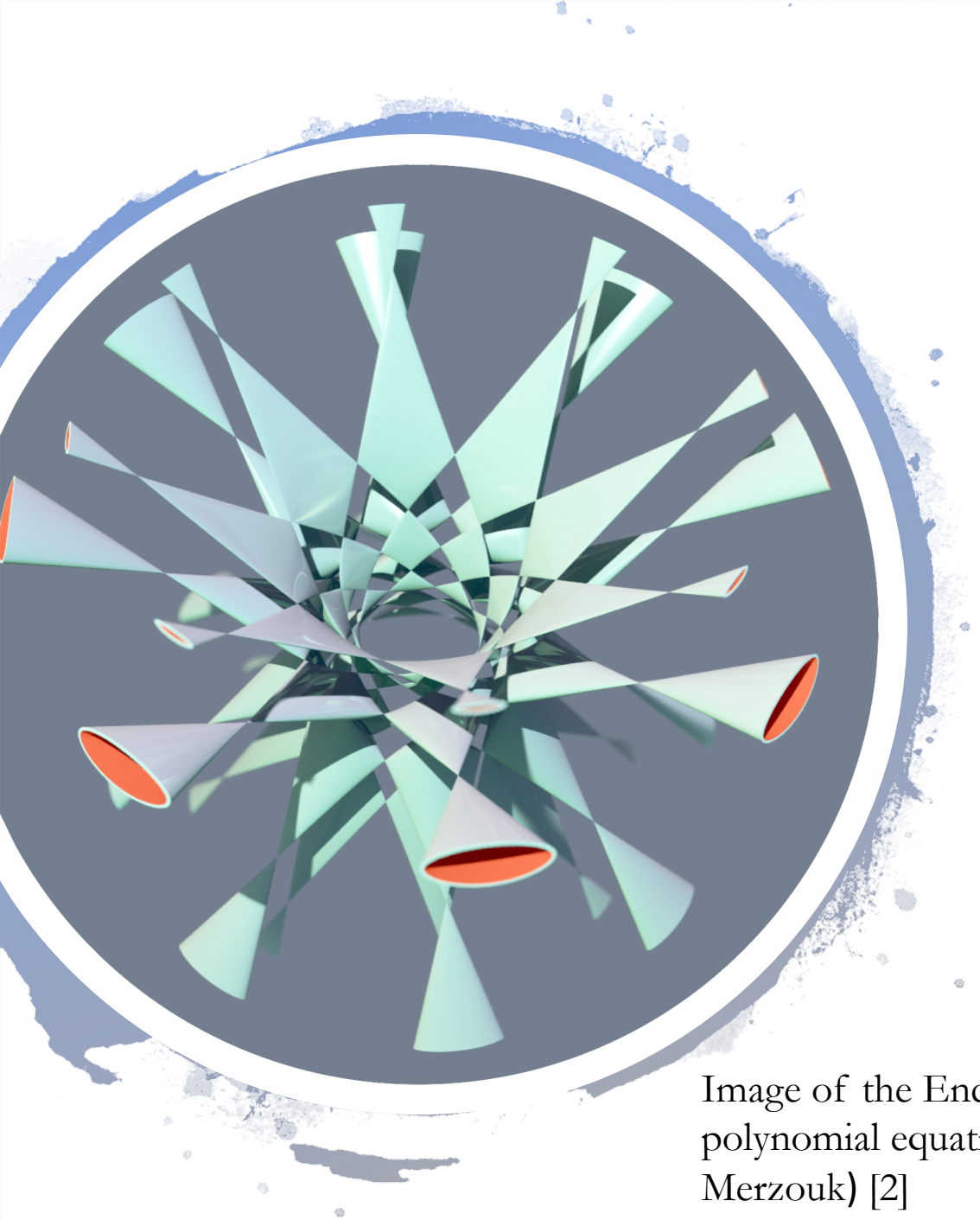


Well, Howdy!

- I'm a third year math PhD student
- Advised by Dr. Andreas Malmendier
- My research is in algebraic geometry
- Translating algebra into geometry – and back! This subject has it all, folks: calculus, linear algebra, differential equations, geometry, topology, and, of course, algebra.

