A Theorem on Homeomorphic Convergence and Some Applications

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A THEOREM ON HOMEOMORPHIC CONVERGENCE
AND SOME APPLICATIONS

by
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TABLE OF CONTENTS

I. INTRODUCTION .................................................. 1
II. DEFINITIONS, NOTATION, AND
    PRELIMINARY RESULTS ...................................... 2
III. A THEOREM ON HOMEOMORPHIC CONVERGENCE ........... 5
IV. APPLICATIONS TO CONTINUA IN $E^2$ ...................... 11
V. AN APPLICATION TO TAME 2-SPHERES IN $E^3$ ............ 16
REFERENCES ..................................................... 18
VITA .............................................................. 19
I. INTRODUCTION

Borsuk [3] has given interesting conditions under which a certain function space is separable (see Theorem 3.1). We give a proof for Borsuk's Theorem here and we show how it can be used to establish a useful theorem on homeomorphic convergence. We illustrate the utility of the theorem on homeomorphic convergence by stating and proving several of its consequences.

For example we show that the plane \((E^2)\) does not contain uncountably many pairwise disjoint continua each of which contains a simple triod (Corollary 4.1). We prove that in an uncountable collection \(G\) of pairwise disjoint simple closed curves in \(E^2\) "almost all" elements of \(G\) must be converged to homeomorphically "from both sides" by sequences of elements of \(G\) (see Theorem 4.3). The same technique allows us to prove the nonexistence of uncountably many pairwise disjoint wild 2-spheres in \(E^3\).

Another interesting consequence of Borsuk's Theorem is Theorem 3.4 which shows that in each set \(G\) consisting of uncountably many compact subsets of a metric space, some element of \(G\) is an element of convergence. Proofs for this theorem do not often appear in the literature, and, as far as the author knows, the proof given here does not appear in the literature.

We wish to emphasize that all the proofs given in this report were constructed by the author without reference to the literature, in fact the author was unaware of the references until after the proofs were given. We given reference at the end of the paper where proofs in the literature can be compared with the proofs given here.

It is assumed throughout this paper that the reader is familiar with the basic concepts in topology. Some theorems that are used in support of this paper, but are not directly part of it, will be stated without proof; however, the proofs can be found in the literature.
II. DEFINITIONS, NOTATION, AND PRELIMINARY RESULTS

Definition: Let $P$ be a metric space, and let $Q$ be a separable metric space, the function space $\pi$ is defined as the set of all continuous functions defined on $P$ and having values in $Q$. The metric topology for $\pi$ is given in Theorem 3.1 to follow.

Notation: Let $(X, d)$ be a metric space, for $x \in X$ and $\epsilon > 0$, $N_d(x, \epsilon) = \{ y \in X : d(x, y) < \epsilon \}$ denotes as the $\epsilon$-neighborhood of $x$ in $(X, d)$.

Definition: A sequence $M_1, M_2, \ldots$ of point sets in a metric space converges homeomorphically to a point set $M$ if, for each positive number $\epsilon$, there exists a positive integer $K$ such that if $n > K$ there exists a homeomorphism $h_n$ from $M_n$ onto $M$ such that $d(x, h_n(x)) < \epsilon$ for all $x \in M_n$.

Definition: Let $(X_n)$ be a sequence of point sets in a space $S$. The set of all points $x$ in $S$ such that every open set containing $x$ intersects all but a finite number of the sets $X_n$ is called the limit inferior of $(X_n)$ and is abbreviated "$\lim\inf X_n$" or "$\underline{\lim} X_n$." The set of all points $y$ in $S$ such that every open set containing $y$ intersects infinitely many sets $X_n$ is called the limit superior of $(X_n)$ and is abbreviated "$\lim\sup X_n$" or "$\overline{\lim} X_n$".

Definition: A sequence $(X_n)$ of sets is said to converge to a set $L$ (abbreviated "$\lim X_n=L$") if $\lim\sup X_n=L=\lim\inf X_n+\phi$.

Definition: A continuum is a compact connected metric space.

Definition: A locally connected continuum is sometimes called a continuous curve or Peano continuum.

Definition: A simple triod is a continuum that is the union of three arcs $AD$, $BD$ and $CD$ such that $D$ is the intersection of each two of them. Sometimes we refer to a simple triod as a triod.
Definition: $E^n$ is defined as the Euclidean $n$-dimensional space.

Definition: For each positive integer $n$ let $S^n = \{(x_1, x_2, \ldots, x_{n+1}) \in E^{n+1} \mid x_1^2 + x_2^2 + \cdots + x_{n+1}^2 = 1\}$. A set $S$ is called an $n$-sphere if $S$ is homeomorphic to $S^n$. If $S$ is a 2-sphere in $E^3$, we denote the bounded component of $E^3 - S$ by $\text{Int} \ S$ and the other component by $\text{Ext} \ S$. 
PRELIMINARY RESULTS

We state here several well known theorems. No proofs are given here, but proofs can be found in some advanced topology texts.

