Sodium and potassium vapor Faraday filters revisited: theory and applications

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A complete theory describing the transmission of atomic vapor Faraday filters is developed. The dependence of the filter transmission on atomic density and external magnetic field strength, as well as the frequency dependence of transmission, are explained in physical terms. As examples, applications of the computed results to ongoing research to suppress sky background, thus allowing Na lidar operation under sunlit conditions, and to enable measurement of the density of mesospheric oxygen atoms are briefly discussed. © 2009 Optical Society of America

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1. INTRODUCTION

Extremely narrowband optical filters are required in many situations for extracting useful signals in the presence of a broadband background. For signals at an atomic resonance, an advantage is attained by using the Faraday effect, which involves a circularly birefringent, dichroic medium between crossed polarizers, as shown in Fig. 1. This medium can be an atomic vapor in an axial magnetic field, causing a rotation of the polarization of light near an atomic resonance while the polarization of off-resonance light is unaffected. Advantages of the atomic vapor Faraday filter include its wide field of view, high background rejection, and high peak transmission [1]. These types of filters are particularly attractive, as the resulting bandwidth is only several gigahertz (1 GHz = 1.2 × 10^12 Hz @ 600 nm) wide, about 400 times narrower than 1 nm bandwidth optical interference filters commercially available.

A Faraday filter of this type was first introduced by Ohman in 1956 [2]. Studies using a variety of atomic species have since been performed [1,3,4]. Specifically, filters for Na were developed by Agnelli et al. and Chen et al., while studies of the relationship between vapor temperature and cell transmission were performed by Hu et al. and Zhang et al. for Na and K filters, respectively [5–8]. Our work has led to a Na vapor Faraday filter deployed in the Colorado State University Na lidar system [9], allowing the measurement of mesopause region (~90 km altitude) temperature and horizontal wind under sunlit conditions, thus permitting studies of the solar atmospheric tides and their variability [10].

Our current interest in the Faraday filter stems from ongoing development of a spectrometer to measure the Na nightglow D2/D1 intensity ratio in the mesopause region. Slanger et al. suggested that the varying D2/D1 results from variation in the ratio of the concentration of atomic oxygen to the concentration of molecular oxygen [O]/[O2] due to competing chemical processes [11]. Measuring the variation between the two chemical pathways requires an extremely high resolution spectrometer—on the order of the bandwidth of a Faraday filter. By using a pair of Faraday filters, with their transmission functions independently optimized as will be shown, we can determine valuable information on the important atomic oxygen concentration.

Designing the spectrometer requires theoretical calculations of filter transmissions. Analysis of available literature showed that most publications are either missing complete derivations or are not easily adaptable to different situations, thus limiting their usefulness. For example, Yeh and Van Baak provide only a limited treatment of the quantum mechanical description of atomic states, ignoring the hyperfine structure, while Dressler et al. give a detailed solution, but only for weak external magnetic fields, which is unsuitable for our high-field application [12–14]. Yin and Shay present a theory valid for arbitrary magnetic field for a cesium Faraday filter operating at the Cs D2 line, but their letter lacks detail to make it useful for others to replicate [15]. This paper not only details the complete calculation of Faraday filter transmission, which is applicable to all values of external magnetic field, but our results are also readily adaptable for other applications.

In Section 2, we detail the classical calculation of the filter transmission and then relate this to the quantum-mechanical derivation of the complex susceptibility in Section 3. Section 4 explains the theoretical results for both Na and K and contains tabulated current values for all relevant constants and coefficients. Appendix A contains the calculation of transition matrix elements.
2. CLASSICAL THEORY: FILTER TRANSMISSION

To calculate filter transmission, we express the optical response of the medium in terms of a complex susceptibility \( \chi_\perp(\omega) = \chi'_\perp + i\chi''_\perp \), where the \( \pm \) represents left- and right-handed circular polarizations of the incident light. The complex susceptibility relates the wavenumber \( k \) and angular frequency \( \omega \) in the dispersion relation, since

\[
k = \frac{\omega}{c} \sqrt{1 + \chi'_\perp + i\chi''_\perp} = \frac{\omega}{c} \left( 1 + \frac{\chi'_\perp}{2} + i\frac{\chi''_\perp}{2} \right),
\]

where \( c \) is the speed of light. This is valid for relatively low vapor density, a situation suitable for most applications. The quantum mechanical derivation of \( \chi_\perp(\omega) \) is given in Section 3.

After the first polarizer of the Faraday filter, the randomly polarized input electric field, entering from the left direction, so the transmission coefficient will be

\[
\hat{x} = -\frac{\hat{+}}{\sqrt{2}}, \quad \text{and} \quad \hat{y} = i\frac{\hat{+} + \hat{-}}{\sqrt{2}}.
\]

The input electric field, \( \mathbf{E}(z,t) = \frac{1}{2} [\mathbf{E}(z,\omega)e^{-i\omega t} + \mathbf{E}^*(z,\omega)e^{i\omega t}] \), enters the vapor cell at \( z = 0 \) and \( t = 0 \). After a cell length \( z = L \) and time \( t \), the light will have passed through the vapor cell and the electric field will be

\[
\mathbf{E}(L,\omega) = -\frac{\mathbf{E}_0}{\sqrt{2}} \left[ \exp \left( i \frac{\omega}{c} (1 + 0.5\chi'_\perp + 0.5i\chi''_\perp)L \right) \hat{+} \right]
- \exp \left( i \frac{\omega}{c} (1 + 0.5\chi'_\perp + 0.5i\chi''_\perp)L \right) \hat{-},
\]

which contains (for each circular polarization) an exponential decay or absorption term depending on \( \chi'_\perp \), and an oscillatory term depending on \( 1 + 0.5\chi'_\perp \)—this is the index of refraction.

