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Measure Theory and Lebesgue Integration

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MEASURE THEORY AND LEBESGUE INTEGRATION

by

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I. INTRODUCTION

The Lebesgue integral is a generalization of the Riemann integral which extends the collection of functions which are integrable.

Lebesgue integration differs from Riemann integration in the way the approximations to the integral are taken. Riemann approximations use step functions which have a constant value on any given interval of the domain corresponding to some partition. Lebesgue approximations use what are called simple functions which, like the step functions, take on only a finite number of values. However, these values are not necessarily taken on by the function on intervals of the domain, but rather on arbitrary subsets of the domain. The integration of simple functions under the most general circumstances possible necessitates a generalization of our notion of length of a set when the set is more complicated than a simple interval. We define the Lebesgue measure "m" of a set $E \in \mathfrak{M}$, where $\mathfrak{M}$ is some collection of sets of real numbers, to be a certain set function which assigns to E a nonnegative extended real number "mE".

This report consists of the solutions of exercises found in "Real Analysis", by H. L. Royden. Quotations from the book are all accompanied by the title "Definition" or "Theorem". The exercises are all entitled "Proposition" and all proofs in this report are my own. All theorems are quoted without proof. The theorems and definitions occur as they are needed throughout the paper, but some of the most basic definitions and theorems are lumped together in section II.

It is assumed in this paper that the reader is familiar with the basic concepts of advanced calculus and set theory.

II. BASIC DEFINITIONS AND THEOREMS

1. Definition: If $\mathfrak{M}$ is any collection of sets of real numbers and $E \in \mathfrak{M}$, a function "m" which assigns a nonnegative, extended real number "mE" to the set E is called a set function.

2. Definition: The Lebesgue outer measure m^*A of a set A of real numbers is defined by $m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n)$, where $l(I_n)$ is the length of the open interval I_n and the sums pertain to a countable covering of A by open intervals I_n.

3. Definition: Let $\mathfrak{M}$ be a collection of sets of real numbers. $\mathfrak{M}$ is called a σ algebra if:

   (a) For every A and B in $\mathfrak{M}$, $A \cap B$ is in $\mathfrak{M}$.

   (b) For every A in $\mathfrak{M}$, $\tilde{A}$ is in $\mathfrak{M}$.

   (c) For every sequence $\langle A_i \rangle_{i=1}^{\infty}$ in $\mathfrak{M}$, $\bigcup_{i=1}^{\infty} A_i$ is in $\mathfrak{M}$.

4. Definition: A set E is said to be Lebesgue measurable if for each set A we have $m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$.

5. Theorem: The set $\mathfrak{M}$ of all measurable sets is a σ algebra.

6. Definition: A set function m whose domain is the collection $\mathfrak{M}$ of all measurable sets is said to be a countably additive measure if for each sequence $\langle E_n \rangle$ of disjoint sets in $\mathfrak{M}$ we have $m(\bigcup E_n) = \sum mE_n$.

7. Definition: The class of Borel sets is the smallest σ algebra containing the open sets.

8. Definition: The family of all countable unions of closed sets is denoted by $\mathfrak{F}_\sigma$.

9. Definition: The family of all countable intersections of open sets is denoted by $\mathcal{G}_\delta$.

10. Definition: The characteristic function χ_E of a set E is defined by $\chi_E(x) = 1$ if $x \in E$, and $\chi_E(x) = 0$ if $x \notin E$.

11. Definition: The function $\phi(x) = \sum_{i=1}^{n} a_i \chi_{E_i}(x)$ is called a simple function if the sets E_i are measurable and $E_i \cap E_j = \phi$ for $i \neq j$, and the values $\{a_1, a_2, \ldots, a_n\}$ are distinct and non zero.

III. MEASURE THEORY

1. Proposition: Let m be a countably additive measure on a σ algebra $\mathfrak{M}$. Then if $A, B \in \mathfrak{M}$ with $A \subset B$, it follows that $mA \leq mB$.

Proof: If $A \subset B$, then $B = (B-A) \cup A$.

$mB = m(B-A) + mA$, and

$m(B-A) \geq 0$.

Therefore $mB \geq mA$.

This property is called monotonicity.

2. Proposition: Let m be a countably additive measure on a σ algebra $\mathfrak{M}$. Then for any sequence $\langle E_n \rangle \in \mathfrak{M}$, $m(\cup E_n) \leq \Sigma mE$.

Proof: Let $F_k^* = E_k - (\bigcup_{i<k} E_i)$.

Then $\cup E_n = \cup E_n^*$ and $E_i^* \cap E_j^* = \phi$ for $i \neq j$.

$m(\cup E_n) = m(\cup E_n^*) = \Sigma mE_n^* = \Sigma m(E_n - (\bigcup_{i<n} E_n))$, but

$(E_n - (\bigcup_{i<n} E_n)) \subset E_n$.

Therefore, $m(E_n - (\bigcup_{i<n} E_n)) \leq mE_n$

which implies $m(\cup E_n) \leq \Sigma mE_n$.

This property is called countable subadditivity.

3. Proposition: Let m be defined as above. If there is a set $A \in \mathfrak{M}$ such that $mA < \infty$, then $m\phi = 0$.

Proof: $A = (A - \phi) \cup \phi$ so

$mA = m(A - \phi) + m\phi$. But

$mA = m(A - \phi)$, which implies

$m\phi = 0$.

4. Theorem: If $\{A_n\}$ is countable, $m^*(\cup A_n) \leq \Sigma\ m^*A_n$.

5. Theorem: If A is countable $m^*A = 0$.

6. Proposition: Let A be the set of rationals between 0 and 1, and let $\{I_n\}$ be a finite collection of open intervals covering A. Then $\sum l(I_n) \geq 1$.

Proof: $[0,1] = \bar{A} \subset \cup \bar{I}_n$.

$1 = m^*[0,1] = m^*\bar{A} \leq m^*(\cup \bar{I}_n) \leq \sum m^*\bar{I}_n = \sum l(\bar{I}_n)$

$= \sum l(I_n)$.

7. Proposition: Given any set A and any $\epsilon > 0$, there is an open set B such that $A \subset B$ and $m^*B \leq m^*A + \epsilon$. There is a $G \in \mathcal{G}_\delta$ such that $A \subset G$ and $m^*A = m^*G$.

Proof: If $m^*A = \infty$, let $B = Re$ and we are done.

If $m^*A < \infty$, then there is a countable collection of open intervals $\{I_n\}$ such that $A \subset \{I_n\}$ and $\sum l(I_n) \leq m^*A + \epsilon$, for any $\epsilon > 0$.

Let $B = \cup\{I_n\}$.

Then $m^*B = \sum l(I_n)$, which implies

$m^*B \leq m^*A + \epsilon$.

Now let $\epsilon = 1/n$.

Then to each n there corresponds a $B_n = \cup\{I_m\}$ such that $\sum l(I_m) \leq m^*A + 1/n$.

$\langle B_n \rangle$ is a countable sequence, and since each B_n is countable, $\{B_n\}$ is countable.

Let $G = \cap\{B_n\}$.

Then $G \in \mathcal{G}_\delta$ and $A \subset G$.

$G = (G-A) \cup A$

$m^*G = m^*(G-A) + m^*A$

Suppose $m^*(G-A) > 0$. Then there is a $d > 0$ such that

$m^*(G - A) = d$.