**Theorem 2.1:** A metric space $S$ is separable if and only if every uncountable subset of $S$ has a limit point.

**Theorem 2.2:** A metric space is compact if and only if it is the continuous image of a Cantor set.

**Theorem 2.3:** Every compact metric space is separable.

**Theorem 2.4:** Every subset of a separable metric space is separable.

**Theorem 2.5:** In a completely separable space, every uncountable subset contains uncountably many limit points of itself.

**Theorem 2.6:** In each uncountable separable metric space $H$ there exists an uncountable subset $T$ of $H$ such that $H-T$ is countable and every open set that interests $T$ contains uncountably many points of $T.$

**Theorem 2.7:** In a metric space $S,$ if $p$ is a limit point of the set $H \subseteq S,$ then there exists a sequence of distinct points of $H$ converging to $p.$

**Theorem 2.8:** If a continuous curve $M$ contains no triod, then $M$ is either an arc, a point or a simple closed curve.

**Theorem 2.9:** Let $S$ be a space having a countable basis. If $G$ is a collection of open sets covering a point set $H \subseteq S,$ then some countable subcollection of $G$ covers $H.$
III. A THEOREM ON HOMEOMORPHIC CONVERGENCE

In this section we first identify a metric for the set $\pi$, then we state and prove an interesting theorem due to Borsuk [3]. Theorem 3.3 is the central theorem of the paper. Apparently Burgess [4] first proved this theorem, although others seemed to know of the theorem in certain special cases (see [5] for example).

Theorem 3.4 is also an interesting consequence of Borsuk's theorem. It shows what conclusions one is able to draw when the disjoint compact sets are not mutually homeomorphic as in Theorem 3.3. Theorem 3.4 was probably first done by R. L. Moore [6] and doesn't appear often in the usual topology texts.

Theorem 3.5 and 3.6 are more general statements of the results in Theorem 3.3 and 3.4, respectively.

**Theorem 3.1:** The set $S$ of all bounded functions from a compact metric space $P$ into a separable metric space $(Q, \rho_Q)$ forms a metric space under the metric $d$ defined as follows: For $f_1, f_2 \in S$, $d(f_1, f_2) = 1$ u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) : x \in P \right\}$.

**Proof:** Suppose $d(f_1, f_2) = 0$. Then 1 u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) : x \in P \right\} = 0$, so it follows that $\rho_Q(f_1(x), f_2(x)) = 0$ for all $x \in P$. Thus $f_1(x) = f_2(x)$ for all $x \in P$, since $\rho_Q$ is a metric for $Q$ and therefore $f_1 = f_2$. The reverse of this argument shows that $f_1 = f_2$ implies $d(f_1, f_2) = 0$.

Since $\rho_Q$ is a metric for $Q$, $\rho_Q(f_1(x), f_2(x)) = \rho_Q(f_2(x), f_1(x))$ for all $x \in P$. Obviously 1 u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) : x \in P \right\}$, so $d(f_1, f_2) = d(f_2, f_1)$.

The triangle inequality follows from the following set of inequalities:

$$d(f_1, f_2) + d(f_2, f_3) = 1$$ u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) : x \in P \right\} + 1$ u. b. $\left\{ \rho_Q(f_2(x), f_3(x)) : x \in P \right\} = 1$ u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) + \rho_Q(f_2(y), f_3(y)) : x, y \in P \right\} \geq 1$ u. b. $\left\{ \rho_Q(f_1(x), f_2(x)) + \rho_Q(f_2(x), f_3(x)) : x \in P \right\} \geq 1$ u. b. $\left\{ \rho_Q(f_1(x), f_3(x)) : x \in P \right\} = d(f_1, f_3)$.

Therefore $d$ is a metric for $S$. 
Theorem 3.2: The metric space \((\pi, d)\) is separable if \(P\) is compact.

Proof: Since every compact metric space is separable by Theorem 2.3, \((P, \rho_P)\) is separable and thus has a countable basis \(U = \{U_i : i \in \mathbb{N}\}\).

Also there is a countable set \(A = \{a_i : i \in \mathbb{N}\}\) in \(Q\) such that \(A\) is dense in \((Q, \rho_Q)\).

Let \(\Sigma\) be the collection of all finite sets of positive integers \(\{n_1, n_2, \ldots, n_k\}\) such that \(P \subseteq U_{n_1} \cup U_{n_2} \cup \cdots \cup U_{n_k}\). Then \(\Sigma \neq \emptyset\) since \(P\) is compact and \(U\) is a cover for \(P\). Also \(\Sigma\) is countable.

