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LAPLACE, FINITE FOURIER SINE
AND COSINE, TRANSFORMATIONS

by

Jan Eugene Wynn

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Jan Eugene Wynn
PART I

Linear Transformations

Fundamental to the concept of a linear transformation is the notion of a linear space. However, before defining a linear space, we need the concept of a field.

**DEFINITION 1.** A field $\mathbb{F}$ is defined to be a triple $\mathbb{F} = \langle F; +, \cdot \rangle$, where $F$ is a non-empty set, and where $+$ and $\cdot$ are each binary operations defined on $F$ satisfying the following postulates. For any $a, b, c \in F$,

1. $a + b = b + a$
2. $(a + b) + c = a + (b + c)$
3. There exists $0 \in F$, called the identity element for $+$, such that for every $x \in F$, $x + 0 = x$.
4. For every $x \in F$, there exists $-x \in F$ such that $x + (-x) = 0$.
5. $a \cdot b = b \cdot a$
6. $(a \cdot b) \cdot c = a \cdot (b \cdot c)$
7. There exists $1 \in F$, $1 \neq 0$, called the identity element for $\cdot$, such that for every $x \in F$, $x \cdot 1 = x$.
8. For every $x \in F$, $x \neq 0$, there exists $x^{-1} \in F$ such that $x \cdot x^{-1} = 1$.
9. $a \cdot (b + c) = a \cdot b + a \cdot c$

The field of real numbers and the field of complex numbers, which shall hereafter be denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively, are two common examples of fields and will be employed throughout this report.
DEFINITION 2. Let $\mathcal{F} = <F;+,\cdot>$ be a field having $+$ and $\cdot$ identities 0 and 1, respectively, let $V$ be a non-empty set on which the binary operation $\oplus$ is defined, and finally let $\ominus$ be the binary operation defined for elements in $F$ with elements in $V$ having values in $V$. Then the system $\mathcal{V} = <V,F;+,\cdot,\oplus,\ominus>$ is called a linear space if and only if the following postulates are satisfied. For any $a,b,c \in F$ and $u,v,w \in V$,

1. $u \oplus v = v \oplus u$

2. $(u \oplus v) \oplus w = u \oplus (v \oplus w)$

3. There exists $0 \in V$, called the identity element for $V$, such that for every $t \in V$, $t + 0 = t$.

4. For every $t \in V$, there exists $-t \in V$ such that $t + (-t) = 0$.

5. $(a + b) \ominus u = (a \ominus u) \oplus (b \ominus u)$

6. $a \ominus (u \oplus v) = (a \ominus u) \oplus (a \ominus v)$

7. $(a \cdot b) \ominus u = a \ominus (b \ominus u)$

8. $1 \ominus u = u$

The collection of Reimann integrable functions on an interval $[a,b]$, and the collection of periodic functions with a fixed period are familiar examples of linear spaces.

If $n$ is a positive integer ( $n \in P.I.$ ) and $R_n = \{(a_1,a_2,\ldots,a_n) : a_i \in R \text{ for } i = 1,2,\ldots,n\}$, then $R_n$ is formed into a linear space over $R$ by the following:
1. If \( \alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and \( \beta = (b_1, b_2, \ldots, b_n) \in \mathbb{R}^n \),
then we define \( \alpha \oplus \beta \) as \( \alpha \oplus \beta = (a_1+b_1, a_2+b_2, \ldots, a_n+b_n) \).

2. If \( \alpha = (a_1, a_2, \ldots, a_n) \in \mathbb{R}^n \) and \( c \in \mathbb{R} \), then we define \( c \odot \alpha \) as \( c \odot \alpha = (ca_1, ca_2, \ldots, ca_n) \).

In particular, \( \mathbb{R}_1 \) is a linear space which is abstract-
ly identical to \( \mathbb{R} \).

It is noted that the term "vector space" is synonymous
with the term "linear space" in the literature.

**DEFINITION 3.** Let \( \mathcal{F} \) be a field over which the two linear
spaces \( \mathcal{U} = (U, F; +, \cdot, \ominus) \) and \( \mathcal{V} = (V, F; +, \cdot, \ominus) \) are de-
defined. A mapping \( T \) that pairs the elements of \( U \) with the
elements of \( V \) is called a **linear transformation** provided
the following conditions are satisfied:

1. \( T(u \oplus v) = T(u) \oplus T(v) \), for all \( u, v \in U \), and

2. \( T(a \odot u) = a \odot T(u) \), for all \( a \in F \) and \( u \in U \).

It is interesting to note that the single condition
\( T[(a \odot u) \oplus (b \odot v)] = [a \odot T(u)] \oplus [b \odot T(v)] \), where \( a, b \in F \) and \( u, v \in U \), is equivalent to the preceding conditions
1 and 2, and is often used to define a linear transformat-
ion. It should be mentioned that some authors prefer the
term "linear mapping" over "linear transformation", and that
each of these terms will be utilized in this report.

Also, for brevity, we shall often shorten the words
"linear space", "linear transformation", and "linear mapping"
by just using the words "space", "transformation", and "map-
ping", respectively.
Example 1. Let $RI[a,b]$ represent the linear space of functions which are Reimann integrable on the interval $[a,b]$. From elementary theorems of calculus it is seen that the operator $\int_a^b$ is a linear transformation which maps the space $RI[a,b]$ into the space $R_1$.

Example 2. The operator $\nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$ is a useful linear mapping, in applied mathematics, which transforms the space of second order differentiable functions of two variables into the space of functions of two variables.

Example 3. A very relevant example of a linear mapping is the general linear integral transformation which will now be formally defined.

**DEFINITION 4.** Let $U$ be a linear space of functions defined on the finite or infinite interval $(a,b)$. The general linear integral transformation of $F(t) \in U$ with respect to the kernel $K(t,s)$ is denoted by $T[F(t)]$ and is defined as

$$(1) \quad T[F(t)] = \int_a^b F(t)K(t,s)dt,$$

whenever this integral exists. In each particular instance, the space $U$, the interval $(a,b)$, the kernel $K(t,s)$, and the conditions on the parameter $s$ have to be prescribed.

The Laplace transformation and the Fourier sine and cosine transformations are special cases of the general
linear integral transformation, and will be discussed in this report.

If \( T \) is a linear mapping that pairs the elements of space \( \mathcal{U} \) with elements of space \( \mathcal{V} \), one naturally wonders if there exists a linear transformation, say \( T^{-1} \), called the inverse of \( T \), which maps the space \( \mathcal{V} \) back into the space \( \mathcal{U} \). The answer is not always affirmative for, in example 1, many different functions in \( \mathbb{R}[a,b] \) may have the same image in \( \mathbb{R}_1 \) and hence the impossibility of defining an inverse mapping.

However, there certainly are examples of linear transformations in which an inverse linear mapping may be defined.

Example 4. If \( f(t) \in C[a,b] \) (the space of all continuous functions on the interval \([a,b]\)), then

\[
T[f(t)] = \int_a^x f(t)\,dt = F(x),
\]

where \( F(x) \) is defined on \([a,b]\), is indeed a linear transformation that maps \( C[a,b] \) into \( C_1[a,b] \) (the space of first order differentiable functions defined on the interval \([a,b]\)). Note that \( f(t) \) may be recovered by the inverse linear mapping \( T^{-1}[F(x)] = \frac{d(F(x))}{dx} \), since

\[
\frac{d}{dx} \int_a^x f(t)\,dt = f(x).
\]

It was mentioned in example 3 that concern will be
given to some special cases of the general linear integral transformation. In each instance the inverse to the transformation is significant and will also be discussed.
PART II

The Laplace Transformation

DEFINITION 1. Let \( F(t) \) be a real-valued function of a positive real variable, and let \( s \in \mathbb{R} \) be a parameter independent of \( t \). The Laplace transform of \( F(t) \), denoted by \( \mathcal{L}[F(t)] \) is defined as

\[
\mathcal{L}[F(t)] = \int_0^\infty F(t)e^{-st}dt,
\]

whenever this integral converges. We see from the definition that the Laplace transform of a function is unique. Also, note that the Laplace transform is nothing but a special case of the general linear integral transformation which was defined in example 3 of part I.

Since the above integral is improper, some sufficient restrictions for its convergence will have to be placed on \( F(t) \). Such restrictions will now be investigated and the results are summarized in Theorem 2 below.

DEFINITION 2. A function \( F(t) \) is said to be sectionally continuous on the closed interval \([a,b]\) if and only if the one sided limits \( F(a^+) \), \( F(b^-) \), \( F(x^+) \), and \( F(x^-) \) exist and are finite for \( x \in (a,b) \), and \( F(x^+) = F(x^-) = F(x) \) for all \( x \in (a,b) \).

* Later in this report we shall let \( s \) assume complex values.
except possibly for a finite number of points in this open interval.

Note that every continuous function is sectionally continuous on the same interval. The unit step function,

\[
S_k(t) = \begin{cases} 
0, & 0 < t < k \\
1, & t > k 
\end{cases}
\]

is another example of a function that is sectionally continuous. The space of all sectionally continuous functions on the finite interval \(0 \leq t \leq T\), for every positive number \(T\), will be denoted hereafter as \(SC[0,T]\).

**DEFINITION 3.** A function \(F(t)\) is said to be of exponential order (E.O.) if there exists constants \(\alpha\), \(M\), and \(T\) such that \(|F(t)| < Me^{\alpha t}\) for all \(t > T\).

An example of a function of E.O. is \(t^2\), for let \(\alpha = M = 1\) and \(T = 0\). But the function \(e^{t^2}\) is not of E.O. since, for any \(\alpha\), \(M\), and \(T\), there exists \(t > T\) such that \(e^{t^2} \geq Me^{\alpha t}\).

**THEOREM 1.** Let \(F(t) \in SC[0,T]\) be of E.O. Then \(L[F(t)]\) converges absolutely for \(s > \alpha\), where the existence of \(\alpha\) is guaranteed in definition 3.

**Proof.** \(F(t) \in SC[0,T]\) implies that \(F(t)\) is bounded. Therefore there exists \(M_1\) such that

\[
|F(t)| < M_1 = (M_1 e^{-\alpha t})e^{\alpha t} \quad (t \geq 0)
\]

Also, \(F(t)\) being of E.O. implies the existence of \(M_2\) and
T such that

\[(4) \quad |F(t)| < M_2 e^{\alpha t} \quad (t > T)\]

But from (3) and (4), we have \(|F(t)| < M_0 e^{\alpha t}\) for all \(t > 0\), where \(M = \max(M_1, M_0 e^{\alpha t}, M_2)\). Hence,

\[
\int_0^b |F(t)| e^{-st} dt < \int_0^b M_0 e^{\alpha t} e^{-st} dt = \frac{M [1 - e^{-(s-\alpha)b}]}{s-\alpha}
\]

and for \(s > \alpha\),

\[
\lim_{b \to \infty} \frac{M [1 - e^{-(s-\alpha)b}]}{s-\alpha} = \frac{M}{s-\alpha}
\]

which means,

\[(5) \quad \lim_{b \to \infty} \int_0^b |F(t)| e^{-st} dt \leq \frac{M}{s-\alpha}.
\]

Since \(|F(t)| e^{-st} > 0\), then \(\int_0^b |F(t)| e^{-st} dt\) is monotonically increasing. Finally by the previous observation coupled with (5),

\[
\lim_{b \to \infty} \int_0^b |F(t)| e^{-st} dt
\]

exists for \(s > \alpha\), which proves the theorem.

COROLLARY. Let \(F(t) \in SC[0, T]\) be of E.O. Then for the \(M\) derived in the above proof and \(s > \alpha\),

\[(6) \quad \lim_{s \to \infty} L[F(t)] = 0 \quad \text{and} \quad sL[F(t)] < M.
\]

In fact,

\[
\left| \int_0^b F(t) e^{-st} dt \right| \leq \int_0^b |F(t)| e^{-st} dt \leq \frac{M}{s-\alpha}.
\]
Hence, letting $b \to \infty$, we have

$$\left| L[F(t)] \right| \leq \frac{M}{s - \alpha},$$

from which each of the relations of (6) follow.

**Theorem 2.** Assume the conditions of Theorem 1. Then $L[F(t)]$ converges uniformly for $s \geq s_0 > \alpha$.

**Proof.** It must be shown for every positive number $\epsilon$ there exists $B = B(\epsilon)$, not dependent upon $s$, such that

$$\left| \int_b^\infty F(t)e^{-st} \, dt \right| < \epsilon \quad (b > B, s > s_0)$$

By Theorem 1, $\int_0^\infty |F(t)|e^{-st} \, dt$ converges for $s > \alpha$, and hence $\int_b^\infty |F(t)|e^{-st} \, dt \to 0$ as $b \to \infty$. In other words, for every $\epsilon > 0$ and $s_0 > \alpha$ there exists $B$ such that $b > B$ implies

$$\int_b^\infty |F(t)|e^{-st} \, dt < \epsilon.$$

But if $s > s_0$, then $e^{-st} < e^{-s_0 t}$ for $t > 0$ which implies that

$$\int_b^\infty |F(t)|e^{-st} \, dt \leq \int_b^\infty |F(t)|e^{-s_0 t} \, dt.$$

Summarizing, for any $s > s_0$,

$$\int_b^\infty |F(t)|e^{-st} \, dt < \epsilon$$

for all $b > B$. But this $B$, chosen for (7), is independent of $s$. The proof is completed by the fact that

$$\left| \int_b^\infty F(t)e^{-st} \, dt \right| \leq \int_b^\infty |F(t)|e^{-st} \, dt.$$

It is observed that

$$L[F(t)] = \int_0^\infty F(t)e^{-st} \, dt.$$
is a function of the parameter $s$. From the following theorem it is apparent that the Laplace transformation is a mapping which pairs sectionally continuous functions of $E.O.$ with continuous functions of parameter $s$ having the properties of (6).

**THEOREM 3.** Assume the conditions of Theorem 1. Then

$$f(s) = \int_0^\infty F(t)e^{-st}dt$$

is a continuous function of $s$ for $s > \alpha$.

Proof. Since $\int_0^\infty F(t)e^{-st}dt$ converges uniformly, for $\varepsilon > 0$ there is a $t_0 > 0$ such that

$$\left| \int_t^{t_0} F(t)e^{-st}dt \right| < \varepsilon/3 \quad (t > t_0)$$

Under the given conditions it can be shown that $\int_0^\infty F(t)e^{-st}dt$ is continuous in $s$. Therefore, for some $\eta > 0$,

$$\left| \int_0^{t_0} F(t)e^{-(s+\Delta s)t}dt - \int_0^{t_0} F(t)e^{-st}dt \right| < \varepsilon/3 \quad (|\Delta s| \leq \eta).$$

Also,

$$f(s) = \int_0^{t_0} F(t)e^{-st}dt + \int_{t_0}^\infty F(t)e^{-st}dt.$$

Thus, $f(s + \Delta s) - f(s) =$

$$\int_0^{t_0} F(t)e^{-(s+\Delta s)t}dt - \int_0^{t_0} F(t)e^{-st}dt + \int_{t_0}^\infty F(t)e^{-(s+\Delta s)t}dt$$

$$- \int_{t_0}^\infty F(t)e^{-st}dt.$$

Moreover,

$$\left| \int_{t_0}^\infty F(t)e^{-st}dt \right| < \varepsilon/3 \quad \text{and} \quad \int_{t_0}^\infty F(t)e^{-(s+\Delta s)t}dt < \varepsilon/3,$$
And finally,
\[ |f(s + \Delta s) - f(s)| < \varepsilon / \delta + \varepsilon / \delta + \varepsilon / \delta = \varepsilon \]
for \( |\Delta s| \leq \eta \), which proves the theorem.

From Theorem 2 it is seen that every function of \( E.C. \) that is a member of \( SC[0,T] \) necessarily has a Laplace image. In other words, the existence of such a transform is guaranteed just as an existence theorem in differential equations may guarantee a solution to a particular differential equation. Although we may know an equation has a solution, it is often very difficult and perhaps even impossible to find the solution. Likewise, even though we may realize the integral \( \int_{0}^{\infty} F(t)e^{-st} dt \) converges for a particular \( F(t) \), it may be too great a task to perform the integration.

Fortunately, there are many useful operational properties of the Laplace transformation that assist one greatly in finding the transform of certain functions. In many cases such properties make it possible to avoid integration entirely. A few of these operational properties will now be presented as theorems.

For the purpose of gaining some tools to work with, let us derive the transform of three basic functions before beginning the theorems.

First, let \( F(t) = 1 \) for \( t > 0 \). Then \( L[F] = \int_{0}^{\infty} e^{-st} dt = \lim_{b \to \infty} \int_{0}^{b} e^{-st} dt = \lim_{b \to \infty} \left[ \frac{-e^{-st}}{s} \right]_{0}^{b} = 1 \) for \( s > 0 \).

Secondly, let \( F(t) = e^{-at} \), where \( t > 0 \) and \( a \) is a constant.
Then $L[F] = \int_0^\infty e^{-at}e^{-st}dt = \lim_{b \to \infty} \int_0^b e^{-(s+a)t}dt = \lim_{b \to \infty} \frac{[-e^{-(s+a)t}]_0^b}{s+a} = \frac{1}{s+a}$ for $s > a$.

Thirdly, let $F(t) = t^p$, where $p > -1$ and $t > 0$. Then $L[F] = \int_0^\infty t^p e^{-st}dt = \frac{1}{s} \int_0^\infty \left( \frac{z}{s} \right)^p e^{-z}dz$, when $t = \frac{z}{s}$.

Now,

$$\int_0^\infty \left( \frac{z}{s} \right)^p e^{-z}dz = \frac{1}{s^{p+1}} \int_0^\infty z^p e^{-z}dz. \tag{8}$$

The so-called gamma function, $\Gamma(x)$, is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt.$$

It is easily shown that $\Gamma(1) = 1$. Using integration by parts one arrives at the recursive formula $x \Gamma(x) = \Gamma(x+1)$ from which it follows immediately that $\Gamma(2) = 1!$, $\Gamma(3) = 2!$, ..., $\Gamma(p+1) = p!$. Also it can be shown that $\Gamma(x)$ converges for $x > 0$.

Returning to the right side of (8),

$$\frac{1}{s^{p+1}} \int_0^\infty z^p e^{-z}dz = \frac{\Gamma(p+1)}{s^{p+1}} \quad (p > -1)$$

Hence, for $s > 0$,

$$L[t^p] = \frac{\Gamma(p+1)}{s^{p+1}} \quad (p > -1)$$

$$= \frac{p!}{s^{p+1}} \quad (p \in \mathbb{F} \cdot \mathbb{I} \cdot \mathbb{I}.)$$

**Theorem 4.** The Laplace transform is indeed a linear mapping.

**Proof.** Let $F_1(t), F_2(t) \in SC[0,T]$ and each be of E.O.

Also let $a \in \mathbb{R}$. Then $L[F_1+F_2] = \int_0^\infty [F_1(t)+F_2(t)]e^{-st}dt = \int_0^\infty L[F_1(t)]e^{-st}dt + \int_0^\infty L[F_2(t)]e^{-st}dt \quad (p \in \mathbb{F} \cdot \mathbb{I} \cdot \mathbb{I}.)$
\[
\int_0^\infty F_1(t)e^{-st}dt + \int_0^\infty F_2(t)e^{-st}dt = L[F_1] + L[F_2]; \text{ and } L[aF] = \int_0^\infty aF(t)e^{-st}dt = a\int_0^\infty F(t)e^{-st}dt = aL[F].
\]

We can use Theorem 4 to obtain the transform of some new functions.

Illustration 1. If \( k \in \mathbb{R} \), then \( L[k] = kL[1] = \frac{1}{s} \) for \( s > 0 \).

Illustration 2. If \( a, b \in \mathbb{R} \), then \( L[a+b] = L[a] + L[b] = \frac{a}{s} + \frac{b}{s} = \frac{a+b}{s} \) for \( s > 0 \).

Illustration 3. If \( b \in \mathbb{R} \) and \( t > 0 \), then \( L[\sinh bt] = \frac{1}{2}L[e^{bt} - e^{-bt}] + \frac{1}{2}L[-e^{-bt}] = \frac{1}{2(s-b)} - \frac{1}{2(s+b)} \) for \( s > |b| \).

THEOREM 5. Let \( F(t) \in C[0,T] \) and be of E.O., and let \( F'(t) \in SC[0,T] \). Then for \( s > \alpha \), \( L[F'(t)] \) exists and
\[
L[F'(t)] = sL[F(t)] - F(c).
\]

Proof. Without loss of generality, for definiteness, let there be a single point, say \( t_0 \), such that \( F'(t_0) \) is discontinuous. Then \( \int_0^T F'(t)e^{-st}dt = \int_0^{t_0} F'(t)e^{-st}dt + \int_{t_0}^T F'(t)e^{-st}dt \). Integrating by parts and letting \( u = e^{-st} \) and \( dv = F'(t)dt \) gives \( \int_0^T F'(t)e^{-st}dt = F(t)e^{-st} \bigg|_0^{t_0} + F(t_0)e^{-st} \bigg|_0^T + s\int_0^T F(t)e^{-st}dt \). But \( F(t) \) is continuous which implies that \( F(t_0^-) = F(t_0^+) \). Hence we now have
\begin{align}
\int_0^T F'(t)e^{-st}dt &= -F(O) + F(T)e^{-sT} + \\
&\quad + s\int_0^T F(t)e^{-st}dt.
\end{align}

Because \(F(t)\) is of E.O., there exist constants \(\alpha\) and \(M\) such that \(|F(t)| < Me^{\alpha t}\) for large \(t\), from which we may conclude that \(|F(T)e^{-sT}| < Me^{-(s-\alpha)T}\). Moreover, for \(s>\alpha\),
\[
\lim_{T \to \infty} Me^{-(s-\alpha)T} = 0.
\]
This means that \(|F(T)e^{-st}| \to 0\) as \(T \to \infty\).