But there exists an integer n such that

$m^*(B_n - A) < 1/n < d$,

and since $G = \bigcap \{B_n\}$, $m^*(G - A) \leq m^*(B_n - A) < 1/n < d$

which contradicts our assumption that $m^*(G - A) = d > 0$.

Therefore $m^*(G - A) = 0$.

It follows that $m^*G = m^*A$.

8. Proposition: m^* is translation invariant.

Proof: For any open interval $I_n = (a_n, b_n)$,

$l(I_n) = b_n - a_n$.

$I_n + x = (a_n + x, b_n + x)$ and

$l(I_n + x) = (b_n + x) - (a_n + x) = b_n - a_n = l(I_n)$.

Also, if $A \subset \bigcup I_n$, then $(A + x) \subset \bigcup (I_n + x)$.

Therefore, $\sum_{A \subset \bigcup I_n} l(I_n) = \sum_{(A+x) \subset \bigcup (I_n + x)} l(I_n + x)$.

Then $m^*A = \inf_{A \subset \bigcup I_n} \sum l(I_n) \leq \sum_{(A+x) \subset \bigcup (I_n + x)} l(I_n + x)$,

that is, m^*A is a lower bound for $\sum_{(A+x) \subset \bigcup (I_n + x)} l(I_n + x)$

which implies $m^*A \leq m^*(A + x)$.

By reversing the roles of A and A+x in the above arguement we obtain $m^*(A + x) \leq m^*(A)$ by the same reasoning. But then,

$m^*(A + x) = m^*A$.

9. Proposition: If $m^*A = 0$, then $m^*(A \cup B) = m^*B$.

Proof: $m^*(A \cup B)$ $m^*A + m^*B$.

$B \subset A \cup B$, which implies $m^*B \leq m^*(A \cup B)$.

Therefore, $m^*(A \cup B) = m^*B$.

Although the Lebesgue outer measure m^* is defined for all sets it is not countably additive as will be demonstrated shortly. However it is countably additive when restricted to a class called measurable sets.

10. Theorem: If $m^*E = 0$, then E is measurable.

11. Theorem: If E_1 and E_2 are measurable, so is $E_1 \cup E_2$.

12. Theorem: The set of measurable sets is a σ algebra.

13. Theorem: Let A be any set, and $E_1, \ldots, E_n$ a finite sequence of disjoint measurable sets. Then,

$m^*(A \cap [\cup E_i]) = \Sigma\, m^*(A \cap E_i)$.

14. Theorem: The interval (a,∞) is measurable.

15. Theorem: Every Borel set is measurable.

16. Proposition: If E is measurable, then $E + y$ is measurable.

Proof: Let A be any set. Then,

$m^*(A + y) = m^*A = m^*(A \cap E) + m^*(A \cap \tilde{E})$

$= m^*((A \cap E) + y) + m^*((A \cap \tilde{E}) + y)$

$= m^*(A + y \cap E + y) + m^*(A + y \cap \tilde{E} + y)$

$= m^*(A + y \quad E + y) + m^*(A + y \quad \widetilde{E + y})$

which implies that $E + y$ is measurable.

17. Proposition: If E_1 and E_2 are measurable, then $m(E_1 \cup E_2) + m(E_1 \cap E_2) = mE_1 + mE_2$.

Proof: $m(E_1 \cup E_2) = m(E_1 \cup E_2 \cap E_1) + m(E_1 \cup E_2 \cap \tilde{E}_1)$

$= mE_1 + m(E_2 \cap \tilde{E}_1)$. And

$mE_2 = m(E_2 \cap E_1) + m(E_2 \cap \tilde{E}_1)$. Then adding, we obtain

$mE_1 + mE_2 = m(E_1 \cup E_2) + m(E_1 \cap E_2)$.

Now we are in a position to define the Lebesgue measure mE to be the outer measure m^*E, where $E \in \mathfrak{M}$

(where $\mathfrak{M}$ is the family of measurable sets).

18. Theorem: $m(\bigcup E_i) = \sum mE_i$ when $\langle E_i \rangle$ is a sequence of pairwise disjoint, measurable sets.

Now we will construct a nonmeasurable set, paraphrasing material in Royden.

Let $x, y \in [0,1]$

Let $x \overset{\circ}{+} y = x + y$ if $x + y < 1$.

Let $x \overset{\circ}{+} y = x + y - 1$ if $x + y \geq 1$.

Let $(\sim)$ be an equivalence relation such that for any $x, y \in [0,1)$, $x \sim y$ iff x and y differ by a rational number. $(\sim)$ splits $[0,1)$ into disjoint equivalence classes. By the axiom of choice there is a set P with one element from each equivalance class.

Let $\langle r_i \rangle$ be a sequence of rational numbers contained in $[0,1)$ such that each rational is exactly one r_i.

Let $P_i = P \overset{\circ}{+} r_i$.

Since every element in P differs by an irrational number, $P_i \cap P_j = \phi$, for $i \neq j$.

If $x \in [0,1)$, $x \sim y$ for some $y \in P$,

$x - y = r_i$ for some i, and

$x = y \overset{\circ}{+} r_i \in \bigcup P_i$.

Therefore, $[0,1) \subset \bigcup P_i$.

Now there is a theorem that states that $m(E + y) = mE$ for a measurable set E and some $y \in [0,1)$.

Then P_i is measurable if P is, and will have the same measure. But then , $1 = m[0,1) = m(\bigcup P_i) = \sum mP_i = \sum mP$. If $mP \neq 0$, then $mP = \infty$. This contradiction shows that P cannot be measurable for any measure m such that $m[0,1) < \infty$.

19. Proposition: If E is measurable and $E \subset P$, Then $mE = 0$.

Proof: Let $E_i = E \dotplus r_i$, for $r_i \in \langle r_i \rangle$ as defined above.

Then $mE_i = mE$.

$\bigcup E_i \subset \bigcup P_i = [0,1)$

$m(\bigcup E_i) = \Sigma mE_i = \Sigma mE \quad m[0,1) = 1$

Now if $mE \neq 0$, $mE = \infty$ which is a contradiction.

Therefore, $mE = 0$.

20. Proposition: It is possible that $m^*(\bigcup E_i) < \Sigma m^*E_i$ for a disjoint sequence $\langle E_i \rangle$.

Proof: Let $E_i = P_i$ where P_i is defined above.

Then $\bigcup P_i = [0,1)$

$1 = m^*[0,1) = m^*(\bigcup P_i) \leq \Sigma m^*P_i = \Sigma m^*P = \infty$, since $\Sigma m^*P = 0$ is a contradiction.

Therefore, $m^*(\bigcup P_i) < \Sigma m^*P_i$.

21. Definition: A function f is said to be Lebesgue measurable if its domain is measurable and the set $\{x: f(x) > \alpha\}$ is measurable for each real number α.

22. Theorem: The set $\{x: f(x) > \alpha\}$ in definition 21 may be replaced by $\{x: f(x) \geq \alpha\}$, $\{x: f(x) < \alpha\}$, or $\{x: f(x) \leq \alpha\}$.

23. Theorem: Let c be a constant and f and g two measurable real valued functions defined on the same domain. Then the functions $f + c$, cf, $f + g$, and fg, are measurable.

