Multiplicative Number - Theoretic Functions

Barney Lee Erickson

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MULTIPLICATIVE NUMBER-THEORETIC FUNCTIONS

by

Barney Lee Erickson

A report submitted in partial fulfillment of the requirements for the degree of

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in

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Plan B

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Barney Lee Erickson
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INTRODUCTION

Quite frequently in the study of number theory we become acquainted with special functions which are defined on the set of positive integers. These functions are known as Number-theoretic functions or Arithmetical functions. This report will give some of the general ideas of the multiplicative number-theoretic functions.

The first part of this report will be devoted to the general development of such functions by means of definitions and theorems. The second part will consist of generalizations of a particular function, the \( \tau \)-function.

Throughout this paper, lower case Greek letters will represent real numbers and lower case English letters will represent integers. Also the basic ideas of summation and product will be assumed as already familiar to the reader.
MULTIPLICATION NUMBER-THEORETIC FUNCTIONS

General theorems

Definition 1.1. A number-theoretic function \( f \) is any function defined on the set of positive integers.

Definition 1.2. A number-theoretic function \( f \) is called multiplicative if \( f(mn)=f(m)f(n) \) whenever \( (m,n)=1 \), i.e., \( m \) and \( n \) are relatively prime. If \( f(mn)=f(m)f(n) \) for all \( m \) and \( n \), then \( f \) is said to be completely multiplicative.

Theorem 1.1. If \( f \) is multiplicative and not identically zero, then \( f(1)=1 \).

Theorem 1.2. \( \sum_{d|n} f(d) = \sum_{d|n} f\left(\frac{n}{d}\right) \).

Theorem 1.3. Let \( n=\prod_{i=1}^{r} p_i^{n_i} \) be the canonical representation of the positive integer \( n \). If \( f \) is multiplicative, then

\[
f(n) = \prod_{i=1}^{r} f(p_i^{n_i}).
\]

Theorem 1.4. If \( f \) and \( g \) are multiplicative, then so are \( F=f \cdot g \) and \( G=f/g \) whenever the latter is defined.

Theorem 1.5. If \( f \) and \( g \) are multiplicative, then so is

\[
F(n) = \sum_{d|n} f(d)g\left(\frac{n}{d}\right).
\]

Proof: If \((m,n)=1\), then in the canonical representation of \( m \) and \( n \) we have \( d|m\) if and only if \( d = d_{1}d_{2} \) where \( d_{1}|m \) and \( d_{2}|n \). Also \((d_{1},d_{2}) = 1 = \left(\frac{m}{d_{1}},\frac{n}{d_{2}}\right) \). Thus, since \( f \) and \( g \) are both multiplicative, we have

\[
F(mn) = \sum_{d|m} f(d)g\left(\frac{mn}{d}\right).
\]
\[ \sum_{d_1|m} \sum_{d_2|n} f(d_1 d_2) g\left(\frac{mn}{d_1 d_2}\right) \]
\[ = \sum_{d_1|m} \sum_{d_2|n} f(d_1) f(d_2) g\left(\frac{m}{d_1}\right) g\left(\frac{n}{d_2}\right) \]
\[ = \sum_{d_1|m} f(d_1) g\left(\frac{m}{d_1}\right) \sum_{d_2|n} f(d_2) g\left(\frac{n}{d_2}\right) \]
\[ = F(m) F(n). \]

Thus \( F \) is multiplicative.

**Corollary 1.5.1.** If \( f \) is multiplicative, then so is

\[ F(n) = \sum_{d|n} f(d). \]

**Proof:** If we let \( g(n) = 1 \) for every \( n \), then \( g \) is multiplicative and

\[ F(n) = \sum_{d|n} f(d) = \sum_{d|n} f(d) \cdot 1 \]
\[ = \sum_{d|n} f(d) g\left(\frac{n}{d}\right). \]

Thus, by theorem 1.5, \( F \) is multiplicative.

**Theorem 1.6.** \( \sum_{d|n} f(d) g(b) = \sum_{d|n} [\sum_{b|d} f(d) g(b)] \)

for every integer \( n \).

**Proof:** If \( d|n \) and \( b|\frac{n}{d} \), then there exist integers \( r \) and \( s \) such that \( n = dr \) and \( \frac{n}{d} = bs \) or \( n = dbs \). Thus \( b|n \) and \( d|\frac{n}{b} \). Therefore every term in the right-hand sum appears in the left-hand sum and vice-versa. Thus
the two expressions are equal as desired.

**Definition 1.3.** The Möbius function $\mu(n)$ is defined as follows:

(i) $\mu(1) = 1$;
(ii) $\mu(n) = 0$ if $p^2/n$ where $p$ is prime.
(iii) $\mu(n) = (-1)^k$ if $n = p_1 p_2 \ldots p_r$ where each $p_i$ is prime and $p_i \neq p_j$ if $i \neq j$.

**Theorem 1.7.** $\mu(n)$ is multiplicative.

Proof is a direct result of Definition 1.3.

**Theorem 1.8.** $\sum_{d|n} \mu(d) = 1$ if $n = 1$

$= 0$ if $n > 1$.

Proof: If $n = 1$, then

$$\sum_{d|n} \mu(d) = \sum_{d|1} \mu(d) = \sum_{d|1} \mu(1) = \mu(1) = 1.$$

from Theorem 1.1.

If $n > 1$, let $n = \prod_{i=1}^{r} p_{i}^{n_i}$ be the canonical representation of $n$. Also, let $H(n) = \sum_{d|n} \mu(d)$. Since $\mu$ is multiplicative by Theorem 1.7, we have by Corollary 1.5.1 that $H$ is also multiplicative. Therefore, by Theorem 1.3,

$$H(n) = \prod_{i=1}^{r} H(p_{i}^{n_i}).$$

But for each $i$,

$$H(p_{i}^{n_i}) = \sum_{d|p_{i}^{n_i}} \mu(d)$$

$$= \mu(1) + \mu(p_{i}) + \mu(p_{i}^2) + \ldots + \mu(p_{i}^{n_i})$$

$$= 1 + (-1) + 0 + \ldots + 0$$

$$= 0.$$
Thus,
\[ \prod_{i=1}^{n_i} = \prod_{i=1}^{n_i} = 0 = 0 = H(n) \]
as desired.

**Theorem 1.9.** (M"obius Inversion Formula). If \( f(n) \) is any number-theoretic function and

\[ F(n) = \sum_{d \mid n} f(d), \]

then

\[ f(n) = \sum_{d \mid n} \mu(d) F\left( \frac{n}{d} \right), \]

and conversely.

**Proof:** Let \( F(n) = \sum_{d \mid n} f(d) \). Then

\[ \sum_{d \mid n} \mu(d) F\left( \frac{n}{d} \right) = \sum_{d \mid n} \left[ \mu(d) \sum_{b \mid \frac{n}{d}} f(b) \right] \]

\[ = \sum_{d \mid n} \sum_{b \mid \frac{n}{d}} \mu(d) f(b) \]

which, by Theorem 1.6, we have

\[ \sum_{b \mid n} \sum_{d \mid \frac{n}{b}} f(b) \]

Now by Theorem 1.8

\[ \sum_{d \mid \frac{n}{b}} \mu(d) = 0 \quad \text{if} \quad \frac{n}{b} > 1 \]

\[ = 1 \quad \text{if} \quad \frac{n}{b} = 1. \]

Thus,

\[ \sum_{b \mid n} \sum_{d \mid \frac{n}{b}} f(b) \sum_{d \mid \frac{n}{b}} \mu(d) = 0 \quad \text{if} \quad \frac{n}{b} > 1 \]
and when \( b = n \)

\[
\sum_{b|n} f(b) \sum_{d|n} \mu(d) = \sum_{b|n} f(b) \cdot 1
\]

\[
= f(n)
\]

as desired.

Conversely, let

\[
f(n) = \sum_{d|n} \mu(d) F\left(\frac{n}{d}\right).
\]

Then

\[
\sum_{d|n} f(d) = \sum_{d|n} f\left(\frac{n}{d}\right)
\]

\[
= \sum_{d|n} \sum_{b|\frac{n}{d}} \mu(d) F\left(\frac{n}{bd}\right).
\]

Now let \( \frac{n}{bd} = c \) so that \( \frac{n}{dc} = b \). Then the above becomes

\[
\sum_{d|n} \sum_{c|\frac{n}{d}} \mu\left(\frac{n}{dc}\right) F(c)
\]

\[
= \sum_{c|n} F(c) \sum_{d|\frac{n}{c}} \mu\left(\frac{n}{dc}\right)
\]

\[
= \sum_{c|n} F(c)
\]

\[
= F(n)
\]

Since \( \sum_{d|\frac{n}{cd}} \mu\left(\frac{n}{cd}\right) = 0 \) if \( \frac{n}{c} > 1 \)
\[ a = 1 \text{ if } \frac{n}{c} = 1. \]

and by Theorem 1.6.

