mathematics

Article

Dynamic Optimization with Timing Risk

Erin Cottle Hunt [1,*] and Frank N. Caliendo [2]

[1] Economics Department, Reed College, Portland, OR 97202, USA
[2] Economics and Finance Department, Huntsman School of Business, Utah State University, Logan, UT 84322, USA; frank.caliendo@usu.edu
* Correspondence: erinc@reed.edu

Abstract: Timing risk refers to a situation in which the *timing* of an economically important event is unknown (risky) from the perspective of an economic decision maker. While this special class of dynamic stochastic control problems has many applications in economics, the methods used to solve them are not easily accessible within a single, comprehensive survey. We provide a survey of dynamic optimization methods under comprehensive assumptions about the nature of timing risk. We also relax the assumption of full information and summarize optimization with limited information, ambiguity, imperfect hedging, and dynamic inconsistency. Our goal is to provide a concise user guide for specialists and nonspecialists alike.

Keywords: dynamic modeling; optimization techniques; timing risk

MSC: 37N40

Citation: Cottle Hunt, E.; Caliendo, F.N. Dynamic Optimization with Timing Risk. *Mathematics* **2024**, *12*, 2654. https://doi.org/10.3390/math12172654

Academic Editors: Hany Guirguis and Hyeon Park

Received: 18 July 2024
Revised: 13 August 2024
Accepted: 22 August 2024
Published: 27 August 2024

1. Introduction

Economic models often involve optimization over an uncertain event. In some settings, the uncertainty is over the *outcome* of the event only—for example, what will the income tax rate be in the future, or how large of a bequest might an individual receive? However, in other settings, the key economic uncertainty is regarding the *timing* of the event itself. For example, *when* might tax policy change, or *when* might an individual receive a bequest? This second type of uncertainty, regarding the risky timing of an economically important event, is the focus of this paper. We provide a textbook summary of timing risk models and methods that can be applied to various economic questions.

Conceptually, timing risk is distinct from outcome, or magnitude, risk. Consider, for example, the risk of becoming (permanently) disabled and losing the ability to work for pay. Becoming disabled at age 40 would have a different impact on lifetime consumption and savings than becoming disabled at age 70. An early shock would be especially devastating, while a later shock becomes progressively less painful. Optimization over such timing risk involves a type of dynamic hedging that ensures optimal decision making before (and after) the timing shock. For a decision maker to optimally hedge timing risk, it requires solving a dynamic stochastic control problem in which every possible timing realization is incorporated into a forward-looking plan. The decision maker must form a set of contingency plans for every possible shock date (i.e., what actions should be taken post shock), and all of these contingency plans must be rolled up into fully specified, time-consistent decision rules that govern choices before the shock hits.

The various methods for solving problems with timing risk are scattered across a number of recent papers in the literature—and each paper provides solution techniques that apply to a specific (and often complicated) timing risk application that has important nuances that distinguish it from other applications. The aim of this paper is to provide a simple, comprehensive summary of the known methods, with the hope that such a summary would be useful to researchers as an accessible and unified user guide.

The timing risk setting that we discuss is best suited to models where the risky event only happens once, or not at all. This is generally true of regime-switching models *without* timing risk, as well. For example, a spouse making a household life-cycle consumption decision faces a risk of their spouse dying. The timing of their spouse passing away is uncertain and will only happen once. Similarly, in the context of disability insurance, an individual faces a risk of becoming (permanently) disabled that either happens once (the individual becomes disabled) or not at all. In contrast, multiple, endogenous switches are considered by [1,2].

We consider two different dimensions of timing risk: the maximum date at which the uncertainty is resolved (bounded vs. unbounded), and the type of shock that occurs (level vs. flow). *Bounded timing risk* refers to problems where the timing risk is resolved within the planning horizon of the economic decision maker, and *unbounded timing risk* refers to problems where the timing risk is (potentially) resolved outside the planning horizon of the decision maker (or the uncertainty is never resolved, i.e., the uncertain event may never occur). An example of bounded timing risk would be the receipt of an accidental bequest from a parent. From the perspective of the child, biological constraints ensure that the bequest will arrive sometime before the end of the child's planning horizon (the maximum age of the child). An example of unbounded timing risk would be the risk of Social Security benefits being reduced: the policy reform may occur after the end of an individual's planning horizon (life-cycle), or never. Thus, from the perspective of an individual solving a consumption/saving problem, the timing uncertainty is unbounded. Notice that in each case, the risk of the event need not be constant over time and the methods summarized in this paper allow for nonstationary timing risk.

Within each type of timing risk (bounded vs. unbounded) we further distinguish between two types of risky events: *level shocks* which are one-time shocks to the level of a state variable, and *flow shocks* which are regime-switching shocks that change the state equation and thus the evolution of the state variable going forward after the date of the shock. An example of a level shock would be the receipt of a lump-sum bequest, which increases an individual's assets. An example of a flow shock would be a change in payroll taxes which increases or decreases take-home pay for an individual every period after the policy change is enacted. We summarize the necessary conditions to solve timing risk problems for all four combinations of bounded vs. unbounded risk with level or flow shocks. We focus on applications where the decision maker faces a *finite* horizon.

Technically speaking, all of the timing risk problems considered in this paper can be solved analytically up to an unknown constant. After summarizing the mathematical solution techniques for timing risk problems in Sections 3 and 4, we outline a computational process to numerically solve each type of problem in Section 5. In Section 6, we discuss welfare, and in Section 7, we show that the optimal hedging of timing risk is inherently time consistent.

Finally, we consider the information set of the decision maker in Section 8. Our baseline analysis follows convention and assumes that the decision maker has full information about the timing uncertainty they face. They know the full distribution of possible shock dates, but they do not have any additional advanced information about when the shock will occur. We consider three alternative information assumptions. First, we consider the case of early information, where the decision maker learns when the shock will occur prior to the shock occurring, thereby giving the decision maker more information than the traditional case. Second, we go to the other extreme case of ambiguity, in which the decision maker knows they face a risk, but they do not have any information about the distribution of that risk. For example, an individual in an aging country may face a risk that their public pension benefit is reduced, but they might not have any (reliable) information about the possible dates of the policy change, since it is difficult to forecast the legislative process. Lastly, we consider imperfect hedging and time inconsistency.

It is our hope that the tools summarized in this survey will be useful to researchers as they seek to answer additional questions with timing uncertainty.

2. Related Literature

The timing risk models considered in this paper are an extension of the two-stage regime-switching optimal control theory, where the switching date is stochastic rather than being known ex ante or a choice variable of the decision maker. Examples of regime-switching models are numerous and date back to early studies on resource extraction [3], operations research [4], and environmental catastrophe [5]. Regime-switching models with structural uncertainty (i.e., uncertainty regarding the characteristics of the new regime, but not the timing of the switch) have been used in the resource extraction literature [6] and later in the technology adoption literature [7–9]. These models, as well as timing uncertainty models, are extensions of two-stage regime-switching models, such as those found in [4,10–12]. Additional applications of regime-switching models include pollution accumulation [13–16], the optimal extraction of natural resources [2,6,17,18], economic growth [19], capital controls [20], technology adoption [9,21,22], marketing and pricing decisions [23,24], and drug policy [25,26]. Regime-switching models also have some parallels to hybrid control theory. Models that use hybrid control theory combine continuous time-driven variable dynamics with discrete, event-driven dynamics, as described in [27] and the references therein. In contrast, regime-switching models generally use multistage control theory.

