A Theorem on Homeomorphic Convergence and Some Applications

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A THEOREM ON HOMEOMORPHIC CONVERGENCE
AND SOME APPLICATIONS

by

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I. INTRODUCTION

Borsuk [3] has given interesting conditions under which a certain function space is separable (see Theorem 3.1). We give a proof for Borsuk's Theorem here and we show how it can be used to establish a useful theorem on homeomorphic convergence. We illustrate the utility of the theorem on homeomorphic convergence by stating and proving several of its consequences.

For example we show that the plane (E^2) does not contain uncountably many pairwise disjoint contina each of which contains a simple triod (Corollary 4.1). We prove that in an uncountable collection G of pairwise disjoint simple closed curves in E^2 "almost all" elements of G must be converged to homeomorphically "from both sides" by sequences of elements of G (see Theorem 4.3). The same technique allows us to prove the nonexistence of uncountably many pairwise disjoint wild 2-spheres in E^3.

Another interesting consequence of Borsuk's Theorem is Theorem 3.4 which shows that in each set G consisting of uncountably many compact subsets of a metric space, some element of G is an element of convergence. Proofs for this theorem do not often appear in the literature, and, as far as the author knows, the proof given here does not appear in the literature.

We wish to emphasize that all the proofs given in this report were constructed by the author without reference to the literature, in fact the author was unaware of the references until after the proofs were given. We given reference at the end of the paper where proofs in the literature can be compared with the proofs given here.

It is assumed throughout this paper that the reader is familiar with the basic concepts in topology. Some theorems that are used in support of this paper, but are not directly part of it, will be stated without proof; however, the proofs can be found in the literature.

II. DEFINITIONS, NOTATION, AND PRELIMINARY RESULTS

Definition: Let P be a metric space, and let Q be a separable metric space, the function space π is defined as the set of all continuous functions defined on P and having values in Q. The metric topology for π is given in Theorem 3.1 to follow.

Notation: Let(X, d) be a metric space, for $x \epsilon X$ and $\epsilon > o$, $N_d(x, \epsilon) = \{ y \epsilon X : d(x,y) < \epsilon \}$ denotes as the ϵ -neighborhood of x in (X, d).

Definition: A sequence M_1, M_2, - - - - - of point sets in a metric space converges homeomorphicaaly to a point set M if, for each positive number ϵ, there exists a positive integer K such that if $n > K$ there exists a homeomophism h_n from M_n onto M such that $\rho(x, h_n(x)) < \epsilon$ for all $x \epsilon M_n$.

Definition: Let (X_n) be a sequence of point sets in a space S. The set of all points x in S such that every open set containing x intersects all but a finite number of the sets X_n is called the limit inferior of (X_n) and is abbreviated "lim inf X_n" or "$\underline{\lim}\, X_n$." The set of all points y in S such that every open set containing y intersects infinitely many sets X_n is called the limit superior of (X_n) and is abbreviated "lim sup X_n or "$\overline{\lim}\, X_n$."

Definition: A sequence (X_n) of sets is said to converge to a set L (abbreviated "lim X_n=L") if lim sup X_n=L=lim inf $X_n \neq \phi$.

Definition: A continuum is a compact connected metric space.

Definition: A locally connected continuum is sometimes called a continuous curve or Peano continuum.

Definition: A simple triod is a continuum that is the union of three arcs AD, BD and CD such that D is the intersection of each two of them. Somethimes we refer to a simple triod as a triod.

<u>Definition:</u> E^n is defined as the Euclidean n-dimensional space.

<u>Definition:</u> For each positive integer n let $S^n = \{(x_1, x_2, ---, x_{n+1}) \in E^{n+1} ; x_1^2 + x_2^2 + ---- + x_{n+1}^2 = 1\}$. A set S is called an n-sphere if S is homeomorphic to S^n. If S is a 2-sphere in E^3, we denote the bounded component of E^3-S by Int S and the other component by Ext S.

PRELIMINARY RESULTS

We state here several well known theorems. No proofs are given here, but proofs can be found in some advanced topology texts.

Theorem 2.1: A metric space S is separable if and only if every uncountable subset of S has a limit point.

Theorem 2.2: A metric space is compact if and only if it is the continuous image of a Cantor set.

Theorem 2.3: Every compact metric space is separable.

