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Axiomatic Set Theory

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AXIOMATIC SET THEORY

by

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INTRODUCTION

An analysis of the well known paradoxes found in intuitive set theory has led to the reconstruction of set theory by axiomatic means. This exposition is devoted to Zermelo-Fraenkel set theory with some changes made by Suppes.

The first order predicate calculus is presupposed. In addition to the usual quantifiers admitted ('$\forall$' universal and '$\exists$' existential), a unique existential quantifier is used (denoted E!). The primitive notions of the set theory are the empty set (denoted 'O') and the two place membership predicate (denoted '$\in$').

Prime formulae shall be of the form '$x \in a$' or '$a = b$' but not all formulae are prime. A recursive definition of a (composite) formula is:

1) Every prime formula is a formula;
2) If P is a formula, then $-P$ is a formula;
3) If P and Q are formulae, then $P \wedge Q$, $P \vee Q$, $P \longrightarrow Q$, and $P \longleftrightarrow Q$ are formulae;
4) If P is a formula and x is a variable, then $(\forall x)P$, $(\exists x)P$ and $(E!x)P$ are formulae;
5) No expression is a formula unless its being follows from a finite string of the above four types.

In 1893 Frege formulated the axiom schema of abstraction. It claims that for every property there is a set having that property. However, in 1901 Russell discovered that the axiom contained a contradictory notion. Frege's axiom was:

$$(\exists y)(\forall x)(x \in y \longleftrightarrow \emptyset(x))$$

where $\emptyset(x)$ is a formula in which y is not free. Clearly the axiom is not a definite assertion but a schema for making many assertions since any formula in which y is not free may replace '$\emptyset(x)$'. Russel let $\emptyset(x)$ be $x \notin x$ and he then had:

$$(\exists y)(\forall x)(x \in y \longleftrightarrow x \notin x)$$

Now since this formula is for all x it is particularly true when $x = y$ so $(\exists y)(y \in y \longleftrightarrow y \notin y)$ which is a self-contradictory notion since this says

$$(\exists y)((y \in y \longrightarrow y \notin y) \wedge (y \notin y \longrightarrow y \in y))$$

or
$$(\exists y)((y \notin y \vee y \notin y) \wedge (y \in y \vee y \in y))$$

or
$$(\exists y)(y \notin y \wedge y \in y)$$

In 1908 Zermelo revised this axiom to form an axiom schema of separation. Zermelo's axiom has the property that it can 'separate off' the elements at a given set that satisfy a given property to form a new set. Thus if the set of all automobiles is known to exist, the axiom schema of separation establishes the existance of all cars made by General Motors. Formally the axiom schema of separation is:

$$(\forall z)(\exists y)(\forall x)(x \in y \longleftrightarrow x \in z \wedge \emptyset(x))$$

where $\emptyset(x)$ is a formula in which y is not free. With this restriction, Russell's paradox cannot be reconstructed according to Stoll.

Any definition is admitted if it satisfies two conditions. First a method of eliminating new symbols from any formula must be given, and second, the definition cannot yield a formula in primitive notation unprovable by previous axioms. These two criterion are stated as:

Criterion of Eliminability. A formula P introducing a new symbol satisfies the criterion of eliminability if and only if: whenever Q_1 is a formula in which the new symbol occurs, then there is a primitive formula Q_2 such that $P \longrightarrow (Q_1 \longleftrightarrow Q_2)$ is derivable from the axioms.

Criterion of Non-Creativity. A formula P introducing a new symbol satisfies the criterion of non-creativity if and only if: there is no primitive formula Q such that $P \longrightarrow Q$ is derivable from the axioms but Q is not.

Now the problem is to provide rules when satisfied implies satisfaction of these two criterion. Rules for defining operation symbols are given below but nominal changes result in rules for defining relation symbols. Proper definitions of operation symbols are either equivalence or identities.

An equivalence P introducing a new n-place operation symbol O is a proper definition if and only if P is of the form: $O(v_1, v_2, \ldots, v_n) = w \longleftrightarrow Q$ and the following are satisfied: i) $v_1, v_2, \ldots, v_n, w$ are distinct variables, ii) Q has no free variables other than $v_1, v_2, \ldots, v_n, w$; iii) Q is a formula in which the only non-logical constants are the primitive or previously defined symbols of set theory; iv) the formula $(E!w)Q$ is derivable from the axioms and preceding definitions.

In iii) reference is made to non-logical constants. Some examples of non-logical constants are: '$\in$', '$\subseteq$' (set inclusion to be defined later), '$\subset$' (proper set inclusion to be defined later), '$\cap$' (set intersection also to be defined later). The only logical constants are:

negation, conjuction, disjunction, implication, and equivalence. In iv) the expression 'preceding definition' implies that the definitions will be given in an order and not simultaneously. This allows new definitions to be in terms of already defined symbols. Another way to express iv) is that performing an operation always yields a unique object.

An identity P introducing a new n-place operation symbol O is a proper definition if and only if P is of the form:

$O(v_1, v_2, \ldots, v_n) = t$ and the following are satisfied: i) $v_1, v_2, \ldots, v_n$ are distinct variables; ii) the term t has no free variables other than $v_1, v_2, \ldots, v_n$; iii) the only non-logical constants in t are primitive symbols and previously defined symbols of set theory.

Some of the definitions in this report do not satisfy the criterion of eliminability. This is because they are conditional in form but appropriate modifications of the two rules can be made.

CHAPTER I

Axioms of Extensionality and Separation

The first definition, that of a set, conforms with the intuitive feeling of what a set is. That is, a set is something that has elements or is the empty set.

Definition 1.

$$(\forall y) \quad y \text{ is a set} \Longleftrightarrow (\exists x)(x \in y \lor y = 0)$$

The axiom of extensionality is an axiom stating when two set are identical.

Axiom 1. The Axiom of Extensionality

$$(\forall x)(x \in A \longleftrightarrow x \in B) \longrightarrow A = B$$

Considering the earlier remarks regarding the axiom schema of separation, no further introductory comment should be necessary.

Axiom 2. The Axiom Schema of Separation

$(\forall A)(\exists B)(\forall x)(x \in B \longleftrightarrow x \in A \land \emptyset(x))$ where B is not free in $\emptyset(x)$

The first theorem of the report states that the empty set contains no elements.

Theorem 1. $(\forall x)(x \notin 0)$

Proof: In the axiom schema of separation, let $\emptyset(x)$ be $x \neq x$ and $A = 0$. Then $(\exists B)(\forall x)(x \in B \longleftrightarrow x \in 0 \land x \neq x)$. Since there exists a set with the above property, call it D. Also since this is true for all x, it is particularly true for an arbitrary z.

Hence $z \in D \longleftrightarrow z \in 0 \wedge z \neq z$

but the conjuction $z \in 0 \wedge z \neq z$

implies $z \neq z$

which is false.

Therefore $z \in D$ is false

or $z \notin D$ is true

Now z was arbitrary.

Therefore $(\forall x)(x \notin D)$

So by definition 1 $D = 0$

Hence $(\forall x)(x \notin 0)$

Corresponding to theorem 1 is a theorem that says if a set does not have any elements then it is the empty set.

<u>Theorem 2.</u> $(\forall x)(x \notin A \longleftrightarrow A = 0)$

The next definition, that of set inclusion or subset, is of fundamental importance throughout the report.

<u>Definition 2.</u>

$$(\forall A)(\forall B)\left[\, A \subseteq B \longleftrightarrow (\forall x)(x \in A \rightarrow x \in B)\right]$$

<u>Theorem 3.</u> $(\forall A)(A \subseteq A)$

The proof of theorem 3 follows very quickly from definition 2. The next theorem's proof utilizes the axiom of extensionality however, and is of noteworthy importance.

<u>Theorem 4.</u> $(\forall A)(\forall B)\left[(A \subseteq B \wedge B \subseteq A) \rightarrow (A = B)\right.$

Proof: Select arbitrary C and D such that $C \subseteq D \wedge D \subseteq C$

$$C \subseteq D \rightarrow (\forall x)(x \in C \rightarrow x \in D)$$

$$\text{and } D \subseteq C \rightarrow (\forall x)(x \in D \rightarrow x \in C)$$

$$(C \subseteq D \wedge D \subseteq C) \longrightarrow (\forall x)(x \in C \rightarrow x \in D) \wedge (\forall x)(x \in D \rightarrow x \in C).$$

$$\longrightarrow (\forall x)\left[(x \in C \rightarrow x \in D) \wedge (x \in D \rightarrow x \in C)\right]$$

$$\longrightarrow (\forall x)\left[x \in C \longleftrightarrow x \in D\right]$$

$$\longrightarrow C = D$$

and since C and D were arbitrary, $(\forall A)$ and $(\forall B)$ the theorem follows.

<u>Theorem 5.</u> $A \subseteq O \rightarrow A = O$

The following theorem states the well known transitivity property for subsets.

<u>Theorem 6.</u> $(\forall A)(\forall B)(\forall C)\left[(A \subseteq B) \wedge (B \subseteq C) \rightarrow A \subseteq C\right]$

Proof: Select arbitrary sets A, B, C such that $A \subseteq B \wedge B \subseteq C$.

$$A \subseteq B \rightarrow (\forall x)(x \in A \rightarrow x \in B)$$

$$B \subseteq C \rightarrow (\forall x)(x \in B \rightarrow x \in C)$$

so $(A \subseteq B \wedge B \subseteq C) \rightarrow (\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in C)$

but $(\forall y)(y \in A \rightarrow y \in B) \rightarrow (x \in A \rightarrow x \in B)$

and $(\forall y)(y \in B \rightarrow y \in C) \rightarrow (x \in B \rightarrow x \in C)$

so

$$\left[(\forall y)(y \in A \rightarrow y \in B) \wedge (\forall y)(y \in B \rightarrow y \in C\right] \rightarrow \left[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in C)\right]$$

$$\rightarrow (x \in A \rightarrow x \in C)$$

therefore

$$\left[(\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in C)\right] \rightarrow \left[(\forall x)(x \in A \rightarrow x \in C)\right] \rightarrow A \subseteq C$$

The following definition is that of proper set inclusion or proper subset.

<u>Definition 3.</u> $A \subset B \rightarrow A \subseteq B \wedge A \neq B$

Anti-reflexivity is asserted in the following theorem.