24. Theorem: Let $\langle f_n \rangle$ be a sequence of measurable functions

with the same domain of definition. Then $\sup\{f_1,\dots,f_n\}$, $\inf\{f_1,\dots,f_n\}$, $\sup_n f_n$, $\inf_n f_n$, $\overline{\lim}\, f_n$, $\underline{\lim}\, f_n$, are all measurable.

25. Theorem: If f is measurable and $f = g$ a.e. (almost everywhere), then g is measurable.

26. Proposition: Let D be a dense set of real numbers, that is, a set of real numbers such that every interval contains an element of D. Let f be an extended real valued function on R such that $\{x: f(x) > \alpha\}$ is measurable for each $\alpha \in D$. Then f is measurable.

Proof: For every $r \in R$, there is an increasing sequence $\langle \alpha_i \rangle$ in D such that $\langle \alpha_i \rangle \to r$.

Every set $\{x: f(x) > \alpha_i\}$ is measurable.

Then $\overline{\lim}_{i\to\infty} \{x: f(x) > \alpha_i\} = \{x: f(x) > r\}$ is measurable.

27. Proposition: (1) $\chi_{A\cap B} = \chi_A \chi_B$, (2) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A\chi_B$, (3) $\chi_{\tilde{A}} = 1 - \chi_A$.

Proof: (1) $x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$.

$x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$ or both.

Therefore, $\chi_{A\cap B} = \chi_A \chi_B$.

(2) $x \in A \cup B \Leftrightarrow$ (a) $x \in A$ or $x \in B$ or (b) both.

(a) $\chi_{A\cup B} = \chi_A + \chi_B$

(b) $\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \chi_B$.

But in case (a), $\chi_A \chi_B = 0$. Therefore,

$\chi_{A\cup B} = \chi_A + \chi_B - \chi_A \chi_B$.

(3) $x \in A \iff x \notin A$

Therefore, $\chi_{\tilde{A}} = 1 - \chi_A$

28. Proposition: The sum and product of simple functions are simple.

Proof: Let $\phi = \sum^{N} \alpha_i \chi_{A_i}$ and $\theta = \sum^{M} \beta_i \chi_{B_i}$

$\phi + \theta = \sum \alpha_i \chi_{A_i} + \sum \beta_i \chi_{B_i}$

$= \sum \alpha_i \chi_{(A_i - (\bigcup^{M} B_j))} + \sum \beta_i \chi_{(B_i - (\bigcup^{N} A_j)}$

$+ \sum_i^N \sum_j^M (\alpha_i + \beta_j) \chi_{A_i \cap B_j}$.

Let $D_i = A_i - (\bigcup^{M} B_j)$ for $i = 1, \ldots, N$.

Let $D_{N+i} = B_i - (\bigcup^{N} A_j)$ for $i = 1, \ldots, M$.

Let $D_{N+M+i+j} = A_i \cap B_j$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$. where j runs consecutively for each i.

Let $\delta_i = \alpha_i$ for $i = 1, \ldots, N$.

Let $\delta_{N+i} = \beta_i$ for $i = 1, \ldots, M$.

Let $\delta_{N+M+i+j} = (\alpha_i + \beta_j)$ for $i = 1, \ldots, N$ and $j = 1, \ldots, M$ where j runs consecutively for each i.

Then $\phi + \theta = \sum \delta_i \chi_{D_i}$ where $i = 1, \ldots, (M+N+MN)$.

The product $\phi\theta = \sum_i \sum_j \alpha_i \beta_j \chi_{A_i \cap B_j}$, $i \leq N$, $j \leq M$ and $i \neq j$ is simple since $(A_i \cap B_j) \cap (A_k \cap B_l) = \phi$ when any one of the subscripts i or j is different from k or l, and $i \neq j$, $k \neq l$.

29. Proposition: Let D and E be measurable sets and f a function with domain $D \cup E$. Then f is measurable iff its restrictions to D and E are measurable.

Proof: If $f|_{D \cup E}$ is measurable then

$D \cap \{x: f(x) > \alpha , f|_{D \cup E}\} = \{x: f(x) > \alpha, f|_D\}$

is measurable since the intersection of two measurable sets is measurable.

Similarly $E \cap \{x: f(x) > \alpha, f|_{D \cup E}\}$

$= \{x: f(x) > \alpha, f|_E\}$ is measurable.

If $f|_D$ and $f|_E$ are measurable, then

$\{x: f(x) > \alpha, f|_D\} \cup \{x: f(x) > \alpha, f|_E\}$

$= \{x: f(x) > \alpha, f|_{D \cup E}\}$ is measurable.

30. Proposition: Let f be a function with measurable domain D. Show that f is measurable iff the function $g(x) = f(x)$ for $x \in D$ and $g(x) = 0$ for $x \notin D$ is measurable.

Proof: If f is measurable, the sets $\{x: g(x) > \alpha\} = \{x: f(x) > \alpha\}$ for $\alpha \geqslant 0$ are measurable, and the sets $\{x: g(x) > \alpha\} = \{x: f(x) > \alpha\} \cup \tilde{D}$ for $\alpha < 0$ are measurable. This implies that g is measurable.

If g is measurable, the sets $\{x: f(x) > \alpha\} = \{x: g(x) > \alpha\}$ for $\alpha \geqslant 0$ are measurable, and the sets $\{x: f(x) > \alpha\} = \{x: g(x) > \alpha\} \cap D$ for $\alpha < 0$ are measurable. Then f is measurable.

31. Proposition: Let f be a measurable function on [a,b] which takes the values $\pm\infty$ only on a set of measure zero. Then given $\epsilon > 0$, there is an M such that $|f| \leqslant M$ except on a set of measure less than $\epsilon/3$.

Proof: Suppose false. Then there is an $\epsilon > 0$ such that for every $M > 0$, $m\{x: |f(x)| > M\} \geqslant \epsilon/3$.

And $m\{x: |f(x)| = \infty\} = m \bigcap_{M \in I_+} \{x: |f(x)| > M\} \geqslant \epsilon/3 \neq 0$ which is a contradiction.

32. Proposition: Let f be a measurable function on $[a,b]$. Given $\epsilon>0$ and M, there is a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ except where $|f(x)| \geq M$. If $m \leq f \leq M$, then we may take ϕ such that $m \leq \phi \leq M$.

Proof: If $\epsilon \geq 2M$, let $\phi(x) = 0$ for all x such that $x \in [a,b] - \{x: |f(x)| \geq M\}$.

If $\epsilon < 2M$, let $N = [2M/\epsilon] + 1$.

Let $\phi(x) = (n\epsilon - M)$ if $(n\epsilon - M) \leq f(x) < ((n+1)\epsilon - M)$ for any integer n such that $0 \leq n < N$ and when $|f(x)| < M$.

Then $|f(x) - \phi(x)| < \epsilon$.

Let $A_n = \{x: \phi(x) = n\epsilon - M\}$.

Then $\phi = \sum_{n=0}^{N} (n\epsilon - M)\chi_{A_n}$.

If $m \leq f \leq M$, let $N = [(M - m)/\epsilon] + 1$.

Then we use m instead of M in the above discussion.