Theorem 2.4: Every subset of a separable metric space is separable.

Theorem 2.5: In a completely separable space, every uncountable subset contains uncountably many limit points of itself.

Theorem 2.6: In each uncountable separable metric space H there exists an uncountable subset T of H such that H-T is countable and every open set that interests T contains uncountably many points of T.

Theorem 2.7: In a metric space S, if p is a limit point of the set $H \subset S$, then there exists a sequence of distinct points of H converging to p.

Theorem 2.8: If a continuous curve M contains no triod, then M is either an arc, a point or a simple closed curve.

Theorem 2.9: Let S be a space having a countable basis. If G is a collection of open sets covering a point set $H \subset S$, then some countable subcollection of G covers H.

III. A THEOREM ON HOMEOMORPHIC CONVERGENCE

In this section we first identify a metric for the set π, then we state and prove an interesting theorem due to Borsuk [3]. Theorem 3.3 is the central theorem of the paper. Apparently Burgess [4] first proved this theorem, although others seemed to know of the theorem in certain special cases (see[5] for example).

Theorem 3.4 is also an interesting consequence of Borsuk's theorem. It shows what conclusions one is able to draw when the disjoint compact sets are not mutually homeomorphic as in Theorem 3.3. Theorem 3.4 was probably first done by R. L. Moore [6] and doesn't appear often in the usual topology texts.

Theorem 3.5 and 3.6 are more general statements of the results in Theorem 3.3 and 3.4, respectively.

Theorem 3.1: The set S of all bounded functions from a compact metric space P into a separable metric space (Q, ρ_Q) forms a metric space under the metric d defined as follows: For f_1, $f_2 \epsilon S$, $d(f_1, f_2) = \text{l.u.b.}\{\rho_Q(f_1(x), f_2(x)): x \epsilon P\}$.

Proof: Suppose $d(f_1, f_2) = 0$. Then $\text{l.u.b.}\{\rho_Q(f_1(x), f_2(x)): x\epsilon P = 0\}$, so it follows that $\rho_Q(f_1(x), f_2(x)) = 0$ for all $x \epsilon P$. Thus $f_1(x) = f_2(x)$ for all $x\epsilon P$, since ρ_Q is a metric for Q and therefore $f_1 = f_2$. The reverse of this argument shows that $f_1 = f_2$ implies $d(f_1, f_2) = 0$.

Since ρ_Q is a metric for Q, $\rho_Q(f_1(x), f_2(x) = \rho_Q(f_2(x), f_1(x))$ for all $x\epsilon P$. Obviously $\text{l.u.b.}\{\rho_Q(f_1(x), f_2(x): x\epsilon P\}$, so $d(f_1, f_2) = d(f_2, f_1)$.

The triangle inequality follows from the following set of inequalities: $d(f_1, f_2) + d(f_2, f_3) = \text{l.u.b.}\{\rho_Q(f_1(x), f_2(x)): x\epsilon P\} + \text{l.u.b.}\{\rho_Q(f_2(x), f_3(x)): x\epsilon P\} = \text{l.u.b.}\{\rho_Q(f_1(x), f_2(x)) + \rho_Q(f_2(y), f_3(y)): x, y \epsilon P\} \geq \text{l.u.b.}\{\rho_Q(f_1(x), f_2(x) + \rho_Q(f_2(x), f_3(x): x\epsilon P\} \geq \text{l.u.b.}\{\rho_Q(f_1(x), f_3(x): x\epsilon P\} = d(f_1, f_3)$.

Therefore d is a metric for S.

Theorem 3.2: The metric space (π, d) is separable if P is compact.

Proof: Since every compact metric space is separable by Theorem 2.3, (P, ρ_p) is separable and thus has a countable basis $U = \{U_i : i \in N\}$. Also there is a countable set $A = \{a_i : i \in N\}$ in Q such that A is dense in (Q, ρ_Q).

Let Σ be the collection of all finite sets of positive integers $\{n_1, n_2, ---- n_k\}$ such that $P \subset U_{n_1} \cup U_{n_2} \cup ---- \cup U_{n_k}$. Then $\Sigma \neq \phi$ since P is compact and U is a cover for P. Also Σ is countable.