A second, crossed polarizer will select the light polarized in the \( \hat{y} \) direction, so the transmission coefficient will be

\[
\mathcal{R}(\nu) = \left| \frac{\mathbf{E} \cdot \hat{y}}{\mathbf{E}_0} \right|^2 = \frac{1}{4} \left[ \exp \left( -\frac{\omega}{c} \chi'_\perp L \right) + \exp \left( -\frac{\omega}{c} \chi''_\perp L \right) 
- 2 \exp \left( -\frac{\omega}{c} \chi'_\perp \chi''_\perp L \right) \cos \left( \frac{\omega}{c} \chi'_\perp \chi''_\perp L \right) \right].
\]

We can define the Faraday rotation \( \theta_F \) as the angle of polarization rotation of the output light relative to the initial linear polarization:

\[
\theta_F = \frac{\omega}{c} \chi''_\perp \frac{L}{\lambda} = \frac{\pi}{\lambda} (\chi''_\perp L) = \pi (\Delta n) L,
\]

where \( \Delta n \) is the difference in the index of refraction of the two circular polarizations (i.e., circular birefringence).

3. QUANTUM-MECHANICAL THEORY

A. Derivation of Susceptibility

Since an atomic vapor is an ensemble of many atoms, its state may be represented by a density matrix: \( \rho = \sum \psi_n \langle \psi_n | \psi_n \rangle \) \[16\]. To model the interaction of the atom in an external magnetic field, perturbation theory is used. The Schrödinger equation for the evolution of the density matrix of a system with Hamiltonian \( H_0 + \gamma H_1 \), including a damping term to model interactions such as collisions, is

\[
\frac{d\rho_{\alpha\beta}(t)}{dt} = \frac{i}{\hbar} \left[ [H_0, \rho(t)]_{\alpha\beta} + [\gamma H_1(t), \rho(t)]_{\alpha\beta} + \frac{\partial \rho_{\alpha\beta}}{\partial \kappa} \right] \text{ random}
\]

\[
= \frac{1}{\hbar} (\omega_{\alpha\beta} \rho_{\alpha\beta} + [\gamma H_1(t), \rho(t)]_{\alpha\beta}) - \Gamma_{\alpha\beta} \rho_{\alpha\beta}(t),
\]

where \( \rho_{\alpha\beta}(t) \) represents an element of the density matrix between energy eigenstates of \( H_0, \alpha, \beta \), where \( \omega_{\alpha\beta} \) is the associated transition frequency. The damping constant \( \Gamma_{\alpha\beta} \) is the natural linewidth of the transition. This means that \( \Gamma_{\alpha\beta}/\pi = A_{\alpha\beta}/(2\pi) \), where \( A_{\alpha\beta} \) is the Einstein coefficient for the transition rate between the two states. \( H_0 \) is the base Hamiltonian, and \( H_1(t) \) is the interaction Hamiltonian, with \( \gamma \) being the strength of the perturbation.

In perturbation theory, \( \rho_{\alpha\beta} \) can be expressed in the form of a power series, \( \rho_{\alpha\beta}(t) = \sum_{n=0}^{\infty} \gamma^n \rho_n^{(i)} \), with each term obtained from a hierarchy of equations; the first-order equation is

\[
\frac{d\rho_{\alpha\beta}^{(1)}(t)}{dt} = \frac{1}{\hbar} (\omega_{\alpha\beta} \rho_{\alpha\beta}^{(1)}(t) + [H_1(t), \rho_{\alpha\beta}^{(0)}]_{\alpha\beta} - \Gamma_{\alpha\beta} \rho_{\alpha\beta}^{(1)}(t)).
\]

For the electric dipole approximation, \( H_1(t) = -e\mathbf{r} \cdot \mathbf{E}(t) \),

\[
\rho_{\alpha\beta}^{(1)}(t) = -\frac{1}{2} [\rho_{\alpha\beta}^{(1)}(t) + \rho_{\alpha\beta}^{(1)}(t)] e^{-i\omega t} + \mathbf{E}(t) e^{i\omega t},
\]

the solution to Eq. (7) is

\[
\rho_{\alpha\beta}^{(1)}(t) = \frac{1}{\hbar} \sum_{k=\text{even}} e^{i(k\omega L)/\hbar} \langle \kappa | \rho_{\alpha\beta}^{(0)} | \kappa \rangle \mathbf{E}_k.
\]

The expectation value of the polarization is
\[ P(t) = \langle -Ne\mathbf{r} \rangle = -Ne\text{Tr}[\rho^{(1)}(t)\mathbf{r}] = \frac{1}{2}[P(\omega)e^{-i\omega t} + P^*(\omega)e^{+i\omega t}]. \]

(10)

To avoid confusion in notation, we decompose the vectors, \( P(\omega), E(\omega), \) and \( \mathbf{r} \) into Cartesian coordinates. For an isotropic medium, both the polarization and electric field are transverse to the propagation direction \( \hat{z} \), so combining Eqs. (9) and (10) gives

\[ P_j(\omega) = -\frac{Ne^2}{\hbar^2} \sum_{\alpha,\beta,k} \langle \rho^{(0)}_{j\alpha\beta} (r^\dagger)_{\beta\beta} \rangle \frac{r^k_{\beta\alpha} E^k(\omega)}{\omega - \omega^{ag} + i\Gamma^{ag}_{\alpha\beta}}. \]

(11)

where \( j \) and \( k \) represent the components (\( \hat{x}, \hat{y} \)).