The methods that we summarize come directly from recent studies in life-cycle consumption and saving, including the death of a spouse [28], receiving a bequest [29], retiring stochastically from the labor force [30], or rare event risks like the Great Depression [31] and other forms of catastrophic shocks.

We also summarize methods that have been used to study the effects of policy uncertainty on economic decisions. A branch of this literature focuses on the effects of specific tax policy uncertainty on the investment decisions of firms, such as [32,33] who both examine the effects of the 2003 Bush Tax Cuts. Several related papers also examine the effects of policy uncertainty more broadly on firms' acquisition decisions [34] or corporate decision making [35].

A second branch of this literature focuses on the effects of policy uncertainty on household decisions and well-being. Several papers examine the effect of Social Security reform uncertainty on the saving-consumption decisions of households [36–39]. In a related vein, ref. [40] examined the economic effects of Medicare reform uncertainty.

A third branch of this literature explores fiscal policy uncertainty more broadly, without narrowing in on a specific policy reforms. Researchers have explored the effect of fiscal policy uncertainty on macroeconomic outcomes [41–43], financial markets and bank-lending [44] or household well-being [45]. With the various timing risk methods already developed and scattered across these and other papers, our goal here is to organize known methods into a single survey, focusing most heavily on our own past work in this area as the basis for this survey.

Outside of optimal control theory, economists have long been interested in risk or uncertainty with a temporal domain. The early literature suggests individuals have a preference for the early resolution of uncertainty, such as [46–48], and more recently, [49]. Additional theoretical papers incorporate timing risk into preferences [50] and derive or calculate a timing risk premium [51,52].

An additional strand of the literature uses empirical evidence to understand individual's preferences regarding timing risk. Ref. [53] use data from business owners and managers and find that uncertainty may be processed similarly in both the time and outcome dimensions. Ref. [54] uses an experiment to show uncertainty may be processed in similar ways along timing and outcome dimensions which may result in behavior that looks like quasi-hyperbolic discounting. Several additional papers, such as, refs. [55–59], use laboratory experiments to elicit individual's preferences over uncertain outcomes. Taken as a whole, the growing empirical literature suggest that the timing of economic events is a meaningful consideration for decision makers.

3. Review: Standard Problem without Timing Risk

Time is continuous and is indexed by t. From the perspective of the decision maker (DM), the finite planning interval starts at $t = 0$ and ends at $t = T$. At each moment in time the DM chooses a control $c(t)$ that generates instantaneous payoff $u(t, c(t))$ that is continuously differentiable in its arguments and is concave in $c(t)$. Note that our writing of the payoff function is general enough to accommodate time discounting and survival risk. The DM is endowed with a state variable $k(0)$ and is constrained by a terminal value $k(T) = 0$ (without loss of generality, assume $k(0) = k(T) = 0$, although any other assumptions will work) and by a state equation

$$\dot{k}(t) = g(t, c(t), k(t)),$$

where g is continuously differentiable in its arguments.

In a standard Pontryagin problem, the function g would either be invariant to t or would vary with t in a manner that is either known ex ante or chosen by the DM. In this case, subject to these constraints, the DM would maximize

$$U = \int_0^T u(t, c(t))dt.$$

4. Timing Risk

We now consider problems where there is a shock to the state system occurring at an unknown time t. The shock date is a continuous random variable with probability density function (PDF) $\phi(t)$ and cumulative density function (CDF) $\Phi(t)$, where $\phi(t)$ is continuously differentiable in t.

The nature of timing risk can differ across applications and we seek to carefully distinguish between different settings that imply subtle changes to the solution techniques. For example, while the timing of the shock is unknown in advance, in some applications, the shock is sure to occur *before* date T. We call this case *bounded timing risk*. Let the latest possible shock date be $t' < T$. Hence, $\Phi(t') = 1$.

In other applications, the shock may occur at the end (or beyond the end) of the planning period. We call this case *unbounded timing risk*. Assume the shock can occur anytime on the interval $[0, \infty]$ and $\int_0^\infty \phi(t)dt = 1$. This includes the case in which the risk is resolved by the terminal planning date T (i.e., $\phi(t) = 0$ for all t beyond T) and for cases in which the shock need not strike before T (i.e., $\phi(t) > 0$ for some t beyond T).

Examples of bounded timing risk include models where the timing risk is sure to resolve itself before the end of the DM's planning horizon. For instance, suppose people face uncertainty about the timing of the receipt of inheritance income because there is uncertainty about the longevity of the prior generation. This occurs in models with specific linkages between parents and children as in [29]. If we assume the parent cannot outlive the child, then the timing risk over inheritance income facing the child is bounded because it would resolve itself before the child reaches the maximum model age. Other examples include models where the individual faces uncertainty about the length of their career as in [30]. If the individual is sure to retire before the maximum model age (say age 100), then the timing risk is again modeled as a bounded process.

Examples of unbounded timing risk include models where the timing risk need not resolve itself before the end of the DM's planning horizon. For instance, in [28], the death of a spouse occurs at an unknown time and there is no guarantee that this death would occur before the maximum model age of the other spouse. Likewise, in [36], the timing of Social Security reform is uncertain and there is no guarantee that it will happen within the DM's lifetime (policy makers can delay endlessly in the model). And in [31], equity holders face the risk of sudden and significant collapse of equity valuations occurring at an unknown time; but again, this may or may not happen within the remaining lifetime of a given DM.

Moreover, within each of these two cases, the shock to the state system could take the form of a permanent *flow shock* to the functional form of the state equation (i.e., a permanent, structural change to the dynamic system), or it could take the form of a one-time *level shock* to the quantity of the state variable. In the case of the flow shock, the function form of the state equation switches from $g(.)$ to $g_2(.)$ at the time of the shock. In the case of a level shock, the quantity of the state variable jumps (up or down) discretely by an amount Δ, while the functional form of the state equation g stays the same before and after the shock. Also, Δ can be large or small. Importantly, in either case, the timing of the shock is unknown ex ante, and that is the key feature of the problems at hand. We carefully distinguish between the two types of shocks below because the first-order conditions differ across the two cases.

Examples of timing risks that come as a flow shock include models where the shock permanently alters the dynamic system. For instance, Social Security reform causes permanent changes to tax and benefit rates in [36]. Likewise, the death of a spouse causes permanent loss of spousal earnings and a permanent shift from couple to single-household Social Security collection rules.

On the other hand, examples of timing risk that come as a level shock include models where the shock has a one-time effect on the state variable. For instance, in [31] a collapse in equity prices is modeled as a one-time sudden destruction in asset holdings. And in [29], an individual receives a one-time inheritance from their parent at an unknown time. In some examples, the distinction between a flow shock and a level shock is more cosmetic than consequential (i.e., the shock can be modeled either way). For instance, in [30], a stochastic departure from the labor force causes a permanent loss in wage earnings and a permanent change in the level of Social Security benefits collected. While the shock is clearly permanent, under the assumption of complete capital markets it is modeled equivalently as a one-time change in the present value of wealth.