For each $s \in \Sigma$, $s = \{n_1, n_2, ----, n_k\}$ we define the collection $\{W_s^1, W_s^2, ----, W_s^k\}$ of disjoint sets as follows: $W_s^1 = U_{n_1}$, $W_s^2 = U_{n_2} - U_{n_1}$, -----, $W_s^k = U_{n_k} - \bigcup_{i=1}^{k-1} (U_{n_i})$. Let π_s be the collection of all functions ϕ such that $\phi(\rho) \subset A$ and $\phi(x)$ is constant over each W_s^i. Then π_s is countable. Let $\pi_\Sigma = \bigcup_{s \in \Sigma} \pi_s$ it follows that π_Σ is a countable collection of bounded functions. Thus it follows from Theorem 3.1 that $\pi' = \pi \cup \pi_\Sigma$ is a metric space. Furthermore from Theorem 2.4 it will follow that π is separable once we show π' is separable. We shall show that π_Σ is a countable dense subset of π'.

Let $f \in \pi$, we shall show the existence of a function $\theta \in \pi_\Sigma$ such that θ is within ϵ of f for an arbitrarily small positive number ϵ.

For each $x \in P$, let G_x be the set of all $y \in P$ such that $\rho_Q(f(x), f(y)) < \frac{\epsilon}{3}$. Then G_x is open is P, hence there exists an integer $n(x)$ such that $x \in U_{n(x)} \subsetneq G_x$. Then there is a finite collection $r = \{n(x_1), n(x_2), ------, n(x_t)\}$ of integers such that $p \subset U_{n(x_1)} \cup ---- \cup U_{n(x_t)}$. Thus $r \in \Sigma$.

Since A is dense in Q, for each i there must be a point $a_i \in A$ such that $\rho_Q(f(x_i), a_i) < \frac{\epsilon}{3}$. By definition of π_Σ, there exist a function θ in π_Σ such that $\theta(x) = a_i$ for each $x \in W_r^i$.

Let $x_o \in P$. Then $x_o \in W_r^i$ for some i.

Since $Wi \subset G_{xi}$, then $\rho_Q(f(x_o), f(x_1)) < \frac{\epsilon}{3}$. Also $\rho_Q(f(x_o), \theta(x_o) \leq \rho_Q(f(x_o), f(x_i)) + \rho_Q(f(x_i) + \theta(x_o)) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}$.

Hence for each $x \in P$, $\rho_Q(f(x), \theta(x)) < \frac{2\epsilon}{3}$. Thus $d(f, \theta) < \epsilon$ and $f \in \overline{\pi_\Sigma}$.

<u>Theorem 3.3:</u> If G is an uncountable collection of mutually homeomorphic compact subsets of a separable metric space (S,ρ), then some sequence of distinct sets of G converges homeomorphically to some element of G.

Proof: Let $G=\{g_\alpha : \alpha \in I, \text{ I is uncountable}\}$. Let $g_\alpha \in G$, and for each $\beta \in I$ let h_β be a homeomorphism of g_α onto g_β. Let $H = \{h_\beta : \beta \in I\}$. Since G is uncountable, H must be uncountable. Let F be the set of all continuous maps from g_α into S. Then it follows from Theorem 3.2 that (F, d) is a separable metric space. Now H is an uncountable subset of F; hence H has a limit point $h \in H$ by Theorem 2.1. Therefore some sequence $\{h_n\}$ of distinct points of H converges to h. Define $k_n = h \circ h_n^{-1}$. Obviously k_n is a homeomorphisim of g_n onto g when g is the image of g_α under the homeomorphism h. Consider the sequence $\{g_n\}$ of distinct points of G.

Let $x_o \in g_n$ then there exists an element y_o of g_α, such that $x_o = h_n(y_o)$ and it follows that $\rho(k_n(x_o), x_o) = \rho(h \circ h^{-1}(x_o), x_o) = \rho(h \circ h_n^{-1}(h_n(y_o)), h_n(y_o)) = \rho(h(y_o), h_n(y_o))$.

Since $\{h_n\}$ converges to h, for each positive number ϵ there exists a positive integer K, such that if $n > K$ $\rho(h(y), h_n(y)) < \epsilon$ for each $y \in g_\alpha$. That is, $\rho(k_n(x_o), x_o) < \epsilon$ if $n > K$. Hence for each $x \in g_\alpha$, $\rho(k_n(x), x) < \epsilon$ if $n > K$. Thus $\{g_n\}$ converges to g homeomorphically.