Due to the axial symmetry of the magnetic field, the circular polarizations are eigenmodes of the system. We can transform the Cartesian coordinates into circular polarizations using (2), and rewriting (11) in terms of the relevant dipole moment, \(-e\mathbf{m}\mathbf{r}^\dagger/\beta\), transition frequency, \(\omega^{ag}_{\alpha\beta}\), and damping constant, \(\Gamma^{ag}_{\alpha\beta}\). We then have:

\[ P_\pm(\omega) = \frac{Ne^2}{\hbar^2} \sum_{\alpha\beta} \langle \rho^{(0)}_{j\alpha\beta} (r^\dagger)_{\beta\beta} \rangle \frac{1}{\omega - \omega^{ag}_{\alpha\beta} + i\Gamma^{ag}_{\alpha\beta}}. \]

(12)

By assuming that in the zero-order, only the ground state is populated, we can rewrite Eq. (12) (replacing \( \beta \) with \( g \) for “ground state”):

\[ \chi^{(1)}(\omega) = \frac{Ne^2}{\hbar^2} \sum_{\alpha\beta} \langle \rho^{(0)}_{g\alpha\beta} (r^\dagger)_{\beta\beta} \rangle \frac{1}{\omega - \omega^{ag}_{\alpha\beta} + i\Gamma^{ag}_{\alpha\beta}} \]

(13)

where \( \omega^{ag}_{\alpha\beta} = -\omega^{ag}_{\beta\alpha} \) and \( \Gamma^{ag}_{\alpha\beta} = \Gamma^{ag}_{\beta\alpha} \) have been assumed.

Since the resonance line is narrow (\( \Gamma^{ag}_{\alpha\beta} \ll \omega \)), and for the range of frequencies of interest, \( \omega + \omega^{ag}_{\alpha\beta} \approx 2\omega^{ag} \gg \Gamma^{ag}_{\alpha\beta} \), Eq. (13) reduces to

\[ \chi^{(1)}(\omega) = \frac{Ne^2}{\hbar^2} \sum_{\alpha\beta} \langle \rho^{(0)}_{g\alpha\beta} (r^\dagger)_{\beta\beta} \rangle \frac{2\omega^{ag}_{\alpha\beta}}{2\omega^{ag} \omega - \omega^{ag} + i\Gamma^{ag}_{\alpha\beta}}. \]

(14)

Including a Doppler broadening due to an atom’s random motion with Gaussian distributed line of sight speed \( v \), Eq. (14) becomes

\[ \chi^{(1)}(v) = \frac{N}{2\pi\hbar^2} \frac{1}{\sqrt{\pi} \sigma_{\text{ag}}} \sum_{\alpha\beta} \langle \rho^{(0)}_{g\alpha\beta} \rangle \frac{2\omega^{ag}_{\alpha\beta}}{2\omega^{ag} \omega - \omega^{ag} + i\Gamma^{ag}_{\alpha\beta}} \exp(-\frac{v^2v^2}{2\sigma_{\text{ag}}^2})dv. \]

(15)

where \( u = \sqrt{2k_B T/m} \), and \( k_B, T, \) and \( m \) are respectively, the Boltzmann constant, temperature, and atomic mass. We have also replaced \( e^2 \left| r^\dagger_{\alpha\beta} \right|^2 \) with \( \left| \rho^{(0)}_{g\alpha\beta} \right|^2 \) for electric dipole moment. The integral in Eq. (15) is the complex Faddeeva function [17] and can be rewritten in real and imaginary components as

\[ \chi^{(1)}(v) = \frac{N}{2\pi\hbar^2} \frac{1}{\sqrt{\pi} \sigma_{\text{ag}}} \sum_{\alpha\beta} \langle \rho^{(0)}_{g\alpha\beta} \rangle \frac{2\omega^{ag}_{\alpha\beta}}{2\omega^{ag} \omega - \omega^{ag} + i\Gamma^{ag}_{\alpha\beta}} \int_{-\infty}^{+\infty} \left| \rho^{(0)}_{g\alpha\beta} \right|^2 \exp(-\frac{v^2v^2}{2\sigma_{\text{ag}}^2})dv. \]

(16)

In practice, the ground state is a multiplet, and the factor \( \rho^{(0)}_{g\alpha\beta} \) takes into account any differences in fractional population in the various ground states. From Maxwell–Boltzmann statistics, \( \rho^{(0)}_{g\alpha\beta} \) is

\[ Z = \sum_{i=1}^{N} \exp\left(\frac{E_i}{k_B T}\right) \rightarrow \rho^{(0)}_{g\alpha\beta} = \frac{\exp\left(\frac{E_i}{k_B T}\right)}{Z}. \]

(18)

where \( E_i \), not to be confused with the electric field, is the ground-state energy eigenvalue from Appendix A.