Returning to (9), we obtain, by taking the limit as \(T \to \infty\),
\[
\mathcal{L}[F'(t)] = s\mathcal{L}[F(t)] - F(O)
\]
which is the assertion.

**COROLLARY 1.** Let \(F(t), F'(t) \in C[O,T]\) and each be of E.O., and let \(F''(t) \in SC[O,T]\). Then for \(s>\alpha\),
\[
\mathcal{L}[F''(t)] = s^2[F(t)] - s[F(O)] - F'(O).
\]

**Proof.** By Theorem 5, \(\mathcal{L}[F'] = s\mathcal{L}[F'] - F'(O)\)
\[
= s[s\mathcal{L}[F] - F(O)] - F'(O) = s^2\mathcal{L}[F] - sF(O) - sF'(O).
\]

**COROLLARY 2.** Let \(F(t), F'(t), \ldots, F^{(n-1)}(t) \in C[O,T]\) and each be of E.O., and let \(F^{(n)}(t) \in SC[O,T]\), where \(n \in \mathbb{F.I.}\). Then
\[
\mathcal{L}[F^{(n)}(t)] = s^n\mathcal{L}[F] - s^{n-1}F(O) - s^{n-2}F'(O) - \ldots - sF^{(n-2)}(0) - F(n-1)(0).
\]

**Proof (by mathematical induction).** Assume that
\[
\mathcal{L}[F^{(k)}] = s^k\mathcal{L}[F] - s^{k-1}F(O) - s^{k-2}F'(O) - \ldots - sF^{(k-2)}(0) - F(k-1)(0).
\]
Then by Theorem 5, \(\mathcal{L}[F^{(k+1)}] = s\mathcal{L}[F^{(k)}] - F^{(k)}(0) = \)
\[
s^{k+1}\mathcal{L}[F] - s^kF(O) - s^{k-1}F'(O) - \ldots - sF^{(k-2)}(0) - sF(k-1)(0)
\]
- F(k)(0). Therefore, (10) is true for all n ∈ P.I.

COROLLARY 3. Let F(T) ∈ SC[0,T] and be of E.O. Then
\[ \mathcal{L}\left[ \int_{0}^{t} F(r) dr \right] = \frac{1}{s^2} \mathcal{L}[F(t)]. \]

Proof. Let G(T) = \int_{0}^{t} F(r) dr. Because F(t) ∈ SC[0,T], then G(t) ∈ C[0,T]. Also, G'(t) = F(t) except where F(t) is discontinuous which implies that G'(t) ∈ SC[0,T]. In order to apply Theorem 5 on G(t) it remains to show that G(t) is of E.O. To verify this fact we know there exists α ∈ O and M such that |F(t)| < Me^{αt} for t ≥ 0. Then
\[ \left| \int_{0}^{t} F(r) dr \right| < \int_{0}^{t} |F(r)| dr < M \int_{0}^{t} e^{αt} dr = \frac{M}{α} (e^{αt} - 1). \]
Hence
\[ G(t) < \frac{M}{α} e^{αt} \text{ and thus } G(t) \text{ is of } E.O. \]
Now by Theorem 5, when s > α, \[ \mathcal{L}[G'(t)] = s \mathcal{L}[G(t)] - G(0). \]
Therefore \[ \mathcal{L}[F(t)] = s \mathcal{L}\left[ \int_{0}^{t} F(r) dr \right] - 0, \] from which Corollary 3 follows.

COROLLARY 4. Under the conditions of Corollary 3,
\[ \mathcal{L}\left[ \int_{0}^{t} \int_{0}^{r} F(\xi) d\xi dr \right] = \frac{1}{s^2} \mathcal{L}[F(t)]. \]

If fact, from Corollary 3,
\[ \mathcal{L}\left[ \int_{0}^{t} \int_{0}^{r} F(\xi) d\xi dr \right] = \frac{1}{s^2} \mathcal{L}\left[ \int_{0}^{t} F(\xi) d\xi \right] = \frac{1}{s^2} \mathcal{L}[F(t)]. \]

Later on, when the inverse of the Laplace transformation is discussed, forms (11) and (12) will be derived differently.

Illustration 4. For \( F(t) = \sin t \), \( t > 0 \)
\[ \mathcal{L}[\sin t] = \mathcal{L}[-F''(t)] = -s^2 \mathcal{L}[\sin t] + s(\sin 0) + \cos 0. \]
Solving for \( \mathcal{L}[\sin t] \) gives
\[(13) \quad L[\sin t] = \frac{1}{s^2 + 1} \quad (s > 0)\]

Also,

\[(14) \quad L[\cos t] = \frac{s}{s^2 + 1} \quad (s > 0)\]

is derived similarly.

In the author's second report, when applications are discussed, Theorem 5 and its corollaries will enable us to find the Laplace transform of certain differential and integral equations.

THEOREM 6. Let \(L[F(t)] = f(s)\) for \(s > \alpha\). Then for \(s-a > \alpha\),

\[L[e^{at}F(t)] = f(s-a).\]

Proof. Substituting \(s-a\) for \(s\) in \(L[F(t)] = f(s)\) we obtain

\[\int_0^\infty F(t)e^{-(s-a)t}dt = f(s-a).\]

But,

\[\int_0^\infty F(t)e^{-(s-a)t}dt = \int_0^\infty e^{at}F(t)e^{-st}dt,\]

which is, by definition, \(L[e^{at}F(t)]\).

Illustration 5. Since \(L[t^n] = \frac{n!}{s^{n+1}}\) for \(s > 0\) and \(n \in \mathbb{P} \cup \{0\}\),

then \(L[e^{at}t^n] = \frac{n!}{(s+a)^{n+1}}\).

Illustration 6. From (13) and Theorem 6 it follows that

\[L[e^{at}\sin t] = \frac{1}{(s+a)^2 + 1} \quad (s > 0)\]

THEOREM 7. Let \(L[F(t)] = f(s)\) for \(s > \alpha\). Then

\[L[F(at)] = \frac{1}{a}f\left(\frac{s}{a}\right) \quad (s > a\alpha \text{ and } a > 0)\]

Proof. By definition

\[L[F(at)] = \int_0^\infty F(at)e^{-st}dt.\]
Letting \( r = \alpha t \), this integral becomes

\[
\frac{1}{\alpha} \int_0^\infty F(r) e^{-(s/\alpha)r} dr,
\]

which is just \( \frac{1}{\alpha} f(\frac{s}{\alpha}) \).

**Illustration 7.** From equation (13) and Theorem 7 we obtain

(15) \[
L[\sin bt] = \left( \frac{1}{b} \right) \frac{\sin b}{s^2 + b^2} = \left( \frac{1}{b} \right) \frac{b}{s^2 + b^2} \quad (b \in \mathbb{R}, s > 0)
\]

Similarly, from equation (14) and Theorem 7 we obtain

(16) \[
L[\cos bt] = \left( \frac{1}{b} \right) \frac{\cos b}{s^2 + b^2} = \left( \frac{1}{b} \right) \frac{s}{s^2 + b^2} \quad (b \in \mathbb{R}, s > 0)
\]

**Theorem 8.** If \( L[F(t)] = f(s) \) for \( s > \alpha \), then

\[
L[tf(t)] = -f'(s) \quad (s > \alpha)
\]

Proof. By hypothesis, \( f(s) = \int_0^\infty F(t)e^{-st} dt \).

Differentiating both sides with respect to \( s \) we have

\[
f'(s) = \frac{d}{ds} \left[ \int_0^\infty F(t)e^{-st} dt \right] = \int_0^\infty F(t)(-t)e^{-st} dt = L[-tF(t)].
\]

**Corollary.** If \( L[F(t)] = f(s) \), then for \( s > \alpha \),

\[
L[t^nF(t)] = (-1)^n f^{(n)}(s).
\]

The proof of this corollary follows from mathematical induction.

**Illustration 8.**

\[
L[tsin t] = -\frac{d}{ds} \left[ \frac{1}{s^2 + 1} \right] = -\frac{2s}{(s^2 + 1)^2} \quad (s > 0)
\]

\[
L[t^2e^{-at}] = (-1)^2 \frac{d^2}{ds^2} \left[ \frac{1}{s+a} \right] = \frac{2}{(s+a)^3} \quad (a \in \mathbb{R}, s > 0)
\]

The results of Theorem 8 and its corollary makes it
possible to find the transform of certain differential and integral equations having polynomial coefficients.

**THEOREM 9.** Let \( F(t) \in SC[0,T] \) and be of E.C., and let the limit of \( \frac{F(t)}{t} \) exist as \( t \to 0^+ \). Then if \( L[F(t)] = f(s) \),

\[
L[F(t)/t] = \int_s^\infty f(x) \, dx \quad (s > \alpha)
\]

**Proof.** We are given that \( \int_0^\infty F(t)e^{-st} \, dt = f(s) \).

Integrating both sides from \( s \) to \( \infty \) and letting \( x \) be a dummy variable on the right, we obtain

\[
\int_s^\infty \int_0^\infty F(t)e^{-st} \, dt \, ds = \int_s^\infty \int_s^\infty f(x) \, dx \, ds.
\]

Reversing the order of integration we have

\[
\int_0^\infty \int_s^\infty F(t)e^{-st} \, ds \, dt = \int_s^\infty \int_s^\infty f(x) \, dx \, ds.
\]

Evaluating the inner integral yields

\[
\int_0^\infty F(t) \left[ -\frac{e^{-st}}{t} \right]_s^\infty \, dt = \int_s^\infty f(x) \, dx.
\]

The left integral reduces to

\[
\int_0^\infty \left[ \frac{F(t)}{t} \right] e^{-st} \, dt,
\]

and since the limit of \( F(t)/t \) exists for \( t \to 0^+ \), it follows that

\[
L[F(t)/t] = \int_s^\infty f(x) \, dx.
\]

**Illustration 9.** \( L[(\sin t)/t] = \int_0^\infty \frac{dx}{x^2 + 1} = \tan^{-1} \left[ \frac{x}{s} \right]_s^\infty = \pi/2 - \tan^{-1} \frac{1}{s}, \)

\( \tan^{-1} \frac{1}{s} = &\cot^{-1} \frac{1}{s}. \)

**THEOREM 10.** Let \( F(t) \) be a periodic function with period \( a > 0 \). Then

\[
L[F(t)] = \frac{\int_0^a F(t)e^{-st} \, dt}{1-e^{-as}}.
\]
Proof. By definition, $L[F(t)] = \int_{0}^{\infty} F(t) e^{-st} dt =$

$= \int_{0}^{a} F(t) e^{-st} dt + \int_{a}^{2a} F(t) e^{-st} dt + \int_{2a}^{3a} F(t) e^{-st} dt + \cdots$

In the second integral substitute $T+a$ for $t$, in the third substitute $T+2a$ for $t$, and in general substitute $T+na$ for $t$ in the $(n+1)$st integral. Hence

$L[F(t)] = \int_{0}^{a} F(T) e^{-sT} dT + \int_{0}^{a} F(T+a) e^{-s(T+a)} dT$

$+ \int_{0}^{a} F(T+2a) e^{-s(T+2a)} dT + \cdots$

$= \int_{0}^{a} F(T) e^{-sT} dT + e^{-as} \int_{0}^{a} F(T+a) e^{-sT} dT$

$+ e^{-2as} \int_{0}^{a} F(T+2a) e^{-sT} dT + \cdots$

But $F(T) = F(T+a) = F(T+2a) = \cdots$ for all values of $T$ since $F(T)$ is periodic with period $a$. Thus we have

$L[F(t)] = (1+e^{-as}+e^{-2as}+\cdots) \int_{0}^{a} F(T) e^{-sT} dT$.

The proof is completed by the fact that the sum of the geometric series $1+e^{-as}+e^{-2as}+\cdots$ is $(1-e^{-as})^{-1}$.

Illustration 10. Find the transform of the saw-tooth wave function shown in Figure 1.

![Figure 1](Note: Figure 1 is not provided in the text. It would typically show a saw-tooth wave function with a period of $\pi$.)

Here the period is $\pi$, and thus, $L[F(t)] =$
\[
\frac{1}{1-e^{-\pi s}} \int_0^\pi 2te^{-st} dt = 2 \left[ \frac{1+\pi s}{s^2} - \frac{\pi}{s(1-e^{-\pi s})} \right].
\]

**THEOREM 11.** Let \( F(t) \in SC[0,T] \) and be of E.O. Then

\[
\mathcal{L}[F_b(t)] = e^{-bs} f(s)
\]

where the transform of \( F(t) \) is \( f(s) \) and \( F_b(t) \) is defined as \( F(t-b), t > 0 \).

**Proof.** By definition we have

\[
\mathcal{L}[F_b(t)] = \int_0^b F(t) dt + \int_b^\infty F(r-b)e^{-sr} dr.
\]

Letting \( t = r-b \), we can write the last integral in the form

\[
\int_0^\infty F(r-b)e^{-sr} dr = \int_0^\infty F(t)e^{-s(t+b)} dt = e^{-bs} \int_0^\infty F(t)e^{-st} dt = e^{-bs} f(s).
\]

**Illustration 11.** Consider the unit step function \( S_k(t) \) as defined in (2). By Theorem 11 we obtain

\[
\mathcal{L}[S_k(t)] = e^{-ks} \mathcal{L}[1] = \frac{e^{-ks}}{s} \quad (t > k, s > 0)
\]

Before closing this section let us allow the parameter \( s \) to be non-real, as promised earlier.

**DEFINITION 4.** A function \( F(t) \) is said to be of the order \( e^{x_0 t} \) \((F(t) \in O(e^{x_0 t}))\) when \( t \geq 0 \) and \( x_0 \in \mathbb{R} \), providing the constant \( M \) exists such that \(|F(t)| < Me^{x_0 t}\).

**DEFINITION 5.** A function \( f(z) \) of a complex variable is said to be analytic at a point \( z_0 \) if its derivative \( f'(z) \) exists at every point of some neighborhood of \( z_0 \).

**DEFINITION 6.** Let \( f(z) = u(x,y) + iv(x,y) \), where \( u \) and \( v \) are real-valued functions. Then \( f(z) \) is said to satisfy
the Cauchy-Riemann conditions if
\[ \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}. \]

**THEOREM 12.** Let \( u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \) be continuous, real valued functions of \( x \) and \( y \) in some neighborhood of the point \( z_0 = (x_0, y_0) \). Then a necessary and sufficient condition that \( f(z) = u + iv \) be analytic at \( z_0 \) is that the Cauchy-Riemann conditions be satisfied at that point.

To prove this theorem would take us far astray from our objectives so the statement of the theorem will have to suffice.

**THEOREM 13.** Let \( s = x + iy \), where \( x, y \in \mathbb{R} \), and let \( F(t) \in SC[0,T] \) and be \( \mathcal{O}(e^{x_0 t}) \) for \( t \geq 0 \). Then the Laplace integral
\[ \int_0^\infty F(t)e^{-st}dt \]
is absolutely convergent for \( x > x_0 \); it is uniformly convergent with respect to \( x \) and \( y \) in each half plane \( x > x_1 \), where \( x_1 > x_0 \). Moreover, \( L[F(t)] \) is an analytic function of \( s \) for \( x > x_0 \).

**Proof.** The Laplace transform of \( F(t) \) is
\[ f(s) = \int_0^\infty F(t)e^{-st}dt = \int_0^\infty F(t)e^{-xt}e^{-iyt}dt \]
But \( e^{-iyt} = \cos yt - i\sin yt \). Thus (18) becomes \( f(s) = u(x,y) + iv(x,y) \) where
\[ u(x,y) = \int_0^\infty F(t)e^{-xt}\cos yt dt \quad \text{and} \quad v(x,y) = -\int_0^\infty F(t)e^{-xt}\sin yt dt. \]
Also, we note that
\[ |F(t)e^{-xt}\cos yt| \leq |F(t)e^{-st}| \quad \text{and} \quad |F(t)e^{-xt}\sin yt| \leq |F(t)e^{-st}|. \]

From the hypothesis, there is an \( M \) such that \( |F(t)| < Me^{x_0t} \) for \( t \geq 0 \). Therefore
\[ |F(t)e^{-st}| = |F(t)| e^{-xt} < Me^{-(x-x_0)t} \leq Me^{-(x_1-x_0)t}, \]
whenever \( x \geq x_1 \), and \( x_1 > x_0 \). Now from (20) and (21) we may conclude that
\[ |F(t)e^{-xt}\cos yt| \leq Me^{-(x-x_0)t}, \quad |F(t)e^{-xt}\sin yt| \leq Me^{-(x_1-x_0)t}, \]
(22)
\[ |F(t)e^{-xt}\cos yt| \leq Me^{-(x-x_0)t}, \quad |F(t)e^{-xt}\sin yt| \leq Me^{-(x_1-x_0)t}, \]
(23)

Since the integral \( \int_0^\infty Me^{-(x-x_0)t} \) converges for \( x > x_0 \), then (17) converges absolutely. Furthermore, by the Weierstrass test for integrals, we establish the uniform convergence of the integrals in (19) with respect to \( x \) and \( y \), whenever \( x \geq x_1 \) and \( x_1 > x_0 \).

It can be shown that \( u, v, \frac{\partial u}{\partial x}, \frac{\partial v}{\partial y} \) are continuous functions of \( x \) and \( y \), that \( \frac{\partial u}{\partial x} = -\frac{\partial v}{\partial y} = \int_0^\infty F(t)(-t)e^{-xt}\cos yt \, dt \) for \( x > x_0 \), and that \( \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} = \int_0^\infty F(t)(-t)e^{-xt}\cos yt \, dt \) for \( x > x_0 \). Thus by Theorem 12, the last assertion of Theorem 13 follows.

**Theorem 14.** Assume the conditions of Theorem 13. Then for \( x > x_0 \),
\[ f^{(n)}(s) = \mathcal{L}(-t)^n F(t) \]\nand
\[ \overline{f(s)} = f(\overline{s}) \]
where \( n \in \mathbb{P}, \mathbb{I} \), \( \overline{f(s)} \) is the complex conjugate of \( f(s) \) and
$\overline{s}$ is the complex conjugate of $s$.

**Proof.** By the previous theorem we know that $f(s)$ is analytic and thus by a theorem from complex variables $f'(s) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}$; that is, $f'(s) = -\int_{0}^{\infty} F(t)e^{-xt}(\cos yt - isin yt)dt = \int_{0}^{\infty} tF(t)e^{-st}dt = L[-tF(t)]$, where $u$ and $v$ are defined in (19), $s = x + iy$, and $x > x_0$. From this, the first claim of Theorem 14 follows by mathematical induction for $n = 2, 3, \ldots$. Moreover,

$$f(s) = \int_{0}^{\infty} F(t)e^{-x(t+iy)} dt = \int_{0}^{\infty} F(t)e^{-xt}(\cos yt + isin yt)dt.$$

And, from (19) we obtain

$$f(s) = u - iv = \int_{0}^{\infty} F(t)e^{-xt}(\cos yt + isin yt)dt.$$

Therefore, $\overline{f(s)} = f(\overline{s})$.

Theorem 13 tells us that $f(s)$ is analytic in the half plane $x > x_0$, where $s = x + iy$. If the integration in the Laplace transform is performed when $s = x$, then a real function, say $\varphi(x)$, is obtained that is identical to $f(s)$ to the right of $x_0$; that is, $\varphi(x) = f(x)$ when $x > x_0$. If $\varphi(s)$ is analytic for $x > x_0$, then it must be identical to $f(s)$, since two different analytic functions cannot be identical along a line.

Thus we see that the properties obtained by assuming $s \in \mathbb{R}$ carry over to the case in which $s \in \mathbb{C}$. It follows that the transforms of particular functions derived for $s \in \mathbb{R}$ are valid when $s \in \mathbb{C}$. One seldom needs the value of $x_0$ which determines the half plane in which $s$ lies; the
existence of the number $x_0$ usually suffices.

We now have sufficient machinery for our purposes to find the Laplace transform of a relatively wide class of functions of E.O. in $SC[0,T]$. The next section of this report is an important discussion on the inverse of the Laplace transform.
PART III

The Inverse of the Laplace Transformation

The Laplace transform is a useful tool in linear system analysis. In particular, this transformation can be used to solve certain linear boundary value problems. Frequently, this Laplace method is shorter and more efficient than other conventional procedures of solving such problems.

Briefly, the Laplace transform method requires one to transform the differential or integral equation into an algebraic equation, solve the transformed equation, and then find the Laplace inverse of the algebraic solution.

In the previous part we were concerned with finding the Laplace transform of certain functions. Since, in the applications, it is necessary to find Laplace inverses, let us now investigate some theory and techniques of finding the functions whose Laplace transforms are given.

In general,

(1) \[ \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\delta-i\beta}^{\delta+i\beta} f(z)e^{zt} dz, \]

the complex inversion integral for the Laplace transformation, pairs the function \( f(s) \) with its inverse \( F(t) \) such that \( L[F(t)] = f(s) \).

The integration of (1) is often difficult to perform
directly for particular \( f(s) \), but there are methods in complex integration theory which reduce the task considerably. Among these, we shall select two to present in this part of the report.

However, before entering into a formal discussion on the inversion integral, a few ideas will be given that enable one to find the Laplace inverse of some transforms without using (1).

From reading part II, the inverses of several transforms should be recognizable. Thus a foundation has been established upon which the following theory can be based.