<u>Theorem 3.4:</u> If G is an uncountable collection of compact sets in a separable metric space S, then some sequence of elements of G converges to an element of G.

Proof: Let $G=\{g_\alpha : \alpha \in I, \text{ I is uncountable}\}$. By Theorem 2.2 for each $\alpha \in I$, there exist a continuous map f_α of a Cantor set C onto g_α.

Let $\pi = \{\theta : \theta \text{ is a continuous map from C into S}\}$. It follows from Theorem 3.2 that (π, d) is a separable metric space. If $F = \{f_\alpha : \alpha \in I\}$, then F is an uncountable subset of π; so F has a limit point $f_r \in F$ by Theorem 2.1. Of course, then some sequence $\{f_n\}$ of distinct points of F converges to f_r. Now consider the sequence $\{g_n\}$ and the set $g_r \in G$.

Let $y_o \epsilon g_r$. Then there exist an element x_o of C such that $f_r(x_o) = y_o$. Since $\{f_n\}$ converges to f_r it follows that for each positive number ϵ there exists an integer K such that $f_n \epsilon N_d(f_r, \epsilon)$ if $n > K$.

Let $n > K$. Then $\rho(f_n(x_o), y_o) = \rho(f_n(x_o), f_r(x_o)) < \epsilon$; that is, $f_n(x_o) \epsilon N\rho(y_o, \epsilon)$. Since $f_n(x_o) \epsilon g_n$, $N\rho(y_o, \epsilon) \cap g_n \neq \phi$, and it follows from the definition of limit inferior that $y_o \epsilon$ lim inf g_n. Since y_o was an arbitrary point of g_r it follows that $g_r \subset$ lim inf g_n.

Assume lim sup $g_n \not\subset g_r$, since lim inf $g_n \subset$ lim sup g_n, it follows that lim sup $g_n \neq \phi$, so there exists a point z in (lim sup g_n)-g_r. Since g_r is closed, $\inf\{\rho(z, x): x \epsilon g_r\} \neq \phi$, so we can choose a number ϵ such that $0 < \epsilon < \inf\{\rho(z, x): x \epsilon g_r\}$. Let $H = \bigcup\{N\rho(x, \frac{\epsilon}{4}): x \epsilon g_r\}$. Since $\{f_n\}$ converges to f_r, for this $\frac{\epsilon}{4}$ there exist an integer K_1 such that if $n > K_1$ $f_n \epsilon N_d(f_r, \epsilon)$; that is, if $n > K_1$ $\rho(f_n(x), y) = \rho(f_n(x), f_r(x) < \frac{\epsilon}{4}$ for all $y = f_r(x) \epsilon g_r$ where $x \epsilon C$. Therefore $g_n \subset H$ for all $n > K_1$. Clearly $N\rho(z, \frac{\epsilon}{4}) \cap H = \phi$. Therefore if $n > K_1$ $N\rho(z, \frac{\epsilon}{4}) \cap g_n = \phi$. It follows that $N\rho(z, \frac{\epsilon}{4})$ intersects at most a finite number of elements of $\{g_n\}$. This implies $z \notin$ (lim sup g_n) thus $z \notin$ (lim sup g_n)-g_r. But this controdicts our assumption.

Therefore (lim sup g_n) $\subset g_r$. Since (lim sup g_n) $\subset g_r \subset$ (lim inf g_n), and (lim inf g_n) $\subset$ (lim sup g_n), it follows that (lim inf g_n) = (lim sup g_n) = g_r. Hence $\{g_n\}$ converges to g_r.

Theorem 3.5: If G is an uncountable collection of mutually homeomorphic compact subsets of a separable metric space (S, ρ), then there is a countable subset G^* of G such that if $g \epsilon G - G^*$ then some sequence of elements of G converges homeomorphically to g.