33. Proposition: Given a simple function ϕ on $[a,b]$, there is a step function g on $[a,b]$ such that $g(x) = \phi(x)$ except on a set of measure less than $\epsilon/3$. If $m \leq \phi \leq M$, then we can take g so that $m \leq g \leq M$.

Proof: Let $\{a_1, \ldots, a_n\}$ be the set of values of ϕ.

Let $A_k = \{x: \phi(x) = a_k\}$.

Then $\bigcup A_k = [a,b]$.

For every $\epsilon > 0$ there exists $\bigcup_{i\geq 1} O_{k,i} \supset A_k$ of disjoint, open intervals, and $m^*(\bigcup_{i\leq 1} O_{k,i} - A_k) < \epsilon/6(m-1)$.

A finite subcover can be picked such that $a \in O_{k,i}$ for some $O_{k,i}$.

Let a_1 be the left end point and b_1 the right one.

Pick the next open interval such that a_2 and b_2 are the end points with $a_2 < b_2$ and b_2 is farther to the right than any other interval with $a_2 \in (a_1, b_1)$. Pick the succeeding intervals the same way. This process must end with some interval (a_m, b_m) since a finite number of intervals cover $[a,b]$. It must be that $b \in (a_m, b_m)$ since if $b = b_m$, b would not be in the cover.

This procedure gives us $a_1 < a_2 < \ldots . < a_m$ and $b_1 < b_2 < \ldots .. < b_m$.

Let $g(x) = a_k$ if $x \in (a_i, b_i) \subset O_{k,i}$ for any i, and $x \notin (a_{i-1}, b_{i-1}) \cap (a_i, b_i)$ and $x \notin (a_i, b_i) \cap (a_{i+1}, b_{i+1})$.

Let $g(x) = 0$ if $x \in (a_{i-1}, b_{i-1}) \cap (a_i, b_i)$ or $x \in (a_i, b_i) \cap (a_{i+1}, b_{i+1})$.

Now $m^*[(\bigcup_{i\geq 1} O_{k,i}) \cap (\bigcup_{i\geq 1} O_{L,i})] \leq$ $m^*[(\bigcup_{i\geq 1} O_{k,i} - A_k) \cup (\bigcup_{i\geq 1} O_{L,i} - A_L)] \leq m^*[\bigcup_{i\geq 1} O_{k,i} - A_k] +$ $m^*[\bigcup_{i\geq 1} O_{L,i} - A_L] < \epsilon/6(m-1) + \epsilon/6(m-1) = \epsilon/3(m-1)$.

In particular, if $(a_i, b_i) \subset \bigcup_{i\geq 1} O_{k,i}$ and $(a_{i+1}, b_{i+1}) \subset \bigcup_{i\geq 1} O_{L,i}$, then $m^*[(a_i, b_i) \cap (a_{i+1}, b_{i+1})] \leq m^*[(\bigcup_{i\geq 1} O_{k,i}) \cap (\bigcup_{i\geq 1} O_{L,i})]$ $< \epsilon/3(m-1)$ for any i.

Then $m^*[\bigcup_{i=1}^{m-1} \{(a_i, b_i) \cap (a_{i+1}, b_{i+1})\}] < \sum_{i=1}^{m-1} \epsilon/3(m-1) = \epsilon/3$.

Then $m^*\{x: g(x) = 0\} < \epsilon/3$.

If $m \leq \phi \leq M$, then in the above definition for g, let $g(x) = m$ whenever the definition would have set $g(x) = 0$ and we have $m \leq g \leq M$.

34. Proposition: Given a step function g on [a,b], there is a continuous function h such that $g(x) = h(x)$ except on

a set of measure less than $\epsilon/3$. If $m \leq g \leq M$, then we may take h so that $m \leq h \leq M$.

Proof: Let $a_0 = x_0 < x_1 < \ldots . < x_{n-1} < x_n = b$ be the partition corresponding to g.

Then let $S = \bigcup_{i=1}^{n-1} \{(x_i - \epsilon'/2(n-1), x_i + \epsilon'/2(n-1))\} - \{x_i : x_i \text{ is an isolated point}\}$.

ϵ' can be made small enough that all elements of S are disjoint. Then $m^*S = \sum_{i=1}^{n-1} m^*(x_i - \epsilon'/2(n-1), x_i + \epsilon'/2(n-1))$ $= \sum_{i=1}^{n-1} \epsilon'/(n-1) = \epsilon' < \epsilon/3$.

Let $h(x) = g(x)$ for $x \in [a,b] - S$.

Let $h(x_i - \epsilon'/2(n-1)) = g(x_i - \epsilon'/2(n-1))$.

Let $h(x_i + \epsilon'/2(n-1)) = g(x_i + \epsilon'/2(n-1))$.

Let $h(x) = g(x_i - \epsilon'/2(n-1)) + [[g(x_i + \epsilon'/2(n-1)) - g(x_i - \epsilon'/2(n-1))]/(\epsilon'/2(n-1))][x - (x_i - \epsilon'/2(n-1))]$.

Then h is continuos except on a set $= \{x_i : x_i \text{ is an isolated point}\}$ which has measure less than $\epsilon/3$.

If $m \leq g \leq M$, then $m \leq h \leq M$.

We can conclude from propositions 31, 32, 33, and 34 that for a given measurable function f, defined on [a,b], which takes on the values $\pm \infty$ on a set of measure zero, then for all $\epsilon > 0$, there is a $M > 0$ such that $|f(x)| < M$ except on a set of measure $< \epsilon/3$. And there is a simple function ϕ such that $|f(x) - \phi(x)| < \epsilon$ except where $|f(x)| \geq M$, which is a set of measure less than $\epsilon/3$. There is a step function $g(x) = \phi(x)$ except on a set of measure less than $\epsilon/3$. Now

$|f(x) - g(x)| = |(f(x) - \phi(x)) + (\phi(x) - g(x))|$

$\leq |f(x)-\phi(x)| + |\phi(x)-g(x)| < \epsilon + 0 = \epsilon$, except on a set of measure less than $\epsilon/3 + \epsilon/3 = 2\epsilon/3 < \epsilon$.

Also, there exists a continuous function h such that $h(x) = g(x)$ except on a set of measure less than $\epsilon/3$.

Then, $|f(x)-h(x)| = |f(x)-g(x) + g(x)-h(x)|$

$\leq |f(x)-g(x)| + |g(x)-h(x)| < \epsilon + 0 = \epsilon$, except on a set of measure less than $2\epsilon/3 + \epsilon/3 = \epsilon$.

Now since we can select ϕ such that $m \leq \phi \leq M$ whenever $m \leq f \leq M$, and we can find a g such that $m \leq g \leq M$ whenever $m \leq \phi \leq M$, and we can find an h such that $m \leq h \leq M$, whenever $m \leq g \leq M$, we have by transitivity, $m \leq g \leq M$ and $m \leq h \leq M$ whenever $m \leq f \leq M$.

We can sum up much of what we have said by stating Littlewood's Three Principles.

(1 Every measurable set is nearly a finite union of intervals.

(2) Every measurable function is nearly continuous.

(3) Every convergent sequence of measurable functions is nearly uniformly convergent.

To illustrate the third principle we give the following theorem.