Let us consider a transform of the form
\[
f(s) = \frac{p(s)}{q(s)},
\]
where \( p(s) \) and \( q(s) \) are polynomials and the degree of \( q(s) \) is greater than the degree of \( p(s) \). In practice this form appears frequently, so hence its consideration.

A corollary to the fundamental theorem of algebra establishes the existence of \( n \) linear factors of an \( r^{th} \) degree polynomial \( q(s) \). Note that in \( q(s) \) there may or may not be repeated factors.

Theorem 1, below, systematizes a procedure for finding inverses of quotients of polynomials where the denominator has a linear factor \( s-a \), not repeated. Theorem 2 handles the situation when the denominator contains a repeated factor \( (s-a)^{n+1} \).

**THEOREM 1.** Let \( f(s) = \frac{p(s)}{q(s)} \), where \( p(s) \) and \( q(s) \) are
polynomials and the degree of \( q(s) \) is greater than the degree of \( p(s) \), and let \( q(s) \) contain the factor \( s-a \) which is not repeated. Then the term in \( L^{-1}[f(s)] \), the inverse of the Laplace transform \( f(s) \), corresponding to the factor \( s-a \) is

\[
(2) \quad \frac{p(s)}{q(s)} \text{ at } q'(a).
\]

Proof. The quotient \( p(s)/q(s) \) can be expressed as

\[
(3) \quad \frac{p(s)}{q(s)} = \frac{A}{s-a} + h(s),
\]

where \( A \) is a constant and \( h(s) \) is the sum of the fractions corresponding to all other factors of \( q(s) \).

Multiplying (3) by \( s-a \) obtains

\[
\frac{(s-a)p(s)}{q(s)} = A + (s-a)h(s).
\]

Taking the limit as \( s \to a \) gives

\[
(4) \quad \lim_{s \to a} \left[ \frac{(s-a)p(s)}{q(s)} \right] = A + \lim_{s \to a} (s-a)h(s).
\]

The factor \( (s-a)/q(s) \) has the form \( 0/0 \) as \( s \to a \); whence, after using L'Hopital's rule in (4) and solving for \( A \) we have

\[
A = \frac{p(a)}{q'(a)}.
\]

The proof is completed by recognizing that

\[
L^{-1} \left[ \frac{A}{s-a} \right] = Ae^{at},
\]

since \( L[Ae^{at}] = \frac{A}{s-a} \).

COROLLARY. If \( q(s) \) is completely factorable into \( n \)
unrepeated linear factors; that is, if
\[ q(s) = (s-a_1)(s-a_2)(s-a_3)\cdots(s-a_n), \]
where these \( a_i \) are distinct, then
\[
L^{-1}[f(s)] = \sum_{k=1}^{n} \frac{p(a_k)}{q'(a_k)} e^{a_k t}.
\]

In fact, this corollary is verified by applying Theorem 1 to each factor of \( q(s) \). The form (5) is sometimes called Heaviside's expansion.

Illustration 1. Find the inverse of \( f(s) = \frac{s}{s^2-a^2} \).

By letting \( p(s) = s \) and \( q(s) = s^2-a^2 = (s-a)(s+a) \), \( f(s) \) can be put in the form \( f(s) = p(s)/q(s) \). Applying the corollary gives us
\[
L^{-1}[f(s)] = \frac{a}{2a} e^{at} + \frac{a}{2a} e^{-at} = \cosh at.
\]

THEOREM 2. Let \( f(s) = p(s)/q(s) \), where \( p(s) \) and \( q(s) \) are polynomials in \( s \) such that the degree of \( q(s) \) is higher than the degree of \( p(s) \), and let \( q(s) \) contain the repeated linear factor \( (s-a)^{n+1} \). Then the terms in \( L^{-1}[f(s)] \) corresponding to this factor are
\[
\sum_{r=0}^{n} \frac{\varphi(n-r)(a)^{n-r} e^{at}}{(n-r)! r!},
\]
where \( \varphi(s) \) is the quotient of \( p(s) \) and all the factors of \( q(s) \) except \( (s-a)^{n+1} \).

Proof. By the definition of \( \varphi(s) \), \( f(s) = \varphi(s)/(s-a)^{n+1} \).

Representing \( f(s) \) as a sum of partial fractions gives
\[
\frac{\varphi(s)}{(s-a)^{n+1}} = \frac{A_0}{s-a} + \frac{A_1}{(s-a)^2} + \cdots + \frac{A_n}{(s-a)^{n+1}} + h(s),
\]
where $A_0, A_1, \ldots, A_n$ are constants and $h(s)$ is the sum of the fractions corresponding to all other factors of $q(s)$.

From equation (7) it follows that

(8) \[ \varphi(s) = A_0 (s-a)^n + A_1 (s-a)^{n-1} + \cdots + A_n + (s-a)^{n+1}h(s). \]

Taking the limit as $s \to a$, we obtain $A_n = \varphi(a)$. To find $A_{n-1}$, we can differentiate each side of (8) with respect to $s$ and then take the limit as $s \to a$. Hence $A_{n-1} = \varphi'(a)$.

Repeating this process we find that

\[ A_r = \frac{\varphi^{(n-r)}(a)}{(n-r)!} \quad (r = 0, 1, \ldots, n). \]

Thus equation (7) becomes

(9) \[ f(s) = \sum_{r=0}^{n} \frac{\varphi^{(n-r)}(a)}{(n-r)!} (s-a)^{n-r} + h(s) \]

The inverse of the first expression on the right of (9) is $\varphi^\prime(s)$ since $L[t^P e^{at}] = \frac{r!}{(s-a)^{r+1}}$. This completes the proof.

Illustration 2. Find the inverse of $f(s) = \frac{s+a}{(s+b)(s+c)^2}$

Let the numerator of $f(s)$ be $p(s)$ and the denominator be $q(s)$, and consider the unrepeated factor $s+b$. From Theorem 1 we realize that the inverse corresponding to this factor is

\[ \frac{p(-b)}{q'(-b)} e^{-bt} \quad a-b \quad e^{-bt} \quad (b-c)^2 \]

Applying Theorem 2 with $\varphi(s) = (s+a)/(s+b)$ we obtain the terms in $L^{-1}[p(s)/q(s)]$ corresponding to the repeated factor $(s+c)^2$. These terms are

\[ \sum_{r=0}^{\infty} \left[ \frac{\varphi^{(1-r)}(-c) t^r e^{-ct}}{(1-r)!} \right] = \left[ (a-c) t - (a-b) \right] e^{-ct}. \]
\[ L^{-1}\left[\frac{s+a}{(s+b)(s+c)^2}\right] = \frac{a-b}{(b-c)^2} e^{-bt} + \left[\frac{a-c}{b-c} - \frac{a-b}{(b-c)^2}\right] e^{-ct}. \]

When \( q(s) \) is a polynomial of degree \( n \), it does indeed have \( n \) linear factors, say \( (s-a_1)(s-a_2)(s-a_3) \cdots (s-a_n) \). Some or all of these \( a_k \) may be non-real in which case the above two theorems are still valid. However, when non-real \( a_k \) appear, the inverse of \( p(s)/q(s) \) contains terms of the form \( t e^{a_k t} \). The reduction of such terms to real forms is sometimes tedious. To avoid such labor, the following theorem is included to handle the cases where there is a pair of conjugate linear factors in \( q(s) \), say \( [s-(-a+bi)][s-(-a-bi)] \), which can be put in the form \( (s+a)^2 + b^2 \), where \( a, b \in \mathbb{R} \). This latter form is called an irreducible quadratic factor.

**Theorem 3.** Let \( f(s) = \frac{p(s)}{q(s)} \), where \( p(s) \) and \( q(s) \) are polynomials in \( s \) with the degree of \( q(s) \) higher than the degree of \( p(s) \), and let \( q(s) \) have an unrepeated, irreducible quadratic factor \( (s+a)^2 + b^2 \) such that \( a, b \in \mathbb{R} \). Then the terms in \( L^{-1}[f(s)] \) corresponding to this factor are

\[ e^{-at} \frac{e^{-bt}}{b} (\varphi_i \cos bt + \varphi_r \sin bt) \]

where \( \varphi_i \) and \( \varphi_r \) are, respectively, the imaginary and real parts of \( \varphi(-a+bi) \) and \( \varphi(s) \) is the quotient of \( p(s) \) and all the factors of \( q(s) \) except \( (s+a)^2 + b^2 \).

**Proof.** In the partial-fraction expansion of \( p(s)/q(s) \),
we have a single fraction of the form \( \frac{As + B}{(s+a)^2 + b^2} \).

Letting \( h(s) \) denote the fractions corresponding to all other factors of \( q(s) \), we can write

\[
\frac{p(s)}{q(s)} = \frac{\varphi(s)}{(s+a)^2 + b^2} + h(s).
\]

Multiplying by \((s+a)^2 + b^2\), we obtain

\[
\varphi(s) = As + B + [(s+a)^2 + b^2] h(s).
\]

Since \( s = -a + bi \) makes \((s+a)^2 + b^2 = 0\), we can write \( \varphi(-a+bi) = (-a+bi)A + B \) or, reducing \( \varphi(-a+ib) \) to its standard complex form \( \varphi_r + i\varphi_i \), gives

\[
(11) \quad \varphi_r + i\varphi_i = (-aA + B) + ibA.
\]

Equating the real and imaginary parts of (11) and solving for \( A \) and \( B \) yields,

\[
A = \frac{\varphi_i}{b} \quad \text{and} \quad B = \frac{b\varphi_r + a\varphi_i}{b}.
\]

Thus

\[
(12) \quad \frac{As + B}{(s+a)^2 + b^2} = \frac{1}{b} \left[ \frac{\varphi_is + b\varphi_r + a\varphi_i}{(s+a)^2 + b^2} \right] = \frac{1}{b} \left[ \frac{(s+a)\varphi_i}{(s+a)^2 + b^2} + \frac{i\varphi_r}{(s+a)^2 + b^2} \right].
\]

From Theorem 6 and equations (15) and (16), of the previous section, we can write the inverse of the right side of (12) as

\[
\frac{1}{b} (\varphi_i e^{-at}\cos bt + \varphi_r e^{-at}\sin bt)
\]

from which the assertion is apparent.

Illustration 3. If \( f(s) = p(s)/q(s) = s/[(s+2)^2(s^2+2s+10)] \), what is \( L^{-1}[f(s)] \)? Considering first the repeated linear factor, and using the notation of Theorem 2 we find

\[
\varphi(s) = \frac{s}{s^2 + 2s + 10}.
\]
Therefore,
\[
\sum_{r=0}^{1} \frac{\varphi(1-r)(-2)^r e^{-2t}}{(1-r)!} = \frac{(3-10t)e^{-2t}}{50}.
\]

For the quadratic factor \( s^2 + 2s + 10 = (s+1)^2 + 3^2 \) we have \( \varphi(s) = s/(s+2)^2 \). Therefore \( \varphi(-a+bi) = \varphi(-1+3i) = (-1+3i)/[(-1+3i)+2]^2 = 13/50 - 9i/50 \), and thus \( \varphi_r = 13/50 \) and \( \varphi_i = -9/50 \). The term in \( L^{-1}[f(s)] \) corresponding to \( s^2 + 2s + 10 \) is therefore
\[
\frac{1}{3} \left[ \frac{e^{-t}(-9\cos 3t + 13\sin 3t)}{50} \right].
\]

Adding the two partial inverses, we finally have
\[
L^{-1}[f(s)] = \frac{(3-10t)e^{-2t}}{50} + \frac{e^{-t}(-9\cos 3t + 13\sin 3t)}{150}.
\]

There is a fourth theorem dealing with repeated irreducible quadratic factors, but because of its limited applications we shall not present it here. The convolution theorem which will be developed next handles many of the simpler transforms involving repeated quadratic factors.

The convolution theorem pairs two functions of \( t \) with the product of their transforms. It follows from this convolution operation that the Laplace inverse of the product of two transforms is given directly in terms of the two functions whose transforms form the given product.

**DEFINITION 1.** The convolution of functions \( F(t) \) and \( G(t) \), denoted as \( F(t) \ast G(t) \), is defined
\[
F(t) \ast G(t) = \int_0^t F(r)G(t-r)dr.
\]
THEOREM 4. Let \( F(t), G(t) \in SC[C, T] \) and each be \( O(e^{\alpha t}) \) as \( t \to \infty \) such that \( L[F(t)] = f(s) \) and \( L[G(t)] = g(s) \). Then for \( s > \alpha \)

\[ L[F(t) \ast G(t)] = f(s)g(s). \]

Proof. Since the integration of \( F(t) \ast G(t) \) starts at \( t = 0 \), we may assume \( F(t) = G(t) = 0 \) for \( t < 0 \). Also, the argument of \( G(t-r) \) changes sign for \( r > t \), making the integrand of (13) zero for such values of \( r \). Therefore, we can write

\[ \int_0^t F(t)G(t-r)dr = \int_0^\infty F(t)G(t-r)dr. \]

(14)

Taking the transform of the right side of (14) gives

\[ L[\int_0^\infty F(t)G(t-r)dr] = \int_0^\infty [\int_0^\infty F(t)G(t-r)dr]e^{-st}dt. \]

But, by changing the order of integration,

\[ \int_0^\infty [\int_0^\infty F(t)G(t-r)dr]e^{-st}dt = \int_0^\infty F(t)dr \int_0^\infty G(t-r)e^{-st}dt. \]

However, \( r > t \) implies \( G(t-r) = 0 \) from which

\[ \int_0^\infty F(t)dr \int_0^\infty G(t-r)e^{-st}dt = \int_0^\infty F(t)dr \int_r^\infty G(t-r)e^{-st}dt. \]

(15)

If we now let \( x = t-r \) in the integral \( \int_r^\infty G(t-r)e^{-st}dt \), the right side of (15) becomes

\[ \int_0^\infty F(t)dr \int_0^\infty G(x)e^{-(x+r)s}dx \text{ or,} \]

\[ \int_0^\infty F(t)e^{-rs}dr \int_0^\infty G(x)e^{-sx}dx. \]

This final form is nothing but \( f(s)g(s) \) which completes the proof.

From Theorem 4 it follows that

\[ L^{-1}[f(s)g(s)] = F(t) \ast G(t). \]
This equation is the one we were seeking; it can be used to find the inverses of many Laplace transforms.

Illustration 4. If \( L[H(t)] = f(s)/s^2 \), what is \( H(t) \)?

Let \( F(t) = L^{-1}[f(s)] \) and \( G(t) = L^{-1}[1/s^2] \). Then

\[
F(t) = \int_0^t f(r)G(t-r)dr = \frac{t}{0} \int f(r)dr - \frac{t}{0} \int rF(r)dr.
\]

Illustration 5. Find the inverse of \( h(s) = k/s(s^2 + k^2) \).

Let \( F(t) = L^{-1}[k/s^2 + k^2] = \sin kt \) and \( G(t) = L^{-1}[1/s] = 1 \).

Then \( L^{-1}[h(s)] = F(t) \times G(t) = \int_0^t \sin kr \, dr = l/k(1 - \cos kt) \).

THEOREM 5. The convolution operation is commutative, distributive, and associative.

Proof. (commutativity) Changing the variable \( r \) to \( t-\delta \) in \( \int_0^t F(r)G(t-r)dr \) gives

\[
\int_0^t F(t-\delta)G(\delta)(-d\delta) = \int_0^t G(\delta)F(t-\delta)d\delta = G(t) \times F(t).
\]

Thus, \( F(t) \times G(t) = G(t) \times F(t) \).

(distributivity) \( F(t) \times [G(t) + H(t) ] =

\[
\int_0^t F(r)[G(t-r)+H(t-r)]dr = \int_0^t F(r)G(t-r)dr + \int_0^t F(r)H(t-r)dr = \]

\( = F(t) \times G(t) + F(t) \times H(t) \).

(associativity) \( F(t) \times (G(t) \times H(t)) = F(t) \times (H(t) \times G(t)) =

\[
= F(t) \times \int_0^t H(\delta)G(t-\delta)\,d\delta = \int_0^t F(\delta) \int_0^t H(\delta)G(t-\delta-r)\,dr\,d\delta = \]

\( = \int_0^t H(\delta) \int_0^t F(\delta)G(t-r-\delta)\,d\delta \,dr = H(t) \times \int_0^t F(\delta)G(t-\delta)\,d\delta =

\( H(t) \times (F(t) \times G(t)) = (F(t) \times G(t)) \times H(t) \).

Illustration 6. In finding the inverse of \( k/s(s^2 + k^2) \) (see
Illustration 5), we may have chosen \( f(s) = 1/s \) and \( g(s) = k/s^2 + k^2 \). Thus \( F(t) \ast G(t) = \int_0^t \sin k(t-r) \, dr \), which is slightly more difficult to evaluate than \( G(t) \ast F(t) = \int_0^t \sin kr \, dr \).

Illustration 7. Find the inverse of \( 1/s^2(s+1)(s+2) \).
It is seen that

\[
\frac{1}{s^2(s+1)(s+2)} = \frac{1}{s^2} \left[ \frac{1}{s+1} - \frac{1}{s+2} \right].
\]

Letting \( f(s) = 1/s^2 \), \( g(s) = 1/s+1 \), and \( h(s) = -1/s+2 \), the desired inverse is \( F(t) \ast \left[ G(t) + H(t) \right] = F(t) \ast G(t) + F(t) \ast H(t) \), where \( L^{-1}[f(s)] = F(t) = t \), \( L^{-1}[g(s)] = G(t) = e^{-t} \), and \( L^{-1}[h(s)] = H(t) = -e^{-2t} \). Evaluating \( G(t) \ast F(t) + H(t) \ast F(t) \), we get

\[
\int_0^t e^{-r}(t-r) \, dr + \int_0^t e^{-2r}(t-r) \, dr = e^{-t} - \frac{3}{2} + e^{-t} - \frac{e^{-2t}}{2}.
\]

THEOREM 6. Let \( F(t) \), \( G(t) \), \( H(t) \) be \( \mathcal{C}[0,T] \) and be \( O(e^{\alpha t}) \) as \( t \to \infty \), and let \( L[F(t)] = f(s) \), \( L[G(t)] = g(s) \), and \( L[H(t)] = h(s) \). Then

(18) \( L^{-1}[f(s) g(s) h(s)] = F(t) \ast \left[ G(t) \ast E(t) \right] \).

Proof. Since \( G(t) \) and \( E(t) \) are each \( O(e^{\alpha t}) \), then

\[
|G(t)| < M_1 e^{\alpha t} \text{ when } t > 0, \text{ and } |H(t-r)| < M_2 e^{\alpha t} \text{ when } t-r > 0,
\]

for some \( M_1, M_2 \in \mathbb{R} \). Therefore, for \( t > 0 \) and \( t-r > 0 \),

\[
|G(t) \ast H(t)| = \left| \int_0^t G(r)H(t-r) \, dr \right| \leq \int_0^t |G(r)||H(t-r)| \, dr < M_2 \int_0^t e^{\alpha r} e^{\alpha(t-r)} \, dr = M_2 t^2 e^{2\alpha t},
\]

where \( M_2 = M_1 M_2 \). Let \( \varepsilon \) be any positive number,
and let \( M_2 = \max(t \rightarrow e^{-\epsilon t}; t \geq 0) \). Then
\[
M_2 e^{2\alpha t} = M_2 e^{-\epsilon t} e^{(2\alpha + \epsilon) t} \leq M e^{(2\alpha + \epsilon) t},
\]
where \( M = M_2 M_4 \). Consequently \(|G(t) \times H(t)| < M e^{(2\alpha + \epsilon) t}\) for \( \varepsilon > 0 \) and \( t > 0 \); that is, \( G(t) \times H(t) \) is \( O(e^{\beta t}) \), for \( \beta = 2\alpha + \epsilon \).

Also, it can be shown that \( G(t) \times H(t) \in SC[0, T] \).

Summarizing, the functions \( F(t) \) and \( G(t) \times H(t) \) satisfy the conditions of Theorem 4 for \( s > \beta \). Therefore
\[
f(s) [g(s) h(s)] = \mathcal{L}[F(t) \times (G(t) \times H(t))].
\]
Taking the inverse of each side produces (18).

An application of Theorem 6 is included in the proof of Theorem 7. This next theorem is often helpful in finding the inverse of a transform containing positive integral powers of \( s \) in the denominator.

**Theorem 7.** Let \( F(t) \in SC[0, T] \) and be of \( O(e^{\alpha t}) \) as \( t \rightarrow \infty \) and \( s > \alpha > 0 \); also let \( \mathcal{L}[F(t)] = f(s) \). Then
\[
\begin{align*}
(19) \quad \mathcal{L}^{-1} [f(s)/s] & = \int_0^t F(r) dr, \\
(20) \quad \mathcal{L}^{-1} [f(s)/s^2] & = \int_0^t F(r) dr d\delta, \\
(21) \quad \mathcal{L}^{-1} [f(s)/s^{n+1}] & = \int_0^t \delta d\delta_{n-1} \cdots \int_0^t F(r) dr d\delta_1 d\delta_2 \cdots d\delta_n.
\end{align*}
\]

**Proof.** If \( F(t) = \mathcal{L}^{-1} [f(s)] \) and \( G(t) = \mathcal{L}^{-1} [1/s] = 1 \), then \( \mathcal{L}^{-1} [f(s) g(s)] = F(t) \times G(t) = \int_0^t F(r) dr \). In proving (20), let \( \mathcal{L}[F(t)] = f(s), \mathcal{L}[G(t)] = g(s) \) and \( \mathcal{L}[H(t)] = h(s) \) where \( g(s) = h(s) = 1/s \). Now from Theorem 6,

\[^*\text{Note that Illustration 4 contains another form for the right side of (20).}\]
\[ I^{-1}[f(s)g(s)h(s)] = (F(t) \ast 1) \ast 1 \]
\[ = \int_0^t F(r)dr \ast 1 \]
\[ = \int_0^t \int_0^\delta F(r)d\delta \]

Formula (21) is proved by mathematical induction.