B. Derivation of Transmission Matrix Elements

The Hamiltonian governing the splitting of individual energy levels due to the hyperfine structure and Zeeman splitting is [18]

\[ H = H_0 + H^I + H_{\text{Zeeman}} \]

\[ H^I = H_{\text{HFS}} + H_{\text{Zeeman}} \]

\[ = A_J \mathbf{I} \cdot \mathbf{J} + \frac{B_J}{2(I(2I-1)+J(2J+1))} \]

\[ + \frac{g_J}{2} \left[ 3(\mathbf{I} \cdot \mathbf{J})^2 + \mathbf{I} \cdot \mathbf{J} - I(I+1)J(J+1) \right] \]

(19)

where \( H_0 \) and \( H^I \) are the base and interaction Hamiltonians, respectively. The parameters \( A_J \) and \( B_J \) represent the strength of the hyperfine magnetic dipole and electric quadrupole interactions; vectors \( \mathbf{I} \) and \( \mathbf{J} \) are the nucleus and electron total angular momentum operators with magnitudes \( I \) and \( J \); \( \mu_B \) and \( \mu_N \) are the Bohr and nuclear magnetons; \( E_0 \) is the external magnetic field strength; and \( g_J \) and \( g_I \) are the Lande-\( g \) factors of the atom and of the nucleus; \( g_J \) may be related to \( J \), orbital angular momentum \( L \), and electron spin \( S \) as [19]

\[ g_J = \frac{3J(J+1) + S(S+1) - L(L+1)}{2J(J+1)}. \]

(20)

In order to evaluate the transition matrix element of the dipole moment in Eq. (15) and the energies of the eigenstates, we must first solve the eigenvalue problem of the individual atomic states. The first step is to choose a
set of basis states; here the unperturbed Hamiltonian includes the Coulomb attraction of the nucleus and the interactions between atomic electrons. Thus the eigenstates of $H_0$ are $|IJm_I m_J\rangle$ with total electronic angular momentum and nuclear spin as good quantum numbers, and they have a degeneracy of $2I+1 \times 2J+1$. Under the influence of $H_{HI}$, some of the degeneracy will be lifted. In the limit of zero (or low) magnetic field, the coupling between the atomic electrons and the nucleus will be dominant, and the eigenstates of $H_0 + H_{HFS}$ are $|IJFm_I m_J\rangle$, with the total angular momentum of the atom ($I+J$) including the nucleus) $F=I+J$; the energy eigenvalues will then depend on the strengths of hyperfine interactions, $A_J$ and $B_J$. In the limit of high magnetic field, its interaction with the atom will cause $I$ and $J$ to align to the external field separately. This would allow the use of $|IJm_I m_J\rangle$ as the eigenstates of $H_0 + H_{Zeeman}$; however, to account for hyperfine interactions, the eigenvalues should include the diagonal contribution from $H_{HFS}$ as an approximation. For a solution valid at all values of magnetic field strength, an appropriate choice would be $|IJ Qm_I m_J\rangle$, which reduces to $|IJFm_I m_J\rangle$ in the low or zero field limit, and to $|IJ m_I m_J\rangle$ in the high field limit. For simplicity we choose to write the $|IJQm_I m_J\rangle$ states as a linear superposition of $|IJm_I m_J\rangle$ states [18]. The total Hamiltonian in this representation is not diagonal; it must then be diagonalized to obtain the eigenstates and eigenvalues of the system. Figure 2 shows an example of splitting in the zero field limit (middle column) and the high field limit (right column) for the sodium $^2P_{3/2}$ excited state.

### Table 1. Atomic Properties$^a$

<table>
<thead>
<tr>
<th></th>
<th>Na</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A$</td>
<td>23</td>
<td>39</td>
</tr>
<tr>
<td>Mass (u)</td>
<td>22.989768</td>
<td>38.963707</td>
</tr>
<tr>
<td>$g_I$</td>
<td>1.475392</td>
<td>0.26099</td>
</tr>
<tr>
<td>Abundance</td>
<td>100%</td>
<td>93.26%</td>
</tr>
<tr>
<td>$A_J$ (GHz)</td>
<td>0.8858130644</td>
<td>0.2308588601</td>
</tr>
<tr>
<td>$B_J$ (GHz)</td>
<td>0.0944</td>
<td>0.027775</td>
</tr>
<tr>
<td>$g_J$</td>
<td>4/3</td>
<td>4/3</td>
</tr>
</tbody>
</table>

$^a$Values for mass are from [22], $g_I$ are from [23], abundance are from [22], and $A_J$ and $B_J$ are from [24–27].

4. SAMPLE RESULTS AND DISCUSSION

A computer program was written in the Interactive Data Language (IDL) [20], using built-in functions to calculate $|IJ Fm_J\rangle$ in the low or zero field limit, and to $|IJ m_I m_J\rangle$ in the high field limit. For simplicity we choose to write the $|IJ Q m_I m_J\rangle$ states as a linear superposition of $|IJ m_I m_J\rangle$ states [18]. The total Hamiltonian in this representation is not diagonal; it must then be diagonalized to obtain the eigenstates and eigenvalues of the system. Figure 2 shows an example of splitting in the zero field limit (middle column) and the high field limit (right column) for the sodium $^2P_{3/2}$ excited state.