Proof: Let $G = \{g_\alpha : \alpha \epsilon I$, I is uncountable$\}$. Let $g_\alpha \epsilon G$, and for each $\beta \epsilon I$ let h_β be a homeomorphism of g_α onto g_β. Let $H = \{h_\beta : \beta \epsilon I\}$. Since G is uncountable, H must be uncountable. Let F be the set of all continuous maps from g_α into S. Then it follows from Theorem 3.2 that (F,d) is a separable metric space. Now H is an uncountable subset of F. and by Theorem 2.4 H is separable, hence every uncountable subset of H has a limit point and it will follow from Theorem 2.6 that there exists an uncountable subset T of H such that H-T is countable and every open set that intersects T contains uncountably many points of T.

Let $h \epsilon T$. Clearly h is limit point of T and also a limit point of H. Therefore some sequence $\{h_n\}$ of distinct points of T converges to h by Theorem 2.7. Define $k_n = h \circ h_n^{-1}$. Clearly k_n is a homeomorphism of g_n onto g where g is the image of g_α under the homeomorphism h.

By the same argument as in the proofs of Theorem 3.3, it is not difficult to show that the sequence $\{g_n = h_n(g_\alpha)\}$ of distinct points of G converges to g homeomorphically.

Let $G' = \{g_r : g_r = h_r(g_\alpha)$ where $h_r \epsilon T\}$. Then $G' \subset G$ and G' is uncountable since T is uncountable, furthermore $G^* = G - G' = \{g_r : g_r = h_r(g_\alpha)$ where $h_r \epsilon H - T\}$ is countable since H-T is countable.

Since for each $g \epsilon G'$ there is some sequence $\{g_n\}$ of distinct points of G converging to $g = h(g_\alpha)$ homeomorphically, G' is uncountable and $G - G' = G^*$ is countable, the theorem follows.

Theorem 3.6: If G is an uncountable collection of compact sets in a metric space S, then, except for at most a countable number of elements of G, each element of G is the limit of some convergent sequence from G.

Proof: Let $G = \{g_\alpha : \alpha \epsilon I$, I is uncountable$\}$. By Theorem 2.2 for each $\alpha \epsilon I$, there exist a continuous map f_α of a cantor set C onto g_α.

Let $\pi = \{\theta : \theta$ is a continuous map fromC into S$\}$. It follows from Theorem 3.2 that (π, d) is a separable metric space.

If $F = \{f_\alpha : \alpha \epsilon I\}$, then F is an uncountable subset of π. Theorem 2.4 implies that F is separable; hence every uncountable subset of F has a limit point. It follows from Theorem 2.6 that there exist an uncountable subset T of F such that F-T is countable and every open set that intersects T contains uncountable many points of T.

Let $f \epsilon T$. Clearly f is a limit point of T and also a limit point of F.

Therefore some sequence $\{f_n\}$ of distinct points of F converges to f. By the same argument as in the proof of Theorem 3.4, it is not difficult to show that the sequence $\{g_n = f_n(C)\}$ converge to $g = f(C)$

Thus for each $g \in G' = \{g_r : g_r = f_r(C)$ where $f_r \in T\}$ there is some sequence $\{g_n = f_n(C)\}$ of distinct points of G which converges to $g=f(C)$. Thus the Theorem follows since G' is uncountable and $G-G' = \{g_r : g_r = f_r(C)$ where $f_r \in F-T\}$ is countable.

IV. APPLICATIONS TO CONTINUA IN E^2

It has been known [5] for many years that there do not exist uncountably many pairwise disjoint triods in E^2. We indicate below how to establish this result using Theorem 3.3. An interesting consequence of the theorem on triods is Corollary 4.3 which has proven useful in resent research in the topology of E^3. (see Theorem 5.4).

We prove (Theorem 4.2) that in an uncountable collection G of disjoint simple closed curves in the plane not only must many elements of G be limits of sequences from G coverging homeomorphically, but G contains a countable collection G' such that each element of G-G' is converged to homeomophically from both sides by elements of G.

It follows from Theorem 3.3 that in any collection G of pairwise disjoint continua in E^2 some sequence of G must converge homeomorphically to an element of G. We were unable to establish the converse — if M is a continuum such that there exists a sequence $\{M_i\}$ of pairwise disjoint continua in E^2-M converging homeomorphically to M, then must there exist uncountably many disjoint copies of M in E^2? We later discovered that this is an unsolved problem [4].

<u>Theorem 4.1:</u> There do not exist uncountably many mutually disjoint triods in E^2.