**Theorem 1.10.** If \( F \) is multiplicative and \( F(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) \), then \( f \) is multiplicative.

**Proof:** By Theorem 1.9

\[ f(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right). \]

Since both \( \mu \) and \( F \) are multiplicative we have \( f \) multiplicative by Theorem 1.5.

**Theorem 1.11.** If \( f \) is a multiplicative number-theoretic function and

\[ F(n) = \sum_{d \mid n} \mu(d) f(d) \]

for every positive integer \( n \), then

\[ F(n) = \prod_{i=1}^{r} \left(1 - f(p_i^n)\right) \]

if \( n = \prod_{i=1}^{r} p_i^{n_i} \) with \( n_i > 1 \) for each \( i \).

**Proof:** Let

\[ F(n) = \sum_{d \mid n} \mu(d) f(d) \]

and let \( n = \prod_{i=1}^{r} p_i^{n_i} \) with \( n_i > 1 \) for each \( i \). Then since both \( \mu \) and \( f \) are multiplicative we have \( F \) multiplicative by Theorem 1.5. Thus

\[ F(n) = \prod_{i=1}^{r} F(p_i^{n_i}). \]

But

\[ F(p_i^{n_i}) = \sum_{d \mid p_i^{n_i}} \mu(d) f(d) \]
\[
\begin{align*}
\mu(1) f(1) + \mu(p_1) f(p_1) + \ldots + \mu(p_1^{n_1}) f(p_1^{n_1}) \\
= f(1) - f(p_1) + 0 + \ldots + 0 \\
= 1 - f(p_1).
\end{align*}
\]

Therefore

\[
F(n) = \frac{r}{i=1} (1 - f(p_i)).
\]

**Theorem 1, 12.** If \(n > 0\) and \(n^2 = \sum_{d \mid n} g(d)\), then

\[
g(n) = \frac{r}{i=1} p_i^{2n_i - 2} (p_i^{2} - 1).
\]

**Proof:** Let \(n = \prod_{i=1}^{r} p_i^{n_i}\) be the canonical representation of \(n\), then

\[
n^2 = \left(\prod_{i=1}^{r} p_i^{n_i}\right)^2 = \prod_{i=1}^{r} p_i^{2n_i}.
\]

By theorem 1.10 and letting \(F(n) = n^2\), which is multiplicative, we have \(g\) multiplicative. Now by Theorem 1.9

\[
g(n) = \sum_{d \mid n} \mu(d) F\left(\frac{n}{d}\right) = \sum_{d \mid n} \mu(d) \frac{n^2}{d^2}
\]

\[
= n^2 \sum_{d \mid n} \frac{\mu(d)}{d^2}.
\]

But since \(g\) is multiplicative

\[
g(n) = \prod_{i=1}^{r} g(p_i^{n_i}).
\]
\[ g(p_i) = p_i \sum_{d\mid p_i} \frac{\mu(d)}{d^2} = p_i \left( \frac{\mu(1)}{1^2} + \frac{\mu(p_i)}{p_i^2} + \ldots + \frac{\mu(p_i^n)}{p_i^{2n_i}} \right) \]

\[ = p_i \left( 1 - \frac{1}{p_i^2} \right) \]

\[ = p_i \left( \frac{p_i^2 - 1}{p_i^2} \right) \]

\[ = p_i^{2n_i - 2} \left( p_i^2 - 1 \right). \]

Thus

\[ g(n) = \prod_{i=1}^{r} p_i^{2n_i - 2} \left( p_i^2 - 1 \right). \]

**Theorem 1.13.** For any number-theoretic function \( f \) and any positive integer \( n \),

\[ s(n) = \sum_{d\mid n} s'(\frac{n}{d}) \]

where

\[ s(n) = \sum_{j=1}^{n} f(j) \text{ and } s'(\frac{n}{d}) = \sum_{k=1}^{\frac{n}{d}} f(kd). \]

\[ (k, \frac{n}{d}) = 1 \]

Proof: Let us first associate each integer, from 1 to \( n \) inclusive, with a divisor \( d \) of \( n \). Now those integers associated with a given \( d \) will be of the form \( kd \) (since every integer between 1 and \( n \) inclusive is a multiple of some divisor \( d \) of \( n \). This divisor may be 1.) with \( 1 \leq k \leq \frac{n}{d} \) and \( (k, \frac{n}{d}) = 1 \). Then the sum \( \sum_{j=1}^{n} f(j) \) may be obtained by summing over
all values of $d$ the sums associated with each $d$. Thus

$$s(n) = \sum_{j=1}^{n} f(j) = \sum_{d|n} \sum_{k=1}^{n/d} f(kd).$$

$$= \sum_{d|n} s'(\frac{n}{d}).$$

**Corollary 1.13.1.** If $f$ is completely multiplicative, then

$$s(n) = f(n) \sum_{d|n} \frac{s^*(\frac{n}{d})}{i(d)},$$

where

$$s(n) = \sum_{j=1}^{n} f(j)$$

and

$$s^*(n) = \sum_{k=1}^{n} f(k)$$

for $n > 0$.

$$(k, n)=1$$

**Proof:** Since $f$ is completely multiplicative, we have

$$f(kd) = f(k)f(d).$$

Thus by definition

$$s^*(\frac{n}{d}) = \sum_{k=1}^{n/d} f(k)f(d)$$

$$(k, \frac{n}{d}) = 1$$

$$n/d$$

$$= f(d) \sum_{k=1}^{n/d} f(k)$$

$$(k, \frac{n}{d})=1$$

$$= f(d) s^*(\frac{n}{d}).$$
Therefore, by Theorem 1.13

\[ s(n) = \sum_{d \mid n} f(d) s^*(\frac{n}{d}) \]

\[ = \sum_{d \mid n} f(\frac{n}{d}) s^*(d). \]

Now we let gd = n. Since f is completely multiplicative,

\[ f(n) = f(gd) = f(g) f(d) \]

and thus

\[ f(\frac{n}{d}) = \frac{f(n)}{f(d)} \]

and therefore we have

\[ s(n) = \sum_{d \mid n} \frac{f(n)}{f(d)} s^*(d) = f(n) \sum_{d \mid n} \frac{s^*(d)}{f(d)} \]

as desired.

**Theorem 1.14.** If f is completely multiplicative and s and s* are as defined in Corollary 1.13.1, then

\[ s^*(n) = \sum_{d \mid n} \mu(d) f(d) s(\frac{n}{d}). \]

**Proof:** From Corollary 1.13.1 we have

\[ f(\frac{n}{d}) = \frac{f(n)}{f(d)}, \]

since f is completely multiplicative and d, n. By applying this result and Theorem 1.9 to
\[
\frac{s(n)}{f(n)} = \sum_{d \mid n} \frac{s^*_d(n)}{f(d)}
\]

from Corollary 1.13.1, we have

\[
\frac{s^*_d(n)}{f(n)} = \sum_{d \mid n} \mu(d) \frac{s\left(\frac{n}{d}\right)}{f\left(\frac{n}{d}\right)}
\]

\[
= \sum_{d \mid n} \mu(d) s\left(\frac{n}{d}\right) \frac{f(d)}{f(n)}
\]

\[
= \frac{1}{f(n)} \sum_{d \mid n} \mu(d) s\left(\frac{n}{d}\right) f(d).
\]

Thus, we have

\[
s^*_n = \sum_{d \mid n} \mu(d) s\left(\frac{n}{d}\right) f(d).
\]

**Theorem 1.15.** If \( f \) is any number-theoretic function and \( n \) is a positive integer, then

\[
s^*_n = \sum_{d \mid n} \mu(d) s\left(\frac{n}{d}\right)
\]

where

\[
s\left(\frac{n}{d}\right) = \sum_{j=1}^{n/d} f(dj)
\]

and \( s^* \) is defined in Corollary 1.13.1.