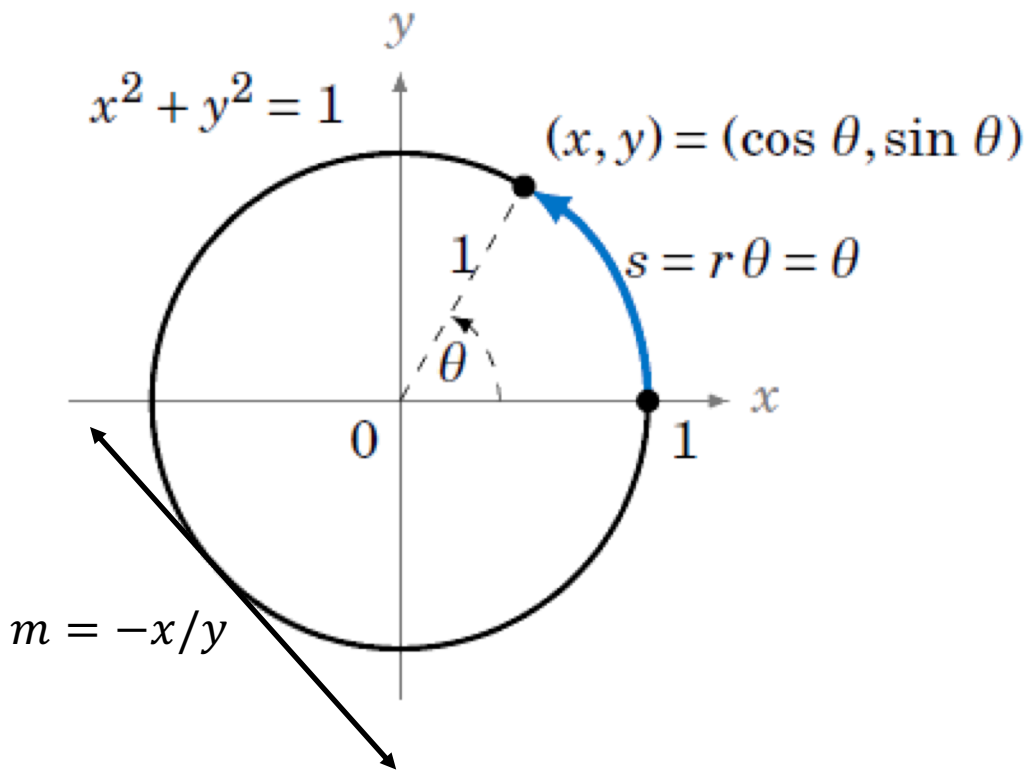
Image of the Endrass Octic – geometric solutions of polynomial equations of degree 8 (Abdelaziz Nait Merzouk) [2]

A Note on Abstraction and Accessibility

- Mathematics is notoriously inaccessible to those who do not speak its many languages
- This can be hugely intimidating – even to someone like myself who has studied extensively
- The goal of this talk is to increase accessibility to those who are blind / low vision by providing image captions and descriptions
- Will provide slides upon request