Note: We do not claim to have a rigorous proof for this theorem. A better proof can be constructed using the axioms of the topology of E^2 but such a proof could be constructed using the outline given below. For a rigorous proof see [5].

Proof: This proof is by contradiction. Suppose there exists a set A consisting of uncountably many mutually disjoint triods in E^2. Let A = $\{T_\alpha : \alpha \in I, \text{ I is uncountable}\}$. Clearly for arbitrary $T_\alpha, T_\beta \in A$, T_α and T_β are homeomorphic and compact. Hence A is an uncountable collection of mutually homeomorphic compact subsets of the separable metric space E^2. Then by Theorem 3.3 some sequence $\{T_i\}$ of distinct sets of A converges homeomorphically to some element T of A. For convenience we assume [7] T is the union of two perpendicular straight line segrements bc and ad where d is the point of intersection (see Figure 1).

Let $\epsilon = \min\{\rho(a, d), \rho(a, b), \rho(d, i)\}$

$N_1 = N\rho(a, \frac{\epsilon}{4})$; $N_2 = N\rho(b, \frac{\epsilon}{4})$;

$N_3 = N\rho(c, \frac{\epsilon}{4})$ and $N = N\rho(d, \frac{\epsilon}{4})$.

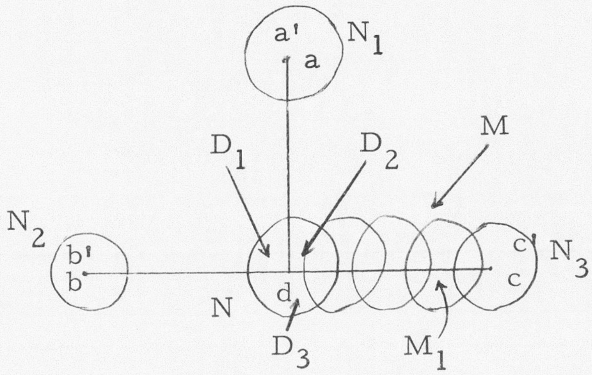

Figure 1

Then there exists an integer K such that if $n>K$, there is a homeomorphism $h_n: T_n \longrightarrow T$ such that $d(x, h(x)) < \frac{\epsilon}{4}$ for all x in T_n. Let i be an integer larger than K, and let T_i consist of the three arcs $a'd'$, $b'd'$ and $c'd'$ where each pair intersects only at d'. Then there exist a homeomorphism $h_i: T_i \longrightarrow T$ such that $d(x, h(x)) < \frac{\epsilon}{4}$ for all x in T_i.

Since h_i is a homeomorphism, it maps end points of T_i onto the end points of T. We assume without loss in generality that $h_i(a') = a$, $h_i(b')=b$ $h_i(c') = c$ and $h_i(d') = d$, then clearly $a' \epsilon N_1$, $b' \epsilon N_2$, $c' \epsilon N_3$ and $d' \epsilon N$.

Obviously, $ad \cup db \cup cd$ divides N-T into three disjoint open sets D_1, D_2 and D_3 as labeled in the picture. By hypothesis T and T_1 are disjoint so that $d' \notin T$.

Supposed $d' \epsilon D_1$. Let $M = \cup\{N\rho(x, \frac{\epsilon}{4}) : x \epsilon dc\} - T$.

Since h_i is a homeomorphism, $h_i(c') = c$, $h_i(d') = d$, $h_i(d'c') = dc$, and furthermore $d(x, h_i(x)) < \frac{\epsilon}{4}$ for all x in the arc $d'c'$. From this and the fact that $d'c' \cap T = \phi$ we see that $d'c' \subset M$, since $x \epsilon N\rho(h_i(x), \frac{\epsilon}{4})$ and $h_i(x) \epsilon dc$. Let $M_1 = M - D_1$, then it is obvious that D_1 and M_1 are two mutually separated sets in M, $d' \epsilon D_1$, and $c' \epsilon M_1$.

But this is impossible since $d'c'$ is a connected subset of $D_1 \cup M_1$

such that $d'c'$ intersects both D_1 and M. This contradiction shows that $d' \notin D_1$.

By a similar argument, we see that $d' \notin D_2$ and $d' \notin D_3$.

Thus, $d' \notin N-T$. This is clearly impossible since $d' \in N=(N \cap T) \cup D_1 \cup D_2 \cup D_3$ and $d' \notin T$.