35. Theorem: Let E be a measurable set of finite measure, and $\langle f_n \rangle$ a sequence of measurable real valued functions such that for each x in E, we have $f_n(x) \rightarrow f(x)$. Then, given $\epsilon > 0$ and $\delta > 0$, there is a measurable set $A \subset E$ with $mA < \delta$ and an integer N such that for all $x \notin A$ and all

$n \geq N, \quad |f_n(x) - f(x)| < \epsilon .$

IV. LEBESGUE INTEGRATION

The Lebesgue integral is a generalization of the Riemann integral in that every Riemann integrable function is Lebesgue integrable, but not conversely, and for such a function the Riemann and Lebesgue integrals are equal. For this reason we recall the definition of the Riemann integral.

1. Definition: Let $a = x_1 < x_2 < \dots < x_n = b$ be a partition of $[a,b]$. Let $M_i = \sup f(x)$ for $x_{i-1} < x \leq x_i$.

Let $m_i = \inf f(x)$ for $x_{i-1} < x \leq x_i$.

Let $S = \sum_{i=1}^{n} (x_i - x_{i-1})M_i$.

Let $s = \sum_{i=1}^{n} (x_i - x_{i-1})m_i$.

We define $R\overline{\int_a^b} f(x)\,dx = \inf S$, and $R\underline{\int_a^b} f(x)\,dx = \sup s$, over all possible subdivisions of $[a,b]$.

If $R\underline{\int_a^b} f(x)\,dx = R\overline{\int_a^b} f(x)\,dx$, we say that f is Rieman integrable, and we define the common value to be $R\int_a^b f(x)\,dx$.

2. Proposition: If $f(x) = 0$ when x is irrational and $f(x) = 1$ when x is rational, then $R\overline{\int_a^b} f(x)\,dx = b - a$ and $R\underline{\int_a^b} f(x)\,dx = 0$

Proof: On any subdivision $M_i = \sup f(x) = 1$, and $m_i = \inf f(x) = 0$.

Then $S = \sum (x_i - x_{i-1})M_i = \sum (x_i - x_{i-1}) = b - a$.

$R\overline{\int_a^b} f(x)\,dx = \inf S = b - a$.

Also, $s = \sum (x_i - x_{i-1})m_i = \sum (x_i - x_{i-1})0 = 0$, so

$R\underline{\int_a^b} f(x)\,dx = \sup s = 0$.

3. Proposition: Construct a sequence $\langle f_n \rangle$ of nonnegative Riemann integrable functions such that $\langle f_n \rangle$ increases monotonically to f.

Let $\langle r_i \rangle_{i=1}^{\infty}$ be a sequence of all rational numbers in [a,b].

Let $f_n(x) = 0$ for all x in $[a,b] - \{r_1, r_2, \ldots, r_n\}$.

Let $f_n(x) = 1$ for all x in $\{r_1, r_2, \ldots, r_n\}$.

Then f_n is Riemann integrable for all n,

$f_n(x) \leq f_{n+1}(x)$, and $\lim f_n(x) = f(x)$.

Now $\lim[\ \inf \sum (x_i - x_{i-1}) \sup\limits_{x_i < x \leq x_{i-1}} f_n(x)\ ] = 0$

$\neq \int_a^b f(x)\,dx = b - a$, which implies that the limiting process cannot be interchanged with the process of integration. This demonstrates some of the difficulties with the Riemann integral.

4. Definition: The function $\phi(x) = \sum^{n} a_i \chi_{E_i}(x)$ is called a simple function if the sets E_i are measurable, $E_i \cap E_j = \phi$, and the set of values $\{a_1, a_2, \ldots, a_n\}$ are distinct and nonzero.

5. Definition: We define the Lebesgue integral of a simple function ϕ by, $\int \phi(x)\,dx = \sum^{n} a_i m A_i$.

6. Theorem: Let $\phi = \sum a_i \chi_{E_i}$, with $E_i \cap E_j = \phi$ for $i \neq j$. Suppose each set E_i is a measurable set of finite measure. Then $\int \phi = \sum a_i m E_i$.

7. Theorem: Let ϕ and ψ be simple functions which vanish outside a set of finite measure. Then $\int (a\phi + b\psi) = a\int\phi + b\int\psi$.

From the preceeding two theorems it follows that if $\phi = \Sigma a_i \chi_{E_i}$, then $\int \phi = \Sigma a_i m_{E_i}$ so the restriction that the sets E_i be disjoint is unnecessary.

A real valued, bounded function f, on a measurable set E is Lebesgue integrable if $\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$ where ϕ and ψ are simple functions.

8. Theorem: Let f be defined and bounded on a measurable set E with mE finite. In order that $\inf_{\psi \geq f} \int_E \psi = \sup_{\phi \leq f} \int_E \phi$ for all simple functions ψ and ϕ, it is necessary and sufficient that f be measurable.

9. Definition: If f is a bounded measurable function on a measurable set E with mE finite, we define the Lebesgue integral of f over E by $\int_E f(x)\,dx = \inf \int_E \psi(x)\,dx$ for all simple functions $\psi \geq f$.

10. Theorem: Let f be a bounded function defined on [a,b]. If f is Riemann integrable on [a,b], then it is measurable and $R\int_a^b f(x)\,dx = \int_a^b f(x)\,dx$.

11. Theorem: If f and g are bounded, measurable functions defined on a set E of finite measure, then:

(1) If f = g a.e., then $\int_E f = \int_E g$.

(2) $\int_E (af + bg) = a\int_E f + b\int_E g$.

(3) If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

(3a) $|\int f| \leq \int |f|$

(4) If $A \leq f(x) \leq B$, then $AmE \leq \int_E f \leq BmE$.

(5) If A and B are disjoint measurable sets of finite measure, then $\int_{A \cup B} f = \int_A f + \int_B f$.

12. Theorem: Let $\langle f_n \rangle$ be a sequence of measurable functions defined on a set of finite measure, and suppose that there is a real number M such that $|f_n(x)| \leq M$ for all n and all x. If $f(x) = \lim f_n(x)$ for each x in E, then $\int_E f = \lim \int_E f_n$.

13. Definition: Let f be a real valued function defined on [a,b]. Then the function $h(y) = \inf_{\delta>0} \sup_{|x-y|<\delta} f(x)$ is called the upper envelope of f.

14. Proposition: Let f be a bounded function on [a,b] and let h be the upper envelope of f. Then $R\overline{\int_a^b} f = \int h$.

Proof: If $\phi \geq f$ is a step function, then $\phi \geq h$ except for a finite number of points, since it may happen that if $a = x_0 < x_1 < \ldots < x_n = b$ is a partition of [a,b], $f(x_i) < h(x_i)$. If $\phi(x_i) = f(x_i)$, then we have $h(x_i) > \phi(x_i)$. However, $h(x) > \phi(x)$ only if $x = x_i$ where x_i is a point in the partition. Therefore the number of such points is finite. Then, since any step function is simple, $R\overline{\int_a^b} f \leq \inf_{\psi \geq f} \int_a^b \psi \leq \inf_{\psi \geq h} \psi = \int_a^b h$.

Now there is a decreasing sequence $\langle \phi_n \rangle$ of step functions such that $\phi_n \downarrow h$ and $\phi_n \geq h$ for all n.

Then $R\overline{\int_a^b} f \leq \int_a^b \phi_n$ for all n.