The solution for energy eigenvalues and eigenvectors and for transition matrix elements is in Appendix A. With the electric dipole matrix transition elements, $|\langle \rho_\nu \rangle_{q,I}|$, evaluated, Eqs. (16) and (17) may be used to calculate $\chi'$ and $\chi''$, from which $\theta_F$ and $\zeta(\nu)$ may be calculated via Eqs. (4) and (5).
eigenvalues and eigenstates, which agreed exactly with longhand calculations. The complex Faddeeva function was calculated using an algorithm reported by Schreier [17].

Filter transmissions and associated $\theta_F$ were calculated for both D$_2$ and D$_1$ transitions of Na and K. Tables 1 and 2 give the necessary constants and coefficients. Values of $g_I$ were calculated using the method detailed in [21].

The presence of different isotopes will affect the values of the $A_J$ and $B_J$ constants and $g_I$ values used in the calculation. Na has only one stable isotope with mass number $A=23$, so the calculation can be done exactly as described. K has two dominant stable isotopes, $A=39$ and $A=41$, and so $\chi$ for K becomes a sum over the two isotopes weighted by their abundances, 93.26% and 6.73%, respectively.

One parameter is yet to be defined: vapor density. The density of the atomic vapor is a function of its temperature and pressure. In our particular vapor cell, we control the temperature of the reservoir (see Fig. 1) containing

<table>
<thead>
<tr>
<th>Ref.</th>
<th>Na</th>
<th>D$_2$</th>
<th>K</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ (nm)</td>
<td>[28]</td>
<td>589.7558</td>
<td>589.1582</td>
</tr>
<tr>
<td>$S=2S_0 \left( e^2a_0^2 \right)$</td>
<td>[29]</td>
<td>37.3</td>
<td>37.3</td>
</tr>
<tr>
<td>$\Delta \nu$ (GHz)</td>
<td>[28]</td>
<td>0.00977</td>
<td>0.00980</td>
</tr>
</tbody>
</table>

$^a$Linestrength $S_0$ is defined in Appendix A.

Fig. 3. (a) D$_2$ and (b) D$_1$ transmission as a fraction of input linear polarization (solid) and Faraday rotation in units of $\pi$ (dashed) vs. frequency (GHz) for a Na vapor Faraday filter optimized for D$_2$. (c) and (d) are the same for a D$_1$ optimized filter. Filter parameters are listed in Table 3. The letters in (a) refer to the three cases described in the text.
solid and liquid Na (or K) to set the vapor pressure on the coexistence curve. Another controller fixes the temperature of the main body of the cell, which is the vapor temperature. From the ideal gas law the vapor density is proportional to this vapor pressure and inversely proportional to the temperature of the vapor. Data for vapor pressure as a function of saturated vapor temperature is given for both solid or liquid Na and K by Honig and Kramer [30], and we derived an equation using a curve-fit method similar to that described in [30]. For Na, the vapor pressure equation is

$$\log_{10}(P_{Na}) = 71.899 - 9217.2(T_{res})^{-1} + 40693000(T_{res})^{-3} + 0.0061264(T_{res}) - 9.6625 \ln(T_{res}),$$

and for K

$$\log_{10}(P_{K}) = 69.53 - 10486(T_{res})^{-1} + 1.8658 \times 10^{6}(T_{res})^{-3} + 0.0027286(T_{res}) - 8.5732 \ln(T_{res}),$$

where $T_{res}$ is the temperature at the solid or liquid reservoir and $P$ is the pressure in Torr.

Figures 3 through 6 show results calculated from the computer program, with the filter parameters listed in Table 3. Notice that the filter temperatures are chosen to respectively optimize the D$_2$ transition in parts (a) and (b) of Figs. 3 and 4 and to optimize the D$_1$ transition in parts (c) and (d) of Figs. 3 and 4. When mesospheric sodium nightglow is sent through the two Na Faraday filters with cell and reservoir temperature settings optimized to minimize error from photon noise, the detected signals may be processed to deduce Na D$_2$/D$_1$ ratio, thereby the ratio between the concentration of atomic oxygen to the concentration of molecular oxygen [O]/[O$_2$].

In the curves for $\chi'$ and $\chi''$, shown in Figs. 5 and 6, there are two groups of curves. Each set of curves is a summation of the curves for the various Zeeman split transitions with $\Delta m_I = \pm 1$, which is the selection rule for the circular $\uparrow$ and $\downarrow$ polarizations defined in Section 2. The central value of each group, indicated by the peak in the curve for $\chi''$ or the zero crossing point of $\chi'$ is labeled as $\nu_0$ for the curves on the left and $\nu_0'$ for the curves on the right, and can be thought of as a resonance frequency for

![D2](image1.png)

![D1](image2.png)

![K_D2](image3.png)

![K_D1](image4.png)

Fig. 4. Same as Fig. 3, except for K. Filter parameters are listed in Table 3.
the circular polarization. This allows $\chi$ to be written as $\chi_\pm$ as was done in Eqs. (16) and (17) of Section 2.