For each \(s \in \Sigma\), \(s = \{n_1, n_2, \ldots, n_k\}\) we define the collection \(\{W^1_s, W^2_s, \ldots, W^k_s\}\) of disjoint sets as follows:

\[
W^1_s = U_{n_1}, \quad W^2_s = U_{n_2} - U_{n_1}, \\
\ldots, \quad W^k_s = U_{n_k} - \bigcup_{i=1}^{k-1} (U_{n_i}).
\]

Let \(\pi_s\) be the collection of all functions \(\phi\) such that \(\phi(\rho) \subseteq A\) and \(\phi(x)\) is constant over each \(W^i_s\). Then \(\pi_s\) is countable.

Let \(\pi = \bigcup_{s \in \Sigma} \pi_s\) it follows that \(\pi\) is a countable collection of bounded functions. Thus it follows from Theorem 3.1 that \(\pi' = \pi \cup \pi\) is a metric space. Furthermore from Theorem 2.4 it will follow that \(\pi\) is separable once we show \(\pi'\) is separable. We shall show that \(\pi\) is a countable dense subset of \(\pi'\).

Let \(x \in \pi\), we shall show the existence of a function \(\theta \in \pi\) such that \(\theta\) is within \(\epsilon\) of \(f\) for an arbitrarily small positive number \(\epsilon\).

For each \(x \in P\), let \(G_x\) be the set of all \(y \in P\) such that \(\rho_Q(f(x), f(y)) < \frac{\epsilon}{3}\).

Then \(G_x\) is open in \(P\), hence there exists an integer \(n(x)\) such that \(x \in U_{n(x)} \subseteq G_x\).

Then there is a finite collection \(r = \{n(x_1), n(x_2), \ldots, n(x_t)\}\) of integers such that \(p \subseteq U_{n(x_1)} \cup \cdots \cup U_{n(x_t)}\). Thus \(r \subseteq \Sigma\).

Since \(A\) is dense in \(Q\), for each \(i\) there must be a point \(a_i \in A\) such that \(\rho_Q(f(x_i), a_i) < \frac{\epsilon}{3}\). By definition of \(\pi\), there exist a function \(\theta\) in \(\pi\) such that \(\theta(x) = a_i\) for each \(x \in W^i_r\).

Let \(x_0 \in P\). Then \(x_0 \in W^i_r\) for some \(i\).

Since \(W^i_r \subseteq G_{x_i}\), then \(\rho_Q(f(x_0), f(x_i)) < \frac{\epsilon}{3}\). Also \(\rho_Q(f(x_0), \theta(x_0)) < \rho_Q(f(x_0), f(x_i)) + \rho_Q(f(x_i), \theta(x_0)) \leq \frac{\epsilon}{3} + \frac{\epsilon}{3} = \frac{2\epsilon}{3}\).

Hence for each \(x \in P\), \(\rho_Q(f(x), \theta(x)) < \frac{2\epsilon}{3}\). Thus \(d(f, \theta) < \epsilon\) and \(f \in \pi\).
Theorem 3.3: If $G$ is an uncountable collection of mutually homeomorphic compact subsets of a separable metric space $(S, \rho)$, then some sequence of distinct sets of $G$ converges homeomorphically to some element of $G$.

Proof: Let $G = \{g_\alpha : \alpha \in I, I \text{ is uncountable}\}$.

Let $g_\alpha \in G$, and for each $\beta \in I$ let $h_\beta$ be a homeomorphism of $g_\alpha$ onto $g_\beta$.

Let $H = \{h_\beta : \beta \in I\}$. Since $G$ is uncountable, $H$ must be uncountable.

Let $F$ be the set of all continuous maps from $g_\alpha$ into $S$. Then it follows from Theorem 3.2 that $(F, d)$ is a separable metric space. Now $H$ is an uncountable subset of $F$; hence $H$ has a limit point $h \in H$ by Theorem 2.1. Therefore some sequence $\{h_n\}$ of distinct points of $H$ converges to $h$. Define $k_n = h \circ h_n^{-1}$. Obviously $k_n$ is a homeomorphism of $g_n$ onto $g$ when $g$ is the image of $g_\alpha$ under the homeomorphism $h$. Consider the sequence $\{g_n\}$ of distinct points of $G$.

Let $x_0 \in g_n$ then there exists an element $y_0$ of $g_\alpha$, such that $x_0 = h_n(y_0)$ and it follows that $\rho(k_n(x_0), x_0) = \rho(h \circ h_n^{-1}(x_0), x_0) = \rho(h \circ h_n^{-1}(h_n(y_0)), h_n(y_0)) = \rho(h(y_0), h_n(y_0))$.