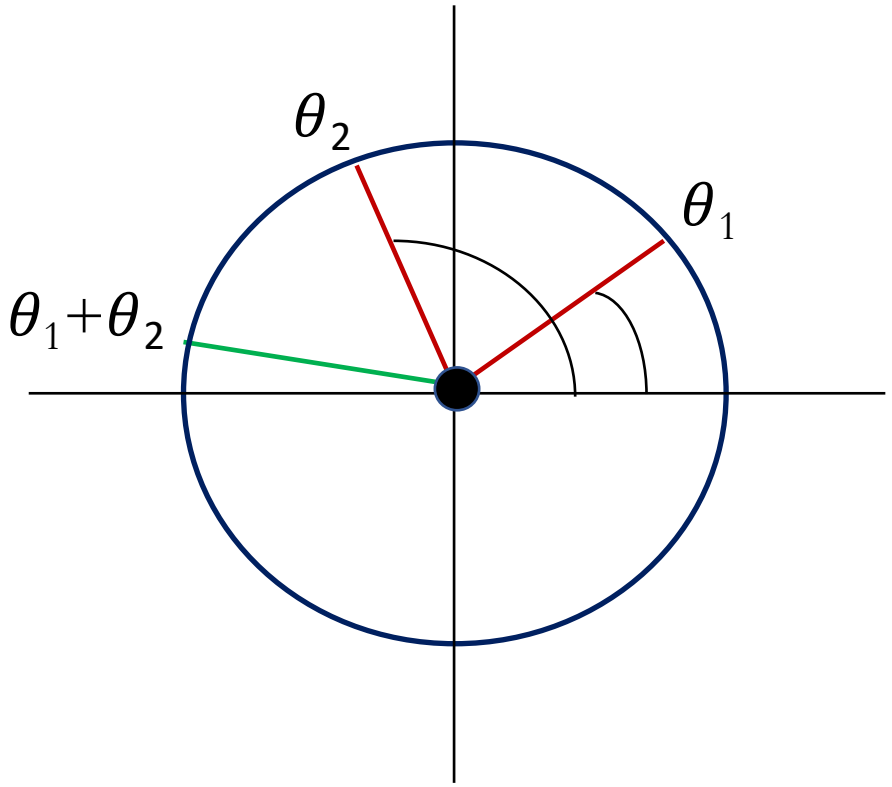
Algebraic Geometry and the Circle

- Geometry
- Algebra
- Calculus
- Topology



The Unit Circle – a geometric figure described by a polynomial equation. The equation encodes geometric and topological information.

Algebra and Topology



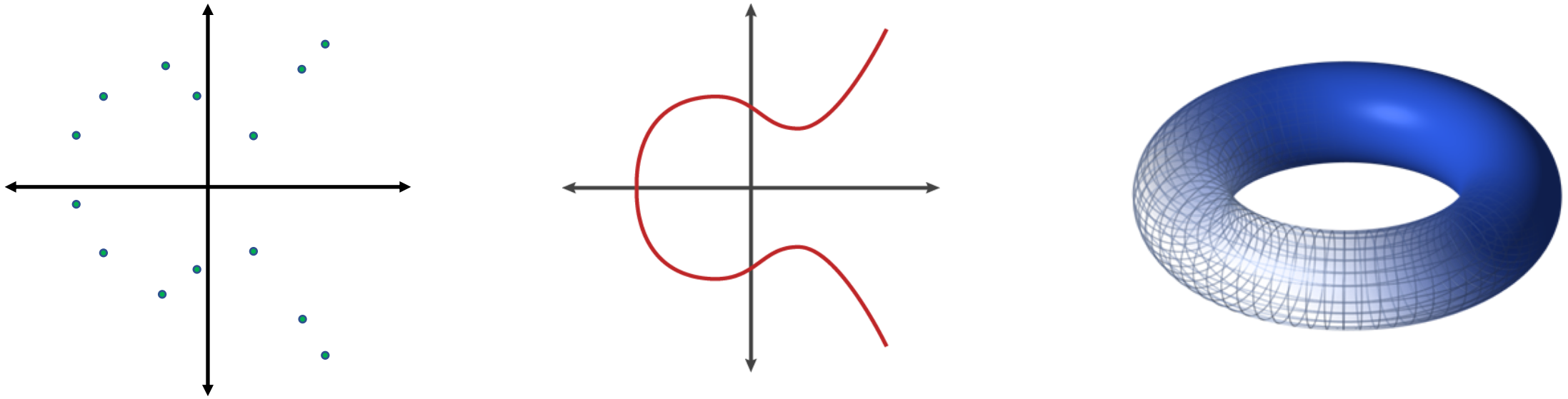
Addition of angles turns the circle into a "group".

Topology studies properties of a space that don't change with continuous deformation – like holes!

Animated image of a coffee cup deforming continuously into a donut – mathematically, a torus. [5] On the left, an circle admits an arithmetic operation by adding angles in a counterclockwise manner. This turns the circle into a "group".

Elliptic Curves – Not ellipses!