Therefore, $R\overline{\int_a^b} f \leq \lim \int_a^b \phi_n = \int_a^b h$, which implies $\int_a^b h = R\overline{\int_a^b} f$.

15. Proposition: A bounded function f on [a,b] is Riemann integrable iff the set of points at which f is discontinuous has measure zero.

Proof: Let K be the lower envelope of f, which is defined by reversing the roles of inf and sup in definition 13.

Then $R\underline{\int_a^b} f = \int_a^b K$ by similar reasoning as in proposition 14.

If f is a bounded, Riemann integrable function,

$\underline{\int_a^b} f = \int_a^b K = \overline{\int_a^b} f = \int_a^b h$.

Now h has a finite number of points $m_1(n)$ such that $h(x_i) > \phi_n(x_i) \geqslant f(x_i)$ for some step function ϕ_n.

Similarly $K(x_i) < \psi_n(x_i) \leq f(x_i)$ for some step function ψ_n , a finite number, $m_2(n)$, of times.

This implies that the total number of discontinuities is $m_1(n) + m_2(n) = N(n)$, where n corresponds to the step functions ϕ_n and ψ_n.

Now N(n) is countable, therefore $m^*[N(n)] = 0$.

If f is discontinuous only on a set of measure zero, $h = k$ except on a set of measure.

Let $E = [a,b]$ and let A be the set of points of discontinuity of f.

Then $\int_E h = \int_{E-A} h + \int_A h = \int_{E-A} h = \int_{E-A} K = \int_{E-A} K + \int_A K = \int_E K$.

Therefore, $R\overline{\int_a^b} f = \int_a^b h = \int_a^b K = R\underline{\int_a^b} f$, which implies that f is Riemann integrable.

16. Definition: If f is a nonnegative measurable function defined on a measurable set E, we define $\int_E f = \sup_{h \leq f} \int_E h$, where h is a bounded function such that $m\{x: h(x) \neq 0\}$ is finite.

17. Theorem: If $\langle f_n \rangle$ is a sequence of nonnegative measurable functions and $f_n(x) \rightarrow f(x)$ a.e. on a set E, then $\int_E f \leq \underline{\lim} \int_E f_n$.

18. Theorem: Let $\langle f_n \rangle$ be an increasing sequence of nonnegative measurable functions, and let $f = \lim f_n$. Then $\int f = \lim \int f_n$.

19. Theorem: Let f be a nonnegative function which is integrable over a set E. Then given $\epsilon > 0$, there is a $\delta > 0$ such that for every set $A \subset E$ with $mA < \delta$, we have $\int_A f < \epsilon$.

20. Proposition: Let f be a nonnegative, measurable function. Then $\int f = 0$ implies that $f = 0$ a.e.

Proof: Assume it is not true that $f = 0$ a.e..

Then $m\{x: f(x) > 0\} > 0$.

Now $\{x: f(x) > 0\} = \bigcup_{n \geq 1} \{x: f(x) > 1/n\}$,

So for some n, $m\{x: f(x) > 1/n\} > 0$.

Let $M = \{x: f(x) > 1/n\}$.

Let $mM = v$ so that $v > 0$.

Let $\psi(x) = 1/n$ if $x \in M$ and $\psi(x) = 0$ if $x \notin M$.

Then $\int \psi = 1/n > 0$,

and since $\psi \leq f$,

$\int f \geq \int \psi > 0$.

21. Definition: A nonnegative measurable function f is called integrable over the measurable set E if $\int_E f < \infty$.

22. Proposition. Let f be a nonnegative, measurable function. Then there is an increasing sequence $\langle \phi_n \rangle$ of nonnegative simple functions, each of which vanishes outside a set of

finite measure, such that $f = \lim \phi_n$.

Proof: Let N be a positive integer.

Let $A_n = \{x: n/N \leq f(x) < (n+1)/N\}$, $n = 0,1,\ldots\ldots(N^2-1)$.

Let $A_{N^2} = \{x: N \leq f(x)\}$.

Let $\phi_N(x) = n/N$ if $x \in A_n$, $n = 0,1,\ldots\ldots,N^2$.

Let E be the domain of f.

Then $\bigcup_{n=0}^{N^2} A_n = E$, and

$\phi_n = \sum_{n=0}^{N^2} n/N \, \chi_{A_n}$.

Then if f is bounded on E, for every $\epsilon > 0$, there is a positive integer N such that $|f(x) - \phi_N(x)| < \epsilon$.

If f is infinite, we have $\phi_N(x) = N \rightarrow \infty$.

Therefore, $\langle \phi_N \rangle \rightarrow f$, and $\phi_N \leq \phi_{N+1}$.

23. Proposition: If f is a nonnegative measurable function, then $\int f = \sup \int \phi$ over all simple functions $\phi \leq f$.

Proof: $\int_E f = \sup \int_E h$ over all bounded functions h such that $h \leq f$, and $m\{x: h(x) \neq 0\}$ is finite.

And $\int_E h = \inf \int_E \psi$ over all simple functions $\psi \geq h$.

Let $\langle \psi_n \rangle$ be a sequence of nonnegative simple functions which vanish outside of E, and let $\lim \psi_n = f$. (This is possible by the preceding proposition).

Then for every $h \geq 0$, there exists an n such that $h \leq \psi_n \leq f$, and $\int_E h \leq \int_E \psi_n \leq \int_E f$.

Therefore $\sup \int_E h = \sup \int_E \psi_n = \int_E f$.

24. Proposition: Let f be a nonnegative integrable function. Then the function defined by $F(x) = \int_{-\infty}^{x} f$ is continuous.

Proof: Let E be the domain of f.

Let x_0 be any point in E.

Let $A_x = (-\infty, x) \cap E$.

If $x \leq x_0$, then $A_{x_0} - A_x = (-\infty, x_0) \cap E - (-\infty, x) \cap E$
$= (x, x_0) \cap E$.

If $x_0 \leq x$, then $A_x - A_{x_0} = (-\infty, x) \cap E - (-\infty, x_0) \cap E$
$= (x_0, x) \cap E$.

Then if $x \leq x_0$, $\left| \int_{A_{x_0}} f - \int_{A_x} f \right| = \left| \int_{(x,x_0) \cap E} f \right| < \epsilon$

whenever $m(x, x_0) < \delta$, for some $\delta > 0$, by theorem 4.19.

When $x_0 \leq x$, $\left| \int_{A_x} f - \int_{A_{x_0}} f \right| < \epsilon$ when $m(x, x_0) < \delta$

for some $\delta > 0$ by the same theorem.

25. Proposition: The inequality $\int_E f \leq \underline{\lim} \int_E f_n$ may be strictly less than.

Proof: Let $f_n(x) = 1$ if $n \leq x < n+1$, and $f_n(x) = 0$ otherwise.

Let E be the positive real numbers.

Now $f = \lim f_n = 0$, since for any $x \in E$, there exists an $n > x$ such that $x \notin [n, n+1]$, which implies that $f(x) = 0$.

However, $\int_E f_n = 1$ for all n, since for any n, $[n, n+1]$ is contained in E.

Therefore, $\lim {}_E f_n = \underline{\lim} {}_E f_n = 1$, and

$0 = \int_E f \leq \underline{\lim} \int_E f_n = 1$.