<u>Theorem 7.</u> $(\forall A)(-A \subset A)$

Proof: Suppose not, that is suppose

$$(\forall A)(A \subset A)$$

$$(\forall A)(A \subset A) \rightarrow (\forall A)(A \subseteq A \wedge A \neq A)$$

$$\rightarrow (\forall A)(A \neq A)$$

which is an obvious contradiction.

The next theorem illustrates that proper inclusion is anti-symetric.

Theorem 8. $A \subset B \rightarrow -(B \subset A)$

Proof: Suppose $B \subset A$

$$B \subset A \rightarrow (\forall x)(x \in B \rightarrow x \in A) \wedge A \neq B$$
$$\rightarrow (\forall x)(x \notin B \vee x \in A) \wedge A \neq B$$
$$\rightarrow (\forall x)(x \notin B \vee x \in A) \quad - - - - - - (1)$$

but
$$A \subset B \rightarrow (\forall x)(x \in A \rightarrow x \in B) \wedge A \neq B$$
$$\rightarrow (\forall x)(x \notin A \vee x \in B) \wedge A \neq B$$
$$\rightarrow (\forall x)(x \notin A \vee x \in B)$$
$$\rightarrow (\forall x)(x \in B \vee x \notin A) \quad - - - - - - (2)$$

but (1) and (2) are contradictory, therefore $-(B \subset A)$ consequently $A \subset B \rightarrow -(B \subset A)$.

Theorem 9. $(A \subset B \wedge B \subset C) \rightarrow A \subset C$

The proof is similiar to theorem 6.

Theorem 10. $A \subset B \rightarrow A \subseteq B$

Proof: $A \subset B \rightarrow (\forall x)(x \in A \rightarrow x \in B) \wedge A \neq B$

$$\rightarrow (\forall x)(x \in A \rightarrow x \in B)$$
$$\rightarrow A \subseteq B$$

Intersection, Union and Difference of Sets

The next theorem asserts the uniqueness of the intersection operation to be defined.

Theorem 11. $(\exists ! C)(\forall x)(x \in C \leftrightarrow x \in A \wedge x \in B)$

The proof is in Suppes and very similar to the proof of theorem 19 (but using the axiom schema of separation in place of the union axiom to be introduced).

Definition 4.

$$A \cap B = y \longleftrightarrow (\forall x)(x \in y \longleftrightarrow x \in A \wedge x \in B) \wedge y \text{ is a set.}$$

Theorem 12. $(\forall x)(x \in A \cap B \longleftrightarrow x \in A \wedge x \in B)$

Proof: In definition 4, let $y = A \cap B$

then $(A \cap B) = (A \cap B) \longleftrightarrow (\forall x)(x \in A \cap B \longleftrightarrow x \in A \wedge x \in B) \wedge (A \cap B)$ is a set.

Commutativity and associativity are established in the following two theorems for intersection.

Theorem 13. $(\forall A)(\forall B)(A \cap B = B \cap A)$

Proof: Select arbitrary A and B and

$$\begin{aligned} (A \cap B) = y &\longleftrightarrow (\forall x)(x \in y \longleftrightarrow x \in A \wedge x \in B) \wedge y \text{ is a set} \\ &\longleftrightarrow (\forall x)(x \in y \longleftrightarrow x \in B \wedge x \in A) \wedge y \text{ is a set} \\ &\longleftrightarrow (B \cap A) = y \end{aligned}$$

therefore $(\forall A)(\forall B)(A \cap B = B \cap A)$

Theorem 14. $(\forall A)(\forall B)(\forall C)[(A \cap B) \cap C = A \cap (B \cap C)]$

Proof: For arbitrary A, B and C, let x be arbitrary such that $x \in [(A \cap B) \cap C]$

$$\begin{aligned} x \in [(A \cap B) \cap C] &\longrightarrow x \in (A \cap B) \wedge x \in C \quad \text{by theorem 12} \\ &\longrightarrow (x \in A \wedge x \in B) \wedge x \in C \quad \text{by theorem 12} \\ &\longrightarrow x \in A \wedge (x \in B \wedge x \in C) \\ &\longrightarrow x \in A \wedge x \in (B \cap C) \\ &\longrightarrow x \in [A \cap (B \cap C)] \end{aligned}$$

Now since x was an arbitrary element of $(A \cap B) \cap C$ then

$$(A \cap B) \cap C \subseteq A \cap (B \cap C)$$

Similiarly $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ for arbitrary A, B and C

Therefore $(\forall A)(\forall B)(\forall C)[(A \cap B) \cap C = A \cap (B \cap C)$ by theorem 4.

Theorem 15. $(\forall A)(A \cap 0 = 0)$

Theorem 16. $(\forall A)(\forall B)(A \cap B \subseteq A)$

Proof: Let x be an arbitrary element of $A \cap B$ for arbitrary A and B.

But $x \in A \cap B \rightarrow x \in A \wedge x \in B$ by theorem 12

$\rightarrow x \in A$ since this is a conjunction

Now since x, A and B were arbitrary, then the theorem follows.

Theorem 17. $(A \subseteq B) \leftrightarrow (A \cap B = A)$

Proof: $\rightarrow$

$$A \subseteq B \rightarrow (\forall x)(x \in A \rightarrow x \in B)$$

therefore $(\forall x)(x \in A \rightarrow x \in A \wedge x \in B)$

or $A \subseteq A \cap B$

but theorem 16 asserts $A \cap B \subseteq A$

hence $(A \subseteq B) \rightarrow (A \cap B = A)$ - - - - - (1)

Proof: $\leftarrow$

$$(A \cap B = A) \rightarrow (A \cap B) \subseteq A \wedge A \subseteq (A \cap B)$$

Now for an arbitrary $x \in A$

$x \in A \rightarrow x \in A \wedge x \in B$ since $A \subseteq A \cap B$

so $x \in B$ conjunction

therefore $(\forall x)(x \in A \rightarrow x \in B)$ since x was arbitrary

hence $A \subseteq B$ by definition 2.

Consequently $(A \cap B = A) \rightarrow (A \subseteq B)$ - - - - - (2)

and from (1) and (2),

$$(A \subseteq B) \leftrightarrow (A \cap B = A)$$

Theorem 18. $(\forall A)(A \cap A = A)$

The following axiom is introduced in order to make the proof of the existance theorem for the union of two sets (as theorem 11 did for intersection) simpler. In the last section of this chapter this axiom, the union axiom, is proven redundant in terms of the rest of the axioms in this chapter.

Axiom 3. The Union Axiom

$$(\exists C)(\forall x)(x \in C \leftrightarrow x \in A \vee x \in B)$$

Theorem 19. $(E!C)(\forall x)(x \in C \leftrightarrow x \in A \vee x \in B)$

Proof: From the union axiom $(\exists C)(\forall x)(x \in C \leftrightarrow x \in A \vee x \in B)$

It is needed that C be unique; so suppose there is a D such that

$$(\forall x)(x \in D \leftrightarrow x \in A \vee x \in B)$$

So $(\forall x)[(x \in C) \leftrightarrow (x \in A \vee x \in B) \leftrightarrow (x \in D)]$

hence by transitivity of implication

$$(\forall x)(x \in C \leftrightarrow x \in D)$$

Consequently $C = D$ by the axiom of extensionality.

Definition 5.

$$(A \cup B = y) \leftrightarrow (\forall x)(x \in y \leftrightarrow x \in A \vee x \in B)_{\wedge} \; y \text{ is a set}$$

The usable theorem for the union operation is:

Theorem 20. $(\forall x)(x \in A \cup B \leftrightarrow x \in A \vee x \in B)$

Proof: In definition 5, let $y = A \cup B$

then $A \cup B = A \cup B \leftrightarrow (\forall x)(x \in A \cup B \leftrightarrow x \in A \vee x \in B)_{\wedge}$ y is a set.

As before, commutativity and associativity of the operation are stated.

Theorem 21. $(\forall A)(\forall B)(A \cup B = B \cup A)$

Proof: For an arbitrary $x \in A \cup B$

$$x \in A \cup B \leftrightarrow x \in A \vee x \in B$$
$$\leftrightarrow x \in B \vee x \in A$$
$$\leftrightarrow x \in (B \cup A)$$

therefore since x was arbitrary, the theorem follows.

Theorem 22. $(\forall A)(\forall B)(\forall C)[(A \cup B) \cup C = A \cup (B \cup C)]$

The proof is similiar to theorem 14.

<u>Theorem 23.</u> $(\forall A)(A \cup A = A)$

Proof: Select an arbitrary x in the union of an arbitrary set B with itself

$$x \in B \cup B \leftrightarrow x \in B \vee x \in B$$
$$\leftrightarrow x \in B$$

therefore $B \cup B = B$ since x was arbitrary and since B was arbitrary the theorem follows.

<u>Theorem 24.</u> $(\forall A)(A \cup 0 = A)$

<u>Theorem 25.</u> $(\forall A)(\forall B)(A \subseteq A \cup B)$

Proof: For an arbitrary $x \in A$

$$x \in A \rightarrow x \in A \vee x \in B$$
$$\rightarrow x \in A \cup B$$

therefore $(\forall A)(\forall B)(A \subseteq A \cup B)$ by definition 2.

<u>Theorem 26.</u> $(A \subseteq B \leftrightarrow A \cup B = B)$

Proof: $\rightarrow$

$$x \in A \cup B \rightarrow x \in A \vee x \in B$$

but $x \in A \rightarrow x \in B$ since $A \subseteq B$

therefore $(x \in A \vee x \in B) \rightarrow x \in B \vee x \in B$

$$\rightarrow x \in B$$

hence $A \cup B \subseteq B$

but $B \subseteq A \cup B$ by theorem 25

therefore $A \subseteq B \rightarrow A \cup B = B$

Proof: $\leftarrow$

$$x \in A \rightarrow x \in A \cup B \text{ by theorem 25}$$
$$\rightarrow x \in B \text{ since } A \cup B = B$$

therefore $A \subseteq B$ by definition 2

<u>Theorem 27.</u> $A \subseteq C \wedge B \subseteq C \rightarrow A \cup B \subseteq C$

The next two theorems are the usual distributive theorems for union and intersection.