**Proof:** By Theorem 1.8

\[
\sum_{d \mid (k, n)} \mu(d) = 1 \quad \text{if } (k, n) = 1
\]
\[ s^*(n) = \sum_{k=1}^{n} f(k) \quad \text{if} \quad (k, n) > 1, \]

Thus,

\[ s^*(n) = \sum_{k=1}^{n} \left[ f(k) \cdot \sum_{d \mid (k, n)} \mu(d) \right] \]

\[ = \sum_{k=1}^{n} \sum_{d \mid (k, n)} f(k) \mu(d). \]

But \( 1 \leq k \leq n \) and \( d \mid (k, n) \) if and only if \( d \mid n, \quad k = jd \) and \( 1 \leq j \leq \frac{n}{d} \). Therefore

\[ \sum_{k=1}^{n} \sum_{d \mid (k, n)} \mu(d) f(k) = \sum_{d \mid n} \frac{n}{d} \sum_{i=1}^{n/d} \mu(d) f(jd) \]

\[ = \sum_{d \mid n} \left[ \sum_{i=1}^{n/d} \left( \frac{n}{d} \right) \right] \]

\[ = \sum_{d \mid n} \mu(d) s\left( \frac{n}{d} \right). \]
\textbf{\(\tau\)-Function}

\textbf{Definition 1.4.} We define \(\tau(n)\) to represent the number of positive divisors of the positive integer \(n\). That is

\[
\tau(n) = \sum_{d|n} 1.
\]

\textbf{Theorem 1.16.} \(\tau(n)\) is multiplicative.

\textbf{Proof is trivial.}

\textbf{Theorem 1.17.} \(\tau(n) = \prod_{i=1}^{r} (n_i + 1)\) where \(n = \prod_{i=1}^{r} p_i^{n_i}\) is the canonical representation of \(n\).

\textbf{Proof is trivial.}

\textbf{Theorem 1.18.} Let \(n = \prod_{i=1}^{r} p_i^{n_i}\) with \(n_i \geq 1\) for each \(i\). Let \(f(n) = \sum_{d|n} \mu(d) \tau(d)\) then

\[
f(n) = (-1)^r
\]

\textbf{Proof:} Since both \(\mu\) and \(\tau\) are multiplicative we have by Theorem 1.4 and Theorem 1.5 that \(f\) is multiplicative. Thus, from theorem 1.3 we have

\[
f(n) = \prod_{i=1}^{r} f(p_i^{n_i}).
\]

Now,

\[
f(p_i^{n_i}) = \sum_{d|p_i^{n_i}} \mu(d) \tau(d)
\]

\[
= \mu(1) \tau(1) + \mu(p_i) \tau(p_i) + \ldots + \mu(p_i^{n_i}) \tau(p_i^{n_i})
\]

\[
= \tau(1) - \tau(p_i) + 0 + \ldots + 0
\]

\[
= 1 - 2
\]

\[
= -1.
\]
Therefore

\[ f(n) = \prod_{i=1}^{r} (-1) = (-1)^r. \]

Theorem 1.19. If \( n > 0 \) and \( \beta(n) = \sum_{d|n} \frac{\mu^2(d)}{\tau(d)} \), then \( \beta(n) = (\frac{3}{2})^r \)

where \( n = \prod_{i=1}^{r} p_i^{n_i} \).

Proof: Since both \( \mu \) and \( \tau \) are multiplicative \( \beta \) is multiplicative.

By Theorem 1.3 we have

\[ \beta(n) = \prod_{i=1}^{r} \beta(p_i^{n_i}). \]

But,

\[ \beta(p_i^{n_i}) = \sum_{d|p_i^{n_i}} \frac{\mu^2(d)}{\tau(d)} \]

\[ = \frac{\mu^2(1)}{\tau(1)} + \frac{\mu^2(p_i)}{\tau(p_i)} + \cdots + \frac{\mu^2(p_i^{n_i})}{\tau(p_i^{n_i})} \]

\[ = \frac{1}{\tau(1)} + \frac{(-1)^2}{\tau(p_i)} + 0 + \cdots + 0 \]

\[ = 1 + \frac{1}{2} \]

\[ = \frac{3}{2}. \]

Thus

\[ \beta(n) = \prod_{i=1}^{r} \left( \frac{3}{2} \right) = \left( \frac{3}{2} \right)^r. \]
**σ Function**

**Definition 1.5.** We define \( σ(n) \) as the sum of the positive divisors of the positive integer \( n \) that is

\[
σ(n) = \sum_{d \mid n} d.
\]

**Definition 1.6.** Define \( σ_k(n) \) as the sum of the \( k \)th powers of the positive divisors of the positive integer \( n \). that is

\[
σ_k(n) = \sum_{d \mid n} d^k.
\]

**Lemma 1.1.** \( σ_k(n) \) is a multiplicative function.

**Proof:** Since \( f(n) = n^k \) is multiplicative, Corollary 1.5.1 says that \( σ_k(n) \) is multiplicative.

**Theorem 1.20.** If \( σ_k(n) = \sum_{d \mid n} d^k \) and \( n = \prod_{i=1}^{r} p_i^{n_i} \) is the canonical representation of \( n \), then

\[
σ_k(n) = \prod_{i=1}^{r} p_i^{\frac{k(n_i+1)}{k} - 1}.
\]

**Proof:** Since \( σ_k(n) \) is multiplicative it follows that

\[
σ_k(n) = \prod_{i=1}^{r} σ_k(p_i^{n_i}).
\]

But

\[
σ_k(p_i^{n_i}) = \frac{1 - p_i^{kn_i}}{1 - p_i^k} = \frac{p_i^{k(n_i+1)} - 1}{p_i^k - 1}.
\]
Thus it follows that
\[ \sigma_k(n) = \frac{k(n_1 + 1)}{\prod_{i=1}^{r} p_i - 1} \]

**Theorem 1.21.** \( \sigma_1(n) = \sigma(n) \) where \( n = \prod_{i=1}^{r} p_i^{n_i} \).

**Proof:** \( \sigma(n) \) is trivially multiplicative, thus
\[ \sigma(n) = \frac{\sigma(p_i^{n_i})}{\prod_{i=1}^{r} p_i^{n_i}}. \]

But
\[ \sigma(p_i^{n_i}) = 1 + p_i + \ldots + p_i^{n_i} = \frac{p_i^{n_i + 1} - 1}{p_i - 1}. \]

Therefore
\[ \sigma(n) = \frac{n_i + 1}{\prod_{i=1}^{r} p_i - 1}. \]

From Theorem 1.20 with \( k = 1 \) we find
\[ \sigma_1(n) = \frac{p_i^{n_i + 1}}{\prod_{i=1}^{r} p_i - 1}. \]

Thus
\[ \sigma(n) = \sigma_1(n). \]

**Theorem 1.22.** If \( g(n) = \sum \mu(d) \sigma(d) \) then \( d \mid n \)
\[ g(n) = (-1)^r \prod_{i=1}^{r} p_i. \]
where \[ n = \prod_{i=1}^{r} p_i^{n_i}. \]

**Proof:** Since both \( \mu \) and \( \sigma \) are multiplicative \( g \) is also multiplicative. Therefore

\[
g(n) = \prod_{i=1}^{r} g(p_i^{n_i}).
\]

But

\[
g(p_i^{n_i}) = \sum_{d \mid p_i^{n_i}} \mu(d) \sigma(d)
\]

\[
= \mu(1) \sigma(1) + \mu(p_i) \sigma(p_i) + \mu(p_i^2) \sigma(p_i^2) + \ldots + \mu(p_i^{n_i}) \sigma(p_i^{n_i})
\]

\[
= 1 \cdot \sigma(1) - 1 \sigma(p_i) + 0 + \ldots + 0
\]

\[
= 1 - (p_i + 1)
\]

\[
= -p_i.
\]

Thus

\[
g(n) = \prod_{i=1}^{r} (-p_i) = (-1)^r \prod_{i=1}^{r} (p_i).\]
**Definition 1.7.** (The Euler $\phi$-function). We define $\phi(n)$ to be the number of positive integers less than or equal to $n$ and relatively prime to $n$.

**Theorem 1.23.** If $n$ is a positive integer, then

$$n = \sum_{d|n} \phi(d).$$

**Proof:** Let $d_1, \ldots, d_k$ be the positive divisors of $n$. We separate the integers between 1 and $n$ inclusive into classes $C(d_1), \ldots, C(d_k)$, putting an integer into the class $C(d_i)$ if its GCD with $n$ is $d_i$. The number of elements in $C(d_i)$ is then

$$\sum_{\substack{1 \leq a \leq n \\ (a,n)=d_i}} 1 = \frac{n}{d_i}.$$

and since every integer up to $n$ is in exactly one of the classes,

$$\sum_{d_i|n} \sum_{\substack{1 \leq a \leq n \\ (a,n)=d_i}} 1 = n.$$

The number of integers $a$ such that $a \leq n$ and $(a,n) = d_i$ is exactly equal to the number of integers $b$ such that $b \leq \frac{n}{d_i}$ and $(b, \frac{n}{d_i}) = 1$; in fact, multiplying the $b$'s by $d_i$ we get the $a$'s. But from the definition of the Euler function, the number of $b$'s is clearly $\phi(\frac{n}{d_i})$. Thus

$$\sum_{d|n} \phi(\frac{n}{d}) = \sum_{d|n} \phi(d) = n.$$
Theorem 1.24. \( \phi(n) \) is a multiplicative function.