The correspondence between (19) and (20) of Part III and the formulae (11) and (12) of Part II is interesting to note.

Illustration 8. The inverse of \( 1/s^2(s^2+1) \) may be obtained via Theorem 7 by letting \( f(s) = 1/s^2+1 \). Then
\[ I^{-1}[1/s^2(s^2+1)] = I^{-1}[f(s)/s^2] = \int_0^t \int_0^\delta \sin r dr d\delta \]
\[ = t - \sin t. \]

The preceding developments in this section are helpful in finding the Laplace inverse of rational functions. For non-rational functions it is necessary to use the inversion integral which will be discussed next.

An extension of Cauchy's integral formula will be stated as a theorem after introducing the notion of order of a function of a complex variable.

**Definition 2.** A function \( f(z) \) is of the order \( \gamma^k \) as \( |z| \) tends to infinity, written
\[ f(z) = O(z^k) \quad \text{as} \quad |z| \to \infty, \]
if there exist constants \( M \) and \( r_0 \) such that \( |f(z)| < M|z|^k \)
whenever \( |z| > r_0 \).

**Theorem 8.** Let \( f(z) \) be analytic when \( \Re(z) \geq \delta \) and let \( f(z) \)
be $O(z^{-k})$ as $|z| \to \infty$ in that half plane, where $\delta, k \in \mathbb{R}$ and $k > 0$. Then if $z_0$ is any complex number such that $\rho_-(z_0) > \delta$,

$$f(z_0) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} \frac{\delta + i\beta}{\delta - i\beta} f(z) dz.$$  

(22)

Since we have not developed a foundation in the theory of functions, the proof of Theorem 8 is omitted.

If a function $F(t)$ belongs to the class $S_0[0,T]$ and is $O(e^{\delta t})$ when $t \to 0$, then by Theorem 13 of Part II we know that $L[F(t)] = f(s)$ is analytic for $x > \delta$, where $s = x + iy$. Also, if $f(s)$ is $O(s^{-k})$ in the half plane $x > \delta$, where $k > 0$, then by Theorem 8, $f(s)$ may be expressed as

$$f(s) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} \frac{\delta + i\beta}{\delta - i\beta} f(z) dz.$$  

(x > \delta)

Taking the inverse of each side and interchanging the order of the operation $L^{-1}$ and the integration, we obtain

$$F(t) = L^{-1} \left[ \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} \frac{\delta + i\beta}{\delta - i\beta} f(z) dz \right].$$

(23)

$$= \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} f(z) e^{zt} dz,$$

since $L^{-1}[1/s-z] = e^{zt}$.

As mentioned earlier, the expression on the right of (23) is called the complex inversion integral for the Laplace transform $f(s)$. Henceforth we shall use the symbol $L^{-1}_1$ to represent this inversion integral; that is,

$$F(t) = L^{-1}_1[f(s)] = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{-\beta}^{\beta} f(z) e^{zt} dz.$$  

Let us next investigate a method for evaluating this
THEOREM 9. Let \( F(t), F'(t) \in SC[0,T] \), and let \( F(t) \) be \( O(e^{-\delta t}) \) for \( t \geq 0 \), where \( L[F(t)] = f(s) \). Then

\[
(24) \quad L^{-1}_1[f(s)] = \frac{e^{\delta t}}{\pi} \int_0^\infty [u(\delta, y) \cos yt - v(\delta, y) \sin yt] dy,
\]

where \( f(\delta + iy) = u(\delta, y) + iv(\delta, y) \) and \( \delta > 0 \).

Proof. By letting \( z = \delta + iy \), where we fix \( \delta \), we obtain

\[
(25) \quad \int_{\delta - i\beta}^{\delta + i\beta} f(z) e^{zt} dt = i e^{\delta t} \left[ \int_0^\beta f(\delta + iy) e^{iyt} dy + \int_{-\beta}^0 f(\delta + iy) e^{iyt} dy \right].
\]

Replacing \( y \) by \( -y \) in the first integral on the right of (25) permits us to write

\[
L_1^{-1}[f(s)] = \frac{e^{\delta t}}{2\pi} \lim_{\beta \to \infty} \int_0^\beta [f(\delta - iy) e^{-iyt} + f(\delta + iy) e^{iyt}] dy.
\]

But, \( f(\delta - iy) e^{-iyt} = [u(\delta, y) - iv(\delta, y)][\cos yt - i\sin yt] \), and \( f(\delta + iy) e^{iyt} = [u(\delta, y) + iv(\delta, y)][\cos yt + i\sin yt] \).

Adding these expressions, we may write

\[
L_1^{-1}[f(s)] = \frac{e^{\delta t}}{\pi} \lim_{\beta \to \infty} \int_0^\beta [u(\delta, y) \cos yt - v(\delta, y) \sin yt] dt
\]
as asserted.

In view of the form (24), we see that the inversion integral is converted into a real integral. However, even for a simple function \( f(s) \), the integration of (24) is generally difficult. Later, a more practical method of evaluating the inversion integral will be presented.

The following theorem contains sufficient conditions on a function \( f(s) \) to guarantee the existence of the
function $F(t)$ such that $\mathcal{L}[F(t)] = f(s)$. Also the theorem states that $F(t)$ is indeed given by the inversion integral.

**Theorem 10.** Let $f(s)$ be an analytic function such that $f(s)$ is $O(s^{-k})$ for all $s = x + iy$ and $x \geq x_0$, where $k > 1$; also let $f(x)$ be real valued for $x \geq x_0$. Then the inversion integral of $f(s)$, along any line $x = \delta$, where $\delta > x_0$, converges to a real-valued function $F(t)$ that is independent of $\delta$,

\begin{equation}
F(t) = L^{-1}_t[f(s)] \quad (\delta > x_0)
\end{equation}

whose Laplace transform is the given function $f(s)$:

\[ \mathcal{L}[F(t)] = f(s) \quad (x > x_0) \]

Furthermore, $F(t)$ is $O(e^{x_0 t})$, is continuous on $(-\infty, \infty)$, and $F(t) = 0$ when $t \leq 0$.

**Proof.** Since analyticity implies continuity, we know that $f(z)$ is continuous. Also, the product of two continuous functions is continuous. Therefore, $f(z)e^{zt}$ is everywhere a continuous function of $y$ and $t$, where $z = \delta + iy$ and $\delta > x_0$. From the hypothesis, $f(s)$ is $O(s^{-k})$ which, by Definition 2, implies the existence of positive numbers $M$ and $y_0$ such that

\begin{equation}
|f(\delta + iy)| \leq \frac{M}{(\delta^2 + y^2)^{k/2}} \leq \frac{M}{|y|^k} \quad (|y| > y_0).
\end{equation}

In form (24), if we define $g(t, y)$ as

\begin{equation}
g(t, y) = u(\delta, y) \cos yt - v(\delta, y) \sin yt
\end{equation}

we can write

\begin{equation}
\pi e^{-\delta t} L^{-1}_t[f(s)] = \int_0^{y_0} g(t, y) dy + \int_{y_0}^{\infty} g(t, y) dy.
\end{equation}

From (27) and (28) it follows that
\[ |g(t,y)| < 2M |y|^{-k} \]

where \( k > 1 \) and \( M \) is independent of \( t \). Therefore, the improper integral in (29) converges uniformly with respect to \( t \) on \((\infty, \infty)\). Both integrals are continuous functions of \( t \) since \( g(t,y) \) is continuous for all \( t \) and \( y \).

We note also that if \( \delta = x_0 \), then the function \( 2|f(x_0 + iy)| \) is independent of \( t \) and integrable. But \( |g(t,y)| \leq 2|f(x_0,y)| \) and hence, the two integrals of (29) are bounded.

To show \( F(t) \) is independent of \( \delta \), let us use another path, say \( x = \delta' \), where \( \delta' > \delta \) (see Figure 2). Now \( f(z)e^{zt} \) is analytic in the rectangle ABCD. Thus by

\[ \int_{\partial R} f(z)e^{zt} \]

from which it follows that

\[ |e^{zt}f(z)| < \frac{Me^{xt}}{p^k}, \]

where \( k > 1 \) and \( M \) is independent of \( t \). Therefore, the improper integral in (29) converges uniformly with respect to \( t \) on \((\infty, \infty)\). Both integrals are continuous functions of \( t \) since \( g(t,y) \) is continuous for all \( t \) and \( y \).

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\[ |e^{zt}f(z)| < \frac{Me^{xt}}{p^k}, \]

where \( k > 1 \) and \( M \) is independent of \( t \). Therefore, the improper integral in (29) converges uniformly with respect to \( t \) on \((\infty, \infty)\). Both integrals are continuous functions of \( t \) since \( g(t,y) \) is continuous for all \( t \) and \( y \).

We note also that if \( \delta = x_0 \), then the function \( 2|f(x_0 + iy)| \) is independent of \( t \) and integrable. But \( |g(t,y)| \leq 2|f(x_0,y)| \) and hence, the two integrals of (29) are bounded.

To show \( F(t) \) is independent of \( \delta \), let us use another path, say \( x = \delta' \), where \( \delta' > \delta \) (see Figure 2). Now \( f(z)e^{zt} \) is analytic in the rectangle ABCD. Thus by

\[ \int_{\partial R} f(z)e^{zt} \]

from which it follows that

\[ |e^{zt}f(z)| < \frac{Me^{xt}}{p^k}, \]
But the right member of the above inequality tends to zero as $\beta \to \infty$ for $k > 0$. Thus the integral of $f(z)e^{zt}$ along $BC = 0$ as $\beta \to \infty$. A similar argument shows that the same integral along side $DA = 0$ as $\beta \to \infty$. Since the integrals around the rectangle, along $BC$ and along $DA$, are all zero, it follows that the sum of the integrals along $AB$ and $CD$ must also vanish as $\beta \to \infty$, that is,

$$\lim_{\beta \to \infty} \left( \int_{\delta + i\beta} f(z)e^{zt}dz + \int_{\delta - i\beta} f(z)e^{zt}dz \right) = 0$$

which implies that

$$\lim_{\beta \to \infty} \int_{\delta + i\beta} f(z)e^{zt}dz = \lim_{\beta \to \infty} \int_{\delta - i\beta} f(z)e^{zt}dz.$$

Thus $F(t)$ is independent of $\delta$.

Next consider the function

$$\varphi(z) = e^{zt}f(z)(z-z_1),$$

where $t$ is nonpositive and fixed, and where $\Re(z_1) > \delta > 0$. Note that $\varphi(z)$ is analytic for $x \geq \delta$. Now $t < 0$ implies $|e^{zt}| \leq 1$; also, $k > 1$ implies $k-1 > 0$, and hence, $\varphi(z)$ is $C(|z|-(k-1))$. By Theorem 8 we can now write

$$\varphi(z_1) = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\delta - i\beta}^{\delta + i\beta} \frac{\varphi(z)dz}{z_1 - z} = L_1^{-1}[f(s)].$$

But by (30), we see that $\varphi(z_1) = 0$, and thus the last assertion of Theorem 10 is established.

Now, since $F(t)$ is continuous and of $\mathcal{E}$., the transform of (26) exists. We write $s = a + bi$, where $a > x_0$ and $z = x_0 + iy$. Then
\[
\mathcal{L}[F(t)] = \lim_{T \to \infty} \int_T^0 e^{-st} \frac{1}{2\pi} \int_{-\infty}^\infty f(z) e^{zt} dy dt.
\]

Changing the order of integration we obtain

(31) \[
\mathcal{L}[F(t)] = \frac{1}{2\pi} \lim_{T \to \infty} \int_0^T \int_{-\infty}^\infty f(z) e^{-(s-z)t} dt dy.
\]

The integrand of this improper integral satisfies the conditions

\[
\left| f(z) \int_0^T e^{-(s-z)t} dt \right| < \frac{M}{|y|^k} \left[ \frac{1-e^{-(a-x_0)T}}{a-x_0} \right] < \frac{M}{a-x_0} \frac{1}{|y|^k},
\]

when \(|y| > y_0 > 0\), and where \(M\) is independent of \(T\) and \(k > 1\).

But \(\lim_{T \to \infty} \int_0^T e^{-(s-z)t} dt = f(z) \int_0^\infty e^{-(s-z)t} dt = \frac{f(z)}{s-z}\).

Summarizing, we can finally write

\[
\mathcal{L}[F(t)] = \frac{1}{2\pi} \int_{-\infty}^\infty \frac{f(z)}{s-z} dz
\]

and by Theorem 8, this last integral is just \(f(s)\). This completes the proof of Theorem 10.

Note that the previous theorem is rather weak in that it does not include the existence of the inverse of \(f(s) = 1/s\), as this function is not \(O(s^{-k})\), where \(k > 1\).

However, the following theorem relaxes the conditions on \(f(s)\) so that the inversion integral applies to a broader class of transforms.

**Theorem 11.** Let \(F(t), F'(t) \in SC[0,T]\), \(F(t) = O(e^{x_0 t})\) for \(t > 0\), and let \(\mathcal{L}[F(t)] = f(s)\). Then the inversion integral of \(f(s)\) along any line \(x = \delta\), where \(\delta > x_0\), exists and represents \(F(t)\):

(32) \[
\mathcal{L}_1^{-1}[f(s)] = F(t) \quad (t > 0)
\]
At any point $t_0$ such that $F(t_0)$ is discontinuous, the inversion integral represents the mean value of $F(t)$:

$$F(t_0) = \frac{1}{2}[F(t_0^+O) + F(t_0^-O)] \quad (t_0 > 0)$$

Furthermore, when $t = 0$, the inversion integral has the value zero, and when $t < 0$ it has the value zero.

Proof. Let $G(t)$ be defined on $\mathbb{R}$ and let $G(t)$, $G'(t) \in SC[0, T]$. Assume the integral $\int_0^\infty |G(t)| \, dt$ exists for large $|t|$. Finally, let

$$G(t_0) = \frac{1}{2}[G(t_0^+O) + G(t_0^-O)]$$

for each point $t_0$ such that $G(t_0)$ is discontinuous. By a theorem of Fourier series, the function $G(t)$ can be represented by the Fourier integral formula

$$G(t) = \frac{1}{\pi} \int_0^\infty \int_0^\infty G(r) \cos y(t-r) \, dr \, dy$$

$$= \frac{1}{\pi} \lim_{\beta \to \infty} \int_0^\beta \int_0^\infty G(r) \left[ e^{iy(t-r)} + e^{-iy(t-r)} \right] \, dr \, dy$$

$$= \frac{1}{\pi} \lim_{\beta \to \infty} \int_0^\beta e^{iyt} \int_0^\infty G(r) e^{-iyr} \, dr \, dy + \frac{1}{\pi} \lim_{\beta \to \infty} \int_0^\beta e^{iyt} \int_0^\infty G(r) e^{iyr} \, dr \, dy$$

$$= \frac{1}{\pi} \lim_{\beta \to \infty} \int_0^\beta e^{iyt} \int_0^\infty G(r) e^{-iyr} \, dr \, dy \quad (-\infty < t < \infty).$$

Now define $G(t)$ as

$$G(t) = \begin{cases} 0 & \text{for } t < 0 \\ F(t)e^{-\delta t} & \text{for } t > 0 \end{cases},$$

where $\delta > x_0$. Therefore on $(-\infty, \infty)$

$$G(t) = \frac{1}{2\pi} \lim_{\beta \to \infty} \int_0^\beta e^{iyt} \int_0^\infty F(r) e^{-(\delta + iy)r} \, dr \, dy.$$

Note that by definition of the Laplace transform,
\[ \int_0^\infty f(r)e^{-(\delta+iy)r}dr \] represents \( f(\delta+iy) \), where \( L[F(t)]=f(s) \).

Letting \( z=\delta+iy \) in equation (35) gives us

\[ (36) \quad e^{\delta t}G(t) = \lim_{\beta \to \infty} \frac{1}{2\pi i} \int_{\delta-i\beta}^{\delta+i\beta} f(z)e^{zt}dz = L_1^{-1}[f(s)]. \]

When \( t<0 \), \( G(t)=0 \) and therefore \( L_1^{-1}[f(s)]=0 \).

When \( t>0 \), \( e^{\delta t}G(t)=F(t) \) and the equation (36) represents \( F(t) \) as \( L_1^{-1}[f(s)] \). But by hypothesis, \( F(\infty)=\frac{1}{2}[F(t_0+0)+F(t_0-0)] \) for each point of discontinuity \( t_0>0 \). Therefore since \( G(t) \) has the mean value (33), the inversion integral converges to \( F(t_0) \). When \( t=0 \), the value of the inversion integral is \( \frac{1}{2}[F(+0)+0] \). This completes the proof of Theorem 11.

Before making an analysis of uniqueness of the inversion integral, let us define a special class of real-valued functions.

**DEFINITION 3.** A function \( F(t) \) is said to belong to the class \( \mathcal{C}(x_0) \) if it is defined for \( t>0 \), is \( O(e^{\delta x_0 t}) \) for \( t>0 \), \( F(t), F'(t) \in SC[0,T] \), and \( F(t_0) = \frac{1}{2}[F(t_0+0)+F(t_0-0)] \) for each point \( t_0 \) of discontinuity while \( F(0) = \frac{1}{2}F(+0) \).

**THEOREM 12.** If \( x_0 \) is finite, no two distinct functions of the class \( \mathcal{C}(x_0) \) have the same transform.

Proof. Suppose \( F(t) \in \mathcal{C}(x_0) \) and \( G(t) \in \mathcal{C}(x_1) \) such that \( L[F(t)]=L[G(t)]=f(s) \).

Then \( L_1^{-1}[f(s)]=F(t) \) when \( t>0 \), where \( \delta>x_0 \); and

\( L_1^{-1}[f(s)]=G(t) \) when \( t>0 \), where \( \delta>x_1 \).

In both inversion integrals choose \( \delta>\max(x_0,x_1) \). Then
\( F(t) = G(t) \).

Lerch's theorem also asserts uniqueness of the inversion integral, only under broader conditions. This theorem states that if two functions, say \( F_1(t) \) and \( F_2(t) \) have the same Laplace transform \( f(s) \), then \( F_2(t) = F_1(t) + N(t) \), where \( N(t) \) is the null function \( \int_0^T N(t)\,dt = 0 \) for every \( T > 0 \). Note that Lerch's theorem implies \( F_1(t) = F_2(t) \) for all \( t > 0 \), if these two functions are continuous. Due to complexity, the proof of Lerch's theorem will be omitted.

Let us next look at a method of evaluating the inversion integral that is more practical than the one given in Theorem 9. Before stating the next theorem, we need the following definitions.

**Definition 4.** If a function is analytic at some point in every neighborhood of a point \( z_0 \), but not at \( z_0 \), then \( z_0 \) is called a singular point of the function. If the function is analytic at all points, except at \( z_0 \), in some neighborhood of \( z_0 \), then \( z_0 \) is said to be an isolated singular point.

If \( z_0 \) is an isolated singular point of \( f(z) \), where \( f(z) \) is analytic on and inside a circle \( C \), and \( z_0 \) is the center of \( C \), then a theorem of the theory of functions states that \( f(z) \) has a Laurent series about \( z = z_0 \) given by

\[
(37) \quad f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n, \text{ where}
\]
DEFINITION 5. The complex number \( \rho = a_{-1} \), the coefficient of \((z-z_0)^{-1}\) in the series (37), is called the residue of \( f(z) \) at the singular point \( z_0 \).

DEFINITION 6. If we can find a positive integer \( n \) such that
\[
\lim_{z \to z_0} (z-z_0)^n f(z) = A \neq 0,
\]
then \( z = z_0 \) is called a pole of order \( n \). If \( n = 1 \), \( z_0 \) is called a simple pole.

To obtain the residue of a function at \( z = z_0 \), where \( z = z_0 \) is a pole of order \( k \), we have the formula
\[
(38) \quad \rho = \lim_{z \to z_0} \frac{1}{(k-1)!} \frac{d^{k-1}}{dz^{k-1}} [(z-z_0)^k f(z)].
\]

THEOREM 13. Let \( f(s) \) be a function for which the inversion integral along a line \( x = \delta \) represents the inverse transform \( F(t) \) of \( f(s) \), and let \( f(s) \) be analytic except for isolated singular points \( s_n \) \( (n = 1, 2, \ldots) \) in the half plane \( x > \delta \). Then the series of residues \( \rho_n(t) \) of \( f(z) e^{zt} \) at \( z = s_n \) converges to \( F(t) \) for each positive \( t \),
\[
F(t) = \sum_{n=1}^{\infty} \rho_n(t) \quad (t > 0)
\]
if the terms in the series corresponding to points \( s_n \) within the ring between successive curves \( C_N \) and \( C_{N+1} \) are grouped as a single term, provided \( f(z) \) satisfies the following conditions: \( C_N \) are the open rectangles (see figure 3) in which \( \beta_N \to \infty \) as \( N \to \infty \), and the numbers \( \delta_N \) and \( M \) exist, where \( M \) is independent of \( N \), such that
\[
|f(x+i\beta_N)| < \delta_N, \quad |f(-\beta_N+iy)| < M
\]
Proof. The singular points \( z = s_n \) are also singular points of the integrand of the inversion integral \( f(z)e^{zt} \). By the residue theorem of the theory of functions, the integral of \( f(z)e^{zt} \) around a closed curve enclosing the points \( s_1, s_2, \ldots, s_N \) has the value

\[
2\pi i \sum_{n=1}^{N} \rho_n(t).
\]

Since our closed path consists of the segment from \( \delta - i\beta_N \) to \( \delta + i\beta_N \) and the open rectangle \( C_N \), then

\[
\frac{1}{2\pi i} \int_{\delta - i\beta_N}^{\delta + i\beta_N} f(z)e^{zt}dz + \frac{1}{2\pi i} \int f(z)e^{zt}dz = \sum_{n=1}^{N} \rho_n(t).
\]

Note that it is not essential that the line \( x = \delta \) and the open rectangle enclose exactly \( N \) poles, as the hypothesis of the theorem implies.