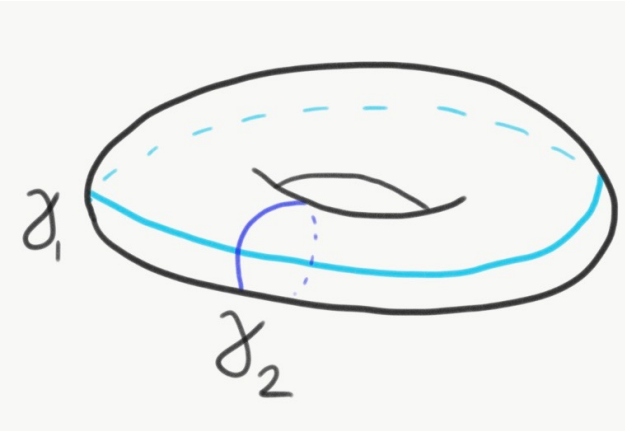
- Solutions of the polynomial equation $y^2 = 4x^3 - g_2x - g_3$
- Letting the variables range over different sets of numbers yield different curves:



Images of elliptic curves over various number fields – from left to right, the rational numbers, the real numbers [6], and the complex numbers. The geometry and topology changes drastically in each case.

Elliptic Curves are… flat?

- Unrolling a torus into a parallelogram allows us to give an elliptic curve an arithmetic structure – that is, an elliptic curve is a group [Husemöller, 7]
- Arithmetic depends on τ.
- Same arithmetic = isomorphism



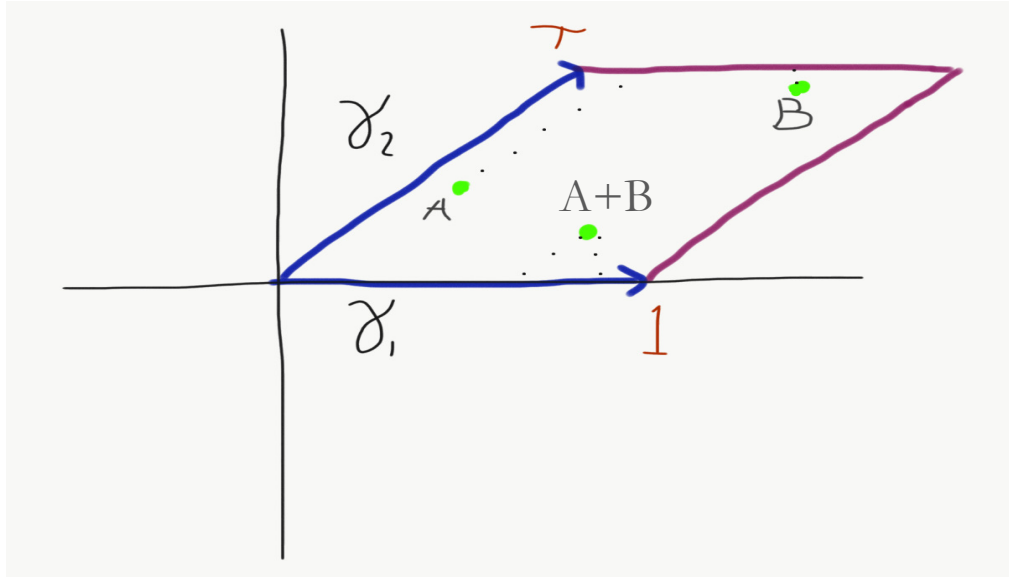
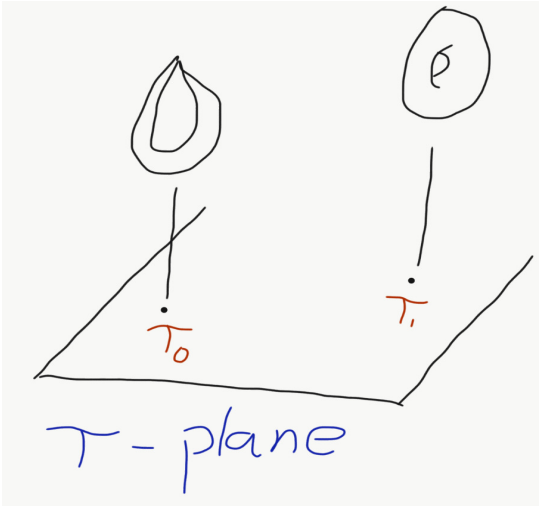


Image depicting how an elliptic curve forms a "group" – a space with an arithmetic structure with nice properties.