26. Proposition: It is not necessarily true that if $\langle f_n \rangle$ is a decreasing sequence of nonnegative, measurable functions with $f = \lim f_n$, then $\int f = \lim \int f_n$.

Proof: Let $f_n(x) = 1$ if $x \geq n$, and $f_n(x) = 0$ if $x < n$.

Then $\langle f_n \rangle \to f = 0$ as $n \to \infty$.

$\int f_n = \infty$ for all n, which implies, $\lim \int f_n = \infty$.

Therefore, $0 = \int f < \lim \int f_n = \infty$.

27. Proposition: If $\langle f_n \rangle$ is a sequence of nonnegative, measurable functions, then $\int \underline{\lim}\, f_n \leq \underline{\lim} \int f_n$.

Proof: $\underline{\lim}\, f_n$ is measurable by theorem 3.24.

Let h be a bounded, measurable function, not greater than $\underline{\lim}\, f_n$, which vanishes outside a set E' of finite measure.

Let $h_n(x) = \min\{h(x), f_n(x)\}$,

then the sequence $\langle h_n \rangle \to h$, and h_n is bounded by the bound of h. Now,

$\int_E h = \int_{E'} h = \lim \int_{E'} h_n \leq \lim \int_{E'} f_n = \underline{\lim} \int_{E'} f_n \leq \underline{\lim} \int_E f_n$.

And sup $h = \underline{\lim}\, f_n$, which implies $\int_E \underline{\lim}\, f_n \leq \underline{\lim} \int_E f_n$.

28. Proposition: Let $\langle f_n \rangle$ be a sequence of nonnegative, measurable functions which converge to f, and suppose $f_n \leq f$ for each n. Then $\int_E f = \lim \int_E f_n$.

Proof: $\int_E f_n \leq \int_E f$ for all n.

Then $\overline{\lim} \int_E f_n \leq \int_E f$ for all n. And by theorem 4.17

$\int_E f \leq \underline{\lim} \int_E f_n$, which implies $\overline{\lim} \int_E f_n \leq \underline{\lim} \int_E f_n$.

Then $\overline{\lim} \int_E f_n = \underline{\lim} \int_E f_n = \lim \int_E f_n = \int_E f$.

29. Proposition: Let f_n be a sequence of nonnegative, measurable functions on $(-\infty, \infty)$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f$. Then for each measurable set E we have $\int_E f_n \to \int_E f$.

Proof: For every $\epsilon > 0$, there exists a positive integer

N such that $|f_n - f| < \epsilon$, whenever $n \geq N$, except on a set of measure zero. Now,

$\lim \int |f_n - f| = \int \lim |f_n - f| = \int 0 = 0$, that is,

$\int |f_n - f| < \epsilon$ whenever $n \geq N$ for some N. Now,

$\int |f_n - f| = \int_E |f_n - f| + \int_{\tilde{E}} |f_n - f|$, which implies

$\int |f_n - f| \geq \int_E |f_n - f|$. Then $\epsilon > \int |f_n - f| \geq \int_E |f_n - f| \geq$

$|\int_E f_n - f| \geq \int_E f_n - f = \int_E f_n - \int_E f$, which implies

$\int_E f_n \to \int_E f$.

30. Definition: $f^+(x) = \max\{f(x), 0\}$, $f^-(x) = \max\{-f(x), 0\}$, $|f| = f^+ + f^-$.

31. Definition: A measurable function f is said to be integrable over E if f^+ and f^- are both integrable over E. In this case we define $\int_E f = \int_E f^+ - \int_E f^-$.

32. Theorem: Let f and g be integrable over E. Then:

(1) cf is integrable over E, and $\int_E cf = c\int_E f$.

(2) The function $f + g$ is integrable over E and $\int_E f + g = \int_E f + \int_E g$.

(3) If $f \leq g$ a.e., then $\int_E f \leq \int_E g$.

(4) If A and B are disjoint, measurable sets contained in E, then $\int_{A \cup B} f = \int_A f + \int_B f$.

33. Theorem: Let g be integrable over E and let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g$ on E, and for almost all x in E we have $f(x) = \lim f_n(x)$. Then $\int_E f = \lim \int_E f_n$.

34. Theorem: Let $\langle g_n \rangle$ be a sequence of integrable functions which converge almost everywhere to an integrable function

g. Let $\langle f_n \rangle$ be a sequence of measurable functions such that $|f_n| \leq g_n$ and $\langle f_n \rangle$ converges a.e.. If $\int g = \lim \int g_n$, then $\int f = \lim \int f_n$.

35. Proposition: If f is integrable over E, then so is $|f|$, and $|\int_E f| \leq \int_E |f|$.

Proof: If f is integrable so are f^+ and f^-. Then, $\int_E f^+ + \int_E f^- = \int_E f^+ + f^- = \int_E |f|$, so $|f|$ is integrable. Now $|\int_E f| = |\int_E f^+ - f^-| \leq |\int_E f^+ + f^-| = |\int_E |f|| = \int_E |f|$. Also, if $|f|$ is integrable the f^+ and f^- must be, and $\int |f| = \int f^+ + f^- \geq \int f^+ - f^- = \int f$, which implies that f is integrable.

36. Proposition: The improper Riemann integral may exist without the function being integrable in the sense of Lebesgue Lebesgue. e.g. if $f(x) = (\sin x)/x$ on $[0, \infty)$. If f is Lebesgue integrable, then the improper Riemann integral is equal to the Lebesgue integral whenever the former exists.

Proof: From advanced calculus we have

$R\int_0^\infty (\sin x)/x \, dx = \pi/2$.

Now $\int_0^\infty (\sin x)/x \, dx = \int_0^\infty [(\sin x)/x]^+ dx - \int_0^\infty [(\sin x)/x]^- dx$

so if $\int_0^\infty |(\sin x)/x| dx = \infty$, then $\int_0^\infty (\sin x)/x \, dx = \infty$,

and hence would not be integrable.

Let $A = \bigcup_{n \geq 0} \{x: (2n+1)\pi/2 - \pi/3 < x < (2n+1)\pi/2 + \pi/3\}$.

Then, if $x \in A$, we have $|\sin x| \geq 1/2$.

Let $h(x) = 1/2$ for all $x \in A$, and $h(x) = 0$ otherwise.

Then $\int_0^\infty |(\sin x)/x| dx \geq \int_0^\infty h(x)/x \, dx = 1/2 \sum_{n \geq 0} \int_{(2n+1)\pi/2 - \pi/3}^{(2n+1)\pi/2 + \pi/3} dx/x$

$$\geq 1/2 \sum_{n\geq 0} \int_{(2n+1)\pi/2 - \pi/3}^{(2n+1)\pi/2 + \pi/3} dx/[(2n+1) + /3]$$

$$= 1/2 \sum_{n\geq 0} (2\pi/3)(1/[(2n+1)\pi/2 + \pi/3]) = \infty .$$

Therefore (sin x)/x is not Lebesgue integrable.

Now if f is Lebesgue integrable, f^+ and f^- are. $\int f^+ = \sup \int h$ over all bounded functions $h \leq f^+$ such that $m\{x: h(x) \neq 0\}$ is finite. Now since all step functions are bounded and defined on a finite interval $R\int_{-\infty}^{\infty} f^+ \leq \sup_{h\leq f^+} \int_{-\infty}^{\infty} h = \int_{-\infty}^{\infty} f^+ \leq R\int_{-\infty}^{\infty} f^+$, but since the function f is Riemann integrable, equality must hold and $R\int_{-\infty}^{\infty} f^+ = \int_{-\infty}^{\infty} f^+$.