Theorem 28.

$$(\forall A)(\forall B)(\forall C)\left[(A \cup B) \cap C = (A \cap C) \cup (B \cap C)\right.$$

Theorem 29.

$$(\forall A)(\forall B)(\forall C)\left[(A \cap B) \cup C = (A \cup C) \cap (B \cup C)\right.$$

Proof: $x \in (A \cap B) \cup C \leftrightarrow x \in (A \cap B) \vee x \in C$

$\leftrightarrow (x \in A \wedge x \in B) \vee x \in C$

$\leftrightarrow (x \in A \vee x \in C) \wedge (x \in B \vee x \in C)$

$\leftrightarrow x \in (A \cup C) \wedge x \in (B \cup C)$

$\leftrightarrow x \in \left[(A \cup C) \cap (B \cup C)\right)$

From this point on in the report the universal quantifies whose scope is the entire formula shall be omitted.

The next theorem establishes the uniqueness of the difference operation. In its proof, the axiom schema of separation is needed using ' $x \notin B$ ' for $\emptyset(x)$.

Theorem 30. $(E!C)(\forall x)(x \in C \leftrightarrow x \in A \wedge x \notin B)$

Definition 6.

$(A - B = y) \leftrightarrow (\forall x)(x \in y \leftrightarrow x \in A \wedge x \notin B) \wedge y$ is a set.

Theorem 31. $x \in A - B \leftrightarrow x \in A \wedge x \notin B$

Theorem 32. $A - A = 0$

Theorem 33. $A - (A \cap B) = A - B$

Theorem 34. $A \cap (A - B) = A - B$

Proof: $x \in A - B \leftrightarrow x \in A \wedge x \notin B$

$\leftrightarrow (x \in A \wedge x \in A) \wedge x \notin B$

$\leftrightarrow x \in A \wedge (x \in A \wedge x \notin B)$

$\leftrightarrow x \in \left[A \cap (A - B)\right]$

Theorem 35. $(A - B) \cup B = A \cup B$

<u>Theorem 36.</u> $(A \cup B) - B = A - B$

Proof: $x \in [(A \cup B) - B] \longleftrightarrow x \in (A \cup B) \wedge x \notin B$

$\longleftrightarrow (x \in A \vee x \in B) \wedge x \notin B$

$\longleftrightarrow (x \in A \wedge x \notin B) \vee (x \in B \wedge x \notin B)$

$\longleftrightarrow x \in A \wedge x \notin B$

$\longrightarrow x \in A - B$

<u>Theorem 37.</u> $(A \cap B) - B = 0$

<u>Theorem 38.</u> $(A - B) \cap B = 0$

The proofs of theorems 37 and 38 are similiar to the proof of theorem 36.

<u>Theorem 39.</u> $A - (B \cup C) = (A - B) \cap (A - C)$

Proof: $x \in [A - (B \quad C)] \longleftrightarrow x \in A \wedge x \notin B \cup C$

$\longleftrightarrow x \in A \wedge - (x \in B \vee x \in C)$

$\longleftrightarrow x \in A \wedge (x \notin B \wedge x \notin C)$

$\longleftrightarrow (x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$

$\longleftrightarrow x \in (A - B) \wedge x \in (A - C)$

$\longleftrightarrow x \in [(A - B) \cap (A - C)]$

The proof of the next theorem is similiar to that of the proof of theorem 39.

<u>Theorem 40.</u> $A - (B \cap C) = (A - B) \cup (A - C)$

<u>Pairing Axiom and Ordered Pairs</u>

Thus far in the report the existance of a set other than the empty set is not known. The pairing axiom establishes a set containing two elements or two individuals.

<u>Axiom 4.</u> <u>Pairing Axiom</u>

$$(\exists A)(\forall z)(z \in A \longleftrightarrow z = x \vee z = y)$$

This axiom is followed by the usual theorem asserting the desired uniqueness property.

Theorem 41. $(\exists! A)(\forall z)\{z \in A \longleftrightarrow (z = x) \vee (z = y)\}$

The proof is similiar to theorem 19.

Definition 7.

$$\{x,y\} = w \longleftrightarrow (\forall z)(z \in w \longleftrightarrow (z = x) \vee (z = y))$$

and w is a set.

Theorem 42. $z \in \{x,y\} \longleftrightarrow (z = x)$ or $(z = y)$

The proof is similiar to theorem 12.

Theorem 43.

$$\{x,y\} = \{u,v\} \longrightarrow ((x = u) \wedge (y = v)) \vee ((x = v) \wedge (y = u))$$

Definition 8.

$$\{x\} = \{x,x\} \ ; \ \{x,y,z\} = \{x,y\} \cup \{z\} \ ; \ \{x,y,z,w\} = \{x,y\} \cup \{x,v\}$$

Theorem 44. $\{x\} = \{y\} \longrightarrow x = y$

Proof: In theorem 43 let $y = x$, $u = v$.

The theorem follows immediately from definition 8.

With the above notion of unordered pair, a new concept is introduced, that of ordered pair. Intuitively, an ordered pair is simply two objects given in a fixed order.

Definition 9. $\langle x,y \rangle = \{\{x\}, \{x,y\}\}$

From the axiom of extensionality it is known that two sets are equal if and only if they contain the same elements. The requirement of equality of ordered pairs is stricter. Two ordered pairs are equal when the first member of one is the same as the first member of the other and the second member of one is the same as the second member of the other. Formally:

Theorem 45. $(\langle x,y \rangle = \langle u,v \rangle) \longrightarrow (x = u \wedge y = v)$

Definition by Abstraction

Earlier it was established that the axiom of abstraction yields a contradiction, yet the idea of 'the set of all objects having the property $\emptyset$ ' is commonly used in numerous branches of mathematics. The objective of this section is to give a formal definition of the abstraction operation

$$\{x: \ \emptyset(x)\}$$

This operation is merely a method of binding variables. It is not a definition introducing a relation symbol, operation symbol, or individual constant.

Definition Schema 10.

$$\{x: \ \emptyset(x)\} = y$$

$$\longleftrightarrow \left[(\forall x)(x \in y \longleftrightarrow \emptyset(x)) \wedge y \ \text{ is a set}\right] \vee \left[y = 0 \wedge -(\exists B)(\forall x)x \in B \longleftrightarrow \emptyset(x))\right.$$

The notion of a definition schema should be familiar to the reader considering the earlier remarks said about axiom schemas. Clearly $\{x: \ \emptyset(x)\}$ is a set but there are occasions where there is no non-empty set whose elements have a property $\emptyset$, hence the definition must allow this possibility as it does.

Theorem Schema 46.

$$y \in \{x: \ \emptyset(x)\} \rightarrow \emptyset(y)$$

Proof: $y \in \{x: \ \emptyset(x)\} \rightarrow \{x: \ \emptyset(x)\}$ is not empty.

So by definition 10 the theorem follows immediately.

Theorem 47. $A = \{x: \ x \in A\}$

Theorem 48. $0 = \{x: \ x \neq x\}$

Proof: Suppose the contrary, that is $y \in \{x: \ x \neq x\}$

Then by theorem 46, $y \neq y$ a contradiction.

The next theorem is the usual theorem contradicting the existance of a universal set.

<u>Theorem 49.</u> $\quad -(\exists A)(\forall x)(x \in A)$

Proof: Suppose the contrary, that is, suppose there does exist a universal set. Call this universal set D , and in the axiom schema of separation, let $\emptyset(x)$ be '$x \notin x$' yielding

$$(\exists B)(\forall x)(x \in B \rightarrow x \in D \quad x \notin x)$$

Since there exists a set B with this property, call it C . Then

$$(\forall x)(x \in C \rightarrow x \in D \wedge x \notin x)$$

but D was the universal set so

$$(\forall x)(x \in C \rightarrow x \notin x)$$

and this is for all x , therefore in particular for C we have

$$C \in C \rightarrow C \notin C$$

which is equivalent to $\quad C \in C \wedge C \notin C$

a contradiction. Hence there does not exist a universal set.

<u>Theorem 50.</u> $\quad 0 = \{x: x \neq x\}$

Proof: Suppose not, that is suppose

$$y = \{x: x = x\} \wedge y \neq 0$$

but by definition schema 10, this is if and only if

$$[(\forall x)(x \in y \longleftrightarrow x = x) \wedge y \text{ is a set}]$$

$$\longleftrightarrow [(\forall x)(x \in y) \longleftrightarrow (\forall x)(x = x)]$$

therefore $\quad (\forall x)(x \in y)$

hence y is a universal set contradicting theorem 49.

With this notation it is not difficult to prove as theorems simple formulas which could be used to define intersection, union and difference of sets; namely

$$\text{i)} \quad A \cap B = \{x: x \in A \wedge x \in B\}$$

(proof follows from theorem 11 and definition 10)

$$\text{ii)}\quad A \cup B = \{x:\ x \in A \vee x \in B\}$$

(proof follows from theorem 19 and definition 11)

$$\text{iii)}\quad A - B = \{x:\ x \in A \quad x \notin B\}$$

(proof follows from theorem 31 and definition 10.

It will be convenient to have a slightly different form of the definition by abstraction later in this paper. Later more complicated expressions will be placed before the colon rather than simply single variables. Hence the following altered form of the definition by abstraction:

Definition Schema 11.

$$\{T(x_1, x_2, \dots, x_u) : \emptyset(x_1, x_2, \dots, x_u)\} =$$

$$\{y:\ (\exists x_1)(\exists x_2) \dots (\exists x_u)(y = T(x_1, x_2, \dots, x_u) \wedge \emptyset(x_1, x_2, \dots, x_u))\}$$

The following theorem schema expressed the important concept that equivalent properties are extensionally identical.

Theorem Schema 51.

$$(\forall x)(\emptyset(x) \longleftrightarrow P(x)) \longrightarrow \{x:\ \emptyset(x)\} = \{x:\ P(x)\}$$

It should be noted that the following is not true.

$$(\forall x)(\emptyset(x) \longrightarrow P(x)) \longrightarrow \{x:\ \emptyset(x)\} \subseteq \{x:\ P(x)\}$$

To see this let $\emptyset(x)$ be $x+3=4$ and $P(x)$ be $x=x$.
Then $\{x:\ x+3=4\} = \{1\}$ and $\{x:\ x=x\}$ is empty by theorem 50.

The Sum Axioms and Families of Sets

Before considering the Sum Axiom, which is the basis of the present section, the notion of the intersection of a family of sets is introduced. To illustrate the notation let $A = \{\{7,8\}, \{4,8\}\}$ then $\bigcap A = \{8\}$. Informally the intersection of A is the set of all things which belong to each member of A. The formal definition uses the abstraction notation introduced in Definition Schema 10.