Since $\{h_n\}$ converges to $h$, for each positive number $\epsilon$ there exists a positive integer $K$, such that if $n > K$, $\rho(h(y), h_n(y)) < \epsilon$, for each $y \in g_\alpha$. That is, $\rho(k_n(x_0), x_0) < \epsilon$ if $n > K$. Hence for each $x \in g_\alpha$, $\rho(k_n(x), x) < \epsilon$ if $n > K$. Thus $\{g_n\}$ converges to $g$ homeomorphically.

Theorem 3.4: If $G$ is an uncountable collection of compact sets in a separable metric space $S$, then some sequence of elements of $G$ converges to an element of $G$.

Proof: Let $G = \{g_\alpha : \alpha \in I, I \text{ is uncountable}\}$. By Theorem 2.2 for each $\alpha \in I$, there exist a continuous map $f_\alpha$ of a Cantor set $C$ onto $g_\alpha$.

Let $\pi = \{0:0 \text{ is a continuous map from } C \text{ into } S\}$. It follows from Theorem 3.2 that $(\pi, d)$ is a separable metric space. If $F = \{f_\alpha : \alpha \in I\}$, then $F$ is an uncountable subset of $\pi$; so $F$ has a limit point $f_\pi \in F$ by Theorem 2.1. Of course, then some sequence $\{f_n\}$ of distinct points of $F$ converges to $f_\pi$. Now consider the sequence $\{g_n\}$ and the set $g_\pi \in G$. 
Let \( y_0 \in g_T \). Then there exist an element \( x_0 \) of \( G \) such that \( f_T(x_0) = y_0 \). Since \( \{|f_n| \} \) converges to \( f_T \), it follows that for each positive number \( \epsilon \) there exists an integer \( K \) such that \( f_n(\epsilon, x_0) N|f_T, \epsilon \) if \( n > K \).

Let \( n > K \). Then \( \rho(f_n(x_0), y_0) = \rho(f_n(x_0), f_T(x_0)) < \epsilon \); that is, \( f_n(x_0) \in N\rho(y_o, \epsilon) \cap g_n \neq \phi \), and it follows from the definition of limit inferior that \( y_0 \in \lim \inf g_n \). Since \( y_0 \) was an arbitrary point of \( g_T \), it follows that \( g_T \subseteq \lim \inf g_n \).

Assume \( \lim \sup g_n \subseteq g_T \), since \( \lim \inf g_n \subseteq \lim \sup g_n \), it follows that \( \lim \sup g_n \neq \phi \), so there exists a point \( z \in (\lim \sup g_n) - g_T \). Since \( g_T \) is closed, \( \rho(z, x) : x \in g_T \neq \phi \), so we can choose a number \( \epsilon \) such that \( 0 < \epsilon < \inf \{ \rho(z, x) : x \in g_T \} \). Let \( H = \bigcup \{ N\rho(x, \frac{\epsilon}{4}) : x \in g_T \} \). Since \( \{|f_n| \} \) converges to \( f_T \), for this \( \frac{\epsilon}{4} \) there exist an integer \( K_1 \) such that if \( n > K_1 \), \( f_n(\epsilon, x_0) N|f_T, \epsilon \); that is, \( \rho(f_n(x), y) = \rho(f_n(x), f_T(x)) < \frac{\epsilon}{4} \) for all \( y = f_T(x) \in g_T \) where \( x \in G \). Therefore \( g_n \subseteq H \) for all \( n > K_1 \). Clearly \( N\rho(z, \frac{\epsilon}{4}) \cap H = \phi \). Therefore if \( n > K_1 \), \( N\rho(z, \frac{\epsilon}{4}) \cap g_n = \phi \). It follows that \( N\rho(z, \frac{\epsilon}{4}) \) intersects at most a finite number of elements of \( \{ g_n \} \). This implies \( z \in (\lim \sup g_n) \) thus \( z \in (\lim \sup g_n) - g_T \). But this contradicts our assumption.

Therefore \( (\lim \sup g_n) \subseteq g_T \). Since \( (\lim \sup g_n) \subseteq g_T \subseteq (\lim \inf g_n) \), and \( (\lim \inf g_n) \subseteq (\lim \sup g_n) \), it follows that \( (\lim \inf g_n) = (\lim \sup g_n) = g_T \). Hence \( \{|g_n| \} \) converges to \( g_T \).

Theorem 3.5: If \( G \) is an uncountable collection of mutually homeomorphic compact subsets of a separable metric space \( (S, \rho) \), then there is a countable subset \( G^* \) of \( G \) such that if \( g \in G - G^* \) then some sequence of elements of \( G \) converges homeomorphically to \( g \).