Timing risk is a stationary process in some applications; that is, the risk of the shock is constant (exponentially distributed). But this would not be the case in general, and a number of examples involve nonstationary timing risk. Therefore, all of the methods summarized in this paper are general enough to handle stationary and nonstationary timing risk.

Beyond the assumptions that are made about the nature of the timing risk itself, the researcher must make assumptions about (1) how much information the DM has about the timing risk and (2) how the DM behaves in the face of this information. Unless we say otherwise, we assume the DM has full information about the distribution of the timing risk and that they behave optimally in the face of this risk.

We seek to provide a simple guide for solving timing risk problems recursively as in a conventional dynamic programming approach. We break the problem into a deterministic problem from the perspective of the time of the shock and then work backward to the dynamic stochastic (time 0) problem.

In all cases, we express the dynamic optimization problem in continuous time and in standard Pontryagin form. This is true of all variations in timing risk that we consider. The necessary conditions for optimality include a Maximum Condition, a Costate Equation (Multiplier Equation), and initial and terminal conditions. The terminal condition is either a fixed endpoint on the state variable or a Transversality Condition.

4.1. Problem A: Bounded Timing Risk

Step 1. Postshock subproblem:

Given shock date $t \leq t' < T$ and state variable $k(t)$, the optimal control path $c(z)$ for $z \in [t, T]$, after the shock has occurred, is the solution to

$$\max_{c(z)_{z\in[t,T]}} : U_2 = \int_t^T u(z, c(z))dz.$$

For the case of a permanent *flow shock* to the state equation, the constraints for the postshock problem are

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given}, k(T) = 0, t \text{ given}.$$

Alternatively, in the case of the one-time *level shock* to the state variable, the constraints are

$$\dot{K}(z) = g(z, c(z), K(z))$$

$$K(t) = k(t) + \Delta$$

$$K(T) = 0$$

$$k(t) \text{ given}, t \text{ given}.$$

Here, $K(t)$ is the state variable post shock and is notationally different from the state variable preshock to accommodate the level shock itself. The change in the state variable is denoted by Δ.

In either case, we denote the solution to this deterministic subproblem as $c_2^*(z, t, k(t))_{z\in[t,T]}$. And solution continuation utility is a function of the timing of the shock t and the state variable at the time of the shock $k(t)$

$$U_2^*(t, k(t)) = \int_t^T u(z, c_2^*(z, t, k(t)))dz.$$

This is nested into the next step of the procedure.

Step 2. Preshock t = 0 subproblem:

Facing random variable t at time 0, the DM faces a dynamic stochastic control problem and maximizes expected utility

$$\max_{c(t)_{t\in[0,t']}} : \int_0^{t'} ([1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)))dt$$

subject to

$$\dot{k}(t) = g(t, c(t), k(t))$$

$$k(0) = 0, k(t') \text{ free}.$$

Note that this is a *free-endpoint* optimal control problem. The uncertainty is sure to resolve itself before the end of the planning horizon, so while the ultimate terminal value of the state variable is fixed, the state variable at the last possible shock date $t' < T$ is free to be chosen optimally and hence the necessary conditions involve a Transversality Condition. Form the Hamiltonian

$$\mathcal{H} = [1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)) + \lambda(t)g(t, c(t), k(t)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial k(t)}$$

$$\lambda(t') = 0.$$

The solution to this subproblem is $(c_1^*(t), k_1^*(t))_{t\in[0,t']}$. The individual follows this contingent timepath as long as the shock has not yet hit. Once the shock strikes at time t, the individual immediately jumps onto the path $c_2^*(z, t, k_1^*(t))_{z\in[t,T]}$ for the remainder of the life cycle. Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a preshock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable at each moment.

4.2. An Important Clarification on the Transversality Condition

Whether we are dealing with flow or level shocks, a technical difficulty arises when solving the preshock subproblem in the setting with bounded timing risk. The challenge is that the first-order conditions—as they are commonly expressed in traditional Pontryagin form—generate a family of solutions rather than a unique solution. Ref. [30] develop a method for recovering a unique solution from the first-order conditions, which can be summarized as follows.

Note that the Maximum Condition is

$$\frac{\partial \mathcal{H}}{\partial c(t)} = [1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t)) + \lambda(t)\frac{\partial}{\partial c(t)}g(t,c(t),k(t)) = 0.$$

Evaluate this condition at $t = t'$

$$[1-\Phi(t')]\frac{\partial}{\partial c(t)}u(t',c(t')) + \lambda(t')\frac{\partial}{\partial c(t)}g(t',c(t'),k(t')) = 0.$$

Note that $\Phi(t') = 1$ by definition, so the condition simplifies to

$$\lambda(t')\frac{\partial}{\partial c(t)}g(t',c(t'),k(t')) = 0.$$

Now, note that imposing the Transversality Condition $\lambda(t') = 0$ causes the above condition to be satisfied automatically for any choice of the terminal pair $(c(t'),k(t'))$. Hence, the Transversality Condition does not appear to provide the needed endpoint condition and, instead, the Maximum Condition and Multiplier Equation generate a family of solution paths rather than a unique solution. However, there is a simple answer to this apparent complication: use the *limiting case* of the first-order conditions, as follows.

Let us rewrite the Maximum Condition as

$$\frac{\frac{\partial}{\partial c(t)}u(t,c(t))}{\frac{\partial}{\partial c(t)}g(t,c(t),k(t))} = -\frac{\lambda(t)}{1-\Phi(t)}.$$

Noting the indeterminate form

$$\frac{\lambda(t')}{1-\Phi(t')} = \frac{0}{0}$$

we apply L'Hôpital's Rule and then impose the Transversality Condition

$$\begin{aligned}
\frac{\frac{\partial}{\partial c(t)}u(t',c(t'))}{\frac{\partial}{\partial c(t)}g(t',c(t'),k(t'))} &= -\lim_{t\to t'}\frac{\lambda(t)}{1-\Phi(t)} \\
&= \lim_{t\to t'}\frac{\dot{\lambda}(t)}{\dot{\Phi}(t)} \\
&= \lim_{t\to t'}\frac{\dot{\lambda}(t)}{\phi(t)} \\
&= \lim_{t\to t'}\frac{-\frac{\partial \mathcal{H}}{\partial k(t)}}{\phi(t)} \\
&= \lim_{t\to t'}\frac{-\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t,k(t)) - \lambda(t)\frac{\partial}{\partial k(t)}g(t,c(t),k(t))}{\phi(t)} \\
&= -\frac{\partial}{\partial k(t)}U_2^*(t',k(t')).
\end{aligned}$$

Hence, written in final form, we have

$$\frac{\frac{\partial}{\partial c(t)}u(t', c(t'))}{\frac{\partial}{\partial c(t)}g(t', c(t'), k(t'))} = -\frac{\partial}{\partial k(t)}U_2^*(t', k(t')).$$

This condition provides the needed endpoint restriction to derive a unique solution from the first-order equations. That is, in addition to the Maximum Condition and Multiplier Equation, the solution path (c, k) must have terminal values $(c(t'), k(t'))$ that satisfy this condition. In previous research, this condition has been referred to as the "Stochastic Continuity Condition" or the "Limiting Case" of the Transversality Condition [29,30].