By comparing the figures with the relevant equations, we can gain physical insight to what occurs in a Faraday filter. To facilitate this appreciation, we further consider the transmission function near three specific frequencies: A, at line center; B, off resonance; and C, at the resonance at either $v_0^+$ or $v_0^-$ [see Fig. 3(a)]. For cases A and B, there is no absorption and the exponential factors in Eq. (4) are all unity, and the transmission reduces to

$$ F = \frac{1}{2}(1 - \cos \theta_F) = \sin^2 \theta_F. $$

For case A, $\theta_F$ should be nonzero, since $\chi_+^+$ and $\chi_+^-$ are of opposite sign, and $\tilde{\chi}(v)$ varies periodically between 0 and near 1 as vapor density varies, so for fixed $B_0$ and cell length, we can easily adjust the vapor density to vary transmission at the line center. For case B, $\theta_F = 0$, since far from resonance both $\chi_+^+$ and $\chi_+^-$ have the same value and $\tilde{\chi}(v) = 0$. For case C, at $v_0^+$ for example, $\chi_+^+$ is zero. While $\chi_+^-$ is nonzero, $\chi_-^+$ is large enough to render

### Table 3. Filter Parameters used to Generate Figs. 2–5, and $v_0^+$ and $v_0^-$ Transition Frequencies

<table>
<thead>
<tr>
<th></th>
<th>Na</th>
<th>K</th>
</tr>
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<tbody>
<tr>
<td>$L$ (cm)</td>
<td>4</td>
<td>4</td>
</tr>
<tr>
<td>$B_0$ (G)</td>
<td>1850</td>
<td>700</td>
</tr>
<tr>
<td>$T_{res}$ (°C)</td>
<td>180</td>
<td>164</td>
</tr>
<tr>
<td>$T_{cell}$ (°C)</td>
<td>186</td>
<td>169</td>
</tr>
<tr>
<td>$D_1 v_0^+$ (GHz)</td>
<td>3.6</td>
<td>1.3</td>
</tr>
<tr>
<td>$D_1 v_0^-$ (GHz)</td>
<td>-3.6</td>
<td>-1.3</td>
</tr>
<tr>
<td>$D_2 v_0^+$ (GHz)</td>
<td>2.5</td>
<td>1.0</td>
</tr>
<tr>
<td>$D_2 v_0^-$ (GHz)</td>
<td>-2.5</td>
<td>-1.0</td>
</tr>
</tbody>
</table>

Fig. 5. $\chi'$ and $\chi''$ curves for (a) Na $D_2$, and (b) $D_1$ lines split by the Zeeman effect due to the 1850 Gauss external magnetic field. Solid gray curve is $\chi'$ for $\sigma_-$, gray dashed curve is $\chi''$ for $\sigma_-$, black solid curve is $\chi'$ for $\sigma_+$, and black dashed curve is $\chi''$ for $\sigma_+$.

Fig. 6. Same as Fig. 5, only for K.
the associated exponential factors zero. In this case, one circular polarization is totally absorbed, and the electric field of the other polarization is reduced by the second polarizer by a factor of $\frac{1}{3}$. Therefore, $\delta(\nu)$ is 0.25 at either $\nu_0$ or $\nu'_0$. In practice, the measured transmission can be scaled to $\frac{1}{4}$ at those frequencies to avoid the more involved normalization measurement described in [6].

5. CONCLUSIONS

We have presented a complete theory of the atomic vapor Faraday filter. In addition to upper atmospheric lidar, the results and associated computer program can now be used for various applications, including our studies of the physics and chemistry of the mesopause region. Other uses abound; Faraday filters can also be used to investigate Na and K atoms in the photosphere of the sun [5] and to investigate lower atmosphere winds with a sodium lidar [31], among other applications. The sample results presented help to further illustrate nuances in the theory. Our computer program can be made available on request. The approach in this paper is readily adaptable to other alkali atoms, which have the same electronic structure but different nuclear spin. For example, cesium has $I = 7/2$ and, therefore, the degeneracy is doubled, but the electronic dipole transition probabilities, Eqs. (A5) and (A6) in Appendix A, are the same as Na and K.

APPENDIX A: TRANSMISSION MATRIX ELEMENT CALCULATIONS

1. Details of the Derivation of Transition Matrix Elements

The $|IJ m_I m_J\rangle$ base states are now abbreviated $|m_I m_J\rangle$. As an example, the states $^2S_{1/2}$ and $^2P_{1/2}$ have $J = 1/2$ ($I = 3/2$ for Na and K). The set of 8 $|m_I m_J\rangle$ eigenstates is

$$\begin{bmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\
3 & 2 & 1 & 1 & 1 & 1 & 1 & 1 \\

\end{bmatrix}$$

and the numbers above the kets will be used as a simpler notation. We now proceed to solve the eigenvalue problem for the total Hamiltonian.

Raising and lowering operators $I_+$ and $J_+$ allows $H'f$ to be written as a sum of three terms, according to the power of raising or lowering operators:

$$H_{40} = g_{J\mu B}S_0J_z - g_{J\mu B}S_0J_z + A_JJ_z + \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ 3(I_+J_+J_z) + \frac{1}{4}(I_JJ_+J_J + I_JJ_+J_J) \right]$$

$$+ \frac{3}{2}I_JJ_z - I(I + 1)J(J + 1) \right],$$

$$H_{41} = \frac{A_J}{2}(I_+J_+I_JJ_J) + \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ I_JJ_+(I_+J_+I_JJ_J) + (I_+J_+I_JJ_J)I_JJ_J + I_JJ_+(I_+J_+I_JJ_J) \right]$$

$$+ \frac{1}{2}(I_+J_+I_JJ_J),$$

$$H_{42} = \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ I_JJ_zI_JJ_z + I_JJ_zI_JJ_z \right].$$