Proof: Since the function \( F(n) = n \) is trivially completely multiplicative and we have

\[
F(n) = n = \sum_{d|n} \phi(d)
\]

by Theorem 1.23, then by Theorem 1.10 we have \( \phi(n) \) is also multiplicative.

Theorem 1.25. If \( n \) is a positive integer, then

\[
\phi(n) = n \sum_{d|n} \frac{\mu(d)}{d} = \sum_{d|n} \frac{n}{d} \frac{\mu(d)}{d} = \sum_{d|n} d \mu \left( \frac{n}{d} \right).
\]

Proof: By the Möbius Inversion Formula in application to

\[
F(n) = n = \sum_{d|n} \phi(d)
\]

we get

\[
\phi(n) = \sum_{d|n} \mu(d) F \left( \frac{n}{d} \right).
\]

\[
= \sum_{d|n} \mu(d) \left( \frac{n}{d} \right)
\]

\[
= \sum_{d|n} \mu \left( \frac{n}{d} \right) d
\]

\[
= n \sum_{d|n} \frac{\mu(d)}{d}.
\]

Theorem 1.26. If \( n = \prod_{i=1}^{r} p_i^{n_i} \) with \( n_i > 1 \) for each \( i \), then

\[
\phi(n) = \prod_{i=1}^{r} p_i^{n_i} (1 - \frac{1}{p_i})
\]
Proof: Since \( \phi \) is multiplicative, we have

\[
\phi(n) = \prod_{i=1}^{r} \phi(p_i^{n_i}).
\]

By Theorem 1.25 we have

\[
\phi(p_i^{n_i}) = p_i^{n_i} \sum_{d|p_i^{n_i}} \frac{\mu(d)}{d}
\]

\[
= p_i^{n_i} \left[ \frac{\mu(1)}{1} + \frac{\mu(p_i)}{p_i} + \ldots + \frac{\mu(p_i^{n_1})}{p_i^{n_1}} \right]
\]

\[
= p_i^{n_i} \left[ 1 - \frac{1}{p_i} \right].
\]

Therefore

\[
\phi(n) = \prod_{i=1}^{r} \phi(p_i^{n_i}) = \prod_{i=1}^{r} p_i^{n_i} \left[ 1 - \frac{1}{p_i} \right].
\]

**Theorem 1.27.** If \( h(n) = \sum_{d|n} \mu(d)\phi(d) \) and \( n = \prod_{i=1}^{r} p_i^{n_i} \) is the canonical representation of \( n \), then

\[
h(n) = \prod_{i=1}^{r} (2 - p_i) .
\]

Proof: Since both \( \mu \) and \( \phi \) are multiplicative, then \( h \) is also. Thus, by Theorem 1.3

\[
h(n) = \prod_{i=1}^{r} h(p_i^{n_i}).
\]
But

\[
\begin{align*}
    h(p_i^{n_i}) &= \sum_{d|p_i^{n_i}} \mu(d) \phi(d) \\
    &= \mu(1)\phi(1) + \mu(p_i)\phi(p_i) + \ldots + \mu(p_i^{n_i})\phi(p_i^{n_i}) \\
    &= 1 - (p_i)(1 - \frac{1}{p_i}) \\
    &= 1 - (p_i)(\frac{p_i - 1}{p_i}) \\
    &= 1 - (p_i - 1) \\
    &= 1 - p_i + 1 \\
    &= 2 - p_i .
\end{align*}
\]

Thus

\[
h(n) = \prod_{i=1}^{r} h(p_i^{n_i}) = \prod_{i=1}^{r} (2 - p_i) .
\]

**Theorem 1.28.** If \( a(n) = \sum_{d|n} \mu^2(d)/\phi(d) \) for every positive integer \( n \), then

\[
a(n) = \frac{n}{\phi(n)} .
\]

**Proof:** Since both \( \mu \) and \( \phi \) are multiplicative so is \( a \). Let

\[
n = \prod_{i=1}^{r} p_i^{n_i}
\]

be the canonical representation of \( n \). Then

\[
a(n) = \prod_{i=1}^{r} a(p_i^{n_i}) .
\]
Then
\[ a(p_i^{n_i}) = \sum_{d|p_i^{n_i}} \mu^2(d)/\phi(d) \]

\[ = \frac{\mu^2(1)}{\phi(1)} + \frac{\mu^2(p_i^1)}{\phi(p_i^1)} + \ldots + \frac{\mu^2(p_i^{n_i})}{\phi(p_i^{n_i})} \]

\[ = 1 + \frac{(-1)^2}{p_i^1 - 1} + 0 + \ldots + 0 \]

\[ = \frac{p_i^1 - 1 + 1}{p_i^1 - 1} \]

\[ = \frac{p_i}{p_i^1 - 1} \]

\[ = \frac{p_i}{p_i^1 - 1} \cdot \frac{p_i}{p_i - 1(p_i^1 - 1)} \]

\[ = \frac{n_i}{p_i^1 - 1} \cdot \frac{p_i}{p_i - 1(p_i^1 - 1)} \]

Therefore
\[ a(n) = \prod_{i=1}^{r} \frac{p_i}{n_i^1 - 1} \cdot \frac{p_i}{p_i - 1(p_i^1 - 1)} \]
\[
\phi(n) \equiv \frac{n}{\phi(n)} 
\]

**Theorem 1.29.** If \( d = (m, n) \), then

\[
\phi(mn) = \frac{d \phi(m) \phi(n)}{\phi(d)}
\]

for any positive integers \( m \) and \( n \).

**Proof:** Let \( m = \prod_{i=1}^{r} p_i^{m_i} \), \( n = \prod_{i=1}^{r} p_i^{n_i} \) and \( d = \prod_{i=1}^{r} p_i^{d_i} \)

be the canonical representations of \( m \), \( n \) and \( d \) with \( d = (m, n) \) and \( m_i > 0 \), \( n_i > 0 \), \( m_i + n_i \neq 0 \) and \( d_i \) equal to the smaller of \( m_i \) and \( n_i \) for each \( i \).

Then for any positive integer \( a \),

\[
\phi(a) = \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right)^{a_i}
\]

\[
= a \prod_{i=1}^{r} \left( 1 - \frac{1}{p_i} \right)^{\delta(a_i)}
\]

where \( a = \prod_{i=1}^{r} p_i^{a_i} \), \( a_i > 0 \) for each \( i \) and we define \( \delta(0) = 0 \) and \( \delta(n) = 1 \) if \( n \neq 0 \).

Therefore, using the above, we have
\[
\frac{d \phi(m) \phi(n)}{\phi(d)} = \frac{d \cdot m \cdot \frac{r}{\prod_{i=1}^{\pi} (1 - \frac{1}{p_i}) \cdot n \cdot \frac{r}{\prod_{i=1}^{\pi} (1 - \frac{1}{p_i})} \cdot \delta(m_i) \cdot \delta(n_i) \cdot \delta(d_i)}{d \cdot \frac{r}{\prod_{i=1}^{\pi} (1 - \frac{1}{p_i})} \cdot \delta(d_i)}
\]

\[
= m \cdot n \cdot \frac{r}{\prod_{i=1}^{\pi} (1 - \frac{1}{p_i})} \cdot \delta(m_i) \cdot \delta(n_i) \cdot \delta(d_i) - \delta(d_i)
\]

\[
= \phi(mn)
\]

Since \(\delta(m_i) + \delta(n_i) - \delta(d_i) = 1\) for each \(i\) and

\[
mn = \frac{\prod_{i=1}^{\pi} (m_i + n_i)}{\prod_{i=1}^{\pi} p_i}
\]

where \(m_i + n_i > 1\) for each \(i\).

**Theorem 1.30.** For \(n \geq 2\), the sum of the positive integers not exceeding \(n\) and relatively prime to \(n\) is \(\phi(n) \cdot \frac{n}{2}\).