4.3. Problem B: Unbounded Timing Risk

Step 1. Postshock subproblem:

If the shock strikes at $t < T$ with state variable $k(t)$, the optimal control path $c(z)$ for $z \in [t, T]$, after the shock has occurred, is the solution to

$$\max_{c(z)_{z\in[t,T]}} : U_2 = \int_t^T u(z, c(z))dz.$$

In the case of a permanent *flow-shock* to the state equation, the constraints for the postshock problem are

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given}, k(T) = 0, t \text{ given}.$$

Alternatively, in the case of the one-time *level shock* to the state variable, the constraints are

$$\dot{K}(z) = g(z, c(z), K(z))$$

$$K(t) = k(t) + \Delta$$

$$K(T) = 0$$

$$k(t) \text{ given}, t \text{ given}.$$

Here, $K(t)$ is the state variable post shock and is notationally different from the state variable preshock to accommodate the level shock itself.

In either case, we denote the solution to this subproblem as $c_2^*(z, t, k(t))_{z\in[t,T]}$. And solution continuation utility is a function of the timing of the shock t and the state variable at the time of the shock $k(t)$

$$U_2^*(t, k(t)) = \int_t^T u(z, c_2^*(z, t, k(t)))dz.$$

This is nested into the next step of the procedure.

Step 2. Preshock t = 0 subproblem:

Facing random variable t at time 0, the DM maximizes expected utility

$$\max_{c(t)_{t\in[0,T]}} : \int_0^T ([1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)))dt$$

subject to

$$\dot{k}(t) = g(t, c(t), k(t))$$

$$k(0) = 0, k(T) = 0.$$

Note that this is a *fixed-endpoint* optimal control problem. Because the distribution of the timing risk extends at least to and potentially beyond the end of the planning horizon, the individual must make an optimal preshock plan that extends all the way to the end of the planning horizon. Hence, the constraint on the terminal value of the state variable provides the needed endpoint condition (rather than a Transversality Condition). Form the Hamiltonian

$$\mathcal{H} = [1 - \Phi(t)]u(t, c(t)) + \phi(t)U_2^*(t, k(t)) + \lambda(t)g(t, c(t), k(t)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(t)} = 0,$$

$$\dot{\lambda}(t) = -\frac{\partial \mathcal{H}}{\partial k(t)}.$$

The solution to this subproblem is $(c_1^*(t), k_1^*(t))_{t \in [0,T]}$. The individual follows this contingent timepath as long as the shock has not yet hit. Once the shock strikes, the individual immediately jumps onto the path $c_2^*(z, t, k_1^*(t))_{z \in [t,T]}$ for the remainder of the life-cycle. Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a preshock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable at each moment.

4.4. Comments

4.4.1. Shock Depends on Shock Date

Although we tried to keep the notation simple, the shock itself can be further generalized to depend on the timing of when the shock is realized. For example, the postshock new regime can depend on the timing of the shock, as in [36], where the longer policymakers wait to reform Social Security, the more severe the necessary increase in taxes, or decrease in benefits, to restore fiscal solvency. The method developed above is well suited to study such applications of timing risk.

4.4.2. Infinite Horizon

We focus on finite planning horizon applications because of our interest in life-cycle overlapping generation models. But the methods above can be applied to infinite horizon models where there are again two cases: the case where the shock is sure to happen before the end of the infinite planning horizon (bounded timing risk), and the case where the shock timing spans the full time horizon (unbounded timing risk). In both cases, the problem can be solved recursively as above. For bounded problems, step 1 is to solve an infinite horizon postshock subproblem from the vantage point of the realization of the shock. And step 2 is to solve a *finite* horizon, preshock subproblem that stretches from time zero up to the last possible finite shock date. For unbounded problems, step 1 is the same. But step 2 is to solve an *infinite* horizon, preshock subproblem spanning the non-negative real line.

4.4.3. Multiple Shocks

Recall that our focus is on applications where a risky event happens only once at an unknown time. In principle, the methods developed above can be generalized to multiple timing shocks, though we do not emphasize our approach as the most useful technique in such cases. In our experience, it is often impossible to maintain an analytical solution under multiple timing shocks, even in the simplest of applications. In fact, even with a single timing shock, computational methods are typically necessary to obtain the solution to the optimization problem, as will be discussed below. And with multiple timing shocks, the continuation utility function (for all shocks that occur before the last shock) is often impossible to write down analytically, which presents a problem for our approach that relies on differentiating the continuation function.

5. Computational Method for the Preshock Subproblem

For Problems A and B, and for both cases (flow shock and level shock), the first-order conditions typically end up being complicated enough (even for the simplest possible applications of timing risk) that we cannot solve for fully closed-form solutions to the preshock subproblem $(c_1^*(t), k_1^*(t))$. However, for typical problems in economics with $\partial g/\partial c = -1$ (as in the case of life-cycle consumption saving problems where an extra unit of consumption reduces the amount saved one for one), we can in fact obtain a closed-form Euler equation $\dot{c}(t)$, and from there, a simple computational method can be used to anchor the Euler equation to an initial condition. We need a computational technique, because we typically cannot write down the control variable in analytical form. However, in many applications in economics, we can write down the optimal rate of growth in the control variable (i.e., the Euler equation), if not the control variable itself. And, we can build a computational technique around the Euler equation.

To obtain the Euler equation, recall the Hamiltonian

$$\mathcal{H} = [1-\Phi(t)]u(t,c(t)) + \phi(t)U_2^*(t,k(t)) + \lambda(t)g(t,c(t),k(t)),$$

with Maximum Condition and Multiplier Equation

$$\frac{\partial \mathcal{H}}{\partial c(t)} = [1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t)) - \lambda(t) = 0$$

$$\dot{\lambda}(t) = -\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t,k(t)) - \lambda(t)\frac{\partial}{\partial k(t)}g(t,c(t),k(t)).$$

Differentiating the Maximum Condition with respect to t

$$\frac{d}{dt}\left([1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t))\right) - \dot{\lambda}(t) = 0,$$

and combining with the Multiplier Equation gives

$$\frac{d}{dt}\left([1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t))\right) + \phi(t)\frac{\partial}{\partial k(t)}U_2^*(t,k(t)) + \lambda(t)\frac{\partial}{\partial k(t)}g(t,c(t),k(t)) = 0.$$

Finally, inserting $\lambda(t) = [1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t))$ from the Maximum Condition gives the final form of the Euler Equation

$$\frac{d}{dt}\left([1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t))\right) = -\phi(t)\frac{\partial}{\partial k(t)}U_2^*(t,k(t)) - [1-\Phi(t)]\frac{\partial}{\partial c(t)}u(t,c(t))\frac{\partial}{\partial k(t)}g(t,c(t),k(t)).$$

This Euler Equation governs the dynamics of the optimal $c(t)$ path. All that remains is to identify the unknown initial value $c(0)$, which can be achieved with a simple computation technique:

Step 1. Guess $c(0)$. The algorithm is robust to a wide range of initial guesses.
Step 2. Use a system of equations (Euler Equation, the law of motion $\dot{k}(t)$, and the initial condition $k(0)$) to simulate the $c(t)$ and $k(t)$ paths.
Step 3.