Definition 12.

$$\bigcap A = \{x : (\forall B)(B \in A \rightarrow x \in B)\}$$

It seems like it could be proven that $(x \in \bigcap A) \leftrightarrow (\forall B)(B \in A \rightarrow x \in B)$, but this is impossible and the reason is fairly obvious.

The right side of the above equivalence is always true and every x is a member of $\bigcap A$ whenever A has no sets as members; but by theorem 49 there is no universal set. It is possible to prove the following more restricted statement however.

Theorem 52.

$$x \in \bigcap A \leftrightarrow [(\forall B)(B \in A \rightarrow x \in B) \wedge (\exists B)(B \in A)]$$

Theorem 53. $\bigcap 0 \neq 0$

Proof: Suppose not, that is $x \in \bigcap 0$, then by theorem 52 there exists a set $B \in 0$ a contradiction. Similiarly $\bigcap \{0\} = 0$ and $\bigcap \{A\} = A$.

Theorem 54. $\bigcap \{A,B\} = A \cap B$

Proof:

$$x \in \bigcap \{A,B\} \leftrightarrow [(\forall C)(C \in \{A,B\} \rightarrow x \in C) \wedge (\exists C)(C \in \{A,B\})]$$

$$\leftrightarrow (x \in A \wedge x \in B) \wedge \{A,B\} \neq 0$$

$$\leftrightarrow x \in A \cap B$$

Theorem 55. $A \subseteq B \wedge (\exists C)(C \in A) \rightarrow \bigcap B \subseteq \bigcap A$

Proof: For an arbitrary $x \in \bigcap B$

$$x \in \bigcap B \leftrightarrow (\forall C)(C \in B \rightarrow x \in C) \wedge (\exists C)(C \in B)$$

but since

$$A \subseteq B$$

$$C \in A \rightarrow C \in B$$

therefore

$$(\forall C)(C \in A \rightarrow x \in A)$$

but given that $(\forall C)(C \in A)$

therefore $(\forall C)(C \in A \rightarrow x \in A) \wedge (\exists C)(C \in A)$

$\longleftrightarrow \quad x = \bigcap A$

<u>Theorem 56.</u> $A \in B \longrightarrow \bigcap B \subseteq A$

Proof follows from theorem 52.

<u>Theorem 57.</u> $A \in B \wedge A \subseteq C \longrightarrow \bigcap B \subseteq C$

Proof follows from theorem 56.

Before introducing the sum axiom which postulates the existance of the union of a family of sets, the notation is illustrated by means of an example. Let $A = \{1, \{2\}, \{3,4\}, \{4,5\}\}$. Then $\bigcup A = \{2,3,4,5\}$. Clearly A is a family of sets together with one individual and the union or sum of A is the set of all things which belong to some member of A.

<u>Sum Axiom.</u> $(\exists C)(\forall x)(x \in C \longleftrightarrow (\exists B)(x \in B \wedge B \in A))$

Again the formal definition uses the abstraction notation.

<u>Definition 13.</u> $\bigcup A = \{x\colon (\exists B)(x \in B \wedge B \in A\}$

It should be noted that in defining anything by abstraction, if the appropriate set of elements does not exist, then $\bigcup A$ is empty. In this instance it has been postulated that there exists a set defined in definition 13 by means of the sum axiom. Hence the second part of the disjunction can be omitted in definition 10.

<u>Theorem 58.</u> $x \in \bigcup A \longleftrightarrow (\exists B)(x \in B \wedge B \in A)$

Proof: $\{x\colon (\exists B)(x \in B \wedge B \in A\} = \bigcup A \longleftrightarrow$

$((\forall x)(x \in \bigcup A \longleftrightarrow (\exists B)(x \in B \wedge B \in A)$ by definition 10.

Again, the more elementary properties of the sum operation are similiar to those of the intersection operation. Some of these are: $\bigcup 0 = 0$, $\bigcup \{0\} = 0$ and $\bigcup \{A\} = A$.

Theorem 59. $\bigcap A \subseteq \bigcup A$

Proof: Let x be arbitrary such that $x \in \bigcap A$

then $(\forall B)(B \in A \rightarrow x \in B) \wedge (\exists B)(B \in A)$ by theorem 52

but from this it follows that

$$(\exists B)(B \in A \wedge x \in B)$$

but this is true $\leftrightarrow$ $x \in \bigcup A$ by theorem 58.

Power Set Axiom and Cartesian Product Set

The subsets of a given set have been considered earlier and it is logical to ask whether or not these subsets constitute a set. The following axiom guarantees the existance of such a set.

Power Set Axiom. $(\exists B)(\forall C)(C \in B \leftrightarrow C \subseteq A)$

In other words, if A is a set, then there exists a set B (called the power set of A and denoted PA) such that $C \in B$ if and only if $C \subseteq A$. For example, if $A = \{4,5\}$ then

$$PA = \{0, \{4\}, \{5\}, \{4,5\}\}$$

Definition 14. $PA = \{B: B \subseteq A\}$

Theorem 60. $B \in PA \leftrightarrow B \subseteq A$

The more obvious facts about power sets are the following:

i) $A \in PA$

ii) $0 \in PA$

iii) $P0 = \{0\}$

iv) $PP0 = \{0, \{0\}\}$

v) $A \subseteq B \leftrightarrow PA \subseteq PB$

The cartesian product of two sets A and B (in symbols: $A \times B$) is the set of all ordered pairs $\langle x,y \rangle$ such that $x \in A$ and $y \in B$. For example, if $A = \{1,2\}$, $B = \{3,4\}$

Then $A \times B = \{\langle 1,3\rangle, \langle 1,4\rangle, \langle 2,3\rangle, \langle 2,4\rangle\}$

Formally the definition of cross-product is:

Definition 15. $A \times B = \{\langle x,y\rangle : x \in A \wedge y \in B\}$

Again the usual theorem must be proven to show that the intuitively appropriate set exists.

Theorem 61.

$$(\exists C)(\forall x)(x \in C \leftrightarrow (\exists y)(\exists z)(y \in A \wedge z \in B \wedge x = \langle y,z\rangle))$$

Proof: In the axiom schema of separation

$$(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \emptyset(x))$$

let $B = C$, $A = PP(A \cup B)$ and $\emptyset(x)$ be $(\exists y)(\exists z)(y \in A \wedge z \in B \wedge x = \langle y,z\rangle)$

Then we have

$$(\exists C)(\forall x)(x \in C \leftrightarrow x \in PP(A \cup B) \wedge (\exists y)(\exists z)(y \in A \wedge z \in B \wedge x = \langle y,z\rangle))$$

If the clause $x \in PP(A \cup B)$ can be eliminated then the theorem is proven.

From the above $x \in C$ but this implies

$$(\exists y)(\exists z)(y \in A \wedge z \in B \wedge x = \langle y,z\rangle)$$

which in turn implies $x = \{\{y\}, \{y,z\}\}$

and since $y \in A$ and $z \in B$ then $\{y\} \subseteq A \cup B$ and $\{y,z\} \subseteq A \cup B$

thus $\{y\} \in P(A \cup B)$ and $\{y,z\} \in P(A \cup B)$

thus $\{\{y\}, \{y,z\}\} \subseteq P(A \cup B)$

hence $x \subseteq P(A \cup B)$ or $x \in PP(A \cup B)$

The following two theorems are direct consequences of the previous theorem.

Theorem 62.

$$x \in A \times B \leftrightarrow (\exists y)(\exists z)(y \in A \wedge z \in B \wedge x = \langle y,z\rangle)$$

Theorem 63. $\langle x,y\rangle \in A \times B \leftrightarrow x \in A \wedge y \in B$

Axiom of Regularity

The final axiom considered in this chapter of the report intuitively says that given any non-empty set A there is a member x of A such that the intersection of A and x is empty.

Axiom of Regularity.

$$A \neq 0 \rightarrow (\exists x)\ [x \in A \wedge (\forall y)(y \in x \rightarrow y \notin A)]$$

Due to the conditional form of the definition, the second part of the disjunction cannot be replaced with the simpler expression '$A \cap x = 0$'. If x is an individual then $A \cap x$ has no meaning; however when it is clear that x is a set, then the simpler notation will be employed.

Theorem 64. $A \notin A$

Proof: Suppose $A \in A$. Since $A \in \{A\}$ then $A \subseteq A \cap \{A\}$

But since $A \in \{A\}$ the axiom of regularity says

$$(\exists x)(x \in \{A\} \wedge A \cap x = 0)$$

But the only element of $\{A\}$ is A, hence contradicting $A \cap \{A\} \neq 0$

Theorem 65. $A \subseteq A \times A \rightarrow A = 0$

The Redundance of the Union Axiom

Meta Theorem 1. The union axiom is derivable from the axiom of extensionality, the pairing axiom and the sum axiom.

Proof: The union axiom says

$$(\exists D)(\forall x)(x \in D \longleftrightarrow x \in A \vee x \in B)$$

From the pairing axiom there exists a set containing two elements, say A and B but by theorem 58

$$x \in \bigcup\{A, B\}$$

$$\longleftrightarrow (\exists D)(D \in \{A, B\} \wedge x \in D)$$

$$\longleftrightarrow (\exists D)((D = A \vee D = B) \wedge x \in D)$$

$$\longleftrightarrow x \in A \vee x \in B$$

Consequently $(\exists D)(\forall x)(x \in D \longleftrightarrow x \in A \vee x \in B)$ which is the union axiom.

CHAPTER II

Operations on Relations

Using the already defined ordered pair concept, the theory of relations can be formulated in set-theoretic language. By a relation we mean here something like marriage (between men and women) or belonging (between elements and sets). More explicitly, what we call a relation is sometimes called a binary relation. An example of a ternary relation is parenthood for people (Adam and Eve are the parents of Cain). In this report we shall have no occasion to treat the theory of relations that are ternary, quaternary or worse.

Definition 16.

$$A \text{ is a relation} \leftrightarrow (\forall x)(x \in A \rightarrow (\exists y)(\exists z)(x = \langle y, z \rangle))$$

It will be convenient to have the useful notation xAy.

Definition 17. $xAy \leftrightarrow \langle x, y \rangle \in A$

It is immediate from the definitions that the following facts are true: i) 0 is a relation; ii) T is a relation $\wedge$ $S \subseteq T \rightarrow S$ is a relation; iii) $T \wedge S$ are relations $\rightarrow T \cap S$, $T \cup S$ and $T - S$ are relations.