Proof: Let \( G = \{ g_a : a \in I, I \text{ is uncountable} \} \). Let \( g_a \in G \), and for each \( \beta \in I \) let \( h_\beta \) be a homeomorphism of \( g_a \) onto \( g_\beta \). Let \( H = \{ h_\beta : \beta \in I \} \). Since \( G \) is uncountable, \( H \) must be uncountable. Let \( F \) be the set of all continuous maps from \( g_a \) into \( S \). Then it follows from Theorem 3.2 that \( (F, d) \) is a separable metric space. Now \( H \) is an uncountable subset
of $F$. and by Theorem 2.4 $H$ is separable, hence every uncountable subset of $H$ has a limit point and it will follow from Theorem 2.6 that there exists an uncountable subset $T$ of $H$ such that $H-T$ is countable and every open set that intersects $T$ contains uncountably many points of $T$.

Let $h \in T$. Clearly $h$ is limit point of $T$ and also a limit point of $H$. Therefore some sequence $\{h_n\}$ of distinct points of $T$ converges to $h$ by Theorem 2.7. Define $k_n = h_n^{-1} h$. Clearly $k_n$ is a homeomorphism of $g_n$ onto $g$ where $g$ is the image of $g_a$ under the homeomorphism $h$.

By the same argument as in the proofs of Theorem 3.3, it is not difficult to show that the sequence $\{g_n = h_n(g_a)\}$ of distinct points of $G$ converges to $g$ homeomorphically.

Let $G^t = \{g^t : g^t = h_r(g_a) \text{ where } h_r \in T\}$. Then $G^t \subset G$ and $G^t$ is uncountable since $T$ is uncountable, furthermore $G^t = G - G^t = \{g^t : g^t = h_r(g_a) \text{ where } h_r \in H - T\}$ is countable since $H - T$ is countable.

Since for each $g \in G^t$ there is some sequence $\{g_n\}$ of distinct points of $G$ converging to $g = h(g_a)$ homeomorphically, $G^t$ is uncountable and $G - G^t = G^s$ is countable, the theorem follows.

**Theorem 3.6:** If $G$ is an uncountable collection of compact sets in a metric space $S$, then, except for at most a countable number of elements of $G$, each element of $G$ is the limit of some convergent sequence from $G$.

**Proof:** Let $G = \{g_a : a \in I, I \text{ is uncountable}\}$. By Theorem 2.2 for each $a \in I$, there exist a continuous map $f_a$ of a cantor set $C$ onto $g_a$.

Let $\pi = \{0 : \theta \text{ is a continuous map from } G \text{ into } S\}$. It follows from Theorem 3.2 that $(\pi, d)$ is a separable metric space.

If $F = \{f_a : a \in I\}$, then $F$ is an uncountable subset of $\pi$. Theorem 2.4 implies that $F$ is separable; hence every uncountable subset of $F$ has a limit point. It follows from Theorem 2.6 that there exist an uncountable subset $T$ of $F$ such that $F - T$ is countable and every open set that intersects $T$ contains uncountable many points of $T$.

Let $f \in T$. Clearly $f$ is a limit point of $T$ and also a limit point of $F$. 
Therefore some sequence \( \{ f_n \} \) of distinct points of \( F \) converges to \( f \).

By the same argument as in the proof of Theorem 3.4, it is not difficult to show that the sequence \( \{ g_n = f_n(C) \} \) converge to \( g = f(C) \).

Thus for each \( g \in G' = \{ g_r : g_r = f_r(C) \text{ where } f_r \in T \} \) there is some sequence \( \{ g_n = f_n(C) \} \) of distinct points of \( G \) which converges to \( g = f(C) \). Thus the Theorem follows since \( G' \) is uncountable and \( G - G' = \{ g_r : g_r = f_r(C) \text{ where } f_r \in F - T \} \) is countable.
IV. APPLICATIONS TO CONTINUOUS IN $E^2$

It has been known [5] for many years that there do not exist uncountably many pairwise disjoint triods in $E^2$. We indicate below how to establish this result using Theorem 3.3. An interesting consequence of the theorem on triods is Corollary 4.3 which has proven useful in resent research in the topology of $E^3$. (see Theorem 5.4).

We prove (Theorem 4.2) that in an uncountable collection $G$ of disjoint simple closed curves in the plane not only must many elements of $G$ be limits of sequences from $G$ converging homeomorphically, but $G$ contains a countable collection $G'$ such that each element of $G - G'$ is converged to homeomorphically from both sides by elements of $G$.

It follows from Theorem 3.3 that in any collection $G$ of pairwise disjoint continua in $E^2$ some sequence of $G$ must converge homeomorphically to an element of $G$. We were unable to establish the converse — if $M$ is a continuum such that there exists a sequence $\{M_i\}$ of pairwise disjoint continua in $E^2-M$ converging homeomorphically to $M$, then must there exist uncountably many disjoint copies of $M$ in $E^2$? We later discovered that this is an unsolved problem [4].