As \( \beta_N \to \infty \) the value of the integral in (39) approaches \( L_1^{-1}[f(s)] \). We need to show that the second integral tends to zero as \( \beta_N \to \infty \). To verify that such is the case, we begin by choosing the \( \beta_N \) (\( N = 1, 2, \ldots \)) so that \( \beta_N \to \infty \) as
Now let the curves $C_N$ with the line $x = \delta$ enclose the points $s_1, s_2, \ldots, s_N$, if the number of singular points is infinite. If the number is finite, let all of them be enclosed when $N$ is greater than some fixed number. Since $|f(x + i\beta_N)| < \delta_N$ for $-\beta_N < x < \delta$, and $\delta_N \to 0$ as $N \to \infty$, then $|f(z)| \to 0$ uniformly on the upper and lower sides of $C_N$ as $N \to \infty$. Hence

$$\left| f(z)e^{zt} \right| = \left| f(z)e^{xt} \right| < \delta_N e^{xt}$$

on these sides which implies that the absolute value of the integrand over each side is less than

$$\delta_N \int_{-\beta_N}^{\beta_N} e^{xt} \, dx = \frac{\delta_N}{t} (e^{\delta t} - e^{-\beta_N t})$$

But the right side of equation (40) tends to zero as $N \to \infty$. Also, if $|f(z)|$ is uniformly bounded on the left side of the rectangle, then there is a constant $M$ such that $|f(-\beta_N + iy)| < M$ for $-\beta_N < y < \beta_N$, where $M$ is independent of $N$. Then we may conclude that the absolute value of the integral of $f(z)e^{zt}$ along that side is less than

$$Me^{-\beta_N t} \int_{-\beta_N}^{\beta_N} e^{yt} \, dy = 2Me^{-\beta_N t},$$

which tends to zero as $N \to \infty$ for $t > 0$. Summarizing, we see that

$$\lim_{N \to \infty} \int_{C_N} f(z)e^{zt} \, dz = 0$$

which concludes the proof.

For some types of functions it may be more convenient to use parabolic or circular arcs instead of open
rectangles for the \( C_N \) part of the path, but we shall not
develop these other possibilities.

From (38), the residue of the function \( f(z)e^{zt} \) at
\( z = s_n \), where \( s_n \) is a pole of order \( m \) of \( f(z)e^{zt} \), is
given by

\[
\rho_n(t) = \lim_{z \to s_n} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z-s_n) f(z)e^{zt}.
\]

From this formula it is seen that the residue of \( f(z)e^{zt} \)
at \( z = s_n \), where \( s_n \) is a simple pole, is given by

\[
\rho_n(t) = \lim_{z \to s_n} (z-s_n)f(z)e^{zt} = e^{sn} \lim_{z \to s_n} (z-s_n)f(z).
\]

If \( f(z) \) is a rational function of the form

\[
f(z) = \frac{p(z)}{q(z)},
\]

where \( p(z) \) and \( q(z) \) are analytic at \( z = s_n \) and \( p(s_n) \neq 0 \),
then the residue at the simple pole \( s_n \) is

\[
\rho_n(t) = \frac{p(s_n)}{q'(s_n)} e^{st}.
\]

And, when all singular points of \( f(z)e^{zt} \) are simple poles
and \( f(z) \) has the form (41), then

\[
F(t) = L^{-1}_{1}[f(s)] = \sum_{n=1}^{\infty} \rho_n(t) = \sum_{n=1}^{\infty} \frac{p(s_n)}{q'(s_n)} e^{st}.
\]

When \( q(s) \) and \( p(s) \) are polynomials, then the number of
poles is finite and formula (42) becomes the Heaviside
expansion, formula (5).

Illustration 9. Find the inverse of \( f(s) = \frac{(2s+1)}{s(s^2+1)} \).

The function \( f(z) \) has simple poles at \( z=0, z=i, \) and
\( z=-i \). Letting \( \rho_1, \rho_2, \rho_3 \) represent the residues of
\[ f(z)e^{zt} \] at \( z=0 \), \( z=i \), and \( z=-1 \), respectively, we have

\[ \rho_1 = e^{it} \lim_{z \to 0} \frac{(z-i)(2z+1)}{z(z+i)(z-1)} = e^{-it} \]
\[ \rho_2 = e^{it} \lim_{z \to i} \frac{(z-i)(2z+1)}{z(z+i)(z-1)} = -\frac{\cos t + \sin t - \cos t - \sin t}{2} \]
\[ \rho_3 = e^{-it} \lim_{z \to -i} \frac{(z+i)(2z+1)}{z(z+i)(z-1)} = -\frac{\cos t + \sin t - \cos t + \sin t}{2} \]

Thus \( L_1^{-1}[f(s)] = \rho_1 + \rho_2 + \rho_3 = 1 - \cos t + 2\sin t \).

Illustration 10. Find the function whose transform is \( f(s) = 1/s^2(s+1) \). The integrand of the inversion integral is \( e^{zt}/z^2(z+1) \) which has a pole of order 2 at \( z = 0 \), and a simple pole at \( z = -1 \). Letting \( \rho_1 \) and \( \rho_2 \) be the residues at \( 0 \) and \( -1 \), respectively, we obtain

\[ \rho_1 = \lim_{z \to 0} \frac{dz}{z} \left[ \frac{(z-i)(2z+1)}{z(z+1)} \right] = t-1 \]
\[ \rho_2 = \lim_{z \to -1} \frac{dz}{z} \left[ \frac{(z-i)(2z+1)}{z(z+1)} \right] = e^{-t} \]

Thus \( L_1^{-1}[f(s)] = \rho_1 + \rho_2 = t-1 + e^{-t} \).

This section is concluded with the following theorem which, perhaps, was anticipated by the reader.

**THEOREM 14.** The operator \( L_1^{-1} \) is indeed a linear transformation.

**Proof.** Let the conditions of Theorem 11 be satisfied for \( f(s) \) and \( g(s) \), where \( L[F(t)] = f(s) \) and \( L[G(t)] = g(s) \), and let \( c \) be a constant. Then

\[ L_1^{-1}[f(s)+g(s)] = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\delta-i\beta}^{\delta+i\beta} [f(z)+g(z)]e^{zt}dz \]
\[ = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\delta-i\beta}^{\delta+i\beta} f(z)e^{zt}dz + \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\delta-i\beta}^{\delta+i\beta} g(z)e^{zt}dz \]
Also,

\[ L_1^{-1}[c \cdot f(s)] = \frac{1}{2\pi i} \lim_{\beta \to \infty} \int_{\delta-i\beta}^{\delta+i\beta} c f(z) e^{zt} dz \]

\[ = \frac{c}{2\pi i} \lim_{\beta \to \infty} \int_{\delta-i\beta}^{\delta+i\beta} f(z) e^{zt} dz \]

\[ = c L_1^{-1}[f(s)]. \]
Finite Fourier Sine and Cosine Transforms

In this part, two other integral transformations will be investigated. Unlike the Laplace transformation which deals with functions defined on infinite intervals, the finite Fourier sine and cosine transforms are mappings of functions defined on finite intervals. It is possible, however, to extend these finite Fourier transforms to functions defined on unbounded intervals, but this extension will not be taken up in this report.

Like the Laplace transform, these two Fourier transforms convert differential forms into algebraic forms involving boundary values.

Definition 1. The finite Fourier sine transform of a function $F(x) \in S[0,\pi]$ is denoted by $S_n[F(x)]$ and is defined as

$$S_n[F(x)] = \int_0^\pi F(x) \sin nx \, dx = f_s(n) \quad (n = 1, 2, \ldots)$$

Note that sin nx is the kernel for this particular integral transformation.

Because there is a one-one correspondence with $f_s(n)$, $n=1,2,\ldots$, and the positive integers, we say that the image of $F(x)$ under this transform is a sequence. Hence, this integral transformation maps a real-valued function into a sequence of real numbers.
DEFINITION 2. The finite Fourier cosine transform of a function \( F(x) \in SC[0,\pi] \) is denoted by \( C_n[F(x)] \) and is defined as

\[
C_n[F(x)] = \int_0^\pi F(x) \cos nx \, dx = f_c(n) \quad (n=0,1,\ldots)
\]

This finite cosine transform and the above finite sine transform have similar properties. Hence, each remark of the preceding paragraph has a corresponding remark applying to the finite Fourier cosine transform.

Since \( \cos nx = 1 \) for \( n=0 \), it follows that

\[
f_c(0) = \int_0^\pi F(x) \, dx,
\]

a relationship that will be used later on.

The finite Fourier sine and cosine transforms exist for any function \( F(x) \in SC[a,b] \), since the finite interval \([a,b]\) is homeomorphic to the interval \([0,\pi]\). In particular, the finite Fourier sine transform of a function \( F(x) \), sectionally continuous on the interval \([0,c]\), can be written in terms of the transform of \( F(x) \) on \([0,\pi]\), the interval in the definition of \( S_n[F(x)] \), by means of the substitution \( r = \pi x/c \) to get:

\[
\int_0^c F(x) \sin \frac{nx}{c} \, dx = \frac{c}{\pi} \int_0^\pi F(cr/\pi) \sin nr \, dr = \frac{c}{\pi} S_n[F(cx/\pi)].
\]

For brevity, the finite Fourier sine and the finite Fourier cosine transform shall often hereafter be referred to as the sine transform and the cosine transform, respectively. A few operational properties of these transforms will now be presented in the following theorems.

THEOREM 1. The sine and cosine transforms are indeed
linear.

In fact,

\[ S_n[F(x)+G(x)] = \int_0^\pi [F(x)+G(x)] \sin nx \, dx \]
\[ = \int_0^\pi F(x) \sin nx \, dx + \int_0^\pi G(x) \sin nx \, dx \]
\[ = S_n[F(x)] + S_n[G(x)], \]

and

\[ S_n[aF(x)] = \int_0^\pi aF(x) \sin nx \, dx = a \int_0^\pi F(x) \sin nx \, dx \]
\[ = aS_n[F(x)], \]

where

\[ F(x), G(x) \in SC[0, \pi] \] and \( a \) is a scalar. The linearity of the cosine transform follows likewise.

Illustration 1. From knowing that \( S_n[1] = \frac{1+(-1)^{n+1}}{n} \) and \( S_n[x] = \frac{n(-1)^{n+1}}{n} \), we may conclude that if \( a, b \in \mathbb{R} \), then

\[ S_n[ax+b] = \frac{a\pi(-1)^{n+1}}{n} + \frac{b[1+(-1)^{n+1}]}{n} \]
\[ = \frac{(ax+b)(-1)^{n+1} + b}{n} \quad (n=1, 2, \ldots) \]

Illustration 2. If \( a \in \mathbb{R} \) and \( n=1, 2, \ldots \), then

\[ C_n[F(x)+a] = C_n[F(x)] + C_n[a] = C_n[F(x)] \], since

\[ C_n[a] = \int_0^\pi \cos nx \, dx = 0. \] For \( n=0 \) we have

\[ C_0[F(x)+a] = C_0[F(x)] + C_0[a] = f_c(0) + \pi a. \]

**Theorem 2.** Let \( F(x) \in SC[0, \pi] \), \( f_s(n) \) be the sine transform of \( F(x) \) and \( f_c(n) \) the cosine transform of \( F(x) \). Then

\[ \lim_{n \to \infty} f_s(n) = 0 \text{ and } \lim_{n \to \infty} f_c(n) = 0. \]

(5)
Proof. Divide the interval \([0, \pi]\) into a finite number of subintervals on each of which \(F(x)\) is continuous. Note that \(F(x)\) is continuous at the end points of these subintervals, if we use the limits from the interior as the values of \(F(x)\) at those points. Let any one of these subintervals be denoted by \([p, q]\). The first equation of (5) will follow providing we can show

\[\lim_{n \to \infty} \int_{p}^{q} F(x) \sin nx \, dx = 0;\]

that is for each \(e > 0\), we need to find an integer \(n_e\) such that

\[\left| \int_{p}^{q} F(x) \sin nx \, dx \right| < e\]

whenever \(n > n_e\).

We begin this task by dividing the interval \([p, q]\) into \(k\) equal parts by the points \(p = x_0, x_1, x_2, \ldots, x_k = q\). Then the integral in (6) can be written as

\[\int_{p}^{q} F(x) \sin nx \, dx = \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} F(x) \sin nx \, dx,\]

or

\[\int_{p}^{q} F(x) \sin nx \, dx = \sum_{i=0}^{k-1} [F(x_i) \int_{x_i}^{x_{i+1}} \sin nx \, dx + \int_{x_i}^{x_{i+1}} (F(x) - F(x_i)) \sin nx \, dx].\]

But,

\[F(x_i) \int_{x_i}^{x_{i+1}} \sin nx \, dx = F(x_i) \cos nx_i - \cos nx_{i+1}.\]

And if \(M\) is the maximum value of \(|F(x)|\) on \([p, q]\), then

\[\sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} F(x) \sin nx \, dx \leq \sum_{i=0}^{k-1} \frac{2M}{n} = \frac{2Mk}{n},\]

since
\[ |\cos nx_i - \cos nx_{i+1}| \leq 2. \]

The function \( F(x) \) is continuous on \([p,q]\) which implies that it is uniformly continuous on that closed interval. Hence there exists a \( \delta \) such that for the \( \varepsilon \) chosen in (6) we have,
\[ |F(x) - F(x_1)| < \frac{\varepsilon}{2(q-p)} \quad \text{whenever} \quad |x-x_1| < \delta. \]

Let \( k = k_\varepsilon \), where the integer \( k_\varepsilon \) is taken so large that the length of each subinterval \((x_i, x_{i+1})\) is less than \( \delta \). Then
\[ \left( \sum_{i=0}^{k-1} \int_{x_i}^{x_{i+1}} |F(x) - F(x_i)| \, dx \right) \leq \frac{\varepsilon}{2(q-p)} \sum_{i=0}^{k-1} (x_{i+1} - x_i) = \frac{\varepsilon}{2} \]

Summarizing, from (7), (8), and (9) we have
\[ \left| \int_{p}^{q} F(x) \sin nx \, dx \right| < \frac{2M \varepsilon}{n} + \frac{\varepsilon}{2} \quad \text{when} \quad \frac{q-p}{k_\varepsilon} < \delta. \]

Now for the fixed \( k_\varepsilon \), select \( n \) so large that \( 2M \varepsilon / n < \varepsilon / 2 \), say \( n > n_\varepsilon \) where \( n_\varepsilon = 4M \varepsilon / \varepsilon \). Then
\[ \left| \int_{p}^{q} F(x) \sin nx \, dx \right| < \varepsilon / 2 + \varepsilon / 2 = \varepsilon \quad \text{whenever} \quad n > n_\varepsilon, \]

which concludes the proof of \( \lim_{n \to \infty} f_s(n) = 0 \). The verification of the second limit of (5) follows similarly.

From Theorem 2 we realize that if the limit of a sequence of real numbers is not zero, then that sequence is not the sine of cosine transform of some function.

**Theorem 3.** Let \( F(x), F'(x) \in C[0,\pi] \) and \( F''(x) \in SC[0,\pi] \). Then
(10) \( S_n[F''(x)] = -n^2f_s(n) + n[F(0) - (-1)^nF(\pi)] \)
for \( n=1,2,\ldots \), and

(11) \( C_n[F''(x)] = -n^2f_c(n) - F'(0) + (-1)^nF'(\pi) \)
for \( n=0,1,2,\ldots \), where \( S_n[F(x)] = f_s(n) \) and \( C_n[F(x)] = f_c(n) \).

**Proof.** Only (11) will be verified since (10) follows similarly.

By definition,
\[
C_n[F''(x)] = \int_0^\pi F''(x) \cos nx \, dx \quad (n=0,1,2,\ldots)
\]
Letting \( u = \cos nx \) and \( dv = F''(x) \, dx \), we get from integrating by parts that
\[
\int_0^\pi F''(x) \cos nx \, dx = \cos nx F'(x) \bigg|_0^\pi + n \int_0^\pi F'(x) \sin nx \, dx
= (-1)^n F'(\pi) - F'(0) + (-1)^n F'(x) \sin nx \, dx.
\]
Now repeating integration by parts on this last integral, letting \( u = \sin nx \) and \( dv = F'(x) \, dx \), gives
\[
\int_0^\pi F''(x) \cos nx \, dx = -n^2 \int_0^\pi F(x) \cos nx \, dx - F'(0) + (-1)^n F'(\pi)
\]
which completes the proof.

**Illustration 3.** Find the sine transform of \( F(x) = e^{cx} \).
From (10) we get \( c^2 S_n[e^{cx}] = -n^2 S_n[e^{cx}] + n[1 - (-1)^n e^{c\pi}] \),
since \( S_n[F''(x)] = S_n[c^2 e^{cx}] = c^2 S_n[e^{cx}] \). Thus
\[
S_n[e^{cx}] = \frac{n[1 - (-1)^n e^{c\pi}]}{n^2 + c^2} \quad (n=1,2,\ldots)
\]

**Illustration 4.** Find the cosine transform of \( F(x) = \sin kx \).
Noting that \( F''(x) = -k^2 \sin kx \), we obtain from (11) that
\[-k^2C_n [\sin kx] = -n^2C_n [\sin kx] - k\cos \theta + (-1)^n k\cos k\pi,\]

which yields,

\[C_n [\sin kx] = \frac{(-1)^n k\cos k\pi - k}{n^2 - k^2}\]

for \(n=0,1,2,\ldots\) and \(k\neq 0,1,2,\ldots\).

**COROLLARY.** Let \(F(x), \ F'(x), \ F''(x), \ F'''(x) \in C[0,\pi]\) and \(F^{(4)}(x) \in SC[0,\pi]\). Then

\[(12) \quad S_n [F^{(4)}(x)] = n^4f_s(n) - n^3[F(0) - (-1)^n F(\pi)] + n[F''(0) - (-1)^n F''(\pi)]\]

and

\[(13) \quad C_n [F^{(4)}(x)] = n^4f_c(n) + n^2 [F'(0) - (-1)^n F'(\pi)] - F''(0) + (-1)^n F'''(\pi),\]

where \(S_n [F(x)] = f_c(n)\) and \(C_n [F(x)] = f_c(n)\).

These forms are verified by replacing \(F(x)\) by \(F''(x)\) in equations (10) and 11).

**THEOREM 4.** Let \(F(x) \in C[0,\pi]\) and \(F'(x) \in SC[0,\pi]\). Then

\[(14) \quad S_n [F'(x)] = -nC_n [F(x)] \quad \quad \quad \quad \quad \quad (n=1,2,\ldots),\]

\[(15) \quad C_n [F'(x)] = nS_n [F(x)] - F(0) + (-1)^n F(\pi) \quad (n=0,1,2,\ldots)\]

The proof of this theorem follows from integration by parts. As each form is derived similarly, we shall only prove (15).

Letting \(u = \cos nx\) and \(dv = F'(x)ds\) we obtain

\[C_n [F'(x)] = \int_0^\pi u \, dv = F(x)\cos nx \bigg|_0^\pi + \int_0^\pi F(x) \sin nx \, dx\]

\[= nS_n [F(x)] - F(0) + (-1)^n F(\pi).\]

**Illustration 5.** From Illustration 4 we, we saw that for \(k\) not an integer, the cosine transform of \(\sin kx\) is \([(-1)^n k\cos k\pi - k]/(n^2 - k^2)\). Therefore from (14) we obtain
Before the next theorem can be given it is necessary to define the odd and even periodic extensions of a function \( F(x) \).

**Definition 3.** If a real-valued function \( F_1(x) = F(x) \) on the open interval \((0, \pi)\), \( F_1(-x) = -F_1(x) \) and \( F_1(x+2\pi) = F_1(x) \) on the interval \((-\infty, \infty)\), then \( F_1(x) \) is said to be the **odd periodic extension**, with period \( 2\pi \), of \( F(x) \).

Similarly, if a real-valued function \( F_2(x) = F(x) \) on \((0, \pi)\), \( F_2(-x) = F_2(x) \) and \( F_2(x+2\pi) = F_2(x) \) on \((-\infty, \infty)\), then \( F_2(x) \) is said to be the **even periodic extension**, with period \( 2\pi \), of \( F(x) \).

**Theorem 5.** Let \( k \) be a constant, \( F(x) \in SC[0, \pi] \), \( f_s(n) \) be the sine transform of \( F(x) \), and \( f_c(n) \) the cosine transform of \( F(x) \). Then

\[
\begin{align*}
(16) \quad f_s(n) \cos nk &= S_n \left[ \frac{1}{2} [ F_1(x-k) + F_1(x+k) ] \right] & (n=1, 2, \ldots) \\
(17) \quad f_s(n) \sin nk &= C_n \left[ \frac{1}{2} [ F_1(x+k) - F_1(x-k) ] \right] & (n=0, 1, \ldots) \\
(18) \quad f_c(n) \cos nk &= C_n \left[ \frac{1}{2} [ F_2(x-k) + F_2(x+k) ] \right] & (n=0, 1, \ldots) \\
(19) \quad f_c(n) \sin nk &= S_n \left[ \frac{1}{2} [ F_2(x-k) - F_2(x+k) ] \right] & (n=1, 2, \ldots)
\end{align*}
\]

where \( F_1(x) \) is the odd periodic extension, with period \( 2\pi \), of \( F(x) \), and \( F_2(x) \) is the even periodic extension, with period \( 2\pi \), of \( F(x) \).