If S is a relation then the domain of S (in symbols: DS) is the set of all things x such that for some y, $\langle x, y \rangle \in S$.

Thus if $S = \{\langle 4,8 \rangle, \langle 3,6 \rangle\}$, $DS = \{4,3\}$

Definition 18. $DA = \{x : (\exists y)(xAy)\}$

Theorem 66. $x \in DA \leftrightarrow (\exists y)(xAy)$

Proof: From the axiom schema of separation

$$(\exists B)(\forall x)(x \in B \leftrightarrow x \in \cup\cup A \wedge (\exists y)(xAy)) \quad - \quad - \quad - \quad (1)$$

Hence it is necessary to show $(\exists y)(xAy) \rightarrow x \in \bigcup\bigcup A$

From definition 17 $(\exists y)(xAy) \rightarrow \langle x,y\rangle \in A$

or $\{\{x\}, \{x,y\}\} \in A$ by definition 9

therefore $\{x\} \in \bigcup A$ and $x \in \bigcup\bigcup A$ by theorem 58

Finally $(\exists B)(\forall x)(x \in B \leftrightarrow (\exists y)(xAy)$ - - - - - (2)

Now in definition schema 10

$$\{x: \emptyset(x)\} = y \leftrightarrow [(\forall x)(x \in y) \leftrightarrow \emptyset(x) \wedge y \text{ is a set}] \vee [y = 0 \wedge -(\exists B)(\forall x)(x \in B \leftrightarrow \emptyset(x)]$$

letting $y = DA$ which is a set and letting $\emptyset(x)$ be $(\exists y)(xAy)$ yields

$$DA = \{x: (\exists y)(xAy)\} \leftrightarrow [(\forall x)(x \in DA \leftrightarrow (\exists y)(xAy)) \vee [-(\exists B)(\forall x)(x \in B \leftrightarrow (\exists y)(xAy)) \wedge DA = 0] \quad \text{- - -(3)}$$

but by definition 17 this yields

$$(\forall x)(x \in DA \leftrightarrow (\exists y)(xAy)) \vee [-(\exists B)(\forall x)(x \in B \leftrightarrow (\exists y)(xAy)) \wedge DA = 0] \quad \text{-(4)}$$

but the second part of the disjunction of (4) contradicts (2). Therefore $(\forall x)(x \in DA \leftrightarrow (\exists y)(xAy))$ the desired theorem.

The following facts can be seen with little difficulty:

i) $D(A \cup B) = DA \cup DB$; ii) $D(A \cap B) \subseteq DA \cap DB$; iii) $DA - DB \subseteq D(A - B)$.

Clearly it is neither the case that $DA \cap DB \subseteq D(A \cap B)$ nor $D(A - B) \subseteq DA - DB$.

To see this let $A = \{\langle 1,2\rangle\}$ and $B = \{\langle 1,3\rangle\}$. In the first case clearly $\{1\} \not\subseteq 0$. In the second case let $A = \{\langle 1,2\rangle, \langle 2,1\rangle\}$ and $B = \{\langle 1,1\rangle, \langle 2,2\rangle\}$. Then certainly $\{1,2\} \not\subseteq 0$.

If S is a relation then the range of S (in symbols: RS) is the set of all things y such that for some x, $\langle x,y\rangle \in S$. Thus if $S = \{\langle 0,1\rangle, \langle 2,3\rangle\}$ then $RS = \{1,3\}$. The notion of range is formally defined as:

<u>Definition 19.</u> $RS = \{y: (\exists x)(xAy)\}$

<u>Theorem 67.</u> $y \in RA \leftrightarrow (\exists x)(xAy)$

The proof is similiar to theorem 66 and theorem 70.

Similiar theorems to those already stated for domain could be stated here for the range but their proofs also parallel so they are not stated here.

Attention is now aimed at the notion of converse operation. The converse of a relation (in symbols: $\breve{S}$) is the relation such that for all x any y, $x\breve{S}y$ if and only if ySx. Given a relation

$$S = \{\langle 1,4\rangle, \langle 3,2\rangle, \langle 6,5\rangle\}$$

then

$$\breve{S} = \{\langle 4,1\rangle, \langle 2,3\rangle, \langle 5,6\rangle\}$$

<u>Definition 20.</u> $\breve{A} = \{\langle x,y\rangle : yAx\}$

<u>Theorem 68.</u> $x\breve{A}y \leftrightarrow yAx$

Proof: In the axiom schema of separation

$$(\forall A)(\exists B)(\forall x)(x \in B \leftrightarrow x \in A \wedge \emptyset(x))$$

let A be (RA×DA) and $\emptyset(x)$ be $(\exists y)(\exists z)(x = \langle y,z\rangle \wedge zAy)$ then the proof is relatively straight forward as in theorem 66 using definition 20.

<u>Theorem 69.</u> $\breve{A}$ is a relation

The more obvious facts about the converse relation are that it distributes over set-theoretic intersection, union and difference.

The next topic for consideration is the relative product of two relations S and T (in symbols: S/T). If S and T are relations then the relative product of S and T is the relation which holds between x and y if and only if there exists a z such that $\langle x,z\rangle$ is in S and $\langle z,y\rangle$ is in T. For example if

$S = \{\langle 0,1\rangle, \langle 2,3\rangle\}$ and $T = \{\langle 1,2\rangle, \langle 3,4\rangle\}$ then

$S/T = \{\langle 0,2\rangle, \langle 2,4\rangle\}$ and $T/S = \{\langle 1,3\rangle\}$. This example clearly illustrates that this operation is not commutative. Formally the definition is:

<u>Definition 21.</u> $A/B = \{\langle x,y\rangle : (\exists z)(xAz \wedge zBy)$

<u>Theorem 70.</u> $x\ A/B\ y \longleftrightarrow (\exists z)(xAz \wedge zBy)$

Proof: In the axiom schema of separation

$$(\forall A)(\exists B)(\forall x)(x \in B \longleftrightarrow x \in A \wedge \emptyset(x))$$

let $A = (DA \times RB)$ and $B = C$

and $\emptyset(x) = (\exists w)(x = \langle y,z\rangle \wedge yAw \wedge wBz)$

then we have

$$(\exists C)(\forall x)(x \in C \longleftrightarrow x \in (DA \times RB) \wedge (\exists w)(x = \langle y,z\rangle \wedge yAw \wedge wBz)$$

and $x \in DA \times RB$ must be eliminated from the above equivalence.

Given $x = \langle y,z\rangle$ and $yAw \wedge wBz \longrightarrow \langle y,z\rangle \in DA \times RB$

$$\longrightarrow x \in DA \times RB$$

therefore $(\exists C)(\forall x)(x \in C \longleftrightarrow (\exists w)(x = \langle y,z\rangle \wedge yAw \wedge wBz))$

The remainder of the proof is similiar to the proof of theorem 66.

<u>Theorem 71.</u> A/B is a relation

<u>Theorem 72.</u> $0/A = 0$

Proof: Suppose not:

$$x\ 0/A\ y \longleftrightarrow (\exists z)(x0z \wedge zAy)$$

$$\longleftrightarrow (\exists z)(\langle x,z\rangle \in 0 \wedge \langle z,y\rangle \in A)$$

and clearly $\langle x,z\rangle \in 0$ is false.

Some facts about the relative product operation are that it is associative and distributes over the union operation. It is also true that: i) $A \subseteq B \wedge C \subseteq D \longrightarrow A/C \subseteq B/D$; ii) $A/(B \cap C) \subseteq (A/B) \cap (A/C)$; iii) $(A/B) - (A/C) \subseteq A/(B - C)$.

The following definition provides the notation for restricting the domain of a relation to a given set.

Definition 22. $S|A =: S \cap (A \times R(S))$

By way of example, let $S = \{<7,8>, <4,5>, <3,4>$ and $A = \{7,4\}$, then $S|A = \{<7,8>, <4,5>\}$.

Theorem 73. $x\ S|A\ y \longleftrightarrow xSy \wedge x \in A$

Proof: $x\ S|A\ y \longleftrightarrow xSy \wedge x(A \times R(S))\ y$ by definition 22 and the rule of biconditional subst.

$\longleftrightarrow xSy \wedge <x,y> \in (A \times R(S))$ by definition 17

$\longleftrightarrow xSy \wedge x \in A$ by theorem 67.

It can be proven that the operation defined in definition 22 distributes over set-theoretic intersection, union and difference.

The following definition introduces the notion of the image of a set under a relation. That is in the above example $S''A$ (read: the image of the set A under S) is $\{8,5\}$. Formally we have:

Definition 23. $S''A =: R(S|A)$

Theorem 74. $y \in S''A \longleftrightarrow (\exists x)(xSy \wedge x \in A)$

Proof: $y \in S''A \longleftrightarrow y \in R(S|A)$ by definition 24

$\longleftrightarrow (\exists x)(x\ S|A\ y)$ by theorem 67

$\longleftrightarrow (\exists x)(xSy \wedge x \in A)$ by theorem 73

The following could be stated as theorems: i) $R''(A \cup B) = R''A \cup R''B$; ii) $R''(A \cap B) \subseteq R''A \cap R''B$; iii) $R''A - R''B \subseteq R''(A - B)$

For convenience the following definition of the field of a relation is introduced due to its usefulness in the next section.

Definition 24. $FA =: DA \cup RA$

Ordering Relations

The conventional notation will be used where appropriate; ' $x,y \in A$ ' for $x \in A \wedge y \in A$; ' $x,y,z \in A$ ' for $x \in A \wedge y \in A \wedge z \in A$; etc.

Beginning with eight basic definitions the concepts of identity relation, minimal element and well ordering follow.

Definition 25.

S is reflexive in $FS \longleftrightarrow (\forall x)(x \in FS \rightarrow xSx)$

*Definition 26.

S is irreflexive in $FS \longleftrightarrow (\forall x)(x \in FS \rightarrow -(xSx)$

Definition 27.

S is symmetric in $FS \longleftrightarrow (\forall x)(\forall y)(x,y \in FS \wedge xSy \rightarrow ySx)$

Definition 28.

S is asymmetric in $FS \longleftrightarrow (\forall x)(\forall y)(x,y \in FS \wedge xSy \rightarrow -(ySx))$

*Definition 29.