- For the case of *bounded timing risk* (Problem A), use the system to simulate $c(t)$ and $k(t)$ up to the timing boundary $t = t'$ and check to see if the "Stochastic Continuity Condition" holds.
- For the case of *unbounded timing risk* (Problem B), use the system to simulate $c(t)$ and $k(t)$ up to the end of the planning horizon $t = T$ and check to see if the terminal constraint on the state variable holds.

Step 4. If the check in Step 3 is met, then stop. If not, go back to Step 1 and repeat.

In our experience, the shooting method described above is a relatively straightforward exercise to implement and is very effective in finding the solution. The Euler equation combines all of the information from the Maximum Condition and Multiplier Equation. Then, using the Euler equation together with the law of motion for the state variable and the initial condition on the state variable, we can shoot forward to simulate the control and state variable timepaths and check to see whether the appropriate boundary condition is satisfied. What we don't know is the initial value of the control variable. But that is a relatively straightforward problem to deal with: the researcher can take a guess on the initial value of the control variable and see if it is correct in the sense that, based on that guess, the terminal condition is satisfied. If so, the guess is correct. If not, then the researcher can guess again and again until the terminal condition is finally satisfied. This guessing process can be performed by a brute force grid search (just guess on many initial values for the control variable and keep the one that works) or through a more elaborate guess-and-update process, though we have had better luck with brute force methods in our own computational work. Finally, while this technique does not provide guidance on how to guess the initial value of the control variable, in our experience the context of the problem often provides sufficient understanding to allow the researcher to make an educated initial guess.

6. Welfare

In the literature, timing risk is studied to answer positive and normative research questions. For instance, ref. [32] considers a positive question: How does uncertainty about the timing of fiscal policy changes affect firm investment spending? In other papers, the research questions are normative and therefore involve the calculation of welfare, such as those in [28,29,31] that examine the insurance role of Social Security when the individual faces various timing uncertainties (the timing of the death of a spouse, the timing of a stock market collapse, and the timing of inheritance income). In some applications, the researcher may wish to measure the welfare effect of a particular government policy. For example, does a particular policy increase the expected lifetime utility of agents in the model compared with an alternative policy? In other applications, the researcher may wish to measure the welfare effect of the timing risk itself. In either case, ex ante expected utility will need to be calculated.

For the case of bounded timing risk (Problem A),

$$\mathbb{E}(U) = \int_0^{t'} \phi(t) \left(\int_0^t u(v, c_1^*(v))dv + \int_t^T u(v, c_2^*(v, t, k_1^*(t)))dv \right) dt.$$

Ex ante, the DM does not know when the shock will hit, so the expected utility is a probability-weighted average of the utility associated with a continuum of shock dates.

For the case of unbounded timing risk (Problem B),

$$\begin{aligned}\mathbb{E}(U) \quad &= \quad \int_0^T \phi(t) \left(\int_0^t u(v, c_1^*(v))dv + \int_t^T u(v, c_2^*(v, t, k_1^*(t)))dv \right) dt \\ &\quad + \left(\int_T^\infty \phi(t)dt \right) \times \left(\int_0^T u(t, c_1^*(t))dt \right).\end{aligned}$$

Here, notice that we must account for the fact that the shock may never strike, in which case the individual will be on the preshock path c_1^* for the entirety of the planning interval.

7. A Remark on Time Consistency

It is worth noting that the timing risk component of the problem is inherently time-consistent by the laws of probability. For example, suppose we are standing at time t_0 and the shock has not yet hit. Further, suppose the individual has been following the optimal preshock path $(c_1^*(t), k_1^*(t))_{t\in[0,t_0]}$. While the results below hold for both Problems A and B, to ease notation, we focus on Problem A.

From the perspective of time t_0, the timing of PDF and CDF, conditional on the shock not yet occurring by date t_0, are

$$\phi_0(t)_{t\in[t_0,t']} = \frac{\phi(t)}{\int_{t_0}^{t'} \phi(t)dt} = \frac{\phi(t)}{1-\Phi(t_0)}$$

$$\Phi_0(t)_{t\in[t_0,t']} = \int_{t_0}^{t} \frac{\phi(v)}{1-\Phi(t_0)}dv = \frac{\Phi(t)-\Phi(t_0)}{1-\Phi(t_0)}.$$

Furthermore,

$$1-\Phi_0(t) = \frac{1-\Phi(t)}{1-\Phi(t_0)}.$$

Given these conditional distributions, if the shock has not yet hit, then the DM standing at time t_0 would seek to solve

$$\max_{c(t)_{t\in[t_0,t']}} : \int_{t_0}^{t'} \left(\frac{1-\Phi(t)}{1-\Phi(t_0)} u(t,c(t)) + \frac{\phi(t)}{1-\Phi(t_0)} U_2^*(t,k(t)) \right) dt.$$

Note the proportionality between this objective functional and the time 0 functional

$$\max_{c(t)_{t\in[0,t']}} : \int_{0}^{t'} ([1-\Phi(t)]u(t,c(t)) + \phi(t)U_2^*(t,k(t)))dt,$$

which in turn implies that the solution to the time t_0 problem will coincide with the solution to the time 0 problem (i.e., the solution is time-inconsistent).

8. Alternative Assumptions about Information

In our baseline model, the DM has full information about the timing of the risk. The DM knows the distribution of possible shock dates but does not know when the shock will strike until it happens. Here, we explore other assumptions about the role of information. Specifically, we consider two alternative assumptions about the information available to the DM: early information and no information.

8.1. Early Information

In our baseline scenario, the DM has no early warning about the timing of the shock. In other words, while the DM has full information about the distribution of the timing risk, the exact timing of the shock cannot be foreseen in advance. Instead, the DM employs an optimal hedging strategy as the solution to the dynamic stochastic problem as described above.

In this section, we consider another possible assumption. Imagine a scenario where a policy change from one state to another is imminent, but the date is uncertain. It is common knowledge that an announcement will be made at time t^*, and the announcement will unveil the future date $t \geq t^*$, at which time the new policy will take effect. That is, the timing of the shock t is revealed early at $t^* \leq t$. Notice the DM still faces ex ante timing risk, but in this case, the timing risk is resolved at the information revelation date t^*. For instance, in 6 weeks, the Fed will announce the length of time for which its current policy position will endure before switching to a new policy (from say expansion to tightening, etc.).

From the perspective of time 0, the DM knows that at time $t^* > 0$ they will learn the timing of the shock. Let us say the shock is distributed over the support $[t^*, T]$, which corresponds to our unbounded timing risk examples above, although other assumptions are possible too. Let us also model a flow shock to lighten notation. As with the other cases, we solve this problem recursively, though this is a simpler problem because all of the timing risk gets resolved at a known, single point. The following is a generic summary of the method derived in [30] for handling such applications.

Step 1. Information Revelation Date:

At the information revelation date t^*, the DM learns the timing of the shock $t \geq t^*$, and given state variable $k(t^*)$, the optimal control path $c(z)$ for $z \in [t^*, T]$, after the information has been revealed, is the solution to

$$\max_{c(z)_{z\in[t^*,T]}} : U_2 = \int_{t^*}^{T} u(z, c(z))dz,$$

subject to

$$\dot{k}(z) = g(z, c(z), k(z)), \text{ for } z \in [t^*, t],$$
$$\dot{k}(z) = g_2(z, c(z), k(z)), \text{ for } z \in [t, T],$$
$$k(t^*) \text{ given}, k(T) = 0, t^* \text{ given}, t \text{ given}.$$

This is a standard two-stage control problem without timing risk, and conventional methods can be applied to find the solution $c_2^*(z, t, t^*, k(t^*))_{z\in[t^*,T]}$. The DM follows this timepath for all points in time z from the information revelation date t^* through the end of the planning horizon T, and note that this path depends on the announced shock date t, as well as the level of the state variable at the announcement date $k(t^*)$.