**Proof:** Using Theorem 1.14, we let \(f(n) = n\) for all \(n\). Then \(s^*(n)\) defined in Corollary 1.13.1 is simply the sum of the positive integers not exceeding \(n\) and relatively prime to \(n\), and

\[
s\left(\frac{n}{d}\right) = \sum_{i=1}^{n/d} f(j) = \frac{n}{2} \left(\frac{n}{d} + 1\right)
\]

Therefore from Theorem 1.14

\[
s^*(n) = \sum_{d|n} \mu(d) f(d) s\left(\frac{n}{d}\right)
\]

\[
= \sum_{d|n} \mu(d) \cdot \frac{n}{d} \cdot \frac{n}{d} \left(\frac{n}{d} + 1\right)
\]
\[
\frac{n}{2} \sum_{d \mid n} \mu(d) \left( \frac{n}{d} + 1 \right)
\]
\[
= \frac{n}{2} \sum_{d \mid n} \frac{n}{d} \mu(d) + \frac{n}{2} \sum_{d \mid n} \mu(d)
\]
\[
= \frac{n}{2} \cdot n \sum_{d \mid n} \frac{\mu(d)}{d} + \frac{n}{2} \sum_{d \mid n} \mu(d)
\]

But \(n > 2\) and thus \(\sum_{d \mid n} \mu(d) = 0\) by Theorem 1.8, and
\[
\frac{n}{2} \sum_{d \mid n} \frac{\mu(d)}{d} = \phi(n)
\]

by Theorem 1.25. Therefore
\[
s^*(n) = \frac{n}{2} \phi(n) + \frac{n}{2} \cdot 0
\]
\[
= \frac{n}{2} \phi(n).
\]

**Theorem 1.31.** For \(k > 0, n > 1,\)
\[
\sum_{h=1}^{kn} h = k^2 \phi(n) \cdot \frac{n}{2}.
\]
\((h, n) = 1\)

**Proof:** Let \(f(n) = n\) for all \(n.\) Then \(f\) is completely multiplicative.

Also let \(s(n) = \sum_{k=1}^{n} f(k).\) Then
\[
\sum_{h=1}^{kn} h = \sum_{d \mid n} \mu(d) f(d) s\left( \frac{kn}{d} \right)
\]
\((h, n) = 1\)
\[\sum_{d|n} \mu(d) \cdot \frac{kn/d}{2} \frac{(kn/d+1)}{2}\]

\[= \frac{kn}{2} \sum_{d|n} \mu(d) \left( \frac{kn}{d} + 1 \right)\]

\[= \frac{kn}{2} \left[ \frac{kn}{2} \sum_{d|n} \frac{\mu(d)}{d} + \sum_{d|n} \mu(d) \right]\]

\[= \frac{kn}{2} \phi(n) + 0\]

\[= \frac{kn}{2} \phi(n)\].

by Theorem 1.30.

**Theorem 1.32.** For \( n > 1 \)

\[\sum_{h=1}^{n} h^2 = \frac{n^2 \phi(n)}{3} + \frac{n}{6} \sum_{i=1}^{r} (1 - p_i)\]

\((h, n) = 1\)

where \( n = \prod_{i=1}^{r} p_i^{n_i} \), with \( n_i > 1 \) for each \( i \), is the canonical representation of \( n \).

**Proof:** From Theorem 1.11 if we let \( f(n) = n \) for each \( n \), then \( f(n) \) is multiplicative and we have

\[\sum_{d|n} \mu(d) f(d) = \sum_{d|n} \mu(d) d = \prod_{i=1}^{r} (1 - p_i).\]
Now let \( g(n) = n^2 \) and \( s(n) = \sum_{k=1}^{n} g(k) \). Then

\[
s\left(\frac{n}{d}\right) = \sum_{k=1}^{\frac{n}{d}} k^2 = \frac{\frac{n}{d}\left(\frac{n}{d} + 1\right)(\frac{2n}{d} + 1)}{6} = \frac{n\left(\frac{n}{d} + 1\right)(\frac{2n}{d} + 1)}{6d}.
\]

Then by Theorem 1.14

\[
\sum_{\frac{n}{d}\mid n} h^2 = \sum_{d\mid n} \mu(d) d^2 s\left(\frac{n}{d}\right) = \sum_{d\mid n} \mu(d) d^2 \left(\frac{n}{6d}\right)(\frac{n}{d} + 1)(\frac{2n}{d} + 1)
\]

\[
= \frac{n}{6} \sum_{d\mid n} \mu(d) d^2 \left(\frac{2n^2}{d^2} + \frac{3n}{d} + 1\right)
\]

\[
= \frac{n^2}{3} \phi(n) + 0 + \frac{n}{6} \sum_{i=1}^{r} \left(1 - p_i\right)
\]

\[
= \frac{n^2}{3} \phi(n) + \frac{n}{6} \sum_{i=1}^{r} \left(1 - p_i\right)
\]
Theorem 1.33.

\[
\sum_{k=1}^{n} k^3 = \frac{n^3 \phi(n)}{4} + \frac{n^2}{4} \prod_{i=1}^{r} (1 - p_i)
\]

if \( n > 1 \) and \( n = \prod_{i=1}^{r} p_i^{n_i} \) is the canonical representation of \( n \).

Proof: Let \( f(n) = n^3 \) and \( s(n) = \sum_{k=1}^{n} f(k) \). Then

\[
s\left(\frac{n}{d}\right) = \sum_{k=1}^{n/d} f(k) = \sum_{k=1}^{n/d} k^3 = \frac{n}{d} \left(\frac{n}{d} + 1\right)^2
\]

Now, by Theorem 1.14

\[
\sum_{k=1}^{n} k^3 = \sum_{d \mid n} \mu(d) f(d) s\left(\frac{n}{d}\right)
\]

\[
= \sum_{d \mid n} \mu(d) d^3 \left(\frac{n}{d}\right) \left(\frac{n}{d} + 1\right)^2
\]

\[
= \frac{n^2}{4} \sum_{d \mid n} \mu(d) d \left(\frac{n^2}{d^2} + \frac{2n}{d} + 1\right)
\]

\[
= \frac{n^2}{4} \left[ n \sum_{d \mid n} \frac{\mu(d)}{d} + 2n \sum_{d \mid n} \mu(d) + \sum_{d \mid n} \mu(d) \cdot d \right]
\]

\[
= \frac{n^2}{4} \left[ n \phi(n) + 0 + \prod_{i=1}^{r} (1 - p_i) \right]
\]
\[ = \frac{n^3 \phi(n)}{4} + \frac{n^2}{4} \sum_{i=1}^{n} (1 - p_i) \]

_Greatest integer function_

**Definition 1.8.** (The greatest integer function). We define \([a]\), where \(a\) is any real number, to be the largest, or greatest, integer not exceeding \(a\). That is, \([a]\) is the integer satisfying the inequality

\[ [a] \leq a < [a] + 1. \]

**Theorem 1.34.** In the following \(a, \beta, \text{ and } \theta\) denote real numbers.

(a) \(a - 1 < [a] \leq a\).

(b) If \(a \leq \alpha\), then \(a \leq [\alpha]\).

(c) If \(a > \alpha\), then \(a \geq [\alpha] + 1 > [\alpha]\).

(d) If \(a \leq \beta\), then \([a] \leq [\beta]\).

(e) If \(\theta = a - [a]\), then 0 \(\leq \theta < 1\).

(f) If \(a = n + \theta\) with \(0 \leq \theta < 1\), then \(n = [a]\).

(g) For any integer \(n\), \([a + n] = [a] + n\).

(h) If \(a = bq + r\) with \(0 \leq r < b\), then \(q = [a/b]\).

The proof is very easy and will be omitted.

**Theorem 1.35.** For any real number \(a\) and any integer \(n > 0\),

\[ \left[ \frac{a}{n} \right] = \left[ \frac{a}{n} \right]. \]

**Proof:** By definition 1.8

\[ \left[ \frac{a}{n} \right] \leq \frac{a}{n} < \left[ \frac{a}{n} \right] + 1. \]
Thus
\[ n\left\lfloor \frac{a}{n} \right\rfloor \leq a < n\left\lfloor \frac{a}{n} \right\rfloor + n. \]

But from Theorem 1.34, part (c) and (d)
\[ n\left\lfloor \frac{a}{n} \right\rfloor \leq \left\lceil a \right\rceil < n\left\lfloor \frac{a}{n} \right\rfloor + n. \]

Therefore
\[ \left\lfloor \frac{a}{n} \right\rfloor \leq \frac{\left\lfloor a \right\rfloor}{n} < \frac{\left\lfloor a \right\rfloor}{n} + 1. \]

But since \( \left\lfloor \frac{a}{n} \right\rfloor \) is an integer, we have by definition 1.8
\[ \left\lfloor \frac{a}{n} \right\rfloor = \left\lfloor \frac{\frac{a}{n}}{n} \right\rfloor. \]

**Theorem 1.36.** \(-\lceil -a \rceil\) is the least integer not less than \(a\).

**Proof:** We must show (1) \(-\lceil -a \rceil \geq a\) and (2) \(-\lceil -a \rceil - a < 1\).

By Theorem 1.34 (a), we have
\[ -a - 1 < \lceil -a \rceil \leq -a \]
or
\[ a + 1 > -\lceil -a \rceil \geq a \]

and thus we have (1).