S is antisymmetric in $FS \longleftrightarrow (\forall x)(\forall y)(x,y \in FS \wedge xSy \wedge ySx \rightarrow x = y)$

Definition 30.

S is transitive in $FS \longleftrightarrow (\forall x)(\forall y)(\forall z)(x,y,z \in FS \wedge xSy \wedge ySz \rightarrow xSz)$

Definition 31.

S is connected in $FS \longleftrightarrow (\forall x)(\forall y)(x,y \in FS \wedge x \neq y \rightarrow xSy \vee ySx)$

*Definition 32.

S is strongly connected in $FS \longleftrightarrow (\forall x)(\forall y)(x,y \in FS \rightarrow xSy \vee ySx)$

*The concepts of irreflexive, antisymmetric and strongly connected are not used later in this paper but the definitions have been included for completeness only.

A need for the identity relation (in symbols: IA) on a given set A is apparent but some care must be practiced recalling that $\{x: x = x\} = 0$ by theorem 50.

Definition 33. $IA = \{\langle x,x\rangle : x \in A\}$

Theorem 75. $x\, IA\, x \longleftrightarrow x \in A$

Proof: From the axiom schema of abstraction

$$(\exists B)(\forall \langle x,x\rangle)(\langle x,x\rangle \in B \longleftrightarrow \langle x,x\rangle \in PPA \wedge x \in A)$$

If $x \in A \rightarrow \langle x,x\rangle \in PPA$ then the theorem is proven.

Now $x \in A \rightarrow x \in PA$

$\rightarrow \{\{x\}\} \in PPA$

but $\langle x,x\rangle = \{\{x\}, \{x,x\}\}$ x

Now to turn to the main topic of this section, namely the topic of a relation well-ordering a set. The concepts of minimal element and that of first element should be distinguished at this point to avoid later confusion. The first element precedes every other element whereas a minimal element has no predecessors. It is not the case that every minimal element is a first element but every first element is minimal.

Definition 34. x is an S-first element of A

$$\leftrightarrow x \in A \wedge (\forall y)(y \in A \wedge x \neq y \rightarrow xSy)$$

Definition 35. x is an S-minimal element of A

$$\leftrightarrow x \in A \wedge (\forall y)(y \in A \rightarrow -(ySx))$$

The distinction is slight by powerful.

Next well-ordering is formally defined.

Definition 36. S well-orders A

$\leftrightarrow$ S is connected in $A \wedge (\forall B)(B \subseteq A \wedge B \neq 0 \rightarrow B$ has an S-minimal element).

With this definition it is not difficult to see that "$<$" (less-than) well-orders the set of positive integers and "$>$" (greater-than) well-orders the set of negative integers.

Theorem 76.

S well-orders $A \rightarrow$ S is asymmetry and transitive in A.

Proof: (1) of asymmetry; suppose not, that is, suppose there are elements $x,y \in A$ such that xSy and ySx. Then $\{x,y\} \subseteq A$ has no S-minimal element, contradicting the given that S well-orders A.

(2) of transitivity; suppose not, that is, suppose there are $x,y,z \in A$

such that xSy and ySz but not xSz. Since S well-orders A then by definition 36, S is connected in A, hence zSx if not xSz. Now what is the minimal element of $\{x,y,z\}$? xSy $\rightarrow$ y is not. ySz $\rightarrow$ z is not and zSx $\rightarrow$ x is not, but this contradicts the fact that S well-orders A.

<u>Theorem 77</u>. S well-orders A $\leftrightarrow$ S is asymmetric and connected in $A \wedge (\forall B)(B \subseteq A \wedge B \neq \emptyset \rightarrow B$ has an S-first element.

Proof: Necessity: S is connected in A by definition 36 and S is asymmetric in A by theorem 76. By definition 36,

$$(\exists x)(x \in A \wedge x \text{ is an S-minimal element of } B).$$

But S is connected in A and thus $x \in A \wedge (\forall y)(y \in A \wedge y \neq x \rightarrow xSy)$.

Sufficiency: We have $(\forall B)(B \subseteq A \wedge B \neq 0 \rightarrow B$ has an S-minimal element). Hence by definition 36 S well-orders A.

<u>Theorem 78</u>. S well-orders $A \wedge A \neq 0 \rightarrow A$ has a unique S-first element.

Proof: Suppose A has two S-first elements, x and y. Then xSy and ySx but this contradicts S is connected in A.

<u>Theorem 79</u>. S well-orders $A \wedge B \subseteq A \rightarrow$ S well-orders B.

Proof: B has an S-first element by theorem 77. Therefore B has an S-minimal element. Now to show S is connected in B, it is given that S well-orders A so S is connected in A or

$$(\forall x)(\forall y)(x,y \in A \rightarrow xSy \vee ySx)$$

Suppose $(\exists a)(\exists b)\left[a,b \in B \wedge a \neq b \rightarrow -(aSb \vee bSa)\right]$

or $(\exists a)(\exists b)\left[a,b \in B \wedge a \neq b \rightarrow -(aSb) \wedge -(bSa)\right]$

or $(\exists a)(\exists b)\left[a,b \in B \wedge a \neq b \rightarrow bSa \wedge aSb\right]$

and $(\exists a)(\exists b)\left[a,b \in A \wedge a \neq b \rightarrow bSa \wedge aSb\right]$

but this contradicts that S is asymmetric in A which follows from theorem 77.

Theorem. S is transitive $\wedge$ symmetric $\rightarrow$ S is reflexive.

Proof: Given $x \in FS$, prove $\langle x,x\rangle \in S$

$$x \in FS \rightarrow x \in (RS \cup DS)$$
$$\rightarrow x \in RS \vee x \in DS$$

two cases:

(1) $$x \in RS \rightarrow (\forall y)(y \in FS) \rightarrow \langle y,x\rangle \in S)$$
$$\rightarrow (\forall y)(y \in FS \rightarrow \langle x,y\rangle \in S) \quad \text{Symmetry}$$

therefore $$(\forall y)(y \in FS \rightarrow \langle x,y\rangle \in S \wedge \langle y,x\rangle \in S)$$

therefore $\langle x,x\rangle \in S$ since S is transitive

(2) $$x \in DS \rightarrow (\forall y)(y \in FS \rightarrow \langle x,y\rangle \in S)$$
$$\rightarrow (\forall y)(y \in FS \rightarrow \langle y,x\rangle \in S) \quad \text{Symmetry}$$
$$\rightarrow (\forall y)(y \in FS \rightarrow \langle x,y\rangle \in S \wedge \langle y,x\rangle \in S)$$
$$\rightarrow \langle x,x\rangle \in S \text{ by transitivity.}$$

Equivalence Relations

Given a set and a relation in that set that is reflexive, symmetric and transitive, then the relation is said to be an equivalence relation in the set. Formally this is defined as:

Definition 37. R is an equivalence relation on FR $\leftrightarrow$ R is reflexive $\wedge$ symmetric $\wedge$ transitive.

The following defines the S-coset of x (written: $S[x]$). The S-coset of x is the set of all things that are equivalent to x when S is an equivalence relation. This is sometimes referred to as the S-equivalence class of x.

Definition 38. $S[x] = \{y: xSy\}$

Theorem 80. $y \in S[x] \leftrightarrow xSy$

The proof of this theorem is similiar to the proof of theorem 70 using the axiom schema of separation.

Theorem 81.

$x,y \in FS$ S is an equivalence relation $\rightarrow ((S[x] = S[y]) \leftrightarrow xSy)$.

This last theorem shows that if S is an equivalence relation then any two elements which stand in relation to each other under S generate identical equivalent classes. The next theorem shows that equivalence classed do not overlap.

Theorem 82. S is an equivalence relation

$$\longrightarrow \quad S[x] = S[y] \ \vee \ S[x] \cap S[y] = 0$$

Partitions

A partition of a given set is a family of non-empty pairwise disjoint subsets of the given set whose union is the given set.

Definition 39. Π is a partition of A

$$\leftrightarrow \bigcup \Pi = A \wedge (\forall B)(\forall C)(B \in \Pi \wedge C \in \Pi \wedge B \neq C \rightarrow B \cap C = 0) \wedge (\forall x)(x \in \Pi \rightarrow (\exists y) y \in x)$$

By way of example, if $A = \{1,2,3\}$ and $\Pi = \{\{1,2\}, \{3\}\}$ then Π is a partition of A.

The following definition establishes a connection between partitions and equivalence relations.

Definition 40. $\Pi(S) = \{B: (\exists x)(B = S[x] \wedge B \neq 0\}$

For example, if $A = \{1,2,3\}$ and $S = \{\langle 1,1\rangle, \langle 2,2\rangle, \langle 3,3\rangle, \langle 2,3\rangle, \langle 3,2\rangle\}$ then $\Pi(S) = \{\{1\}, \{2,3\}\}$. The partition $\Pi(S)$ is clearly a partition of A and it is said to be generated by S

Theorem 83. $C \in \Pi(S) \leftrightarrow (\exists x)(C = S[x] \wedge C \neq 0)$

Proof: From the axiom schema of separation

$$(\exists B)(\forall C)(C \in B \leftrightarrow C \in PFS \wedge (\exists x)(C = S[x] \wedge C \neq 0)$$

Again the crucial step in the proof is to show that

$$(\exists x)(C = S[x] \wedge C \neq 0) \rightarrow C \in PFS$$

$$(\exists x)(C = S[x] \wedge C \neq 0) \rightarrow C = \{z: (\exists x)(xSz)\}$$
$$\rightarrow C = RS$$
$$\rightarrow C \subseteq FS$$
$$\rightarrow C \in PFS$$

therefore $(\exists B)(\forall C)(C \in B) \longleftrightarrow (\exists x)(C = S[x] \wedge C \neq 0)$

After routine manipulation of definition by abstraction and utilizing definition 40, the theorem follows:

Theorem 84. S is an equivalence relation on $A \rightarrow \pi(S)$ is a partition of A.

A relation generated by a partition (in symbols: $S(\pi)$) is the following definition.

Definition 41.

$$S(\pi) = \{\langle x, y\rangle : (\exists B)(B \in \pi \wedge x \in B \wedge y \quad B)\}$$

This definition is followed by the usual theorem whose proof is similiar to theorem 83.