The Universal Bundle of Elliptic Curves

- The collection of all isomorphism classes is parameterized by the sphere
- Attaching a representative elliptic curve above each point gives an elliptic fibration over the sphere – the universal bundle over the j-sphere



$$\tau \to j(\tau)$$

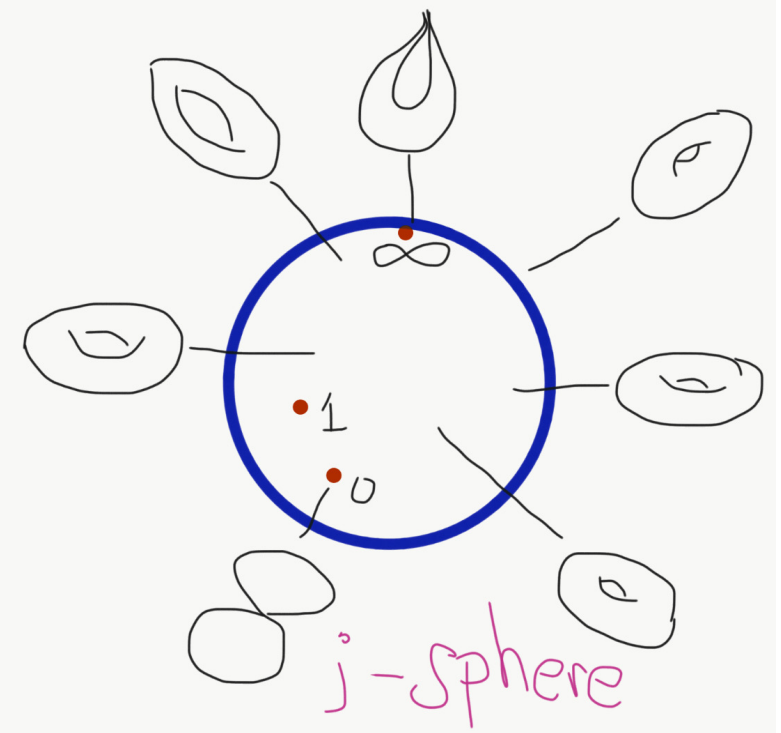
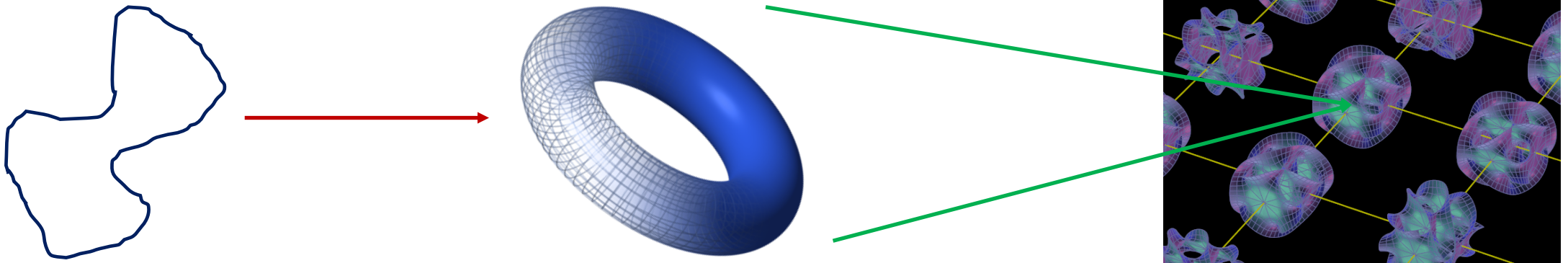


Image depicting elliptic curves associated to their isomorphism classes, parameterized by the sphere.

Elliptic Curves are everywhere!