Solution utility as of date t^* is a function of the shock date t and the level of state variable at the announcement date $k(t^*)$

$$U_2^*(t, t^*, k(t^*)) = \int_{t^*}^{T} u(z, c_2^*(z, t, t^*, k(t^*)))dz.$$

Step 2. Preshock t = 0 subproblem:

Facing random variable t, at time 0 the DM maximizes expected utility

$$\max_{c(v)_{v\in[0,t^*]}} : \int_0^{t^*} u(v, c(v))dv + \int_{t^*}^{T} \phi(t) U_2^*(t, t^*, k(t^*))dt,$$

subject to

$$\dot{k}(v) = g(v, c(v), k(v)), \text{ for } v \in [0, t^*],$$
$$k(0) = 0, k(t^*) \text{ free}.$$

Note that this is a *free-endpoint* optimal control problem because the DM is free to choose the level of the state variable at the information revelation date. In selecting this amount, the DM is conscious of all of the potential shock dates and accordingly forms a probability-weighted continuation value of the state variable.

To solve this problem, form the Hamiltonian

$$\mathcal{H} = u(v, c(v)) + \lambda(v) g(v, c(v), k(v)),$$

with necessary conditions

$$\frac{\partial \mathcal{H}}{\partial c(v)} = 0$$
$$\dot{\lambda}(v) = -\frac{\partial \mathcal{H}}{\partial k(v)}$$

and the Transversality Condition

$$\lambda(t^*) = \frac{\partial}{\partial k(t^*)} \int_{t^*}^{T} \phi(t) U_2^*(t, t^*, k(t^*))dt.$$

The solution to this subproblem is $(c_1^*(v), k_1^*(v))_{v\in[0,t^*]}$. The DM follows this contingent timepath up to the information revelation date t^*, at which time they learn the shock date $t \geq t^*$ and immediately jumps onto path c_2^* for the remainder of the life-cycle (the DM does not wait for the shock to actually materialize to jump onto the c_2^* path). Hence, optimal dynamic hedging involves a contingent plan for every possible shock date c_2^* as well as a preshock path (c_1^*, k_1^*) that takes all of these possible contingencies into account according to the probability-weighed continuation value of the state variable.

8.2. No Information (Ambiguity)

At the other extreme, another possibility is that the DM has *no information* about the distribution of timing risk. Suppose the DM knows there will be a regime switch at some point, but the probabilities are unknown. For instance, suppose fiscal policy reform is necessary to balance the government's budget, but it is not possible to know the probability that reform will happen at a given date.

To lighten the notation, let us focus on the case of the flow shock (switch in the law of motion of the state variable from g to g_2). Let us further suppose for the moment that the switch to g_2 is a bad thing from the DM's perspective, and therefore the later the shock occurs the better. Then, in a Maximin fashion, the DM standing at t would rationally plan for the shock to happen *immediately* (which is the worst-case scenario) and optimize accordingly.

Standing at time $t < T$ with state variable $k(t)$, the DM optimizes under a worst-case scenario of the shock happening at that moment and therefore g_2 rather than g governing the law of motion for the state variable for the remainder of the planning interval. Therefore, the optimal control path $c(z)$ for $z \in [t, T]$ is the solution to

$$\max_{c(z)_{z\in[t,T]}} : U_2 = \int_t^T u(z, c(z))dz,$$

subject to

$$\dot{k}(z) = g_2(z, c(z), k(z))$$

$$k(t) \text{ given}, k(T) = 0, t \text{ given}.$$

Denote the solution to this subproblem as $c_2^*(z, t, k(t))_{z\in[t,T]}$.

But unlike the baseline problems studied above, this problem is *time-inconsistent*. If the shock does not hit immediately as planned at date t, then the DM will need to reoptimize at the next moment in time because the state variable will have evolved according to g rather than g_2 as planned. The DM will follow the control plan only at the moment the plan is made t and will need to reoptimize in the next moment. Hence, the DM's actual control choice (for as long as the shock has not yet hit) at time t is $c_2^*(t, t, k(t))$, where $k(t)$ follows the preshock law of motion and boundary constraints

$$\dot{k}(t) = g(t, c_2^*(t, t, k(t)), k(t))$$

$$k(0) = 0, k(T) = 0.$$

Note that the DM's actual control path is the *envelope* of initial values of the many planned control paths. Each planned path, including the initial value of that planned path, is a utility-maximizing path based on the worst-case scenario of an immediate regime shift from g to g_2. Yet the state variable continues to evolve according to the preshock regime g as long as the shock has not yet hit. Eventually, at some point, the shock does hit, and the decision maker is finally correct at that moment and stays on the postshock path from there forward rather than reoptimizing. In sum, Maximin behavior under timing ambiguity involves solving a sequence of time-inconsistent, deterministic control problems, as opposed to our baseline case with risk only (and no ambiguity about that risk) where the DM solves an ex ante control problem that is time-consistent.

Other examples are possible too. Suppose the switch from g to g_2 is a good thing in the eyes of the DM. If the DM has no information about the distribution of the timing risk, then a Maximin strategy would be to assume the switch to g_2 never comes. Now, the optimization problem is time-consistent during the preshock period because the DM's predictions are correct so far. Then, when the shock eventually hits, the DM will need to reoptimize to factor in the good news. Hence, an initial, deterministic control problem based on the worst case is followed until good news hits, and then a new, deterministic control problem governs choices from that point forward.

8.3. No Hedging and Imperfect Hedging

In some applications, it may make sense to explore the possibility that the DM does not hedge the timing risk they face. This situation may arise for behavioral reasons. Perhaps the DM is simply unaware of the risk they face, or perhaps the solution to the dynamic hedging problem is too complicated to solve and implement. Alternatively, a lack of hedging may arise out of information constraints. Perhaps the DM lacks information about the distribution of the timing risk they face, which can arise if there is no historical data from which to infer the risk at hand or the historical data is otherwise incomplete, thereby making it difficult to measure the risk distribution. For example, an equity holder may ignore rare-event risk and focus only on the average long-run return of a risky asset [31]. Or it may be that historical (ex post) data are unrelated to ex ante risk facing the DM.

Whatever the particular motivation, one way to model a lack of hedging is to make the extreme assumption that the DM initially follows the solution to a deterministic problem that abstracts from timing risk, and then when the shock strikes, they immediately respond optimally to the shock and reoptimize from that point forward. In other words, the DM optimizes along every margin (besides timing risk) for which they have information, but they are otherwise taken by surprise when the timing shock hits.

Other departures from optimal hedging are possible too. For instance, it may be that the DM recognizes that they face timing risk, but that they base their decisions on the expected value of the timing shock. In other words, they optimize based on the expected risk that they face but ignore the variance. For example, a financial planner may develop a retirement saving plan that is based on expected retirement dates (and also on expected longevity dates), without explicitly taking the full distribution of these random variables into account in devising a plan.