Continuing with the $^2S_{1/2}$ (or $^2P_{1/2}$) example, since there are 8 states, the Hamiltonian will be an 8 x 8 matrix in block-diagonal form with the subscripts using the notation of Eq. (A1):

$$H_{40} = \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ I_JJ_zI_JJ_z + I_JJ_zI_JJ_z \right],$$

$$H_{41} = \frac{A_J}{2}(I_+J_+I_JJ_J) + \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ I_JJ_+(I_+J_+I_JJ_J) + (I_+J_+I_JJ_J)I_JJ_J + I_JJ_+(I_+J_+I_JJ_J) \right]$$

$$+ \frac{1}{2}(I_+J_+I_JJ_J),$$

$$H_{42} = \frac{B_J}{2(I - 1)J(2J - 1)}$$

$$\times \left[ I_JJ_zI_JJ_z + I_JJ_zI_JJ_z \right].$$

The values for the Hamiltonian matrix elements for these states of Na/K are listed here:

$$H_{1,1} = \frac{3}{4}A_J + \frac{1}{2}g_{J\mu B}E_0 - \frac{3}{2}g_{J\mu B}E_0,$$

$$H_{6,6} = \frac{3}{4}A_J - \frac{1}{2}g_{J\mu B}E_0 + \frac{1}{2}g_{J\mu B}E_0,$$

$$H_{3,2} = \frac{3}{4}A_J - \frac{1}{2}g_{J\mu B}E_0 + \frac{3}{2}g_{J\mu B}E_0,$$

$$H_{3,3} = \frac{1}{4}A_J + \frac{1}{2}g_{J\mu B}E_0 - \frac{1}{2}g_{J\mu B}E_0,$$

$$H_{3,3} = \frac{1}{4}A_J - \frac{1}{2}g_{J\mu B}E_0 - \frac{1}{2}g_{J\mu B}E_0.$$
\[ H_{6,6} = \frac{3}{4} A_J - i \frac{1}{2} g_J \mu_B E_0 - \frac{3}{2} g_I \mu_n E_0, \]

\[ H_{4,4} = - \frac{1}{4} A_J - i \frac{1}{2} g_J \mu_B E_0 - \frac{1}{2} g_I \mu_n E_0, \]

\[ H_{3,3} = H_{3,2} = H_{1,6} = \frac{\sqrt{3}}{2} A_J, \]

\[ H_{5,5} = - \frac{1}{4} A_J + i \frac{1}{2} g_J \mu_B E_0 + \frac{1}{2} g_I \mu_n E_0, \]

\[ H_{4,5} = H_{5,4} = A_J. \]  
(A4)

The blocks in the Hamiltonian matrices, at most 2 x 2, can be solved independently and easily, yielding energy eigenvalues \( E \) and eigenstates with their associated coefficients. The \(^2P_{3/2}\) state can be solved similarly; however, it is a 16 x 16 block-diagonal matrix with a maximum block size of 4 x 4.

With the energy levels and eigenstate coefficients determined, we can calculate the transition frequencies and probabilities of the allowed transitions. The transitions of interest are electric dipole with selection rules \( \Delta \ell = 0 \) and \( \Delta m_J = 1 \) for absorption or emission of a circularly polarized photon in the filter. The transition probability is given by the square of the transition matrix element \( \langle m_J'm_j' | p_s | m_J m_j \rangle \) where primed and unprimed represent excited and ground states. This may be written as product of 3-j symbols and the reduced matrix element \( \langle I' J'|p|I J \rangle \) \([32]\). This can be further reduced to the reduced matrix element between states in the “basic” model of the atom using 6-j symbols \([33]\). This basic-model reduced matrix element, \( \langle I'|p|I \rangle \), is known as the line strength, \( S_0 \), of the transition; it is the total intensity of the line and is experimentally determined and tabulated in a National Bureau of Standards (NBS) publication \([29]\) and reproduced in Table 2. The total transition probability of each Zeeman transition is given by

\[ \langle m_J'm_j' | p_s | m_J m_j \rangle \rangle^2 = \frac{|a|^2|b|^2(2J' + 1)(2J + 1)}{S_0} \]

\[ \times \left( J' \begin{array}{ccc|c} 1 & 1 & J & S^2 \end{array} \right) \left( \begin{array}{c} m_j' \pm 1 \end{array} \right) \left( J \begin{array}{ccc|c} J \end{array} \right) \]

\[ = F_{II} \left( 2J' + 1 \right) \left( 2J + 1 \right) \left( I' \begin{array}{ccc|c} J' \end{array} \right) \left( \begin{array}{c} J \end{array} \right) \]

where \( a \) and \( b \) are the expansion coefficients of the appropriate eigenstate for the ground and excited states as defined above. The 3-j symbol is

\[ \begin{array}{ccc|ccc} J' & 1 & J & S \end{array} \] 

\[ \begin{array}{cc} m_j' \pm 1 & m_j \end{array} \]