Theorem 85. $xS(\pi)y \longleftrightarrow (\exists B)(B \in C \wedge x \in B \wedge y \in B)$

Theorem 86. π is a partition of $A \rightarrow S(\pi)$ is an equivalence relation on A.

The proof of theorem 86 is not difficult since reflexivity, symmetry and transitivity follow from definition 41.

Functions

Definition 42. f is a function $\longleftrightarrow$ f is a relation

$$\wedge (\forall x)(\forall y)(\forall z)(xfy \wedge xfz \rightarrow y \quad z)$$

The next definition introduces the standard functional notation: $f(x) = y$.

Definition 43.

$$f(x) = y \longleftrightarrow [(E!z)(xfz) \wedge (xfy)] \vee [-(E!z)(xfz) \wedge y = 0]$$

It is noteworthy that definition allows the notation ' $f(x) = y$ ' for relations f that are not functions. For example, if

$$f = \{\langle 7,7\rangle, \langle 7,10\rangle, \langle 8,9\rangle\}$$

then $$f(7) = 0 \text{ and } f(8) = 9.$$

However, if f is a function then the first part of the disjunction of definition 43 is all that is needed.

The composition of two functions f and g (denoted: $f \circ g$) is defined in terms of the relative product of f and g (see definition 21).

Definition 44. $\quad f \circ g = g/f$

Clearly if f and g are functions, the $f \circ g$ and $f \cap g$ are functions.

Theorem 87. f and g are functions $\quad (f \circ g)(x) \quad f(g(x))$

Just as the domain of a relation was restricted (definition 22), so can the domain of a function be restricted.

Theorem 88. $\quad (f \circ g) \mid A = f \circ (g \mid A)$

The concept of a 1-1 function is defined in terms of the converse of a relation (definition 20).

Definition 45. $\quad f$ is 1-1 $\leftrightarrow f$ and $\breve{f}$ are functions.

Theorem 89. $\quad f$ is 1-1 $\wedge\ x \in Df \wedge\ y \in Df \rightarrow (f(x) = f(y) \leftrightarrow x = y)$.

Given a 1-1 function f, the inverse of f is defined in the natural way.

Definition 46. $\quad f$ is 1-1 $\rightarrow f^{-1} = \breve{f}$

Theorem 90. $\quad f$ is 1-1 $\rightarrow (f^{-1}(y) = x \leftrightarrow f(x) = y)$

Theorem 91. $\quad f$ is 1-1 $\wedge\ x \in Df \rightarrow f^{-1}(f(x)) \quad x$

Clearly theorem 91 is also true if Df is replaced by Rf and f is interchanged with f^{-1} in the conclusion.

Theorem 92.

$$f \wedge g \text{ are 1-1} \wedge Df \cap Dg = 0 \wedge Rf \cap Rg = 0 \rightarrow f \cup g \text{ is 1-1}$$

For the sake of completeness some of the standard mathematical language is introduced after which this chapter is concluded by considering a notion which is useful in many branches of mathematics.

Definition 47. i) f is a function from A into $B \leftrightarrow f$ is a function $\wedge Df = A \wedge Rf \subseteq B$; ii) f is a function from A onto $B \leftrightarrow f$ is a function $\wedge Df = A \wedge Rf = B$; iii) f maps A into $B \leftrightarrow f$ is a 1-1 function $\wedge Df = A \wedge Rf \subseteq B$; iv) f maps A onto $B \leftrightarrow f$ is a 1-1 function $\wedge Df = A \wedge Rf = B$.

The final definition of this chapter introduces the notion of the set of all functions from B to A (in symbols: A^B).

Definition 48.

$$A^B = \{f: f \text{ is a function from } B \text{ into } A\}$$

Theorem 93.

$$f \in A^B \leftrightarrow f \text{ is a function} \wedge Df = B \wedge Rf \subseteq A$$

Proof: From the axiom schema of separation

$$(\exists C)(\forall f)(f \in C \leftrightarrow f \in P(BxA) \wedge f \text{ is a function from } B \text{ into } A)$$

Now using the fact that $M \subseteq N \rightarrow DxM \subseteq DxN$ then $DfxRf \subseteq BxA$. But $f \subseteq DfxRf$, hence $f \subseteq BxA$. But $f \subseteq BxA \rightarrow f \in P(BxA)$.

CHAPTER III

Equipollence

Two sets A and B are said to be equipolent or have the same power (in symbols: $A \approx B$) if there exists a 1-1 function between them.

Definition 49.

i) $A \approx B$ under $f \leftrightarrow f$ is a 1-1 function $\wedge$ $Df = A \wedge Rf = B$

ii) $A \approx B \leftrightarrow (\exists f)(A \approx B$ under $f)$

Theorems 94, 95 and 96 establish that equipollence is an equivolence relation.

Theorem 94. $A \approx A$

Proof: The identity function clearly the desired function.

Theorem 95. $A \approx B \rightarrow B \approx A$

Proof: Let f be a 1-1 function establishing that $A \approx B$. Then f^{-1} is the desired function establishing $B \approx A$.

Theorem 96. $(A \approx B \wedge B \approx C) \rightarrow A \approx C$

Proof: Let f be a 1-1 function establishing that $A \approx B$ and g a 1-1 function establishing $B \approx C$. Then the desired function establishing $A \approx C$ is f o g.

The following theorems will be employed in the development of the cardinal numbers.

Theorem 97.

$$(A \approx B \wedge C \approx D \wedge A \cap C = 0 \wedge B \cap D = 0) \rightarrow A \cup C \approx B \cup D$$

Proof: Since $A \approx B \wedge C \approx D$ then there exists 1-1 functions f and g such that $Df = A$, $Rf = B$, $Dg = C$, $Rg = D$. But $A \cap C = 0 \wedge B \cap D = 0$ so

$$Df \cap Dg = 0 \wedge Rf \cap Rg = 0$$

so $f \cup g$ is 1-1 and $A \cup C \approx B \cup D$ under $f \cup g$.

Theorem 98. $(A \approx B \wedge C \approx D) \rightarrow A \times C \approx B \times D$

Proof: Since $A \approx B \wedge C \approx D$, let f and g be the corresponding desired 1-1 functions. Then the function h such that $x \in A$ and $y \in C$ is $h(\langle x,y \rangle) = \langle f(x), g(y) \rangle$ establishes the equipollence of $A \times C$ and $B \times D$.

The next two theorems establish the familiar commutative and associative properties for equipollence.

Theorem 99. $A \times B \approx B \times A$

Proof: For $x \in A$ and $y \in B$ the function f such that

$$f(\langle x,y \rangle) = \langle y,x \rangle$$

establishes the desired equipollence.

Theorem 100. $A \times (B \times C) \quad (A \times B) \times C$

Theorem 101. $(A \times \{y\} \approx A) \wedge (\{y\} \times A \approx A)$

Proof: Define f as follows: if $z \in A$ then $f(\langle z,y \rangle) \approx z$. For the proof the second half of the theorem, it is merely necessary to apply theorem 99 and the first half of this theorem.

Theorem 102. $(\exists C)(\exists D)(A \approx C \wedge B \approx D \wedge C \cap D = 0)$

Proof: Define $C = A \times \{0\}$ and $D = B \times \{0\}$

Then $C \approx A$ and $B \approx D$ by theorem 101 and it is easily seen $C \cap D = 0$.

Theorem 103. $(A \approx B \wedge C \approx D) \rightarrow A^C \approx B^D$

Theorem 104. $(B \cap C = 0) \rightarrow A^{B \cup C} \approx A^B \times A^C$

Theorem 105. $(A \times B)^C \approx A^C \times B^C$

Theorem 106. $(A^B)^C \approx A^{B \times C}$

The following definition is that of the relation $\preceq$ (read: being equal to or having less power).

Definition 50. $A \preceq B \leftrightarrow (\exists C)(A \approx C \wedge C \subseteq B)$

Theorem 107. $A \approx B \rightarrow A \preceq B$

Proof: Immediate since $(\exists B)(A \approx B \wedge B \subseteq B \rightarrow A \preceq B)$

Theorem 108. $A \subseteq B \rightarrow A \precsim B$

Proof: Immediate since $(\exists A)(A \approx A \wedge A \subseteq B \rightarrow A \precsim B)$

Theorem 109. $(A \precsim B \wedge B \precsim C) \longrightarrow A \precsim C$

The next theorem is probably the most fundamental theorem on the power of sets. Its proof is found in many texts including Suppes, but due to the theorems' importance, its proof is included here.

Theorem 110. Bernstein Theorem

$$(A \precsim B \wedge B \precsim A) \rightarrow A \approx B$$

Proof: $A \precsim B \rightarrow (\exists f)(f \text{ maps } A \text{ onto } B_1 \subseteq B)$

$$B \precsim A \rightarrow (\exists g)(g \text{ maps } B \text{ onto } A_1 \subseteq A)$$

It will be proven that A and B have the same power if there exists a subset K of A such that g maps $B - f''K$ onto $A - K$. This is true since defining h as: $h = (f|K) \cup (g|(A - K))$

then $$Dh = K \cup (A - K) = A$$

and
$$\begin{aligned} Rh &= (f''K) \cup (\breve{g}''(A - K)) \\ &= (f''K) \cup (B - f''K) \\ &= B \end{aligned}$$

Consequently a subset K of A is needed such that

$$g''(B - f''K) = A - K$$

Define $$D = \{ C: C \subseteq A \wedge g''(B - f''C) \subseteq A - C \}$$

then I need to show $\bigcup D = K$.

Clearly $C_1 \subseteq A \wedge C_2 \subseteq A \wedge C_1 \subseteq C_2 \rightarrow g''(B - f''C_2) \subseteq g''(B - f''C_1)$

so $$A - g''(B - f''C_1) \subseteq A - g''(B - f''C_2) \quad \text{- - - -(1)}$$

Also since $C \in D$ and from the above definition of D we have

$$C \subseteq A - g''(B - f''C) \quad \text{- - - - - - - (2)}$$

Now since every $C \in D$ is a subset of $\bigcup D$

then $$C \subseteq A - g''(B - f'' \bigcup D) \quad \text{- - - - - - -(3)}$$

Consequently $\bigcup D \subseteq A - g''(B - f''\bigcup D)$ - - - - - - (4)

Defining $F = A - g''(B - f''\bigcup D)$ - - - - - - (5)

Then by (1), (4), and (5)

$$A - g''(B - f''\bigcup D) \subseteq A - g''(B - f''F)$$

so $F \subseteq A - g''(B - f''F)$

Consequently $F \in D$

so $A - g''(B - f''\bigcup D) \subseteq D$ - - - - - - - (6)

therefore from (4) and (6)

$$\bigcup D = A - g''(B - f'' \quad D)$$

and letting $K = \bigcup D$ we achieve the required results

$$g''(B - f''K) = A - K$$

Theorem 111. $A \preceq B \wedge C \preceq D \rightarrow$

i) $(B \cap D = 0) \rightarrow (A \cup C \preceq B \cup D)$; ii) $A \times C \preceq B \times D$; iii) $A^C \preceq B^D$

The next definition is that of the relation $\prec$ (read: having less power).