The researcher may find it useful to explore these or other departures from the benchmark model with full information and fully rational optimization. For instance, the welfare implications of various economic policies may depend on the assumptions that we make about whether the DM does or does not hedge the timing risk that they face.

9. Conclusions

This paper is a user guide to modeling timing uncertainty—the date at which an economically significant event takes place is unknown from the perspective of the decision maker. We outline the steps to solve a model with timing risk recursively. The first step is to solve the postshock, deterministic subproblem. The second step is to embed the solution to the postshock subproblem into the dynamic stochastic (time 0) problem. We provide details on how to solve models where the shock occurs within the planning horizon of the decision maker (bounded risk) and models where the shock potentially occurs outside the planning horizon of the decision maker (unbounded risk). We distinguish between flow shocks and stock shocks which either change the state equation or the level of the state variable. We also provide guidance on how to solve the model computationally, how to conduct welfare analysis, and how changing the information assumptions alters the problem. Our hope is that this user guide will be helpful for researchers who want to write down an economic model with timing risk. The tools summarized in this paper can be applied to many economic disciplines including economic growth, natural resource extraction, firm investment decisions and valuations, life-cycle consumption and saving, fiscal policy analysis, monetary policy analysis, and political economics.

Author Contributions: Conceptualization, E.C.H. and F.N.C.; Methodology, E.C.H. and F.N.C.; Formal analysis, E.C.H. and F.N.C.; Writing—original draft, E.C.H. and F.N.C.; Writing—review & editing, E.C.H. and F.N.C. All authors have read and agreed to the published version of the manuscript.

Funding: This research received no external funding.

Data Availability Statement: This research did not use any data.

Conflicts of Interest: The authors declare no conflicts of interest.

References

1. Moser, E.; Seidl, A.; Feichtinger, G. History-dependence in production-pollution-trade-off models: A multi-stage approach. *Ann. Oper. Res.* **2014**, *222*, 457–481. [CrossRef]
2. Boucekkine, R.; Pommeret, A.; Prieur, F. Technological vs. Ecological Switch and the Environmental Kuznets Curve. *Am. J. Agric. Econ.* **2013**, *95*, 252–260. [CrossRef]
3. Dasgupta, P.; Heal, G. The Optimal Depletion of Exhaustible Resources. *Rev. Econ. Stud.* **1974**, *41*, 3–28. [CrossRef]
4. Kamien, M.I.; Schwartz, N.L. Sufficient Conditions in Optimal Control Theory. *J. Econ. Theory* **1971**, *3*, 207–214. [CrossRef]
5. Clarke, H.R.; Reed, W.J. Consumption/Pollution Tradeoffs in an Environment Vulnerable to Pollution-related Catastrophic Collapse. *J. Econ. Dyn. Control* **1994**, *18*, 991–1010. [CrossRef]
6. Hoel, M. Resource Extraction when a Future Substitute has an Uncertain Cost. *Rev. Econ. Stud.* **1978**, *45*, 637–644. [CrossRef]
7. Hugonnier, J.; Morellec, E.; Pommeret, A. Technology Adoption under Uncertainty in General Equilibrium. In *2006 Meeting Papers 692*; Society for Economic Dynamics: Stonybrook, NY, USA, 2006.
8. Pommeret, A.; Schubert, K. Abatement Technology Adoption under Uncertainty. *Macroecon. Dyn.* **2009**, *13*, 493–522. [CrossRef]
9. Abel, A.B.; Eberly, J.C. Investment, Valuation, and Growth Options. *Q. J. Financ.* **2012**, *2*, 1250001. [CrossRef]
10. Kemp, M.C.; Long, N.V. Optimal Control Problems with Integrands Discontinuous with respect to Time. *Econ. Rec.* **1977**, *53*, 405–420. [CrossRef]
11. Tomiyama, K. Two-stage Optimal Control Problems and Optimality Conditions. *J. Econ. Dyn. Control* **1985**, *9*, 317–337. [CrossRef]
12. Makris, M. Necessary Conditions for Infinite-horizon Discounted Two-stage Optimal Control Problems. *J. Econ. Dyn. Control* **2001**, *25*, 1935–1950. [CrossRef]
13. Tahvonen, O.; Withagen, C. Optimality of Irreversible Pollution Accumulation. *J. Econ. Dyn. Control* **1996**, *20*, 1775–1795. [CrossRef]
14. Tsur, Y.; Zemel, A. Accounting for Global Warming Risks: Resource Management under Event Uncertainty. *J. Econ. Dyn. Control* **1996**, *20*, 1289–1305. [CrossRef]
15. Tsur, Y.; Zemel, A. Pollution Control in an Uncertain Environment. *J. Econ. Dyn. Control* **1998**, *22*, 967–975. [CrossRef]
16. Ren, B.; Polasky, S. The Optimal Management of Renewable Resources under the Risk of Potential Regime Shift. *J. Econ. Dyn. Control* **2014**, *40*, 195–212. [CrossRef]
17. Amit, R. Petroleum Reservoir Exploitation: Switching from Primary to Secondary Recovery. *Oper. Res.* **1986**, *34*, 534–549. [CrossRef]
18. Boucekkine, R.; Pommeret, A.; Prieur, F. Optimal Regime Switching and Threshold Effects. *J. Econ. Dyn. Control* **2013**, *37*, 2979–2997. [CrossRef]
19. Dogan, E.; Le Van, C.; Saglam, C. Optimal Timing of Regime Switching in Optimal Growth Models: A Sobolev Space Approach. *Math. Soc. Sci.* **2011**, *61*, 97–103. [CrossRef]
20. Boucekkine, R.; Pommeret, A.; Prieur, F. On the Timing and Optimality of Capital Controls: Public Expenditures, Debt Dynamics and Welfare. *Int. J. Econ. Theory* **2013**, *9*, 101–112. [CrossRef]
21. Boucekkine, R.; Saglam, C.; Vallée, T. Technology Adoption Under Embodiment: A Two-Stage Optimal Control Approach. *Macroecon. Dyn.* **2004**, *8*, 250–271. [CrossRef]
22. Saglam, C. Optimal Pattern of Technology Adoptions under Embodiment: A Multi-stage Optimal Control Approach. *Optim. Control. Appl. Methods* **2011**, *32*, 574–586. [CrossRef]
23. Seidl, A.; Caulkins, J.P.; Hartl, R.F.; Kort, P.M. Serious strategy for the makers of fun: Analyzing the option to switch from pay-to-play to free-to-play in a two-stage optimal control model with quadratic costs. *Eur. J. Oper. Res.* **2018**, *267*, 700–715. [CrossRef]
24. Caulkins, J.P.; Feichtinger, G.; Grass, D.; Hartl, R.F.; Kort, P.M.; Seidl, A. When to Make Proprietary Software Open Source. *J. Econ. Dyn. Control* **2013**, *37*, 1182–1194. [CrossRef]
25. Grass, D.; Behrens, D.A.; Caulkins, J.P.; Feichtinger, G.; Tragler, G. *Optimal Control of Nonlinear Processes: With Applications in Drugs, Corruption, and Terror*; Springer: Berlin/Heidelberg, Germany, 2008.