Theorem 4.1: There do not exist uncountably many mutually disjoint triods in $E^2$.

Note: We do not claim to have a rigorous proof for this theorem. A better proof can be constructed using the axioms of the topology of $E^2$ but such a proof could be constructed using the outline given below. For a rigorous proof see [5].

Proof: This proof is by contradiction. Suppose there exists a set $A$ consisting of uncountably many mutually disjoint triods in $E^2$. Let $A = \{ T_\alpha : \alpha \in I, I \text{ is uncountable} \}$. Clearly for arbitrary $T_\alpha, T_\beta \in A, T_\alpha$ and $T_\beta$ are homeomorphic and compact. Hence $A$ is an uncountable collection of mutually homeomorphic compact subsets of the separable metric space $E^2$. Then by Theorem 3.3 some sequence $\{T_i\}$ of distinct sets
of $A$ converges homeomorphically to some element $T$ of $A$. For convenience we assume \[ T \] is the union of two perpendicular straight line segments $bc$ and $ad$ where $d$ is the point of intersection (see Figure 1).

Let $\epsilon = \min\{\rho(a,d), \rho(a,b), \rho(d,i)\}$

\[
N_1 = N_\rho(a, \frac{\epsilon}{4}); N_2 = N_\rho(b, \frac{\epsilon}{4}); N_3 = N_\rho(c, \frac{\epsilon}{4}) \text{ and } N = N_\rho(d, \frac{\epsilon}{4}).
\]

Then there exists an integer $K$ such that if $n > K$, there is a homeomorphism $h_n: T_n \rightarrow T$ such that $d(x, h(x)) < \frac{\epsilon}{4}$ for all $x$ in $T_n$. Let $i$ be an integer larger than $K$, and let $T_i$ consist of the three arcs $a'd'i$, $b'd'i$ and $c'd'i$ where each pair intersects only at $d'i$. Then there exist a homeomorphism $h_i: T_i \rightarrow T$ such that $d(x, h(x)) < \frac{\epsilon}{4}$ for all $x$ in $T_i$.

Since $h_i$ is a homeomorphism, it maps end points of $T_i$ onto the end points of $T$. We assume without loss in generality that $h_i(a^i) = a$, $h_i(b^i) = b$, $h_i(c^i) = c$ and $h_i(d^i) = d$, then clearly $a^i \epsilon N_1$, $b^i \epsilon N_2$, $c^i \epsilon N_3$ and $d^i \epsilon N$.

Obviously, $ad \cup b \cup c$ divides $N-T$ into three disjoint open sets $D_1$, $D_2$ and $D_3$ as labeled in the picture. By hypothesis $T$ and $T_1$ are disjoint so that $d^i \epsilon T$.

Supposed $d^i \epsilon D_1$. Let $M = \bigcup\{N_\rho(x, \frac{\epsilon}{4}): x \in dc\} - T$.

Since $h_i$ is a homeomorphism, $h_i(c^i) = c$, $h_i(d^i) = d$, $h_i(d^i c^i) = dc$, and furthermore $d(x, h_i(x)) < \frac{\epsilon}{4}$ for all $x$ in the arc $d^i c^i$. From this and the fact that $d^i c^i \cap T = \phi$ we see that $d^i c^i \subset M$, since $x \in N_\rho(h_i(x), \frac{\epsilon}{4})$ and $h_i(x) \in dc$. Let $M_1 = M - D_1$, then it is obvious that $D_1$ and $M_1$ are two mutually separated sets in $M$, $d^i \epsilon D_1$, and $c^i \epsilon M_1$.

But this is impossible since $d^i c^i$ is a connected subset of $D_1 \cup M_1$. 
such that \( d'c' \) intersects both \( D_1 \) and \( M \). This contradiction shows that \( d' \notin D_1 \).

By a similar argument, we see that \( d' \notin D_2 \) and \( d' \notin D_3 \).

Thus, \( d' \notin N-T \). This is clearly impossible since \( d' \notin (N \cap T) \cup D_1 \cup D_2 \cup D_3 \) and \( d' \notin T \).

**Corollary 4.1:** If a continuum \( M \) in \( E^2 \) contains a triod, then there do not exist uncountably many disjoint copies of \( M \) in \( E^2 \).

**Proof:** This result follows directly from Theorem 4.1.

**Corollary 4.2:** If a nondegenerate continuous curve \( M \) is neither an arc nor a simple closed curve, then there do not exist uncountably many copies of \( M \) in the plane.