and

<p>| Table 4. ( D_1 ) Allowed Transition F(<em>1) Value and Polarization |
|------------------|------------------|------------------|------------------|
| (^2S</em>{1/2}) | (^2P_{1/2}) | ( F_1 ) | Polarization |
| [|3/2, 1/2)]| [b_2, b_3, b_4, b_5, b_6, b_7)]| [b_2, b_3, b_4, b_5, b_6, b_7)]| [\pm] |
| [a_2, |3/2, −1/2) + a_2, |1/2, 1/2)]| [|3/2, 1/2)]| [a_2, b_2, b_3, b_4, b_5, b_6, b_7)]| [\pm] |
| [a_3, |3/2, −1/2) + a_3, |1/2, 1/2)]| [|3/2, 1/2)]| [a_3, b_3, b_4, b_5, b_6, b_7)]| [\pm] |
| [a_4, |1/2, −1/2) + a_4, |−1/2, 1/2)]| [|3/2, 1/2)]| [a_4, b_4, b_5, b_6, b_7, b_8)]| [\pm] |
| [a_5, |1/2, −1/2) + a_5, |−1/2, 1/2)]| [|3/2, 1/2)]| [a_5, b_5, b_6, b_7, b_8)]| [\pm] |
| [a_6, |1/2, −1/2) + a_6, |−3/2, 1/2)]| [|3/2, 1/2)]| [a_6, b_6, b_7, b_8)]| [\pm] |
| [a_7, |1/2, −1/2) + a_7, |−3/2, 1/2)]| [|3/2, 1/2)]| [a_7, b_7)]| [\pm] |
| [|−3/2, −1/2)]| [|−3/2, 1/2)]| [b_8, b_9)]| [\pm] |</p>
<table>
<thead>
<tr>
<th>$^2S_{1/2}$</th>
<th>$^2P_{3/2}$</th>
<th>$F_1$, $F_2$</th>
<th>Pol.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[3/2, 1/2]$</td>
<td>$[3/2, 3/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2]$</td>
<td>$b_{a_{2}}[1/2] + 1/4 + b_{a_{6}}[1/2] + b_{a_{7}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2] + b_{a_{2}}[1/2] + b_{a_{3}}[1/2]$</td>
<td>$b_{a_{4}}[1/2] + b_{a_{5}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$a_{2}[3/2, -1/2] + a_{2}[1/2, 1/2]$</td>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2] + b_{a_{2}}[1/2] + b_{a_{3}}[1/2]$</td>
<td>$b_{a_{4}}[1/2] + b_{a_{5}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$a_{4}[1/2, -1/2] + a_{4}[1/2, 1/2]$</td>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2] + b_{a_{2}}[1/2] + b_{a_{3}}[1/2]$</td>
<td>$b_{a_{4}}[1/2] + b_{a_{5}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$a_{6}[1/2, -1/2] + a_{6}[1/2, 1/2]$</td>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2] + b_{a_{2}}[1/2] + b_{a_{3}}[1/2]$</td>
<td>$b_{a_{4}}[1/2] + b_{a_{5}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$a_{7}[1/2, -1/2] + a_{7}[1/2, 1/2]$</td>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
<tr>
<td>$b_{a_{0}}[3/2] + b_{a_{1}}[1/2] + b_{a_{2}}[1/2] + b_{a_{3}}[1/2]$</td>
<td>$b_{a_{4}}[1/2] + b_{a_{5}}[1/2]$</td>
<td>1/4</td>
<td>+</td>
</tr>
</tbody>
</table>
is the 6-j symbol; their values are from Tables 5 and 6 in Edmonds [34]. $F_1$ and $F_2$ are defined as

$$F_1 = |a|^2 |b|^2,$$

$$F_2 = \left( \begin{array}{ccc} J' & 1 & J \\ -m_j' & \pm 1 & m_j \end{array} \right)^2 \cdot$$  (A6)

These probabilities are tabulated for the Na and K Zeeman-split hyperfine transitions for each circular polarization component of the $^3P_{1/2} \rightarrow ^2S_{1/2}$ ($D_1$) and $^3P_{3/2} \rightarrow ^2S_{1/2}$ ($D_2$) transitions, as discussed below. Note that with the choice of the $|l, J, m_I, m_J>$ base states, the electronic dipole moments of the transitions are independent from the nuclear spin of the system.

2. D$_1$ Transition Probabilities of Allowed Transitions

For this transition $J'=1/2$, $J=1/2$, $l'=1$ (P-state), $l=0$ (S-state), and $S=1/2$. By inserting these values into Eq. (A5), the transition probability is found to be

$$|\langle m_l m_j' | p_2 | m_l m_j \rangle|^2 = F_1 F_2 S_0, \quad (A7)$$

with $F_2 = 1/3$ from Eq. (A6). Table 4 shows the values of $F_1$ of the allowed transitions, as well as for which $\sigma_+\times \sigma_-$ circular polarization. The subscripts on the $a$ and $b$ coefficients represent which state and $m_J$ value they correspond to.

3. D$_2$ Transition Probabilities of Allowed Transitions

For this transition $J'=3/2$, $J=1/2$, $l'=1$ (P-state), $l=0$ (S-state), and $S=1/2$. By inserting these values into Eq. (A5), the transition probability is found to be

$$|\langle m_l m_j' | p_2 | m_l m_j \rangle|^2 = \frac{4}{3} F_1 F_2 S_0, \quad (A8)$$

where $F_2 = 1/12$ or 1/4, depending on whether $|m_J|=1/2$, or 3/2, respectively, as indicated in Table 5. Table 5 shows the values of $F_1 F_2$ of the allowed transitions, as well as the $\sigma_+$ circular polarization. The subscripts on the $a$ and $b$ coefficients represent which state and $m_J$ value they correspond to.

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