26. Caulkins, J.P.; Feichtinger, G.; Hartl, R.F.; Kort, P.M.; Novak, A.J.; Seidl, A. Long Term Implications of Drug Policy Shifts: Anticipating and Non-anticipating Consumers. *Annu. Rev. Control* **2013**, *37*, 105–115. [CrossRef]
27. Zhu, F.; Antsaklis, P.J. Optimal Control of Hybrid Switched Systems: A Brief Survey. *Discret. Event Dyn. Syst.* **2015**, *25*, 345–364. [CrossRef]
28. Cottle Hunt, E.; Caliendo, F.N. Social Security and Longevity Risk: An Analysis of Couples. *J. Public Econ. Theory* **2022**, *24*, 547–579. [CrossRef]
29. Cottle Hunt, E.N.; Caliendo, F.N. Social Security and Longevity Risk: The Case of Risky Bequest Income. *Macroecon. Dyn.* **2022**, *26*, 1647–1678. [CrossRef]
30. Caliendo, F.N.; Casanova, M.; Gorry, A.; Slavov, S. Retirement Timing Uncertainty: Empirical Evidence and Quantitative Evaluation. *Rev. Econ. Dyn.* **2023**, *51*, 226–266. [CrossRef]
31. Cottle Hunt, E.; Caliendo, F.N. Social Security Safety Net with Rare Event Risk. *Macroecon. Dyn.* **2024**, *28*, 495–515. [CrossRef]
32. Stokey, N.L. Wait-and-See: Investment Options under Policy Uncertainty. *Rev. Econ. Dyn.* **2016**, *21*, 246–265. [CrossRef]
33. Àbrahàm, A.; Brendler, P.; Cárceles-Poveda, E. Capital Tax Reforms with Policy Uncertainty. *Int. Econ. Rev.* **2024**, *65*, 75–116. [CrossRef]
34. Ferguson, A.; Hu, C.W.; Lam, P. 'Know when to Fold 'em': Policy Uncertainty and Acquisition Abandonment. *Account. Financ.* **2024**, *64*, 1277–1309. [CrossRef]
35. Campello, M.; Kankanhalli, G. Chapter 15: Corporate decision-making under uncertainty: Review and future research directions. In *Handbook of Corporate Finance*; Edward Elgar Publishing: Cheltenham, UK, 2024. [CrossRef]
36. Caliendo, F.N.; Gorry, A.; Slavov, S. The Cost of Uncertainty About the Timing of Social Security Reform. *Eur. Econ. Rev.* **2019**, *118*, 101–125. [CrossRef]
37. Kitao, S. Policy Uncertainty and Cost of Delaying Reform: The Case of Aging Japan. *Rev. Econ. Dyn.* **2018**, *27*, 81–100. [CrossRef]
38. Cottle Hunt, E. Adaptive Learning, Social Security Reform, and Policy Uncertainty. *J. Money Credit. Bank.* **2021**, *53*, 677–714. [CrossRef]
39. Nelson, J. Welfare Implications of Uncertain Social Security Reform. *Public Financ. Rev.* **2020**, *48*, 425–466. [CrossRef]
40. Michelangeli, V.; Santoro, M. Households' Uncertainty about Medicare Policy. *BE J. Macroecon.* **2013**, *13*, 151–186. [CrossRef]
41. Bi, H.; Leeper, E.M.; Leith, C. Uncertain Fiscal Consolidations. *Econ. J.* **2013**, *123*, F31–F63. [CrossRef]
42. Davig, T.; Leeper, E.M.; Walker, T.B. "Unfunded liabilities" and Uncertain Fiscal Financing. *J. Monet. Econ.* **2010**, *57*, 600–619. [CrossRef]
43. Davig, T.; Foerster, A.T. Uncertainty and Fiscal Cliffs. *J. Money Credit. Bank.* **2018**, *51*, 1857–1887. [CrossRef]
44. Biswas, S.; Zhai, W. Economic Policy Uncertainty and Cross-border Lending. *J. Corp. Financ.* **2021**, *67*, 101867. [CrossRef]
45. Bachmann, R.; Bai, J.H.; Lee, M.; Zhang, F. The Welfare and Distributional Effects of Fiscal Volatility: A Quantitative Evaluation. *Rev. Econ. Dyn.* **2020**, *38*, 127–153. [CrossRef]
46. Kreps, D.M.; Porteus, E.L. Temporal Resolution of Uncertainty and Dynamic Choice Theory. *Econometrica* **1978**, *46*, 185–200. [CrossRef]
47. Chew, S.H.; Epstein, L.G. The Structure of Preferences and Attitudes Towards the Timing of the Resolution of Uncertainty. *Int. Econ. Rev.* **1989**, *30*, 103–117.
48. Epstein, L.G.; Zin, S.E. Substitution, Risk Aversion, and the Temporal Behavior of Consumption and Asset Returns: A Theoretical Framework. *Econometrica* **1989**, *57*, 937–969. [CrossRef]
49. Ergin, H.; Sarver, T. Hidden Actions and Preferences for Timing of Resolution of Uncertainty. *Theor. Econ.* **2015**, *10*, 489–541. [CrossRef]
50. DeJarnette, P.; Dillenberger, D.; Gottlieb, D.; Ortoleva, P. Time Lotteries and Stochastic Impatience. *Econometrica* **2020**, *88*, 619–656. [CrossRef]
51. Epstein, L.G.; Farhi, E.; Strzalecki, T. How Much Would You Pay to Resolve Long-Run Risk? *Am. Econ. Rev.* **2014**, *104*, 2680–2697. [CrossRef]
52. Ebert, S. Decision Making When Things Are Only a Matter of Time. *Oper. Res.* **2020**, *68*, 1564–1575. [CrossRef]
53. Chesson, H.W.; Viscusi, W.K. Commonalities in Time and Ambiguity Aversion for Long-term Risks. *Theory Decis.* **2003**, *54*, 57–71. [CrossRef]
54. Epper, T.; Fehr-Duda, H.; Bruhin, A. Viewing the Future through a Warped Lens: Why Uncertainty Generates Hyperbolic Discounting. *J. Risk Uncertain.* **2011**, *43*, 169–203. [CrossRef]
55. Onay, S.; Öncüler, A. Intertemporal Choice under Timing Risk: An Experimental Approach. *J. Risk Uncertain.* **2007**, *34*, 99–121. [CrossRef]
56. Coble, K.H.; Lusk, J.L. At the Nexus of Risk and Rime Preferences: An Experimental Investigation. *J. Risk Uncertain.* **2010**, *41*, 67–79. [CrossRef]
57. Andreoni, J.; Sprenger, C. Risk Preferences Are Not Time Preferences. *Am. Econ. Rev.* **2012**, *102*, 3357–3376. [CrossRef]

58. Lanier, J.; Miao, B.; Quah, J.K.H.; Zhong, S. Intertemporal Consumption with Risk: A Revealed Preference Analysis. *Rev. Econ. Stat.* **2024**, 1–15. [CrossRef]
59. Ikink, I.; Roelofs, K.; Figner, B. The Effect of Time Ambiguity on Choice Depends on Delay and Amount Magnitude. *J. Behav. Decis. Mak.* **2024**, *37*, e2354. [CrossRef]