**Proof.** Only the proof of (18) will be given, as (16), (17) and (19) are established analogously. By Definition 2 we have
\[ f_0(n) \cos nk = \int_0^\pi F(x) \cos nx \cos nk \, dx \]
\[ = \int_0^\pi F_2(x) \cos nx \cos nk \, dx \]
\[ = \frac{1}{2} \int_{-\pi}^\pi F_2(x) \cos nx \cos nk \, dx \]
\[ = \frac{1}{4} \int_{-\pi}^\pi F_2(x) \left[ \cos n(x-k) + \cos n(x+k) \right] \, dx \]
\[ = \frac{1}{4} \int_{-\pi}^\pi F_2(x) \cos n(x-k) \, dx + \frac{1}{4} \int_{-\pi}^\pi F_2(x) \cos n(x+k) \, dx. \]

(20)

Letting \( \eta = x-k \), the first integral on the right of (20) becomes

\[ \frac{1}{4} \int_{-\pi-k}^{\pi-k} F_2(\eta+k) \cos n \eta \, d\eta, \]

(21)

and letting \( \eta = x+k \), the second integral on the right of (20) becomes

\[ \frac{1}{4} \int_{-\pi+k}^{\pi+k} F_2(\eta-k) \cos n \eta \, d\eta. \]

(22)

But the integrands of (21) and (22) are periodic functions of \( \eta \), so the limits \(-\pi-k\) to \(\pi-k\) and \(-\pi+k\) to \(\pi+k\) may each be replaced by \(-\pi\) to \(\pi\). Thus, we may write \( f_0(n) \cos nk \) as

\[ \frac{1}{4} \int_{-\pi}^{0} F_2(\eta+k) \cos n \eta \, d\eta + \frac{1}{4} \int_{0}^{\pi} F_2(\eta+k) \cos n \eta \, d\eta + \frac{1}{4} \int_{-\pi}^{0} F_2(\eta-k) \cos n \eta \, d\eta + \frac{1}{4} \int_{0}^{\pi} F_2(\eta-k) \cos n \eta \, d\eta. \]

(23)

By changing the variable \( \eta \) to \(-\delta\) in the first and third integrals of (23) we have

\[ \frac{1}{4} \int_{-\pi}^{0} F_2(\eta+k) \cos n \eta \, d\eta = \frac{1}{4} \int_{-\pi}^{\pi} F_2(-\delta+k) \cos n \delta \, d\delta \]
and
\begin{align*}
\frac{1}{4} \int_{-\pi}^{\pi} F_2(\eta-k) \cos n\eta \, d\eta &= \frac{1}{4} \int_{0}^{\pi} F_2[-(\delta+k)] \cos n\delta \, d\delta.
\end{align*}

Hence we may now write \( f_c(n) \cos nk \) as
\begin{align*}
\frac{1}{4} \int_{0}^{\pi} F_2(\delta-k) \cos n\delta \, d\delta + \frac{1}{4} \int_{0}^{\pi} F_2(\eta+k) \cos n\eta \, d\eta + \\
\frac{1}{4} \int_{0}^{\pi} F_2(\delta+k) \cos n\delta \, d\delta + \frac{1}{4} \int_{0}^{\pi} F_2(\eta-k) \cos n\eta \, d\eta.
\end{align*}

The proof of (18) is completed by unifying the dummy variables of integration and then adding.

No illustration of Theorem 5 is displayed at this point, but this theorem will be used to prove an important convolution theorem in the following part. However, a useful corollary follows from Theorem 5.

**COROLLARY.** Let \( F(x) \in SC[0,\pi], S_n[F(x)] = f_s(n), \) and \( C_n[F(x)] = f_c(n). \) Then
\begin{align*}
(24) \quad f_s(n)(-1)^{n+1} &= S_n[F(\pi-x)] \quad \text{and} \\
(25) \quad f_c(n)(-1)^n &= C_n[F(\pi-x)].
\end{align*}

In fact, when \( k = \pi \) in formula (18) we get
\begin{align*}
f_c(n) \cos n\pi &= f_c(n)(-1)^n = C_n \left[ \frac{1}{2} [F_2(x-\pi) + F_2(x+\pi)] \right] \\
&= C_n \left[ \frac{1}{2} [F_2(x-\pi) + F_2(x-\pi+2\pi)] \right] \\
&= C_n[F_2(x-\pi)] \quad \text{which proves (25).}
\end{align*}

Formula (24) is verified similarly.

**Illustration 6.** If \( S_n[x] = (-1)^{n+1}\pi/n, \) find \( S_n[\pi-x]. \)

From (24) we procure
\begin{align*}
S_n[\pi-x] &= \frac{(-1)^{n+1}\pi(-1)^n+1}{n} = \frac{\pi}{n} \quad (n=1, 2, \ldots)
\end{align*}

In the succeeding part we shall examine the inverses of these finite Fourier sine and cosine transforms.
PART V

The Inverse of the Finite Fourier Sine and Cosine Transforms

Analogous to the Laplace transform method in solving a boundary value problem, it would be rolling the stone of Sisyphus if we did not know how to find the solution function from the algebraic solution of the transformed equation. Hence, the necessity of the present part of this report.

Let us begin the discussion by stating a classical theorem of Fourier series. From this theorem the definitions of our desired inverses will be gleaned.

THEOREM 1. Let \( F(x) \in SC[0, \pi] \), and for each point of discontinuity \( x_0 \in (0, \pi) \) let \( F(x_0) = \frac{1}{2}[F(x_0 + 0) + F(x_0 - 0)]. \)

Then at each point \( x \in (0, \pi) \) where the right hand derivative and the left hand derivative exist, \( F(x) \) is represented by its Fourier sine series

\[
(1) \quad F(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} \int_{0}^{\pi} f(\eta) \sin nx \, d\eta,
\]

and also by its Fourier cosine series

\[
(2) \quad F(x) = \frac{1}{\pi} \int_{0}^{\pi} F(x) \, dx + 2 \sum_{n=1}^{\infty} \int_{0}^{\pi} f(\eta) \cos nx \, d\eta.
\]

Due to the lack of machinery, no attempt will be made to prove this theorem.

Equation (1) can be written in the form

\[
(3) \quad F(x) = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx,
\]
where \( f_s(n) \) is the finite Fourier sine transform of \( F(x) \).

It can now be recognized that (3) is an inversion formula which recovers the function \( F(x) \) from its sine transform. Also, we may notice that this inversion formula produces a unique function, since a convergent series has one and only one sum. Thus, the inverse of \( f_s(n) \), denoted as \( S_n^{-1}[f_s(n)] \) is indeed given by (3). It may also be correctly said that \( S_n^{-1}[f_s(n)] \) represents the function whose sine transform is \( f_s(n) \).

Equation (2) can be written in the form

\[
F(x) = \frac{1}{\pi} f_c(0) + \frac{2}{\pi} \sum_{n=1}^{\infty} f_c(n) \cos nx,
\]

where \( f_c(n) \) is the finite Fourier cosine transform of \( F(x) \). Hence (4) is an inversion formula that produces the function whose cosine transform is \( f_c(n) \). We denote the inverse of the cosine transform \( f_c(n) \) as \( C_n^{-1}[f_c(n)] \).

Like the inverse of the sine transform, \( C_n^{-1}[f_c(n)] \) is also unique.

The following theorems exemplify some properties of these two inverse transformations.

**THEOREM 2.** \( S_n^{-1}[f_s(n)] \) and \( C_n^{-1}[f_c(n)] \) are linear transformations.

**Proof.** Let \( f_s(n) \) and \( g_s(n) \) be the sine transforms of \( F(x) \) and \( G(x) \), respectively. Then

\[
S_n^{-1}[f_s(n)+g_s(n)] = \frac{2}{\pi} \sum_{n=1}^{\infty} [f_s(n)+g_s(n)] \sin nx
\]

\[
= \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx + \frac{2}{\pi} \sum_{n=1}^{\infty} g_s(n) \sin nx
\]
\[ s_n^{-1}[f_s(n)] + s_n^{-1}[g_s(n)]. \]

Also,
\[ s_n^{-1}[af_s(n)] = \frac{2}{
\sum_{n=1}^{\infty} a f_s(n) \sin nx = \frac{2}{\pi} \sum_{n=1}^{\infty} f_s(n) \sin nx = a s_n^{-1}[f_s(n)]. \]

The linearity of \( c_n^{-1}[f_c(n)] \) follows analogously.

Illustration 1. Find the inverse of the sine transform \( \frac{1}{n} \).

We can express \( \frac{1}{n} \) as
\[ \frac{1}{n} = \frac{1-(-1)^n}{n} - \frac{(-1)^{n+1}}{n}. \]

Then
\[ s_n^{-1}[\frac{1}{n}] = s_n^{-1}\left[ \frac{1-(-1)^n}{n} \right] + s_n^{-1}\left[ \frac{(-1)^{n+1}}{n} \right]. \]

since \( s_n^{-1}[1] = \frac{1-(-1)^n}{n} \) and \( s_n^{-1}\left[ \frac{x}{n} \right] = \frac{(-1)^{n+1}}{n} \).

THEOREM 3. Let \( F(x) \in S[-\pi, \pi] \), \( f_s(n) \) be the sine transform of \( F(x) \), and \( f_c(n) \) be the cosine transform of \( F(x) \). Then

(5) \( s_n^{-1}\left[ \frac{f_s(n)}{n^2} \right] = \frac{x}{\pi} \int_0^x F(r) dr dt + \frac{x^t}{\pi} \int_0^x F(r) dr dt \) and

(6) \( c_n^{-1}\left[ \frac{f_c(n)}{n^2} \right] = \frac{x}{\pi} \int_0^x F(r) dr dt + \frac{f_c(0)}{2\pi} (x-\pi)^2 + A, \)

where \( A \) is a constant because \( f_c(n)/n^2 \) is not defined when \( n=0 \).

**Proof.** Let us prove (5) first.

Define \( Y(x) \) as

(7) \[ Y(x) = \frac{x}{\pi} \int_0^x F(r) dr dt - \frac{x}{\pi} \int_0^x F(r) dr dt. \]

Then
\[ Y(x) = \frac{x}{\pi} B - \int_0^x G(t) dt, \]

where \( B \) is a constant and \( G'(r) = F(r) \). Now \( Y'(x) = \frac{B}{\pi} - G(x) \)
and \( Y''(0) = 0 - G'(x) = -F(x) \). Noting that \( Y(0) = Y(\pi) = 0 \) and using (10) of Part 4 we obtain

\[
S_n[-F(x)] = -n^2 S_n[Y(x)], \text{ or }
\]

\[-f_s(n) = -n^2 y_s(n) \text{ where } S_n[Y(x)] = y_s(n).\]

Therefore, \( y_s(n) = \frac{f_s(n)}{n^2} \) and finally

\[
Y(x) = S_n^{-1}[y_s(n)] = S_n^{-1}[\frac{f_s(n)}{n^2}].
\]

The proof of (6) is begun by defining \( Y(x) \) as

\[
(8) \quad Y(x) = \int_0^\pi F(r)dr + f_c(0)(x-\pi) + A,
\]

where \( A \) is a constant. Now (8) can be put in the form

\[
Y(x) = \int_0^x H(t)dt + f_c(0)(x-\pi) + A, \text{ where }
\]

\( H(t) = G(\pi) - G(t) \) and \( G'(r) = F(r) \). Differentiation gives us

\[
(9) \quad Y'(x) = [H(x) = G(\pi) - G(x)] + \frac{f_c(0)(x-\pi)}{\pi} \text{ from which }
\]

\[
Y'(\pi) = G(\pi) - G(\pi) + \frac{f_c(0)(\pi-\pi)}{\pi} = 0 \text{ and }
\]

\[
Y'(0) = G(\pi) - G(0) - f_c(0).
\]

But \( G(\pi) - G(0) = \int_0^\pi F(r)dr \) and previously we saw that

\[
f_c(0) = \int_0^\pi F(r)dr. \text{ Thus } Y'(0) = 0.
\]

From (9) above we get

\[
Y''(x) = -G'(x) + \frac{f_c(0)}{\pi} = -F(x) + \frac{f_c(0)}{\pi}.
\]

Using (11) of Part 4 yields
\[ C_n[-F(x) + \frac{f_c(0)}{\pi}] = -n^2 C_n[Y(x)]. \]

But from Illustration 2 of Part 4, it is clear that
\[ C_n[-F(x) + \frac{f_c(0)}{\pi}] = C_n[-F(x)] \quad (n \neq 0). \]

Hence \(-f_c(n) = -n^2 y_c(n)\) where \(C_n[Y(x)] = y_c(n)\). Finally, since \(y_c(n) = \frac{f_c(n)}{n^2}\), we obtain
\[ Y(x) = C_n^{-1}[y_c(n)] = C_n^{-1}[\frac{f_c(n)}{n^2}]. \]

This completes the proof of Theorem 3.

Illustration 2. Find the inverse of the sine transform
\[ y_s(n) = \frac{1 - (-1)^n}{n^2}. \] We can write \(y_s(n)\) as
\[ y_s(n) = \frac{1 - (-1)^n}{n^2} = \frac{1 - (-1)^n}{n} \cdot \frac{1}{n^2}. \] Knowing that
\[ S_n^{-1}[\frac{1 - (-1)^n}{n}] = 1 \] we have the form \(S_n^{-1}[\frac{f(s(n))}{n^2}]\) where \(F(x) = 1\) and \(S_n[F(x)] = f_s(n)\). Applying formula (5) gives us
\[ S_n^{-1}[\frac{1 - (-1)^n}{n^2}] = \frac{x}{\pi} \int_0^\pi \int_0^x \frac{t}{\pi} \frac{\pi}{\pi} \frac{\pi}{\pi} = \frac{x}{\pi^2} \frac{\pi}{\pi^2} = \frac{x}{\pi^2} (\pi - x). \]

Illustration 3. Find the function whose cosine transform is \(y_c(n) = \frac{(-1)^n}{n^2}\), \(y_c(0) = 0\). Writing \(\frac{(-1)^n}{n^2}\) as \(\frac{(-1)^n}{n^2} \cdot \frac{1}{n^2}\) and knowing that \(C_n^{-1}[\frac{(-1)^n}{n^2}] = \frac{x^2}{2\pi}\), we have the form \(C_n^{-1}[\frac{(-1)^n}{n^2}] = C_n^{-1}[\frac{f_c(n)}{n^2}]\), where \(F(x) = \frac{x^2}{2\pi}\) and
\[ C_n[F(x)] = f_c(n). \] Also we have \(f_c(n) = \frac{(-1)^n}{n^2}\) for \(n=1, 2, \ldots\)
and \( f_c(0) = \frac{\pi^2}{6} \) for \( n = 0 \). Applying formula (6) gives us

\[ c_n^{-1}\left[\frac{(-1)^n}{n^4}\right] = \int_0^T \frac{\pi^2}{2\pi} \frac{r^2}{12} (x-\pi)^2 + A. \]

To solve for \( A \) we use the fact that \( y_c(0) = 0 \) and get

\[ A = \frac{-37\pi^3}{360}. \]

Thus, after integrating and putting in the value of \( A \), we obtain

\[ c_n^{-1}\left[\frac{(-1)^n}{n^4}\right] = \frac{1}{24\pi} (2\pi^2x^2 - x^4 - \frac{7\pi^4}{15}). \]

This part shall be concluded with an important convolution theorem. This theorem is useful in finding the inverse of the sine and cosine transforms, just as the convolution theorem of Part 3 is applicable in finding the inverse of certain Laplace transforms.

However, before presenting this convolution theorem, let us gather some machinery via a definition and a few lemmas.

**Definition 1.** Let the functions \( P(x) \) and \( Q(x) \) be defined on the interval \((-2\pi, 2\pi)\). Then the function

\[ P(x) \star Q(x) = \int_{-\pi}^{\pi} P(x-r)Q(r)dr, \]

when this integral exists, is called the convolution of \( P \) and \( Q \) on the interval \((-\pi, \pi)\).

**Lemma 1.** The convolution of \( P \) and \( Q \) as defined in (10) is an even function if \( P \) and \( Q \) are both even or both odd, and is odd if \( P \) is odd and \( Q \) is even or if \( P \) is even and \( Q \) is odd.

**Proof.** Only part of this lemma will be established
as the proof of the remaining portion is similar.

Let $F$ and $Q$ both be odd functions; i.e., $F(-x) = -F(x)$ and $Q(-x) = -Q(x)$. We need to show that $F(-x) \times Q(-x) = F(x) \times Q(x)$. By definition,

(11) \[ P(-x) \times Q(-x) = \int_{-\pi}^{\pi} P(-x-r)Q(r)dr = -\int_{-\pi}^{\pi} P(x+r)Q(r)dr.\]

Changing the variable of integration $r$ to $-\eta$ we obtain

\[ \int_{-\pi}^{\pi} P(x+r)Q(r)dr = \int_{-\pi}^{\pi} P(x-\eta)Q(-\eta)d\eta\]
\[ = \int_{-\pi}^{\pi} P(x-\eta)Q(\eta)d\eta\]
\[ = P(x) \times Q(x).\]

Let $P$ be odd and $Q$ be even; i.e., $P(-x) = -P(x)$ and $Q(-x) = Q(x)$. We need to show that $P(-x) \times Q(-x) = -[P(x) \times Q(x)]$.

From (11) above,

\[ P(-x) \times Q(-x) = -\int_{-\pi}^{\pi} P(x+r)Q(r)dr.\]

Changing the variable of integration $r$ to $-\eta$ we obtain

\[ \int_{-\pi}^{\pi} P(x+r)Q(r)dr = \int_{-\pi}^{\pi} P(x-\eta)Q(-\eta)d\eta\]
\[ = \int_{-\pi}^{\pi} P(x-\eta)Q(\eta)d\eta\]
\[ = -[P(x) \times Q(x)].\]

**Lemma 2.** If $P$ and $Q$ are periodic, with period $2\pi$, then

\[ P \times Q = Q \times P.\]

**Proof.** Letting $x-\eta = \eta$ we obtain

\[ P \times Q = \int_{-\pi}^{\pi} P(x-r)Q(r)dr = -\int_{-\pi}^{x-\pi} P(\eta)Q(x-\eta)d\eta\]
\[ = \int_{x+\pi}^{\pi} P(\eta)Q(x-\eta)d\eta\]
\[ = \int_{-\pi}^{\pi} P(\eta)Q(x-\eta)d\eta.\]
= \int_{-\pi}^{x+\pi} Q(x-\eta)P(\eta)\,d\eta. \text{ This last integral is just}
\int_{-\pi}^{\pi} Q(x-\eta)P(\eta)\,d\eta = Q \ast P,

since x is a parameter and the period of the integrand is 2\pi.

**Lemma 3.** Let \(F(x), G(x) \in SC[0, \pi]\), \(F_1(x)\) and \(G_1(x)\) denote the odd periodic extensions, with period 2\pi, of \(F(x)\) and \(G(x)\), respectively; and let \(F_2(x)\) and \(G_2(x)\) denote the even periodic extensions, with period 2\pi, of \(F(x)\) and \(G(x)\), respectively. Then

\[
(12) \quad F_2 \ast G_2 = \int_{0}^{\pi} G(r)[F_2(x-r)+F_2(x+r)]\,dr = I_1 + I_2 + I_3 + I_4,
\]

\[
(13) \quad F_1 \ast G_1 = \int_{0}^{\pi} G(r)[F_1(x-r)-F_1(x+r)]\,dr = I_1 - I_2 - I_3 + I_4, \text{ and}
\]

\[
(14) \quad F_1 \ast G_2 = \int_{0}^{\pi} G(r)[F_1(x-r)+F_1(x+r)]\,dr
\]

\[
= \int_{0}^{\pi} F(r)[G_2(x-r)-G_2(x+r)]\,dr = I_1 + I_2 - I_3 - I_4,
\]

where
\[
I_1 = \int_{0}^{\pi} F(r)G(x-r)\,dr, \quad I_2 = \int_{0}^{\pi} F(r)G(r-x)\,dr, \quad I_3 = \int_{0}^{\pi} F(r)G(x+r)\,dr,
\]

and
\[
I_4 = \int_{\pi-x}^{\pi} F(r)G(2\pi-x-r)\,dr.
\]

**Proof.** Let us verify (14) and realize that (12) and (13) are established similarly.

By definition, we have

\[
F_1 \ast G_2 = \int_{-\pi}^{\pi} F_1(x-r)G_2(r)\,dr = \int_{-\pi}^{\pi} G_2(r)F_1(x-r)\,dr + \int_{0}^{\pi} G_2(r)F_1(x-r)\,dr.
\]
Letting $r = -\eta$ in the first integral on the right we obtain
\[
\int_{-\pi}^{\pi} G_2(r)F_1(x-r)dr = \int_{0}^{\pi} G_2(-\eta)F_1(x+\eta)d\eta
\]
\[
= \int_{0}^{\pi} G_2(\eta)F_1(x+\eta)d\eta,
\]
from which the first part of (14) follows. To show the second part, let us begin by observing that
\[
F_1 \times G_2 = G_2 \times F_1 = \int_{-\pi}^{\pi} G_2(x-r)F_1(r)dr
\]
\[
= \int_{-\pi}^{0} F_1(r)G_2(x-r)dr + \int_{0}^{\pi} F_1(r)G_2(x-r)dr.
\]
Letting $r = -\eta$ in the first integral on the right of the preceding we obtain
\[
\int_{-\pi}^{0} F_1(r)G_2(x-r)dr = \int_{0}^{\pi} F_1(-\eta)G_2(x+\eta)d\eta
\]
\[
= -\int_{0}^{\pi} F_1(\eta)G_2(x+\eta)d\eta,
\]
from which the second part of (14) follows. Lastly, the third part of (14) is justified by breaking up this latter form of $F_1 \times G_2$; that is,
\[
F_1 \times G_2 = \int_{0}^{\pi} F(r)[G_2(x-r)-G_2(x+r)]dr
\]
\[
= \int_{0}^{\pi} F(r)G_2(x-r)dr - \int_{0}^{\pi} F(r)G_2(x+r)dr.
\]
But,
\[
\int_{0}^{\pi} F(r)G_2(x-r)dr = \int_{0}^{x} F(r)G_2(x-r)dr + \int_{x}^{\pi} F(r)G_2(x-r)dr,
\]
\[
\int_{x}^{\pi} F(r)G_2(x-r)dr,
\]
since
\[ G_2(x-r) = G_2(r-x), \text{ for } G \text{ is even}. \] Also,

\[
(17) \quad -\int_0^\pi F(r)G_2(x+r)dr = \int_0^{\pi-x} F(r)G_2(x+r)dr - \int_{\pi-x}^\pi F(r)G_2(2\pi-x-r)dr, \text{ since } G_2(x+r) = G_2(-x-r) = G_2(2\pi-x-r); \text{ for } G_2 \text{ is even and periodic, with period } 2\pi.
\]

Summarizing, the first and second integrals on the right of (16) are \( I_1 \) and \( I_2 \), respectively, and the first and second integrals on the right of (17) are \( -I_3 \) and \( -I_4 \), respectively. This completes the proof of Lemma 3.