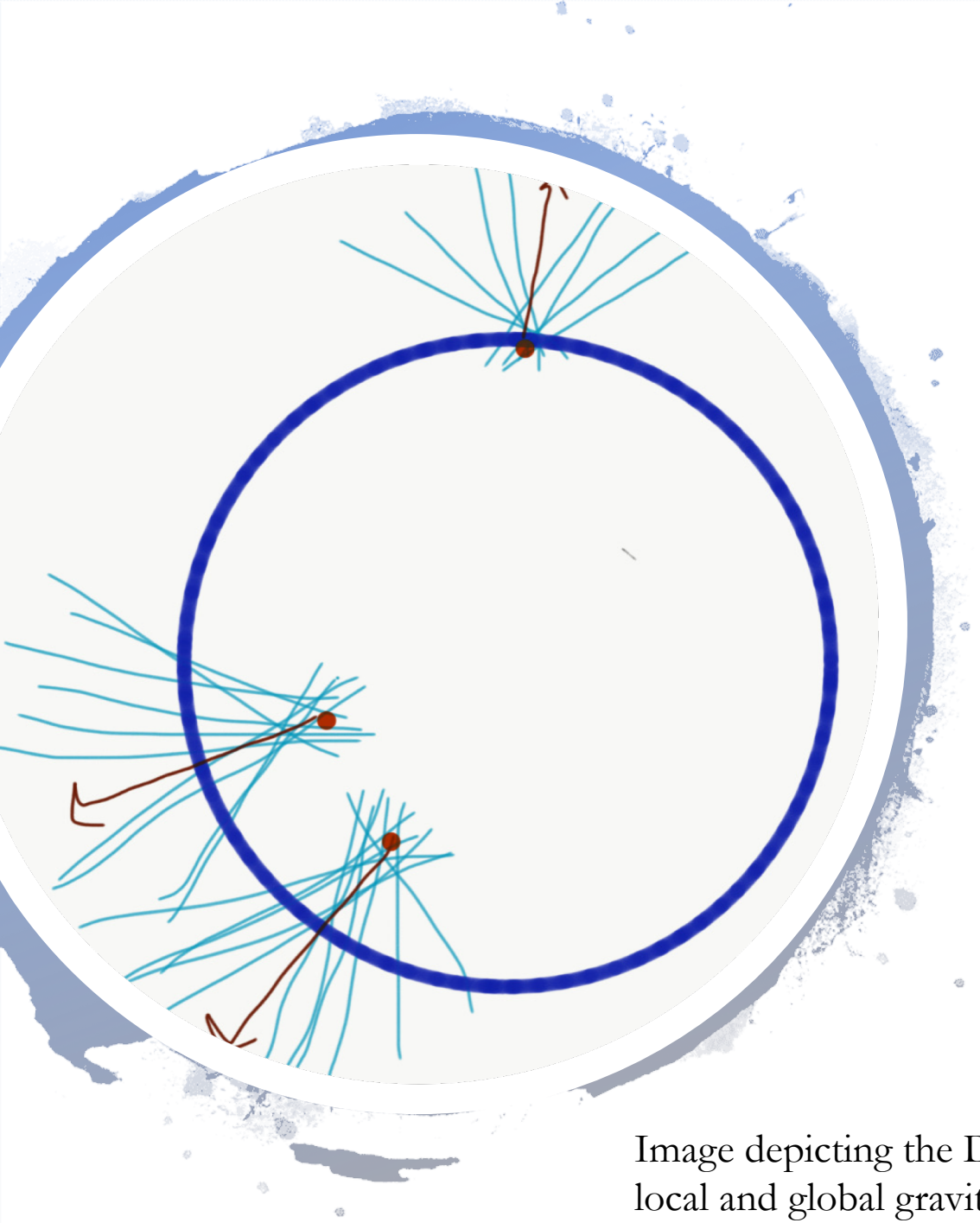
- Cryptography [9a]
- Fermat's Last Theorem [Wiles, 9b]
- Birch & Swinnerton-Dyer Conjecture – solve for $1M! [9c]
- String Theory / F-Theory [Aspinwall, 9d]



Images show how elliptic curves arise in string theory – propagating strings can sweep out a torus, which then embed into a higher dimensional spacetime

Physics on an Elliptic Curve

- In string theory, we study matter fields called fermions (electrons, quarks, etc) that live on elliptic curves. [Alvarez, et. al 10a]
- The zero modes of these fields are measured by a differential operator D_τ , the Dirac operator [Alvarez, 10b]
- There are always a finite number of zero modes
- The index ind(D_τ) is a measure of this number – may change with τ



Gravitational Anomalies

- As the parameter j varies, zero modes of the Dirac operator vary as well – the index $\mathrm{ind}(D_\tau)$ gives rise to a geometric structure on the sphere called the Determinant Line Bundle [Bost, 11]
- If the number of zero modes jump, the theory has anomalies
- Anomalies are gravitational, because they don't arise from matter fields
- Local anomaly is geometrically represented by curvature
- Global anomaly represented by impulse functions called current contributions

Image depicting the Determinant Line Bundle, encoding local and global gravitational anomalies

Interpretation and Resolution

- Local anomaly represents failure of partition function of the associated quantum field theory to generate fields in a consistent way [Freed, 12a] – can be resolved
- Global anomaly represents where elliptic curves have increased symmetry – more fundamental, harder to resolve
- Our research, applying results from [Friedman, et al, 12b] shows how to resolve the anomalies – this means fixing each D_τ operator.