Similarly $R\int_{-\infty}^{\infty} f^- = \int_{-\infty}^{\infty} f^-$, and by subtracting we obtain,

$$R\int_{-\infty}^{\infty} f^+ - R\int_{-\infty}^{\infty} f^- = \int_{-\infty}^{\infty} f^+ - \int_{-\infty}^{\infty} f^- , \text{ or}$$

$$R\int_{-\infty}^{\infty} f = \int_{-\infty}^{\infty} f.$$

37. Proposition: For a simple function ϕ, the two definitions

(1) $\int \phi = \sum^{n} a_i m A_i$

(2) $\int \phi = \int \phi^+ - \int \phi^-$

are equivalent.

Proof: $\int \phi^+ = \int(\phi \vee 0) = \sum^{k} b_i m F_i$, where $b_i > 0$ for all i, and $F_i = \{x: \phi^+(x) = b_i\}$.

And $\int \phi^- = \int(-\phi \vee 0) = \sum^{L} c_i m G_i$, where $c_i > 0$, and $G_i = \{x: \phi(x)^- = c_i\}$.

Let $a_i = b_i$ and $E_i = F_i$ for $i = 1,\dots,k$.

Let $a_{k+i} = -c_i$ and $E_{k+i} = G_i$ for $i = 1,\dots,L$.

Let $n = k + L$.

Then $\int\phi = \int\phi^+ - \int\phi^- = \sum^k b_i mF_i - \sum^L c_i mG_i$
$= \sum^n a_i mE_i$.

38. Proposition: Let g be an integrable function on a set E and suppose $\langle f_n \rangle$ is a sequence of measurable functions such that $|f_n(x)| \leq g(x)$ a.e. on E. Then
$\int_E \underline{\lim}\, f_n \leq \underline{\lim}\int_E f_n \leq \overline{\lim}\int_E f_n \leq \int_E \overline{\lim}\, f_n$.

Proof: $g + f_n$ and $g - f_n$ are nonnegative measurable functions, and $\int_E \underline{\lim}(g + f_n) \leq \underline{\lim}\int_E (g + f_n)$.

Now $\int_E g + \int_E \underline{\lim}\, f_n \leq \int_E g + \underline{\lim}\int_E f_n$.

This implies that $\int_E \underline{\lim}\, f_n \leq \underline{\lim}\int_E f_n$.

And $\int_E \underline{\lim}(g-f_n) \leq \int \underline{\lim}\,_E(g-f_n)$.

Now $\underline{\lim}(g-f_n) = g + \underline{\lim}(-f_n) = g - \overline{\lim}\, f_n$,

and $\underline{\lim}\int_E (g-f_n) = \underline{\lim}[\int_E g - \int_E f_n] = \int_E g - \overline{\lim}\int_E f_n$.

Therefore, $\int_E g - \int_E \overline{\lim}\, f_n \leq \int_E g - \overline{\lim}\int_E f_n$,

which implies $\overline{\lim}\int_E f_n \leq \int_E \overline{\lim}\, f_n$.

Therefore it follows that

$\int_E \underline{\lim}\, f_n \leq \underline{\lim}\int_E f_n \leq \overline{\lim}\int_E f_n \leq \int_E \overline{\lim}\, f_n$.

39. Proposition: Let f be integrable over E. Then given $\epsilon > 0$, there is a simple function ϕ such that $\int_E |f-\phi| < \epsilon$.

Proof: We have extablished the fact that there exists an increasing sequence of simple functions such that $\langle\phi_n\rangle \uparrow f^+$ and similarly there is an increasing sequence of simple functions such that $\langle\psi_n\rangle \uparrow f^-$.

Then for every $\epsilon/2mE > 0$, there is an n such that $|f^+ - \phi_n| < \epsilon/2mE$. And there exists an m such that $|f^- - \psi_m| < \epsilon/2mE$.

Then $\int_E |f^+ - \phi_n| < \int_E \epsilon/2mE = \epsilon/2$, and $\int_E |f^- - \psi_n| < \epsilon/2$.

Then $\int_E |(f^+-O_n) - (f^- - \Psi_n)| \leq \int_E |f^+-\phi_n| + |f^- - \Psi_n| < \epsilon$.

But $f^+-f^- = f$, and letting $\lambda_n = \phi_n - \Psi_n$, which has been shown to be a simple function, $\int_E | f-\lambda_n| < \epsilon$,

40. Proposition: Let f be integrable over E. Then given $\epsilon > 0$, then there is a step function g such that $\int_E |f-g| < \epsilon$.

Proof: From the proof of the previous proposition we have $|f-\phi| < \epsilon/2mE$ for some simple function ϕ.

By proposition 3.33 there is a step function $g(x) = \phi(x)$ except on a set A of measure less than $\epsilon/2B$. Then,

$|f-g| = |(f-\phi) + (\phi-g)| \leq |f-\phi| + |\phi-g| = |f-0|_{\text{on } E-A}$

$< \epsilon/2mE$.

Since g is a step function and ϕ is a simple function, they are both bounded. Let B be a bound for $|\phi-g|$.

Then $\int_E |f-g| \leq \int_E |f-\phi| + |\phi-g|$

$= \int_{E-A} |f-\phi| + \int_A |f-\phi| + |\phi-g|$

$= \int_E |f-\phi| + \int_A |\phi-g|$

$< \int_E \epsilon/2mE + \int_A B$

$= \epsilon/2 + BmA$

$< \epsilon/2 + B\epsilon/2B = \epsilon/2 + \epsilon/2 = \epsilon$.

That is $\int_E |f-g| < \epsilon$.

41. Proposition: Let f be integrable over E. Then there is a contiouous function h vanishing outside a finite interval such that $\int_E |f-h| < \epsilon$.

Proof: From the proof of the previous proposition we have a step function g such that $|f-g| < \epsilon/3mE$ except for a set A of measure less than $\epsilon/6B$.

By proposition 3.34, there is a continuos function

$h(x) = g(x)$ except on a set C of measure less than

$/6D$.

Now $|f-h| = |(f-g) + (g-h)| \leq |f-g| + |g-h|$, and

$\int_E |f-h| \leq \int_E |f-g| + |g-h| = \int_{E-(A \cup C)} |f-g|$

$+ \int_A |f-g| + |g-h| + \int_C |f-g| + |g-h|$

$= \int_{E-(A \cup C)} |f-g| + \int_A |f-g| + \int_A |g-h| + \int_C |f-g| + \int_C |g-h|$.

$< \int_E \epsilon/2mE + \int_A B + \int_{A \cap C} D + \int_{A \cap C} B + \int_C D$,

where B and D are bounds of $|f-g|$ on A and $|g-h|$ on C, respectively,

$< \epsilon/3 + 2BmA + 2DmC$

$< \epsilon/3 + 2B\epsilon/6B + 2D\epsilon/6D$

$= \epsilon/3 + \epsilon/3 + \epsilon/3 = \ .$

Then $\int |f-h| < \epsilon$.

V. REFERENCES

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VI. VITA

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