**Proof:** If a nondegenerate continuous curve is neither an arc nor a simple closed curve it must contain a triod (see Theorem 2.8). Thus Corollary 4.2 follows from Corollary 4.1 above.

**Corollary 4.3:** If \( G \) is an uncountable set of mutually disjoint nondegenerate continuous curves in \( E^2 \), then all but countably many curves of \( G \) are either arcs or simple closed curves.

**Definition:** Let \( J \) be a simple closed curve in \( E^2 \). The interior of \( J \), denoted by \( I(J) \), is the bounded component of \( E^2 - J \). The exterior \( E(J) \) is \( E^2 - (J \cup I(J)) \).

**Definition:** A simple closed curve \( J \) will have property \( Q_4 \) relative to an uncountable set \( G \) of simple closed curve if there exists a simple closed curve \( J_0 \in G \) such that

1. \( J_0 \subseteq E(J) \)
2. \( J \subset I(J_0) \)
3. There exist a homeomorphism \( f \) such that \( f(J_0) = J \) and \( \rho(f(x), x) < \frac{1}{i} \) for all \( x \in J_0 \).

Furthermore a set \( G' \) of simple closed curves will have property \( Q_4 \) relative to an uncountable set \( G \) of simple closed curves if for every element \( J \in G' \), \( J \) has property \( Q_4 \)
relative to $G$. A simple closed curve $J$ is said to have property $P$ relative to $G$ if $J$ has the properties as in Theorem 4.2 which follows. Furthermore a subset $G_0$ of $G$ is said to have property $P$ relative to $G$ if for each $J \in G_0$, $J$ has property $P$ relatives to $G$.

**Theorem 4.2:** If $G$ is an uncountable collection of disjoint simple closed curves $\{J_a\}$ in $E^2$, then there exists an element $J$ of $G$ and two sequences $\{J_i\}$ and $\{J^i_1\}$ of elements of $G$ such that:

1. $\{J_i\}$ converges homeomorphically to $J$,
2. $\{J^i_1\}$ converges homeomorphically to $J$,
3. $J^i_1 \subseteq I(J)$ for each $i$, and
4. $J^i_1 \subseteq E(J)$ for each $i$.

**Proof:** As in previous proofs it can be shown that there is an uncountable subset $G'$ of $G$ such that $G - G'$ is countable and for each $g \in G'$ there exists a sequence $\{g_i\}$ from $G$ converging homeomorphically to $g$.

Since for every element $J$ in $G'$ there exists a sequence that converges to $J$ homeomorphically, we can let $G' = G^I \cup G^E$ where $G^I = \{ J :$ there exists a sequence $\{ J_i \}$ converges to $J$ such that $J_i \in I(J) \}$ and $G^E = \{ J :$ there exists a sequence $\{ J^i_1 \}$ converges to $J$ such that $J^i_1 \in E(J) \}$. Since $G'$ is uncountable, one of $G^I$ and $G^E$ must be uncountable. We assume without loss in generality that $G^I$ is uncountable.

Let $J_0 \in G^I$; and let $\{J^I_i\}$ be a sequence of elements of $G^I$ converging to $J_0$ such that $J^I_i \subseteq I(J_0)$ for each $i$. Then $J_o \subseteq E(J^I_i)$ for each $i$. For each $i$, let $f_i$ be a homeomorphism of $J^I_i$ onto $J_0$ satisfying definition of homeomorphic convergence.

There exists an integer $N_1$ such that $i > N_1$ implies $f_i$ moves no point of $J^I_i$ more than a distance 1. Hence $i > N_1$ implies $J^I_i$ has property $Q_1$. 
relative to $G_1^I$. Then it follows from Theorem 2.9 and the argument above that there exists at most a countable subset of $G_1^I$ that doesn't have property $Q_1$ relative to $G_1^I$.

Let $G_1'$ be the set of all elements $J$ of $G_1^I$ such that $J$ doesn't have property $Q_1$ relative to $G_1^I$. Then $G_1'$ is countable and $G_1 = G_1^I - G_1'$ is uncountable.

Similarly we examine $G_1$ relative to property $Q_2$. There is a subset $G_2$ of $G_1$ such that $G_2$ is uncountable $G_1 - G_2$ is countable, and each element of $G_2$ has property $Q_2$ relative to $G_1$. For each $i$, define an uncountable collection $G_i$ of simple closed curves such that (1) $G_i$ has property $Q_1$ relative to $G_{i-1}$, (2) $G_i \subset G_{i-1}$, (3) $G_1 - G_{i-1}$ is countable.

Let $G_o = \bigcap_{i=1}^{\infty} G_i$. Since we take out at most a countable number of elements from $G_i$ at each stage, it is obvious that $G_o = G_1 - [G_2 \cup G_3 \cup \ldots]$ is not empty and, moreover, it is uncountable.