Corollary 4.1: If a continuum M in E^2 contains a triod, then there do not exist uncountably many disjoint copies of M in E^2.

Proof: This result follows directly from Theorem 4.1.

Corollary 4.2: If a nondegenerate continuous curve M is neither an arc nor a simple closed curve, then there do not exist uncountable many copies of M in the plane.

Proof: If a nondegenerate continuous curve is neither an arc nor a simple closed curve it must contain a triod (see Theorem 2.8). Thus Corollary 4.2 follows from Corollary 4.1 above.

Corollary 4.3: If G is an uncountable set of mutually disjoint nondegenerate continuous curves in E^2, then all but countably many curves of G are either arcs or simple closed curves.

Definition: Let J be a simple closed curve in E^2. The interior of J, denoted by $I(J)$, is the bounded component of E^2-J. The exterior $E(J)$ is $E^2-(J \cup I(J))$.

Definition: A simple closed curve J will have property Q_i relative to an uncountable set G of simple closed curve if there exists a simple closed curve $J_o \in G$ such that

(1) $J_o \subset E(J)$

(2) $J \subset I(J_o)$

(3) There exist a homeomorphism f such that $f(J_o)=J$ and $\rho(f(x), x) < \frac{1}{i}$ for all $x \in J_o$.

Furthermore a set G' of simple closed curves will have property Q_i relative to an uncountable set G of simple closed curves if for every element $J \in G'$, J has property Q_i

relative to G. A simple closed curve J is said to have property P relative to G if J has the properties as in Theorem 4.2 which follows. Furthermore a subset G_o of G is said to have property P relative to G if for each $J \epsilon G_o$ J has property P relatives to G.

Theorem 4.2: If G is an uncountable collection of disjoint simple closed curves $\{J_\alpha\}$ in E^2, then there exists an element J of G and two sequences $\{J_i\}$ and $\{J_i\}$ of elements of G such that:

(1) $\{J_i\}$ converges homeomorphically to J,

(2) $\{J_i'\}$ converges homeomorphically to J,

(3) $J_i \subset I(J)$ for each i, and

(4) $J_i' \subset E(J)$ for each i.

Proof: As in previous proofs it can be shown that there is an uncountable subset G' of G such that G-G' is countable and for each $g \epsilon G'$ there exists a sequence $\{g_i\}$ from G converging homeomorphically to g.

Since for every element J in G' there exists a sequence that converges to J homeomorphically, we can let $G' = G^I \cup G^E$ where $G^I = \{J$: there exists a sequence $\{J_i\}$ converges to J such that $J_i \epsilon I(J)\}$ and $G^E = \{J$: there exists a sequence $\{J_i\}$ converges to J such that $J_i \epsilon E(J)\}$.

Since G' is uncountable, one of G^I and G^E must be uncountable. We assume without loss in generality that G^I is uncountable.

Let $J_o \epsilon G^I$; and let $\{J_i^I\}$ be a sequence of elements of G^I converging to J_o such that $J_i^I \subset I(J_o)$ for each i. Then $J_o \subset E(J_i^I)$ for each i. For each i, let f_i be a homeomorphism of J_i^I onto J_o satisfying definition of homeomorphic convergence.

There exists an integer N_1 such that $i > N_1$ implies f_i moves no point of J_i^I more than a distance 1. Hence $i > N_1$ implies J_i^I has property Q_1

relative to G^I. Then it follows from Theorem 2.9 and the argument above that there exists at most a countable subset of G^I that doesn't have property Q_1 relative to G^I.

Let G_1' be the set of all elements J of G^I such that J doesn't have property Q_1 relative to G^I. Then G_1' is countable and $G_1 = G^I - G_1'$ is uncountable.

Similarly we examine G_1 relative to property Q_2. There is a subset G_2 of G_1 such that G_2 is uncountable $G_1 - G_2$ is countable, and each element of G_2 has property Q_2 relative to G_1. For each i, define an uncountable collection G_i of simple closed curves such that (1) G_i has property Q_i relative to G_{i-1}, (2) $G_i \subset G_{i-1}$, (3) $G_i - G_{i-1}$ is countable.