We are now ready to deliver the promised convolution theorem, which will conclude this report.

**THEOREM 4.** Let \( F(x), G(x) \in SC[0, \pi] \), \( S_n[F(x)] = f_s(n) \), \( S_n[G(x)] = g_s(n) \), \( C_n[F(x)] = f_c(n) \), and \( C_n[G(x)] = g_c(n) \). Then

\[
(18) \quad f_s(n)g_s(n) = C_n[\frac{1}{2}(F_1 \times G_1)]
\]

\[
(19) \quad f_s(n)g_c(n) = S_n[\frac{1}{2}(F_1 \times G_2)], \text{ and}
\]

\[
(20) \quad f_c(n)g_c(n) = C_n[\frac{1}{2}(F_2 \times G_2)]
\]

where \( F_1, F_2, G_1, \) and \( G_2 \) are the previously defined extensions of \( F(x) \) and \( G(x) \).

**Proof.** Once again, due to the similarity of the proofs for (18), (19), and (20), let us verify just one of these, say (19).

Consider the product
\[(21) \quad f_{s}(n)g_{c}(n) = f_{s}(n)\int_{0}^{\pi} G(r)\cos nr \, dr = \int_{0}^{\pi} G(r)f_{s}(n)\cos nr \, dr.
\]

From (16) of the previous part we know that
\[f_{s}(n)\cos nr = S_{n}\left[\frac{1}{2}(F_{1}(x-r)+F_{1}(x+r))\right].\]
Therefore
\[(22) \quad f_{s}(n)g_{c}(n) = \int_{0}^{\pi} G(r)\left[\frac{1}{2}(F_{1}(x-r)+F_{1}(x+r))\right] \sin nx \, dxdr.
\]

By changing the order of integration the right side of (22) becomes
\[\int_{0}^{\pi} \int_{0}^{\pi} \sin nx \left[\frac{1}{2}\right] G(r)[F_{1}(x-r)+F_{1}(x+r)] drdx.
\]

However, from equation (14), the inner integral of the preceding expression is just $F_{1} \ast G_{2}$. Thus we can write
\[f_{s}(n)g_{c}(n) = \int_{0}^{\pi} \left[\frac{1}{2}\right][F_{1} \ast G_{2}] \sin nx \, dx = S_{n}\left[\frac{1}{2}(F_{1} \ast G_{2})\right],
\]

which completes the proof.

Illustration 4. Find the inverse of the sine transform $[(-1)^{n}-1]/n^{3}$. Letting
\[f_{s}(n) = \frac{(-1)^{n+1}}{n} \quad \text{and} \quad g_{c}(n) = \frac{(-1)^{n}-1}{n^{3}},
\]

we can show that
\[\frac{(-1)^{n}-1}{n^{3}} = f_{s}(n)g_{c}(n).
\]

Now letting $F(x) = S_{n}^{-1}[f_{s}(n)] = \frac{X}{\pi}$ and $G(x) = C_{n}^{-1}[g_{c}(n)] = X$, we apply (19) to obtain
\[f_{s}(n)g_{c}(n) = S_{n}\left[\frac{1}{2}(F_{1} \ast G_{2})\right] \text{ or}
\]
\[ S_n^{-1}[f_s(n)g_c(n)] = \frac{1}{2}(F_1 \times G_2). \]

But \( \frac{1}{2}(F_1 \times G_2) = \frac{1}{2}(I_1 + I_2 - I_3 - I_4) \), where these \( I_k \) are defined in Lemma 3. Evaluating these four integrals gives us \( \frac{1}{2}(F_1 \times G_2) = \frac{x(x-n)}{2} \). Hence

\[ S_n^{-1}\left[\frac{(-1)^{n-1}}{n^2}\right] = \frac{x(x-n)}{2}. \]
BIBLIOGRAPHY


APPLICATIONS OF LAPLACE AND FINITE
FOURIER SINE TRANSFORMATIONS

by

Jan Eugene Wynn

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INTRODUCTION

The Laplace and finite Fourier sine transforms can be used to solve certain boundary value problems. Also, the Laplace transform is a useful tool in solving some integral and integrodifferential equations. This report is composed of transform solutions of seventeen such applied problems, while the author's first report is focused towards the theoretical aspect of these transforms.

Several different types of problems are solved in this report. Among these are Bessel's classical differential equation of index n, two electrical circuit problems, a beam problem, a vibrating string problem, a heat flow problem, and a temperature gradient problem.

One of the objectives of this report is to illustrate several operational properties of the Laplace and finite Fourier sine transforms. Therefore, various methods of inverting transforms are employed to provide diversification.
Problem 1.

Solve the following boundary value problem.

\[ Y''(t) + Y'(t) = e^{2t}; \quad Y(0) = Y'(0) = Y''(0) = 0. \]

Solution: Letting \( y(s) \) denote the Laplace transform of \( Y(t) \) we apply the transformation to the differential equation to obtain

\[ s^3y(s) - s^2Y(0) - sY'(0) - Y''(0) + sy(s) - Y(0) = \frac{1}{s-2}. \]

Using the boundary conditions and solving for \( y(s) \) gives

\[ y(s) = \frac{1}{s(s^2 + 1)(s-2)} = \frac{1}{s(s-i)(s+i)(s-2)}. \]

According to the theory of partial fractions, there exist constants \( A, B, C, \) and \( D \) such that

\[ \frac{1}{s(s-i)(s+i)(s-2)} = \frac{A}{s} + \frac{B}{s-i} + \frac{C}{s+i} + \frac{D}{s-2}, \]

or

\[ A(s^2+1)(s-2) + Bs(s+i)(s-2) + Cs(s-i)(s-2) + Ds(s^2+1) = 1. \]

We can solve for \( A, B, C, \) and \( D \) by letting \( s=0, s=i, s=-i, \)

and \( s=2, \) respectively, to get \( A = -1/2, B = (2+i)/10, \)

\( C = (2-i)/10, \) and \( D = 1/10. \) Hence

\[ y(s) = \frac{1}{2s} + \frac{(2+i)/10}{s-i} + \frac{(2-i)/10}{s+i} + \frac{1/10}{s-2}, \]

and thus

\[ Y(t) = L^{-1}\left[\frac{1}{2s}\right] + L^{-1}\left[\frac{(2+i)/10}{s-i}\right] + L^{-1}\left[\frac{(2-i)/10}{s+i}\right] + L^{-1}\left[\frac{1/10}{s-2}\right] \]

\[ = -\frac{1}{2} + \frac{2+i}{10}e^{it} + \frac{2-i}{10}e^{-it} + \frac{1}{10}e^{2t} \]

\[ = -\frac{1}{2} + \frac{2+i}{10}(\cos t + isin t) + \frac{2-i}{10}(\cos t - isin t) + \]
\[
\frac{1}{10}e^{2t} = -\frac{1}{10} + \frac{2}{5}\cos t - \frac{1}{10}\sin t + \frac{1}{10}e^{2t}.
\]

Problem 2.

Solve the system of ordinary differential equations

\begin{align*}
(1) & \quad Y'(t) - Z'(t) - 2Y(t) + 2Z(t) = 1 - 2t, \\
(2) & \quad Y''(t) + 2Z'(t) + Y(t) = 0, \text{ having the initial conditions} \\
(3) & \quad Y(0) = Z(0) = Y'(0) = 0.
\end{align*}

Solution: Let \( y(s) \) and \( z(s) \) represent the respective transforms of \( Y(t) \) and \( Z(t) \). Applying the Laplace transform to (1) and (2) gives us the algebraic equations

\begin{align*}
(4) & \quad sy(s) - Y(0) - (sz(s) - Z(0)) - 2y(s) + 2z(s) = \frac{1}{s} - \frac{2}{s^2} \\
(5) & \quad s^2y(s) - sy(0) - y'(0) + 2sz(s) - Z(0) + y(s) = 0.
\end{align*}

Employing the conditions of (3) and solving the algebraic system (4) and (5) obtains

\[ y(s) = \frac{2}{s(s+1)^2} \quad \text{and} \quad z(s) = -\frac{(s^2+1)}{s^2(s+1)^2} = \frac{-1}{(s+1)^2} - \frac{1}{s^2(s+1)^2}. \]

In finding the Laplace inverse of \( y(s) \) we note that the function

\[ g(z) = \frac{2e^{zt}}{z(z+1)^2} \]

has a simple pole at \( z=0 \) and a pole of order 2 at \( z=-1 \). If \( \rho_1 \) and \( \rho_2 \) denote the residues of \( g(z) \) at \( z=0 \) and \( z=-1 \), then

\[
\rho_1 = \lim_{z \to 0} \left[ \frac{(z-0)(z)}{(z)(z+1)^2} \right] = 2 \quad \text{and} \\
\rho_2 = \lim_{z \to -1} \left[ \frac{1}{(2-1)!} \frac{d}{dz} \frac{(z-(-1))^2 2e^{zt}}{z(z+1)^2} \right] = \lim_{z \to -1} \frac{d}{dz} \left( \frac{2e^{zt}}{z} \right) = \]
-2te^{-t} - 2e^{-t}. Hence, Y(t) = \rho_1 + \rho_2 = 2 - 2e^{-t} - 2te^{-t}.

Let us use the convolution theorem in solving for Z(t). Consider the expression \(-1/s^2(s+1)^2\), the second term of z(s). Denoting \(f(s)\) and \(g(s)\) as
\[ f(s) = \frac{1}{(s+1)^2} \quad \text{and} \quad g(s) = -\frac{1}{s^2} \]
we obtain
\[ F(t) = L^{-1}[f(s)] = te^{-t} \quad \text{and} \quad G(t) = L^{-1}[g(s)] = -t. \]
Thus the convolution \(F \ast G\) is
\[
F \ast G = \int_0^t F(r)G(t-r)dr
= \int_0^t re^{-r}(r-t)dr
= \int_0^t r^2 e^{-r}dr - t\int_0^t e^{-r}dr
= 2 - te^{-t} - 2e^{-t} - t.
\]
The first term of z(s) is \(-1/(s+1)^2\) whose Laplace inverse is \(-te^{-t}\). Finally, adding the two parts of \(L^{-1}[z(s)]\) gives
\[ Z(t) = 2 - 2te^{-t} - 2e^{-t} - t. \]

Problem 3.

Solve the differential equation
\[ Y''(t) - 2kY'(t) + k^2Y(t) = F(t), \]
having no prescribed boundary conditions.

Solution: If \(L[Y(t)] = y(s)\) and \(L[F(t)] = f(s)\), then transformed equation is
\[ s^2y(s) - sy(0) - Y'(0) - 2k(sy(s) - Y(0)) + k^2y(s) = f(s). \]
Solving this algebraic equation yields

\[ y(s) = \frac{f(s)+sy(0)+y'(0)-2ky(0)}{(s-k)^2}. \]

Letting \( A = y'(0)-2ky(0) \) and \( B = y(0) \) allows us to write

\[ y(s) = \frac{f(s)}{(s-k)^2} + \frac{ Bs}{(s-k)^2} + \frac{ A}{(s-k)^2}. \]

Using convolution on the first term by agreeing that

\[ L^{-1}[f(s)] = F(t) \text{ and } L^{-1}[g(s) = \frac{1}{(s-k)^2}] = [G(t) = te^{kt}] \]

we have

\[ L^{-1}[f(s)g(s)] = F \ast G = \int_0^t F(r)(t-r)e^{k(t-r)}dr \]

\[ = e^{kt}\int_0^t F(r)(t-r)e^{-kr}dr. \]

Now the second term in \( y(s) \) can be inverted by residues.

In fact, the function \( Bz^2/(z-k)^2 \) has a pole of order 2 at \( z=k \). The residue \( \rho \) at this point is

\[ \rho = \lim_{z \to k} \frac{d}{dz} \left[ \frac{(z-k)^2Bz^t}{(z-k)^2} \right] \]

\[ = Bk^t + Be^{kt}. \]

The inversion of the third term in \( y(s) \) is \( Ate^{kt} \).

Summarizing, we obtain the inverse of \( y(s) \); that is,

\[ Y(t) = L^{-1}[y(s)] = e^{kt}\int_0^t F(r)(t-r)e^{-kr}dr + Bkte^{kt} + Be^{kt} + Ate^{kt} \]

\[ = e^{kt}\left[ \int_0^t F(r)(t-r)e^{-kr}dr + B + (Bk+A)t \right]. \]

Problem 4.

Solve the integrodifferential equation

\[ \int_0^t Y(r)dr - Y'(t) = t, \text{ having the initial condition } Y(0)=2. \]

Solution: Letting \( L[Y(t)] = y(s) \) and taking the
Laplace transform of each side of the given equation yields

\[ \frac{1}{s} y(s) - (sy(s) - Y(0)) = \frac{1}{s^2}, \text{ or} \]

\[ y(s) = \frac{2s^2 - 1}{s(s^2 - 1)}. \]

If we let \( y(s) = \frac{p(s)}{q(s)} \) where \( p(s) = 2s^2 - 1 \) and \( q(s) = s(s^2 - 1) \), then we can use the formula

\[ L^{-1}\left[ \frac{p(s)}{q(s)} \right] = \sum_{n=1}^{3} \frac{p(a_n)}{q'(a_n)} e^{a_n t}, \]

where the \( a_n \) are the zeros of \( q(s) \); i.e., \( q(s) = (s-a_1)(s-a_2)(s-a_3) \). Our particular \( q(s) \) is factored into such linear factors by \( q(s) = (s-0)(s-(-1))(s-1) \). Now if \( a_1 = 0, a_2 = -1, \) and \( a_3 = 1, \) we can write

\[ L^{-1}[y(s)] = \frac{p(0)}{q'(0)} e^{0t} + \frac{p(-1)}{q'(-1)} e^{-t} + \frac{p(1)}{q'(1)} e^{t}. \]

Hence, \( Y(t) = 1 + \frac{1}{2} e^{-t} + \frac{1}{2} e^{t}. \)

Problem 5.

Solve the Bessel equation of index \( n \) \( (n=1, 2, \ldots) \)

\[ (1) \quad t^2 Y''(t) + tY'(t) + (t^2 - n^2)Y(t) = 0. \]

Solution: Using the substitution

\[ Y(t) = t^{-n}Z(t) \] on (1) we obtain a new differential equation

\[ (2) \quad tZ''(t) + (1-2n)Z'(t) + tZ(t) = 0. \]

Letting \( L[Z(t)] = z(s) \) and taking the Laplace transform of (2) yields

\[ \frac{d}{ds} \left( s^2 z(s) - sz(0) - Z'(0) \right) + \frac{1}{2}(1-2n)(sz(s) - Z(0)) - \frac{dz(s)}{ds} = 0, \]
which reduces to \( \frac{dz}{z} = -\frac{(1+2n)sds}{s^2+1} \). Integrating each side of this last equation gives us

\[
\ln z = -\frac{(1+2n)}{2} \ln(s^2+1) + C, \quad \text{or}
\]

\[
z(s) = (s^2+1)^{-\frac{(1+2n)}{2}} e^C = \frac{A}{(s^2+1)^{n+1/2}},
\]

where \( C \) is a constant and \( A = e^C \). Note that

\[
(1) \quad \frac{1}{(1+s^2)^{n+1/2}} = \frac{1}{s^{2n+1}(1+1/s^2)^{n+1/2}}.
\]

Next, consider the binomial expansion of \((1+1/s^2)^{-n-1/2}\); i.e.,

\[
(1+1/s^2)^{-n-1/2} = 1 + \frac{(-n-1/2)}{1!} \frac{1}{s^2} + \frac{(-n-1/2)(-n-1/2-1)}{2!} \left(\frac{1}{s^2}\right)^2 + \frac{(-n-1/2)(-n-1/2-1)(-n-1/2-2)}{3!} \left(\frac{1}{s^2}\right)^3 + \ldots
\]

The general term of this expansion is

\[
\begin{align*}
(\frac{-n-1/2}{n-1/2-1}) \cdots \frac{1}{n-1/2-(k-1)} \frac{1}{k!} \frac{1}{s^{2k}},
\end{align*}
\]

(k=0,1,..).

Through several manipulations it can be shown that this general term reduces to

\[
\begin{align*}
\frac{(-1)^k[2(n+k)]!}{2^{n+2k} k! (n+k)! [1\cdot3\cdot(2n-3)(2n-1)] s^{2k}}.
\end{align*}
\]

Hence

\[
\begin{align*}
K_n = \frac{1}{1\cdot3\cdot(2n-3)(2n-1)}.
\end{align*}
\]

From (3) and (4) we can now write \( z(s) \) as
(5) \[ z(s) = \sum_{k=0}^{\infty} \frac{(-1)^k [2(n+k)]!}{2^{n+2k} k! (n+k)! s^{2(n+k)+1}}. \]

Taking the Laplace inverse of (5) gives

\[ Z(t) = \sum_{k=0}^{\infty} \frac{(-1)^k t^{2(n+k)}}{2^{n+2k} k! (n+k)!}, \]

since

\[ L[t^{2(n+k)}] = \frac{[2(n+k)]!}{s^{2(n+k)+1}}. \]

Finally,

\[ Y(t) = t^{-n} Z(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t)^{n+2k}}{k! (n+k)! (2)}, \]

or

\[ Y(t) = BJ_n(t), \]

where \( B = AK_n \) and \( J_n(t) \) is Bessel's equation of the first kind defined by

\[ J_n(t) = \sum_{k=0}^{\infty} \frac{(-1)^k (t)^{n+2k}}{k! (n+k)! (2)} \quad (n=0, 1, \ldots). \]

Problem 6.

Solve the integral equation

\[ Y(t) = \sin t - 2 \int_0^t Y(r) \cos (t-r) \, dr. \]

Solution: If \( L[Y(t)] = y(s) \), the transformed equation becomes

\[ y(s) = \frac{\alpha}{s^2+1} - 2[I(t) * \cos t] \]

\[ = \frac{\alpha}{s^2+1} - 2y(s) \frac{s}{s^2+1}, \]

since \( L[\cos t] = \frac{s}{s^2+1} \). And solving for \( y(s) \) we obtain

\[ y(s) = \frac{s}{s^2+1} \text{.} \]

Taking the inverse of each side gives the solution

\[ Y(t) = ate^{-t}. \]
Problem 7.

Solve the integro-differential equation

\[ F'(t) + k^2 \int_0^t F(x) \cosh k(t-x) \, dx = 0, \]

having no prescribed initial condition.

Solution: The transformed equation is

\[ sf(s) - F(0) + k^2[F \ast \cosh kt] = 0. \]

But \( F \ast \cosh kt = f(s)[s/(s^2-k^2)] \), and upon solving (2) for \( f(s) \), we find that

\[ f(s) = \frac{F(C)(s^2-k^2)}{s^3} = F(C)[\frac{1}{s^3} - \frac{k^2}{s^3}]. \]

Therefore,

\[ F(t) = L^{-1}[f(s)] = F(C)L^{-1}[\frac{1}{s^3}] - F(C)L^{-1}[\frac{k^2}{s^3}] = F(C)[1 - \frac{k^2t^2}{2}]. \]

Problem 3.

Consider the electrical circuit shown in Figure 1, where \( Q \) is the charge accumulated in the capacitor \( C \) at time \( t \), \( I \) is the current, \( R \) is the resistance of the circuit, \( L \) is the inductance of a coil, and where \( I(t) = Q'(t) \).