Since \(a + 1 > -\lceil -a \rceil\), we have \(1 > -\lceil -a \rceil - a\) as desired for (2). Thus our proof is complete.

**Theorem 1.37.** No integer is nearer \(a\) than \(\left\lfloor a + \frac{1}{2} \right\rfloor\). If two integers are equally near, then \(\left\lfloor a + \frac{1}{2} \right\rfloor\) is the larger of the two integers.

**Proof:** We must show \(\left| \left\lfloor a + \frac{1}{2} \right\rfloor - a \right| \leq \frac{1}{2}\) for all \(a\). By Theorem 1.34(a) we have
\[ a + \frac{1}{2} - 1 \leq [a + \frac{1}{2}] \leq a + \frac{1}{2} \]

and thus

\[ -\frac{1}{2} < \lfloor a + \frac{1}{2} \rfloor - a \leq \frac{1}{2} \]

which says

\[ \left| \lfloor a + \frac{1}{2} \rfloor - a \right| \leq \frac{1}{2} \]

as desired.

Now suppose two integers are equally near \( a \). Then \( n + \frac{1}{2} = a \) for some integer \( n \) and \( a + \frac{1}{2} = n + 1 \), and \( \lfloor a + \frac{1}{2} \rfloor = n + 1 \). Thus, since \( n + 1 > n \) we have \( \lfloor a + \frac{1}{2} \rfloor \) the larger of the two integers.

**Theorem 1.38.** If \( n \) is a positive integer and \( p \) is a prime, then \( p \) appears in the canonical representation of \( n! \) with the exponent

\[ e = \sum_{i=1}^{r} \left[ \frac{n}{p^i} \right] \]

where \( r \) is determined by the inequality \( p^r \leq n < p^{r+1} \).

**Proof:** For any given integer \( k \), the multiples of \( p^k \) which do not exceed \( n \) are \( p^k, 2p^k, \ldots, gp^k \) where \( gp^k \leq n \). Now this implies that \( g \) is the largest integer not exceeding \( \frac{n}{p^k} \). That is, \( g = \left[ \frac{n}{p^k} \right] \). Therefore, \( \left[ \frac{n}{p^k} \right] \) gives the number of positive multiples of \( p^k \) which do not exceed \( n \). If \( 1 \leq m \leq n \), then \( m = gp^k \) with \( (g, p) = 1 \), \( 0 \leq k \leq r \), and \( m \) contributes precisely \( k \) to the total exponent \( e \) with which \( p \) appears in the canonical representation of \( n! \). Also, \( m \) is counted precisely \( k \) times by the sum

\[ \left[ \frac{n}{p} \right] + \left[ \frac{n}{p^2} \right] + \ldots + \left[ \frac{n}{p^r} \right] \]
Once as a multiple of \( p \), once as a multiple of \( p^2 \), ..., once as a multiple of \( p^k \), and no more. If \( k=0 \), then \( m \) is not counted. Thus, the above sum accounts for the contribution of each \( m \) between 1 and \( n \) to the exponent \( e \) as claimed.

**Theorem 1.39.** \( [a] + [a + \frac{1}{2}] = [2a] \) for every real number \( a \).

Proof: By Theorem 1.34(f), \( a = [a] + \theta \) with \( 0 \leq \theta < 1 \) for every real number \( a \). Now consider \( 0 \leq \theta < \frac{1}{2} \). We then have

\[
[a + \frac{1}{2}] = [a]
\]

and thus

\[
[a] + [a + \frac{1}{2}] = [a] + [a] = 2[a]
\]

and

\[
[2a] = [2([a] + \theta)] = 2[a] + 2\theta = 2[a]
\]

Now assume

\[
\frac{1}{2} \leq \theta < 1.
\]

Then \( [a + \frac{1}{2}] = [a] + 1 \)

and

\[
[a] + [a + \frac{1}{2}] = [a] + [a] + 1 = 2[a] + 1.
\]

But since \( \frac{1}{2} \leq \theta < 1 \), we have

\[
[2a] = [2([a] + \theta)]
\]

\[
= [2[a] + 2\theta]
\]

\[
= 2[a] + [2\theta]
\]

\[
= 2[a] + 1
\]

The above by Theorem 1.34(g). And thus in any case...
\[ [a] + [a + \frac{1}{2}] = [2a]. \]

**Theorem 1.40.** If \( a \) and \( \beta \) are real numbers, then

\[ [a] + [\beta] \leq [a + \beta]. \]

**Proof:** By definition, \([a] \leq a\) and \([\beta] \leq \beta\). thus

\[ [a] + [\beta] \leq a + \beta. \]

But by Theorem 1.34(d) we have

\[ [[a] + [\beta]] \leq [a + \beta]. \]

But

\[ [[a] + [\beta]] = [a] + [\beta], \]

by Theorem 1.34(g) and thus

\[ [a] + [\beta] \leq [a + \beta]. \]

**Theorem 1.41.** \( \frac{(a + b)!}{a! b!} \) is an integer for any positive integers \( a \) and \( b \).

**Proof:** We must show that every prime which divides the denominator divides the numerator at least \( r \) times, or by Theorem 1.37 we must show

\[ \sum_{i=1}^{r} \left[ \frac{a+b}{p_i} \right] \geq \sum_{i=1}^{r} \left[ \frac{a}{p_i} \right] + \sum_{i=1}^{r} \left[ \frac{b}{p_i} \right]. \]
Theorem 1.40 says

\[
\left[ \frac{a}{i} \right] + \left[ \frac{b}{i} \right] \leq \left[ \frac{a+b}{i} \right].
\]

Thus

\[
\sum_{i=1}^{r} \left[ \frac{a}{i} \right] + \sum_{i=1}^{r} \left[ \frac{b}{i} \right] \leq \sum_{i=1}^{r} \left[ \frac{a+b}{i} \right]
\]

as desired.

Now let us look at some other inversion formulas involving the Greatest Integer Function.

Theorem 1.42. If

\[
s^*(m,n) = \sum_{k=1}^{m} \sum_{j=1}^{(k,n)=1} f(k)
\]

where \(f\) is any number-theoretic function, then

\[
s^*(m,n) = \sum_{d|n} ([m/d] \sum_{j=1}^{\mu(d)} f(\sigma j)) .
\]

Proof: Using the same argument we did in Theorem 1.15 we have

\[
s^*(m,n) = \sum_{k=1}^{m} f(k) = \sum_{k=1}^{m} \left( f(k) \sum_{d|(k,n)} \mu(d) \right)
\]

\(\text{(k,n)=1} \)

\[
= \sum_{k=1}^{m} \sum_{d|(k,n)} \mu(d) f(k)
\]

\(\text{d|(k,n)}\)
\[
\left\lfloor \frac{m}{d} \right\rfloor = \sum_{d|n} \sum_{j=1}^{\left\lfloor \frac{m}{d} \right\rfloor} f(dj) \mu(d)
\]

\[
= \sum_{d|n} \left( \mu(d) \sum_{j=1}^{\left\lfloor \frac{m}{d} \right\rfloor} f(dj) \right)
\]

**Corollary 1.42.1.** Let \( \phi(m, n) \) denote the number of positive integers not exceeding \( m \) which are relatively prime to \( n \). Then

\[
\phi(m, n) = \sum_{d|n} \mu(d) \left\lfloor \frac{m}{d} \right\rfloor.
\]

**Proof:** In theorem 1.42 we take \( f(n) = 1 \) for every \( n \). Then \( s^*(m, n) = \phi(m, n) \).

\[
\sum_{j=1}^{\left\lfloor \frac{m}{d} \right\rfloor} f(dj) = \sum_{j=1}^{\left\lfloor \frac{m}{d} \right\rfloor} 1 = \left\lfloor \frac{m}{d} \right\rfloor.
\]

Therefore, we have

\[
\phi(m, n) = \sum_{d|n} \left( \mu(d) \sum_{j=1}^{\left\lfloor \frac{m}{d} \right\rfloor} f(dj) \right)
\]

\[
= \sum_{d|n} \mu(d) \left\lfloor \frac{m}{d} \right\rfloor.
\]