Let $G_o = \bigcap_{i=1}^{\infty} G_i$. Since we take out at most a countable number of elements from G_i at each stage, it is obvious that $G_o = G_1 - [G_2' \cup G_3' \cup \ldots]$ is not empty and, moreover, it is uncountable.

For every $J \in G_o$, $J \in G^I$ and J has property Q_i relative to G for every i. It is not difficult to show that J has property P relative to G. Therefore the uncountable subset G_o has property P relative to G.

Remark: In the proof of above theorem, if G^E is also uncountable then we can use the same technique to prove that there exist an uncountable subset G_o' of G^E such that G_o' has property P relative to G and $G^E - G_o'$ is countable. In case G^E is countable, let $G_o' = \phi$. Then the set $G_o \cup G_o'$ is uncountable and has property P relative to G. Furthermore $G - (G_o \cup G_o')$ is countable, so we can state a stronger theorem than Theorem 4.2 as follows:

Theorem 4.3: If $\{J_\alpha\}$ is an uncountable collection G of disjoint simple closed curves in E^2, then there exist a uncountable subset G_o of G such that G_o has property P relative to G and $G - G_o$ is countable.

V. AN APPLICATION TO TAME 2-SPHERES IN E^3

We indicate here one more application of our main theorem. A 2-sphere S in E^3 is said to be tamely embedded in E^3 (or tame in E^3) if and only if there is a homeomorphism h of E^3 onto itself such that h(S) is a round sphere (S^2). A 2-sphere that is not tame is called wild. Wild 2-spheres are known to exist in E^3.

Bing [2] has given a characterization of tame 2-spheres which we state below as Theorem 5.1. The proof is apparently difficult and has not been studies by the author. Based on Theorem 5.1 we show that each collection of disjoint wild 2-spheres is at most countable.

Theorem 5.1: (Bing[2]) A 2-sphere S in E^3 is tame in E^3 if for each component V of E^3-S there exists a sequence $\{S_i\}$ of 2-sphere in V converging homeomorphically to S.

Note: Since there are only two components Int S and Ext S of E^3-S the same technique as given in the proof of Theorem 4.2 can be used to prove the following theorem.

Theorem 5.2: If G is an uncountable collection of disjoint 2-spheres in E^3, then there exists an uncountable subset G_o of G such that for every $S \epsilon G_o$ there exist two sequence $\{S_i\}$ and $\{S'_i\}$ of elements of G such that :

(1) $\{S_i\}$ and $\{S_i\}$ both converges homeomorphically to S,

(2) $S_i \subset$ Int S, and $S'_i \subset$ Ext S.

Furthermore $G-G_o$ is countable.

Remark: Suppose there exist an uncountably collection G of disjoint wild 2-spheres in E^3. Then it follows from Theorem 5.2 and Theorem 5.1 that uncountably many 2-sphere in G are tame. This contradiction gives us the following:

<u>Theorem 5.3:</u> There do not exist uncountably many disjoint wild 2-sphere in E^3.

<u>Remark:</u> In E^3, let $P = \{p_t : t \in [-1, 1]\}$ be the collection of all planes that are parallel to xy-plane and let S be a 2-sphere contained in the union of the planes from $Z = 1$ to $Z = -1$. Suppose for every $p_t \in P$, $M_t = p_t \cap S$ is a locally connected continuum. Then it is obvious that for each $t \in (-1, 1)$ M_t is neither a point, nor an arc. Let $G = \{M_t : M_t$ is not a simple closed curve and $t \neq -1, 1\}$. Suppose G is uncountable. Then it follows from Theorem 2.8 that for each $M_t \in G$, there is a triod $T_t \in M_t$. Let $H = \{T_t : T_t \subset M_t,\ M_t \in G\}$. Then H is an uncountable collection of pairwise disjoint triods in S. If we remove one point p from the intersection of $z=1$ with S, then $S - \{p\}$ is homeomorphic to the plane E^2, but this contradicts the fact that there doesn't exist uncountable many triods in E^2. This contradiction gives us the following:

<u>Theorem 5.4:</u> In E^3, let $P = \{p_t : t \in [-1, 1]\}$ be the collection of all planes that are parallel to xy-plane and let S be a 2-sphere that is contained in $\cup p_t$. If for every $p_t \in P$ $M_t = p_t \cap S$ is a locally connected continuum, then all but countable number of M_t are simple closed curves.

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