![Figure 1](image)

If the positive sense of flow \( I(t) \) of positive charge is
taken in the clockwise direction as indicated in the figure, then $Q(t)$ measures the charge on the upper plate of the capacitor. Adding the three voltage drops and equating the sum to zero, we have

\begin{align*}
LI'(t) + \frac{1}{C}Q(t) + RI(t) &= 0, \\
LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) &= 0.
\end{align*}

If the resistance is neglected, the capacitor has an initial charge of $Q_0$ ($Q(0) = Q_0$) and the initial current is $I_0$ ($I(0) = Q'(0) = I(0)$), whence the transformed equation of (1) becomes

$$L(s^2q(s) - sQ(0) - I_0) + \frac{1}{C}q(s) = 0.$$ 

Solving for $q(s)$, we have

$$q(s) = \frac{Q_0s}{s^2 + w_o^2} + \frac{I_0}{s^2 + w_o^2},$$

where $w_o = \sqrt{L/C}$. Taking the Laplace inverse of each side of equation (2) gives the solution

$$Q(t) = Q_0\cos w_o t + \frac{I_0}{w_o} \sin w_o t.$$ 

Problem 9.

The electric current $I$ and the charge $Q$ on the capacitor in the circuit shown in Figure 2 are functions of $t$ that satisfy the equations

$$LI'(t) + RI(t) + \frac{Q}{C} = E_o, \quad Q = \int_0^t I(r)dr,$$

t is the time after closing the switch $K$, and where $Q$ and $I$ are initially zero ($Q(0)=0$, $I(0)=0$). Let the electromotive force $E_o$ be constant and solve for $I(t)$. 
Solution: If $L[I(t)] = i(s)$, the transform of the above differential equation yields the algebraic equation

$$L(si(s) - I(0)) + Ri(s) + \frac{i(s)}{Cs} = \frac{E_0}{s},$$

which reduces to

$$i(s) = \frac{E_0/L}{s^2 + \frac{R}{L} + \frac{1}{LC}} = \frac{E_0/L}{[s(-b)]^2 + w_1^2},$$

where $w_1 = \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$

and $b = R/2L$. Taking the inverse of our algebraic solution gives

$$I(t) = \frac{E_0}{lw_1} e^{-bt} \sin w_1 t \quad (w_1^2 > 0).$$

Problem 10.

A beam of length $2c$ is fixed at the left end and pinned at the right end. The part $0 < x < c$ is unloaded but the part $c < x < 2c$ has a uniform load of $w_0$ units per unit length, so that the total load is $W_0 = w_0 c$ (see Fig. 3).

The internal bending moment $M(x)$ is given by the equation
where $E$ is Young's modulus, $I$ is the moment of inertia of the cross-sectioned area with respect to the neutral axis, $Y(x)$ is the displacement, and $Y'(x)$ is the slope of deflection. The internal shearing force, $F(x)$, is given by the equation

$$F(x) = E I Y'''(x),$$

since $M'(x) = F(x)$. Also, if $W(x)$ represents the transverse load per unit length along the beam, then

$$W(x) = E I Y^{(4)}(x).$$

Find the shearing force exerted by the beam on the pin, and the shearing force and bending moment exerted on the wall at $x=0$.

Solution: The displacement and slope of deflection at $x=0$ are each zero, which gives us two end conditions ($Y(0)=Y'(0)=0$). The displacement and bending moment at the end $x=2c$ are each zero which gives us two more end conditions ($Y(2c)=Y'(2c)=0$). Equation (3), for this particular problem, becomes

$$Y^{(4)}(x) = a w_o S_c(x),$$

where $a^{-1} = E I$ and $S_c(x) = \begin{cases} 0 & 0 < x < c \\ 1 & x \geq c \end{cases}$. The Laplace transform of (4) is

$$s^4 Y(s) - s^3 Y(0) - s^2 Y'(0) - s Y''(0) - Y'''(0) = \frac{aw_o e^{-cs}}{s},$$

which has the solution

$$Y(s) = \frac{Y''(0)}{s^3} + \frac{Y'''(0)}{s^4} + \frac{aw_o e^{-cs}}{s^5}.$$
Taking the inverse of the preceding equation gives us

\[ Y(x) = \frac{1}{2} Y''(0)x^2 + \frac{1}{6} Y'''(0)x^3 + \frac{1}{24} a w_o(x-c)^4 s_c(x). \]

Using the conditions \( Y(2c) = Y''(2c) = 0 \), for \( x > c \), we can solve for \( Y''(0) \) and \( Y'''(0) \), which makes equation (5) become

\[ Y(x) = \frac{7}{64} a w_o c^2 x^2 - \frac{23}{384} a w_o c x^3 + \frac{1}{24} a w_o (x-c)^4 s_c(x). \]

Now

\[ F(x) \bigg|_{x > c} = \frac{1}{2} Y''(x) \bigg|_{x > c} = - \frac{23}{64} w_o c + w_o (x-c). \]

Hence, the shearing force at \( x = 2c \) is

\[ F(2c) \bigg|_{x > c} = - \frac{23}{64} w_o c + w_o (2c-c) = \frac{41}{64} w_o c = \frac{41}{64} w_o. \]

But \( F(x) \bigg|_{x < c} = \frac{1}{2} Y''(x) \bigg|_{x < c} = - \frac{23}{64} w_o c. \)

Hence, the shearing force at \( x = 0 \) is

\[ F(0) \bigg|_{x < c} = - \frac{23}{64} w_o c = - \frac{23}{64} w_o. \]

Finally,

\[ M(x) \bigg|_{x < c} = \frac{1}{2} Y''(x) \bigg|_{x < c} = \frac{7}{32} w_o c^2 - \frac{23}{64} w_o c x. \]

Hence the bending moment at \( x = 0 \) is

\[ M(0) \bigg|_{x < c} = \frac{7}{32} w_o c^2 = \frac{2}{32} w_o. \]

Problem 11.

Solve the second order partial differential equation

\[ Y_{xx}(x,t) - 2Y_{xt}(x,t) + Y_{tt}(x,t) = 0 \quad (0 < x < 1, t > 0) \]

having the boundary conditions
\[ Y(x,0) = Y_t(x,0) = Y(0,t) = 0, \quad Y(1,t) = F(t) \quad (t > 0). \]

If we denote \( y(x,s) \) as the Laplace transform of \( Y(x,t) \), then the transform of (1) with respect to \( t \) becomes

\[
\frac{d^2[y(x,s)]}{dx^2} - 2 \frac{dy(x,s)}{dx} + s^2y(x,s) = -sY(x,0) = 0, \quad \text{or}
\]

\[
(2) \quad \frac{d^2y}{dx^2} - 2s \frac{dy}{dx} + s^2y = 0,
\]

where \( y(0,s) = 0 \) and \( y(1,s) = f(s) \). Letting \( L[y(x,s)] = u(p,s) \) and applying the Laplace transformation to equation (2) with respect to \( x \) we obtain

\[
p^2u(p,s) - py(0,s) - \frac{dy}{dx}(0,s) - 2s(pu(p,s) - y(0,s)) + s^2u(p,s) = 0, \quad \text{or}
\]

\[
p^2u - 2spu + s^2u = A, \quad \text{where} \quad A = \frac{dy}{dx}(0,s). \quad \text{Solving the}
\]

algebraic equation in \( u \) gives us \( u(p,s) = \frac{A}{(p-s)^2} \) which inverted gives \( y(x,s) = Axe^{sx} \). Using the condition \( y(1,s) = f(s) \) we find that \( A = f(s)/e^s \). Hence

\[
y(x,s) = \frac{f(s)xe^{sx}}{e^s} = xf(s)e^{-(1-x)s}. \]

Taking the inverse of \( y(x,s) \) gives us our desired solution

\[
L^{-1}[y(x,s)] = Y(x,t) = xf[t-(1-x)] = xf(t+x-1) \quad (t+x-1 > 0)
\]

\[= 0 \quad \text{for} \quad (t+x-1 < 0). \]

**Problem 12.**

Solve the partial differential equation

\[
xY_x(x,t) + Y_t(x,t) + Y(x,t) = xF(t),
\]
having the prescribed conditions
\[ Y(x,0) = Y(0,t) = 0. \]

Solution: The transformed boundary value problem is

\[ \frac{xd[y(x,s)]}{dx} + sy(x,s) - Y(x,0) + y(x,s) = xf(s); \ y(0,s) = 0. \]

Multiplying this linear first-order differential equation by the integrating factor \( \exp \left( \frac{\int x^2 ds}{x} \right) = x^{s+1} \), we obtain

\[ x^{s+1} \frac{dy}{dx} + (s+1)x^s y = x^{s+1} f(s), \text{ or} \]

\[ \frac{d}{dx}[x^{s+1} y] = x^{s+1} f(s). \]

Integrating each side with respect to \( x \) gives

\[ x^{s+1} y(x,s) = \frac{x^{s+2}}{s+2} f(s) + C, \]

where \( C \) is a constant of integration. Using the fact that \( y(0,s) = 0 \), we find that \( C = 0 \). Hence, the solution of the transformed problem is

\[ y(x,s) = \frac{xf(s)}{s+2}. \]

Letting \( g(s) = 1/(s+2) \) and \( L^{-1}[g(s)] = G(t) = e^{-2t} \) we find that

\[ L^{-1}[y(x,s)] = xF(t) * G(t) = \int_0^t F(r)G(t-r)dr = \]

\[ \int_0^t F(r)e^{-2(t-r)}dr. \]

Therefore, the solution to the problem is

\[ Y(x,t) = xe^{-2t} \int_0^t e^{2r} F(r) dr. \]
Problem 13.

A string is displaced into the curve \( Y = b \sin \frac{\pi x}{c} \) and released from rest (see Figure 4). Set up and solve the boundary value problem for the displacement \( Y(x,t) \).

\[
\begin{align*}
\frac{\partial^2 Y}{\partial t^2} (x,t) &= a^2 \frac{\partial^2 Y}{\partial x^2} (x,t) \quad (x>0, t>0),
\end{align*}
\]

where \( a^2 = H/\rho \); \( H \) is the horizontal tension in the string and \( \rho \) is the mass per unit length. Also, we see that

\[
Y(x,0) = b \sin \frac{\pi x}{c}, \quad Y(0,t) = Y_x(x,0) = Y(c,t) = 0,
\]

where \( Y(x,t) \) represents the velocity of the string.

The transformed equation of (1) is

\[
\begin{align*}
\mathcal{S}^2 y(x,s) - s Y(x,0) - Y_t(x,0) &= a^2 \mathcal{S}^2 y_{xx}(x,s), \quad \text{or}
\end{align*}
\]

(2)

\[
\begin{align*}
\mathcal{S}^2 y(x,s) - s b \sin \frac{\pi x}{c} &= a^2 \mathcal{S}^2 y_{xx}(x,s).
\end{align*}
\]

Letting \( u(p,s) \) denote the transform of \( y(x,s) \) with respect to \( x \), where \( p \) is the new parameter, we write the new transformed equation of (2)

(3)

\[
\begin{align*}
\mathcal{S}^2 u(p,s) - \frac{b \pi \sqrt{c}}{p^2 + (\frac{\pi}{c})^2} &= a^2 \left[ p^2 u(p,s) - s y(0,s) - y_x(0,s) \right].
\end{align*}
\]

But \( Y(0,t) = 0 \) implies \( y(0,s) = 0 \), and letting \( y_x(0,s) = A \) equation (3) becomes

\[
\begin{align*}
\mathcal{S}^2 u - \frac{b \pi \sqrt{c}}{p^2 + (\frac{\pi}{c})^2} &= a^2 p^2 u - A a^2.
\end{align*}
\]
Solving for \( u \) we obtain

\[
(4) \quad u(p, s) = \frac{A}{p^2 - \left(\frac{s}{a}\right)^2} - \frac{bsn/2ca^2}{[p^2 - \left(\frac{s}{a}\right)^2][p^2 + \left(\frac{s}{C}\right)^2]}
\]

The inverse of the first term on the right of (4) is \( Aa/s \sinh sx/a \). Using convolution on the second term by letting

\[
f(s) = \frac{1}{p^2 - \left(\frac{s}{a}\right)^2} \quad \text{and} \quad g(s) = \frac{1}{p^2 + \left(\frac{s}{C}\right)^2}, \quad \text{or}
\]

\[
F(x) = \mathcal{L}^{-1}[f(s)] = \frac{a}{s} \sinh \frac{sx}{a} \quad \text{and} \quad G(x) = \mathcal{L}^{-1}[g(s)] = \frac{c}{\pi} \sin \frac{\pi x}{c},
\]

we have

\[
\frac{bsn}{2ca^2}(F \ast G) = \frac{bsn}{2ca^2} \int_0^x \left[\frac{a}{s} \sinh \frac{sx}{a}\right]\left[\frac{c}{\pi} \sin \frac{\pi x}{c}-(x-r)\right] dr
\]

\[
= \frac{b}{a} \int_0^x \sinh \left(\frac{s}{a}r \sin \frac{\pi x}{c}(x-r)\right) dr.
\]

Performing the integration (5) gives us the inversion of (4)

\[
(6) \quad y(x, s) = \frac{Aa}{s} \sinh \frac{sx}{a} - \frac{bs}{ca} \sinh \left(\frac{s}{a}x\right) + \frac{bs \sin \frac{\pi x}{c}}{\left(\frac{s}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}.
\]

But \( Y(c, t) = 0 \) implies \( y(c, s) = 0 \), and using this fact we find that

\[
A = \frac{b\pi}{ca^2} \frac{s}{\left(\frac{s}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2}.
\]

Putting this constant into equation (6) and cancelling we obtain

\[
y(x, s) = \frac{b}{a^2} \sin \left(\frac{\pi x}{c}\right) \frac{s}{\left(\frac{s}{a}\right)^2 + \left(\frac{\pi}{c}\right)^2} = b \sin \left(\frac{\pi x}{c}\right) \frac{s}{s^2 + \left(\frac{\pi a}{c}\right)^2}.
\]

We finally invert \( y(x, s) \) to get the solution
\[ Y(x,t) = b \sin \frac{\pi x}{c} \cos \left( \frac{\pi a}{c} t \right). \]

Problem 14.

The lateral surface of a bar of unit length is insulated while its ends \( x=0 \) and \( x=1 \) are kept at zero temperature. Heat is generated throughout the bar at a constant rate of \( B \) units per unit volume. The initial temperature is \( B(x-x^2)/2K \) (see Figure 5). Set up the boundary value problem and derive the temperature formula.

\[ \begin{align*}
U &= 0 \\
U(x,0) &= B(x-x^2)/2K \\
U(0,t) &= U(1,t) = 0
\end{align*} \]

Figure 5

Solution: The temperature satisfies the heat equation

\[ (1) \hspace{1cm} U_t(x,t) = U_{xx}(x,t) + \frac{B}{K}, \quad (0<x<1, t>0). \]

having the boundary conditions

\[ U(x,0) = \frac{Bx}{2K} - \frac{Bx^2}{2K}, \text{ for } 0<x<1, \text{ and } U(0,t) = U(1,t) = 0 \text{ for } t>0. \]

The transform of equation (1) with respect to \( t \) is

\[ su(x,s) - \frac{Bx}{2K} + \frac{Bx^2}{2K} = \frac{d^2[u(x,s)]}{dx^2} + \frac{B}{Ks}, \text{ or} \]

\[ \frac{d^2u}{dx^2} - su = \frac{Bx}{Ks} - \frac{Bx}{2K} + \frac{Bx^2}{2K}. \]

This latter differential equation has a complementary solution

\[ u_c = c_1 e^{-\frac{Bx}{Ks}} + c_2 e^{\frac{Bx}{Ks}}, \]

and a particular solution

\[ u_p = a + bx + cx^2. \]
Using the method of undetermined coefficients we find that $a = 0$, $b = B/2Ks$, and $c = -B/2Ks$.

Hence,

$$u(x, s) = c_1 e^{-V_s x} + c_2 e^{V_s x} + \frac{Bx}{2Ks} - \frac{Bx^2}{2Ks}.$$ 

But $U(0, t)$ implies $u(0, s) = 0$ and $U(1, t) = 0$ implies $u(1, s) = 0$. Using these two prescribed conditions of $u(x, s)$ we can solve for the constants $c_1$ and $c_2$ and find that $c_1 = c_2 = 0$. Hence the solution of the transformed problem is

$$u(x, s) = \frac{Bx}{2Ks} - \frac{Bx^2}{2Ks}.$$ 

and upon inverting we arrive at the expected solution to the problem

$$U(x, t) = \frac{B(x-x^2)}{2K}.$$ 

Problem 15.

Using the finite Fourier sine transform, solve the differential equation

$$(1) \quad F'(x) + F(x) = -\frac{1}{6\pi} \left[ (6-n^2)x^2 \right],$$

where $F(0) = F(\pi) = 0$.

Solution: Applying the sine transform on equation (1) gives

$$-n^2f_s(n) + n[f(0) - (-1)^n f(\pi)] + f_s(n) = \frac{(-1)^n n^2 (-1)^{n+1}}{n} \frac{n^2 - \pi^2}{6n} \frac{6 - \pi^2}{n},$$

where $f_s(n) = S_n[F(x)]$. This transformed equation in $f_s(n)$ has the solution
The solution of (1) is obtained by finding the function whose sine transform is \((-l)^{n+1}/n^3\); that is
\[
F(x) = S_n^{-1}[(-l)^{n+1}/n^3] = \frac{x(n^2-x^2)}{6\pi}.
\]

Problem 16.

A steady-state temperature function \(V\) satisfies the conditions
\[
\begin{align*}
V_{xx}(x,y) + V_{yy}(x,y) &= 0 \\
V(0,y) &= 0, \quad V(\pi,y) = A \\
V(x,0) &= B, \quad (0<x<\pi, y>0), \\
V(\pi, y) &= A, \quad (0<y<\pi).
\end{align*}
\]
Also, \(V(x,y)\) is bounded. (see Figure 6).

[Figure 6]

Derive the formula for \(V(x,y)\).

Solution: Taking the sine transform of \(V_{xx} + V_{yy} = 0\) with respect to \(x\), where \(S_n[V(x,y)] = V_s(n,y)\), yields

\[
-n^2V_s(n,y) + n[V(0,y)-(-l)^nV(\pi,y)] + \frac{d^2}{dy^2}[V_s(n,y)] = 0,
\]

or

\[
\frac{d^2V}{dy^2} - n^2V = (-l)^nA_n.
\]

Solving this latter differential equation we obtain

\[
V_s(n,y) = c_1e^{-ny} + c_2e^{ny} - (-l)^nA_n/n.
\]
But \( \lim_{n \to \infty} v_s(n,y) = 0 \) which implies that \( c_2 \) is necessarily zero. Also, we use the fact that \( V(x,0) = B \) implies that \( v_s(n,0) = B_n[1 - (-1)^n]/n \) to solve for \( c_1 \). We finally obtain

\[
v_s(n,y) = \frac{B[1 - (-1)^n]}{n} e^{-ny} + \frac{A(-1)^n}{n} e^{-ny} + \frac{(-1)^n + 1}{n}.
\]

Therefore,

\[
V(x,y) = BS_n^{-1}[1 - (-1)^n] e^{-ny} + AS_n^{-1}[- e^{-y}]^n + AS_n^{-1}[-1]^n.
\]

Using tables we find that

\[
V(x,y) = \frac{2B}{\pi} \arctan \frac{2e^{-y} \sin x}{\pi} - \frac{2A}{\pi} \arctan \frac{e^{-y} \sin x}{1 - e^{-2y}} + \frac{Ax}{\pi}\cos x.
\]

Problem 17.

Solve the boundary value problem

\[
(1) \quad \frac{\partial^2 Y(x,t)}{\partial t^2} = -a^2 \frac{\partial^4 Y(x,t)}{\partial x^4} + F(x)
\]

\[
Y(0,t) = Y(\pi,t) = Y_{xx}(0,t) = Y_{xx}(\pi,t) = Y(\pi,0) = 0.
\]

Solution: The sine transform of (1) with respect to \( x \) is

\[
\frac{d^2[y_s(n,t)]}{dt^2} = -a^2 [n^2y_s(n,t) - n^2Y(\pi,t) + n^2(-1)^nY(0,t) + nY_{xx}(0,t)] + f_s(n), \quad \text{or}
\]

\[
\frac{d^2y}{dt^2} + a^2 n^4 y = f_s(n).
\]

This second order linear differential equation has the
solution

\[ y_s(n,t) = c_1 \sin an^2 t + c_2 \cos an^2 t + \frac{f_s(n)}{a^2 n^4}. \tag{2} \]

But from the given boundary conditions we see that 
\[ y_s(n,0) = 0. \] 
Employing this fact into (2) we find that 
\[ c_2 = -\frac{f_s(n)}{a^2 n^4}. \] 
Also, the condition \( y_t(x,0) = 0 \) implies that 
\[ \frac{d}{dt} [y_s(n,t)] \bigg|_{t=0} = 0, \] 
which enables us to realize that 
\[ c_1 = 0. \] 
Hence,

\[ y_s(n,t) = \frac{f_s(n)}{a^2 n^4} - \frac{f_s(n) \cos an^2 t}{a^2 n^4}. \]

Inverting gives the desired solution

\[ Y(x,t) = S_n^{-1}[y_s(n,t)] = \frac{1}{a^2 S_n} \left[ \frac{f_s(n)}{n^4} \right] - \frac{1}{a^2 S_n} \left[ \frac{f_s(n) \cos an^2 t}{n^4} \right] \]

\[ = \frac{1}{a^2} G(x) - \frac{2}{\pi a^2} \sum_{n=1}^{\infty} \frac{f_s(n) \cos an^2 t \sin nx}{n^4}, \]

where \( g_s(n) = \frac{f_s(n)}{n^4} \), \( S_n^{-1}[g_s(n)] = G(x) \), \( G^{(4)}(x) = F(x) \),

\( G(0) = G(\pi) = G''(0) = G''(\pi) = 0. \)