For every $J \in G_o$, $J \in G_1^I$ and $J$ has property $Q_1$ relative to $G$ for every $i$. It is not difficult to show that $J$ has property $P$ relative to $G$. Therefore the uncountable subset $G_o$ has property $P$ relative to $G$.

Remark: In the proof of above theorem, if $G^E$ is also uncountable then we can use the same technique to prove that there exist an uncountable subset $G_o'$ of $G^E$ such that $G_o'$ has property $P$ relative to $G$ and $G^E - G_o'$ is countable. In case $G^E$ is countable, let $G_o' = \emptyset$. Then the set $G_o \cup G_o'$ is uncountable and has property $P$ relative to $G$. Furthermore $G - (G_o \cup G_o')$ is countable, so we can state a stronger theorem than Theorem 4.2 as follows:

**Theorem 4.3:** If $\{J_a\}$ is an uncountable collection $G$ of disjoint simple closed curves in $E^2$, then there exist a uncountable subset $G_o$ of $G$ such that $G_o$ has property $P$ relative to $G$ and $G - G_o$ is countable.
V. AN APPLICATION TO TAME 2-SPHERES IN $E^3$

We indicate here one more application of our main theorem. A 2-sphere $S$ in $E^3$ is said to be tamely embedded in $E^3$ (or tame in $E^3$) if and only if there is a homeomorphism $h$ of $E^3$ onto itself such that $h(S)$ is a round sphere ($S^2$). A 2-sphere that is not tame is called wild. Wild 2-spheres are known to exist in $E^3$.

Bing [2] has given a characterization of tame 2-spheres which we state below as Theorem 5.1. The proof is apparently difficult and has not been studies by the author. Based on Theorem 5.1 we show that each collection of disjoint wild 2-spheres is at most countable.

**Theorem 5.1:** (Bing[2]) A 2-sphere $S$ in $E^3$ is tame in $E^3$ if for each component $V$ of $E^3 - S$ there exists a sequence $\{S_i\}$ of 2-sphere in $V$ converging homeomorphically to $S$.

**Note:** Since there are only two components $\text{Int } S$ and $\text{Ext } S$ of $E^3 - S$ the same technique as given in the proof of Theorem 4.2 can be used to prove the following theorem.

**Theorem 5.2:** If $G$ is an uncountable collection of disjoint 2-spheres in $E^3$, then there exists an uncountable subset $G_0$ of $G$ such that for every $S \subseteq G_0$ there exist two sequence $\{S_i\}$ and $\{S'_i\}$ of elements of $G$ such that:

1. $\{S_i\}$ and $\{S'_i\}$ both converges homeomorphically to $S$,
2. $S_i \subseteq \text{Int } S_i$ and $S'_i \subseteq \text{Ext } S_i$.

Furthermore $G - G_0$ is countable.

**Remark:** Suppose there exist an uncountably collection $G$ of disjoint wild 2-spheres in $E^3$. Then it follows from Theorem 5.2 and Theorem 5.1 that uncountably many 2-sphere in $G$ are tame. This contradiction gives us the following:
Theorem 5.3: There do not exist uncountably many disjoint wild 2-sphere in $E^3$.

Remark: In $E^3$, let $P = \{ p_t : t \in [-1, 1] \}$ be the collection of all planes that are parallel to $xy$-plane and let $S$ be a 2-sphere contained in the union of the planes from $Z = 1$ to $Z = -1$. Suppose for every $p_t \in P$, $M_t = p_t \cap S$ is a locally connected continuum. Then it is obvious that for each $t \in (-1, 1)$, $M_t$ is neither a point, nor an arc. Let $G = \{ M_t : M_t$ is not a simple closed curve and $t \neq -1, 1 \}$. Suppose $G$ is uncountable. Then it follows from Theorem 2.8 that for each $M_t \in G$, there is a triod $T_t \in M_t$. Let $H = \{ T_t : T_t \subset M_t, M_t \in G \}$. Then $H$ is an uncountable collection of pairwise disjoint triods in $S$. If we remove one point $p$ from the intersection of $z=1$ with $S$, then $S - \{ p \}$ is homeomorphic to the plane $E^2$, but this contradicts the fact that there doesn't exist uncountably many triods in $E^2$. This contradiction gives us the following:

Theorem 5.4: In $E^3$, let $P = \{ p_t : t \in [-1, 1] \}$ be the collection of all planes that are parallel to $xy$-plane and let $S$ be a 2-sphere that is contained in $\bigcup p_t$. If for every $p_t \in P$, $M_t = p_t \cap S$ is a locally connected continuum, then all but countable number of $M_t$ are simple closed curves.
REFERENCES


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