**Theorem 1.43.** If \( m \) and \( n \) are positive integers, let \( g(m, n) \) denote the sum of the positive integers not exceeding \( m \) and relatively prime to \( n \). Then

\[
g(m, n) = \sum_{d|n} d \mu(d) \left\lfloor \frac{m}{d} \right\rfloor \left\lfloor \frac{m}{d} + 1 \right\rfloor / 2
\]
Proof: Let \( f(n) = n \) for every \( n \) in theorem 1.42. Then \( s^*(m, n) = g(m, n) \),

\[
\frac{[m/d]}{d} \sum_{j=1}^{[m/d]} f(dj) = d \sum_{j=1}^{[m/d]} \frac{dj}{d} = d \sum_{j=1}^{[m/d]} j
\]

\[
= d \left[ \frac{m}{d} \right] \left[ \frac{m}{d} + 1 \right] \frac{2}{2}.
\]

Thus

\[
g(m, n) = \sum_{d | n} \mu(d) \left( \sum_{j=1}^{[m/d]} f(dj) \right)
\]

\[
= \sum_{d | n} \mu(d) d \left[ \frac{m}{d} \right] \left[ \frac{m}{d} + 1 \right] \frac{2}{2}.
\]

**Theorem 1.44.** If \( f \) is completely multiplicative and \( k > 0 \) then

\[
s^*(kn, n) = \sum_{d | n} \mu(d) f(d) s\left( \frac{kn}{d} \right)
\]

where

\[
s(n) = \sum_{i=1}^{n} i(i) \text{ and } s^*(n) = \sum_{k=1}^{n} f(k) \text{ for } n > 0.
\]

Proof: From Theorem 1.42

\[
s^*(kn, n) = \sum_{d | n} \mu(d) \left( \sum_{i=1}^{[kn/d]} f(dj) \right)
\]

and since \( f \) is completely multiplicative, the above becomes
\[
\sum_{d|n} \left[ \frac{kn}{d} \right] = \sum_{d|n} \left( \mu(d) \sum_{j=1}^{\left\lfloor \frac{kn}{d} \right\rfloor} f(d) f(j) \right)
\]

\[
= \sum_{d|n} \mu(d) f(d) s\left( \frac{kn}{d} \right)
\]

**Theorem 1.45.** If \( k > 0, \ n > 0 \) then

\[
\phi(kn, n) = k \phi(n).
\]

**Proof:** By Corollary 1.42.1

\[
\phi(kn, n) = \sum_{d|n} \mu(d) \left[ \frac{kn}{d} \right]
\]

and since \( d|n \) we have \( d|kn \) and thus \( \frac{kn}{d} \) is an integer. Therefore

\[
\frac{kn}{d} = \left[ \frac{kn}{d} \right].
\]

Thus

\[
\phi(kn, n) = \sum_{d|n} \mu(d) \frac{kn}{d}
\]

\[
= kn \sum_{d|n} \frac{\mu(d)}{d}
\]

\[
= k \phi(n)
\]

by Theorem 1.25.
GENERALIZATION OF $\tau$-FUNCTION

Definition 2.1. $\tau_k(N)$ is to be defined as the number of all possible different factorizations of $N$ into a product of $k$ factors. Different permutations of the same factors will be considered as different factorizations.

For example, $\tau_3(12)$ would be all possible factorizations of 12 into a product of 3 factors.

Lemma 2.1. $\tau(N) = \tau_2(N)$.

Proof: $\tau(N) = \sum_{d|N} 1$

which means $\tau(N)$ is the number of ways of writing $N$ as a product of 2 factors since if $d | N$ then $\frac{N}{d} | N$, and

$$d \cdot \frac{N}{d} = N = \frac{N}{d} \cdot d.$$ 

Thus

$$\tau(N) = \tau_2(N).$$

Lemma 2.2. $\tau(N)$ is the sum of all the numbers of all possible factorizations of the divisors of $N$ into a product of 2-1 factors.

Proof: For any $N$, if we take the divisors of $N$ into a product of 2-1 factors, we will always get the divisor of $N$, and this can be done in exactly $\tau(N)$ different ways.

Lemma 2.3. $\tau_k$ is a multiplicative function.

Proof: Assume $p, q$ are natural numbers such that $(p, q)=1$. We must show that

$$\tau_k(pq) = \tau_k(p) \tau_k(q).$$
If either $p$ or $q$ or both cannot be factored into $k$ factors, then we will use repeated multiplication by 1 to make up the deficiency in factors.

Now consider a factorization of $p$ into $k$ factors and also one for $q$; say

\[ p = f_1 f_2 \cdots f_k \]
\[ q = g_1 g_2 \cdots g_k . \]

Then $pq = f_1 f_2 \cdots f_k g_1 g_2 \cdots g_k = (i_1 g_1)(f_2 g_2) \cdots (f_k g_k)$. Now if we have a factorization of $pq$ into $k$ factors, say $pq = h_1 h_2 \cdots h_k$, then we have the following equalities:

\[ h_i = f_i g_i \quad (k = 1, 2, \ldots, k) \]

which uniquely determine the $f$'s and $g$'s. Thus any factorization of $pq$ into $k$ factors can be expressed as a product of $k$ factors of $p$ and $k$ factors of $q$ and vice versa. This proves the Lemma.

**Lemma 2.4.** For all natural numbers $n$,

\[ \sum_{i=0}^{n} \frac{(i + r)!}{i! \cdot r!} = \frac{(n + r + 1)!}{n! \cdot (r + 1)!} \]

**Proof:** By induction on $n$.

When $n = 1$ we have

\[ \sum_{i=0}^{1} \frac{(i + r)!}{i! \cdot r!} = \frac{r!}{0! \cdot r!} + \frac{(1 + r)!}{1! \cdot r!} \]

\[ = 1 + 1 + r \]

\[ = 2 + r \]
\[= \frac{(1 + r + l)!}{l! \ (r + 1)!}.\]

Thus true for \(n = 1\).

Assume true for \(n = k\), i.e.,

\[
\sum_{i=0}^{k} \frac{(i + r)!}{i! \ r!} = \frac{(k + r + l)!}{k! \ (r + l)!}\]

and show true for \(n = k + 1\).

Now

\[
\sum_{i=0}^{k+1} \frac{(i + r)!}{i! \ r!} = \sum_{i=0}^{k} \frac{(i + r)!}{i! \ r!} + \frac{(k + 1 + r)!}{(k + 1)! \ r!}
\]

\[= \frac{(k + r + l)!}{k! \ (r + 1)!} + \frac{(k + r + l)!}{(k + 1)! \ r!}\]

\[= \frac{(k + 1)(k + r + l)! + (r + 1)(k + r + 1)!}{(k + 1)! \ (r + 1)!}\]

\[= \frac{(k + r + l)! \ (k + r + 2)}{(k + 1)! \ (r + 1)!}\]

Thus, true for \(n = k + 1\), and thus by the induction principle, true for all \(n\).

**Lemma 2.5.** \(\tau_k(p_i^{a_i}) = \frac{(a_i + k - 1)!}{a_i! \ (k - 1)!}\)

**Proof:** By induction on \(k\) when \(k = 1\), then

\[
\tau_1(p_i^{a_i}) = 1 = \frac{(a_i + 1 - 1)!}{a_i! \ (1 - 1)!}
\]
when \( k=2 \) then

\[
\tau_2 (p_i^{a_i}) = a_i + 1 = \frac{(a_i + 2 - 1)!}{a_i!(2-1)!}
\]

Thus true for \( k=1, k=2 \).

Assume true for \( k=m-1 \), i.e.,

\[
\tau_{m-1} (p_1^{a_i}) = \frac{(a_i + m - 2)!}{a_i!(m-2)!}
\]

and show true for \( k=m \), i.e.,

\[
\tau_m (p_1^{a_i}) = \frac{(a_i + m - 1)!}{a_i!(m-1)!}
\]

Now, to obtain all factorizations of \( p_1^{a_i} \) into a product of \( m \) factors, the easiest way is to take each of the factorizations of \( p_1^{a_i} \) into \( m-1 \) factors and to factorize in each case in all possible ways, the first of these factors into 2 factors.