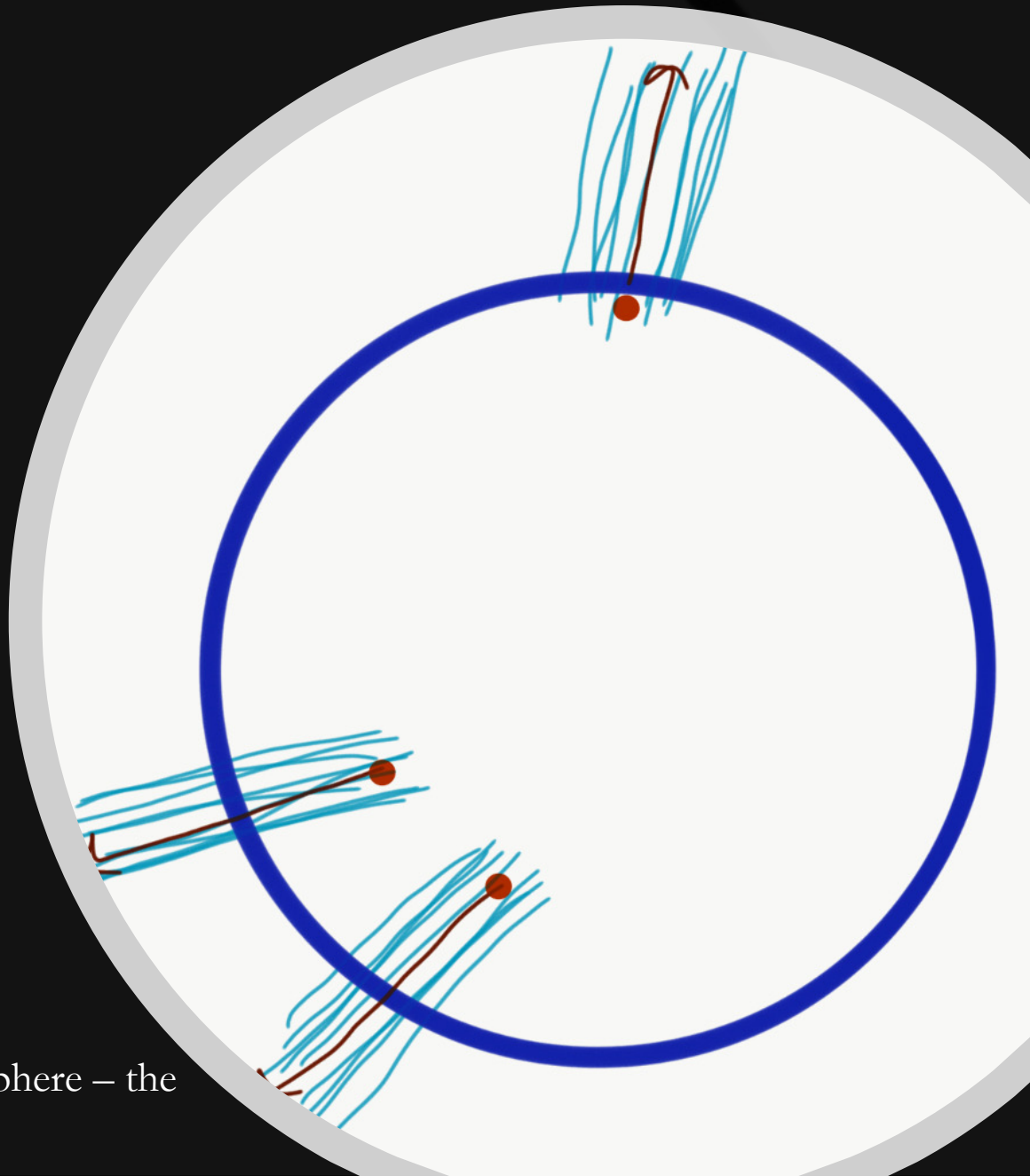
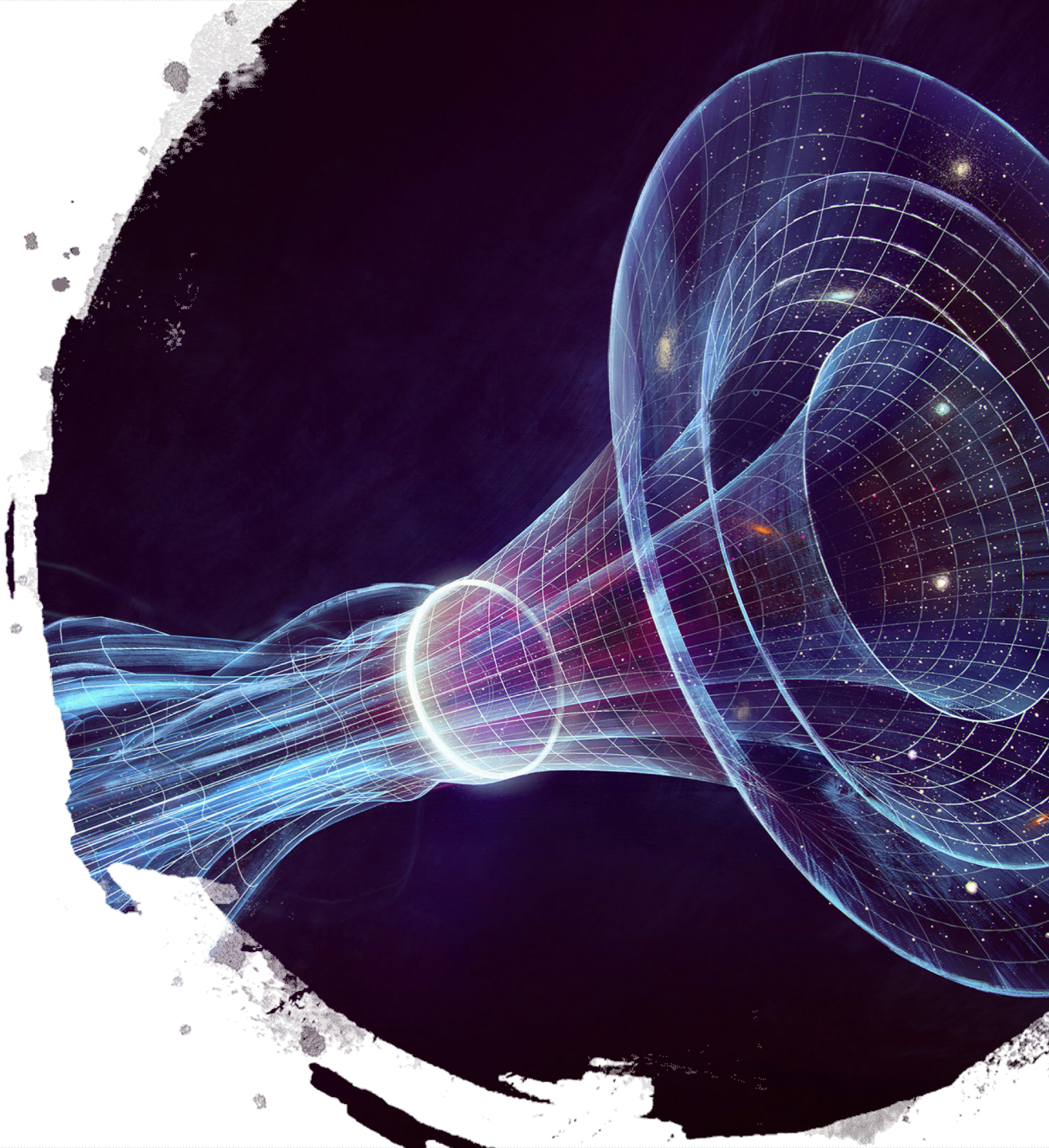


Image depicting a "flat" line bundle on the sphere – the curvature has been resolved.

Significance & Implications

- Testable model for understanding mathematical mechanism for anomaly cancellation
- Potential use for F-Theory backgrounds for Type II-B string theory (8-D string backgrounds with nontrivial axio-dilaton field) [12b]
- Sections of these elliptic manifolds may have use in F-theory / heterotic string duality [7c]



Thank You!

References

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