Of the factorizations of \( p_1^{a_i} \) into \( m \) factors we distinguish those where the first factor is \( p_1^e \) where \( e \) is arbitrary but fixed, between 0 and \( m-1 \), the limits included. There are

\[
\tau_1 (p_1^e) \cdot \tau_{m-1} (p_1^{a_i-e})
\]

factorizations of this kind. Giving \( e \) the successive values 0, 1, \ldots, \( a_i \) we find by summation all factorizations of \( p_1^{a_i} \) into a product of \( m \) factors. Therefore

\[
\tau_m (p_1^{a_i}) = \sum_{e=0}^{a_i} \tau_1 (p_1^e) \cdot \tau_{m-1} (p_1^{a_i-e})
\]

and since \( \tau_1 (p_1^e) = 1 \), we have
\[ \tau_m(p_i) = \sum_{e=0}^{a_i-e} \tau_{m-1}(p_i) \]

\[ = \sum_{e=0}^{a_i} \frac{(a_i - e + m - 2)!}{(a_i - e)! (m - 2)!} \]

and from Lemma 2.4, with \( a_i - e \) playing the role of \( i \) and \( m - 2 \) the role of \( r \), we have

\[ \sum_{e=0}^{a_i} \frac{(a_i - e + m - 2)!}{(a_i - e)! (m - 2)!} = \frac{(a_i + m - 1)!}{a_i! (m - 1)!} \]

which means the Lemma is true for \( k=m \) and thus by induction is true for all \( k \).

**Theorem 2.1.** The number of all possible factorizations of \( N=p_1^{a_1}p_2^{a_2} \ldots p_n^{a_n} \) into a product of \( k \) factors equals

\[ \tau_k(N) = \frac{1}{[(k-1)!]^n} \prod_{i=1}^{n} \frac{(a_i + k - 1)!}{a_i!} \]

**Proof:** From Lemma 2.3, \( \tau_k \) is multiplicative. Thus

\[ \tau_k(N) = \tau_k(p_1^{a_1} \ldots p_n^{a_n}) = \prod_{i=1}^{n} \tau_k(p_i^{a_i}) \]

Now by Lemma 2.5

\[ \tau_k(p_i^{a_i}) = \frac{(a_i + k - 1)!}{a_i! (k-1)!} \]

Therefore

\[ \tau_k(N) = \prod_{i=1}^{n} \frac{(a_i + k - 1)!}{a_i! (k-1)!} \]
\[
\frac{1}{(k-1)!} \sum_{i=1}^{n} \frac{(a_i + k-1)!}{a_i!}
\]

**Corollary 2.1.1.** \( \tau_k(N) = \sum_{d|n} \tau_{k-1}(N/d) \) where \( N = \prod_{i=1}^{n} p_i^{a_i} \).

**Proof:** We have

\[
\tau_k(p_i^{a_i}) = \sum_{e=0}^{a_i} \tau_{k-1}(p_i^{a_i-e})
\]

from Lemma 2.5. But this is the same as saying

\[
\tau_k(p_i^{a_i}) = \sum_{d_1|p_i^{a_i}} \tau_{k-1}(p_i^{a_i/d_1}) = \sum_{d_1|p_i^{a_i}} \tau_{k-1}(d_1).
\]

In analogy we have relations of this kind for \( d_2, d_3, \ldots, d_n \). After multiplication of these relations and applying Lemma 2.3 we have

\[
\tau_k(N) = \prod_{i=1}^{n} \left( \sum_{e=0}^{a_i} \tau_{k-1}(p_i^{a_i-e}) \right).
\]

\[
= \prod_{i=1}^{n} \left( \sum_{d_i|p_i^{a_i}} \tau_{k-1}(d_i) \right).
\]

But now, by multiplying out, regrouping and using the fact that \( \tau_k \) is multiplicative, we have

\[
\tau_k(N) = \sum \tau_{k-1}(p_i^{a_i-e_1} \cdots p_n^{a_n-e_n})
\]

\[0 \leq e_1 \leq a_1 \cdots 0 \leq e_n \leq a_n\]
\[ = \sum_{d \mid N} \tau_{k-1}^{(N/d)} \]

\[ = \sum_{d \mid N} \tau_{k-1}(d) \]

since for every \( d \) such that \( d \mid N \) we have a situation above and for every \( a_{1} \) we have a \( d \). Thus the Corollary is true.

From the above Corollary and the Möbius Inversion Formula, we have

\[ \tau_{k-1}(N) = \sum_{d \mid N} \mu(d) \tau_{k}^{(N/d)} \]

**Definition 2.2.** From the integers 1, 2, ..., \( N \) we choose the numbers \( t \), which have the properties: \( t \nmid N \), \( (t, N) \neq 1 \). The number of numbers of this kind is denoted by \( \xi(N) \).

Example: \( \xi(12) = 3 \).

**Definition 2.3.** \( X(N) \) is the sum of the numbers of \( \xi(N) \).

**Lemma 2.6.** For each integer \( N \) we have

\[ N = \tau(N) + \phi(N) + \xi(N) - 1. \]

Proof: \( \tau(N) \) accounts for every integer less than or equal to \( N \) that divides \( N \). \( \phi(N) \) accounts for every integer less than \( N \) which is relatively prime to \( N \) and \( \xi(N) \) accounts for all the rest of the integers less than \( N \). But \( \tau(N) \) and \( \phi(N) \) both have 1. Thus

\[ N = \tau(N) + \phi(N) + \xi(N) - 1. \]
Since 1 is counted twice we need to subtract it once.

**Corollary 2.6.1.** If \(d \mid N\) then

\[
d = \tau_2(N) + \phi(d) + \xi(d) - 1.
\]

**Proof:** By Lemma 2.1

\[
\tau_2(d) = \tau(d).
\]

Thus the proof is Lemma 2.6.

**Theorem 2.2.** If \(N = p_1^{a_1} \ldots p_n^{a_n}\) is the canonical representation of \(N\), then

\[
X(N) = \sigma(N) + \tau(N) - \frac{1}{2} \sum_{i=1}^{n} \frac{(a_i + 2)!}{a_i!} - N.
\]

**Proof:** From Corollary 2.6.1 we have

\[
d = \tau_2(d) + \phi(d) + \xi(d) - 1.
\]

Therefore by summing over all divisors of \(N\) we have

\[
\sum_{d \mid N} d = \sum_{d \mid N} (\tau_2(d) + \phi(d) + \xi(d) - 1)\]

\[
= \sum_{d \mid N} \tau_2(d) + \sum_{d \mid N} \phi(d) + \sum_{d \mid N} \xi(d) - \sum_{d \mid N} 1.
\]

From Definition 2.3

\[
X(N) = \sum_{d \mid N} \xi(d),
\]

\[
\sum_{d \mid N} \phi(d) + \sum_{d \mid N} \xi(d) - \sum_{d \mid N} 1.
\]
thus from Lemma 2.4 and Theorem 2.1

\[ \sum_{d | N} \tau_2(d) = \tau_3(d) = \left( \frac{1}{2} \right)^n \prod_{i=1}^{n} \frac{(a_i + 2)!}{a_i!} \]

Therefore

\[ \sum_{d | N} d = \left( \frac{1}{2} \right)^n \prod_{i=1}^{n} \frac{(a_i + 2)!}{a_i!} + \sum_{d | N} \phi(d) + X(N) - \sum_{d | N} 1. \]

Thus by Definition 1.5, Definition 1.4 and Theorem 1.23

\[ \sigma(N) = \left( \frac{1}{2} \right)^n \prod_{i=1}^{n} \frac{(a_i + 2)!}{a_i!} + N + X(N) - \tau(N). \]

Therefore

\[ X(N) = \sigma(N) + \tau(N) - \left( \frac{1}{2} \right)^n \prod_{i=1}^{n} \frac{(a_i + 2)!}{a_i!} - N. \]

Corollary 2.2.1. \( X(N) \) is non multiplicative.

Proof: Proof can easily be seen in Theorem 2.2. But an example is as follows:

\[ X(4, 3) = X(12) + \sigma(12) = \tau(12) - \left( \frac{1}{2} \right)^2 \cdot \frac{4!}{2!} \cdot \frac{3!}{1!} - 12 \]

\[ = 28 + 6 - \frac{1}{4} (4, 3, 3, 2) - 12 \]

\[ = 4 \]

\[ X(4) \cdot X(3) = (\sigma(4) + \tau(4) - \frac{1}{2} \cdot \frac{4!}{2!} - 4)(\sigma(3) + \tau(3) - \frac{1}{2} \cdot \frac{3!}{1!} - 3) \]
\[(7 + 3 - 6 - 4)(4 + 2 - 3 - 3)\]

\[= 0.0\]

\[= 0\]

Therefore \(X(4.3) \neq X(4) \cdot X(3)\).
BIBLIOGRAPHY


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