Definition 51. $A \prec B \leftrightarrow (A \preceq B \wedge -(B \preceq A))$

The next theorem states that '$\prec$' is irreflexive, asymmetric and transitive.

Theorem 112.

i) $-(A \prec A)$; ii) $A \prec B \rightarrow -(B \prec A)$; iii) $(A \prec B \wedge B \prec C) \rightarrow A \prec C$

Theorem 113.

i) $A \preceq B \rightarrow -(B \prec A)$; ii) $A \prec B \wedge B \preceq C \rightarrow A \prec C$; iii) $A \preceq B \wedge B \prec C \rightarrow A \prec C$;

iv) $A \preceq B \leftrightarrow (A \simeq B \vee A \prec B)$

Proof: The proof of each part follows quickly from the preceding definitions and theorems but part (iv) is proven here:

$\longrightarrow$ given $A \preceq B$, suppose $-(A \prec B)$

Prove $A \simeq B$

By the contrapositive of definition 51, it follows that $-(A \preceq B) \vee B \preceq A$

but from given, it is known that $A \preceq B$ so $B \succeq A$, therefore from the Bernstein Theorem $A \approx B$

$\leftarrow$ given $A \approx B \vee A \prec B$, prove $A \preceq B$

by theorem 107 $A \approx B \rightarrow A \preceq B$

by definition 51 $A \prec B \rightarrow A \preceq B$

Theorem 114. $A \prec PA$

It should be noted that a theorem on the comparability of the power of two sets is not stated. According to Suppes such a theorem not only requires the axiom of choice, but is equivalent to it. Since this is beyond the scope of this paper, it is omitted.

Finite Sets

A set X is finite according to Tarski if any non-empty family of subsets of X has a member of which no other member of the family is a proper subset. This definition is conveniently in terms of the concept of minimal element defined earlier and later that of maximal element will be useful. They are defined jointly as:

Definition 52.

i) x is a minimal element of $A \leftrightarrow x \in A \wedge x$ is a set $\wedge$ $(\forall B)(B \in A \rightarrow -(B \subset A)$; ii) x is a maximal element of $A \leftrightarrow x \in A \wedge$ x is a set $\wedge (\forall B)(B \in A \rightarrow -(x \subset B)$

Formally Tarski's definition of a finite set is:

Definition 53.

A is finite $\leftrightarrow (\forall B)(B \neq 0 \wedge B$ is a family of subsets of $A \wedge B$ has a minimal element).

Theorem 115. 0 is finite

Theorem 116. $\{x\}$ is finite

Proof: Since there is only one non-empty family of subsets of $\{x\}$, namely $\{x\}$, and $\{x\} \subseteq \{x\}$ then $\{x\}$ is finite.

Theorem 117. A is finite $\wedge$ $B \subseteq A \rightarrow B$ is finite

Proof: Define F to be a non-empty family of subsets of B. Since $B \subseteq A$ then F is a non-empty family of subsets of A. But A is finite so F has a minimal element. Therefore B is finite.

Theorem 118. A is finite $\rightarrow$ $A \cap B \wedge A - B$ is finite

Proof: By theorem 16 $A \cap B \subseteq A$ and since A is finite $A \cap B$ is finite by theorem 117. Similiarly for the second half of the theorem $A - B \subseteq A$ and again since A is finite $A - B$ is finite by theorem 117.

The next theorem's proof is quite lengthy and can be found in Suppes.

Theorem 119. $A \wedge B$ finite $\rightarrow$ $A \cup B$ is finite

Theorem 120. A is finite $\rightarrow$ $A \cup \{x\}$ is finite

Proof: Since $\{x\}$ is finite by theorem 116 and it is given that A is finite, then it follows that $A \cup \{x\}$ is finite by theorem 119.

Theorem 121. Every non-empty family of subsets of a finite set has a maximal element.

Theorem 122. If every non-empty family of subsets of a set, A , has a maximal element than A is finite.

The next theorem is the first of its nature thus presented, that of induction for finite sets.

Theorem Schema 123. If i) A is finite; ii) $\emptyset(0)$; iii) $(\forall x)(\forall B)(x \in A \wedge B \subseteq A \wedge \emptyset(B) \rightarrow \emptyset(B \cup \{x\}))$

Then $\emptyset(A)$

Proof: Define $K = \{B: B \subseteq A \wedge \emptyset(B)\}$ - - - - - - -(1)

then K is not empty since $0 \subseteq A$ and by ii) $\emptyset(0)$ so $0 \in K$. Now since A is finite, K must have a maximal element, say B by theorem 121. It is needed that $B = A$ for then $\emptyset(A)$ will follow. Suppose

$A \not\subseteq B$, however by (1) $B \subsetneq A$ so $A - B \neq 0$. Let $x \in A - B$. Then $B \cup \{x\} \subseteq A$ but in iii) it is known that $\phi(B \cup \{x\})$ so $B \cup \{x\} \in K$ which cannot hold since B is the maximal element of K.

Theorem 124. If i) A is finite; ii) $0 \in K$; iii) $(\forall x)(\forall B)(x \in A \wedge B \subsetneq A \wedge B \in K \rightarrow B \cup \{x\} \in K)$

Then

$$A \in K$$

Proof: In theorem 123 let $\phi(B)$ be ' $B \in K$ '.

Theorem 125. A is finite $\leftrightarrow$ A belongs to every set K satisfying (ii) and (iii) in theorem 124.

Proof: $\rightarrow$ follows immediately from theorem 124. $\leftarrow$ Given A belongs to every set K satisfying (ii) and (iii) in theorem 124. Define K_1 to be the family of all finite subsets of A. Clearly $0 \in K_1$ by theorem 115. Also if $B \in K_1$ and $x \in A$ then $B \cup \{x\} \in K_1$ by theorem 120. Consequently we have $A \in K_1$ so A is finite.

Theorem 126. If A is finite and f is a function such that $Df = A$ and $Rf = B$ then B is finite.

Proof: Define $K = \{C: C \subseteq A \wedge f''C$ is finite$\}$

Using induction theorem 125 it is needed to prove $A \in K$ so that B will be finite since $f''A = B$. Clearly $0 \in K$ since $0 \subseteq A$ and $f''0 = 0$. Now assuming $x \in A$ and $C \in K$ all that is needed to complete the proof is to show $C \cup \{x\} \in K$. Clearly $C \cup \{x\} \subseteq A$. Now since $\{x\}$ is finite and f is a function then $f''\{x\}$ is finite. Also since $C \in K$ then $f''C$ is finite. Consequently $(f''C) \cup f''\{x\}$ is finite but

$$(f''C) \cup f''\{x\} = f''(C \cup \{x\})$$

so $f''(C \cup \{x\})$ is finite and by the definition of K it is known that $C \cup \{x\} \in K$.

Theorem 127. If A is finite and every set which is a member of A is finite then $\cup A$ is finite.

Theorem 128. A is finite $\wedge$ $A \approx B \rightarrow B$ is finite

Proof: Immediate from theorem 126 and definition 49.

Theorem 129. A is finite $\wedge$ $B \preceq A \rightarrow B$ is finite

Proof: Since $B \preceq A$ then there exists a $C \subseteq A$ such that $B \approx C$ but A is finite so by theorem 117 C is finite and by theorem 128 it follows that B is finite.

At the end of the last section it was noted that given any two sets, it is now known that they are comparable without the axiom of choice. However, the following theorem states that if one of the two sets is finite, they are comparable.

Theorem 130. A is finite $\rightarrow A \prec B \vee B \prec A \vee B \approx A$

Proof: Using induction on the subsets of A, define

$$K = \{C: C \subseteq A \wedge (C \prec B \vee C \approx B \vee B \prec C)\}$$

and the proof is relatively straight forward.

Theorem 131. A is finite and B is not $\rightarrow A \prec B$

Proof: B is not finite $\rightarrow A$ is not finite or $-(B \preceq A)$ by the contrapositive of theorem 129. Since it is given that A is finite then it follows that $A \prec B$.

The following definition is that of a finite set in sense of Dedikind. A set is Dedikind finite if and only if it is not equipolent to any of its proper subsets. It is proven in Suppes that if a set is finite (Tarski is finite) then it is Dedikind finite.

Definition 54. A set A is Dedikind finite $\leftrightarrow (\forall B)(B \subset A \rightarrow -(B \approx A))$

Theorem 132. If a set is Tarski finite then it is Dedikind finite.

In the future unless specifically stated 'finite' will refer to 'Tarski finite' as before.

Theorem 133. A is finite $\wedge$ $B \subset A \rightarrow B \prec A$

Proof: A is finite $\Leftrightarrow$ A is Dedikind finite by theorem 132

$\longrightarrow -(A \approx B)$ by definition 54

$\longrightarrow (A \prec B \vee B \prec A)$ by theorem 130

but $B \subset A \rightarrow B \preceq A$ by theorem 108

therefore not $A \prec B$

consequently $B \prec A$

Theorem 134.

$$A \wedge B \wedge C \text{ finite} \wedge A \prec B \wedge B \cap C = 0 \longrightarrow A \cup C \prec B \quad C$$

Theorem 135.

$$A \wedge B \wedge C \wedge D \text{ finite} \wedge A \prec B, C \prec D \wedge B \cap D = 0 \longrightarrow A \cup C \prec B \cup D$$

Tarski has proven that theorem 135 without the hypotheses that A, B, C, and D be finite is equivalent to the axiom of choice.

Theorem 136. A is finite $\wedge x \notin A \longrightarrow A \prec A \cup \{x\}$

Proof: A is finite $\wedge x \notin A \rightarrow A$ is finite $\wedge A \subset A \cup \{x\}$.

$\longrightarrow A \prec A \cup \{x\}$ by theorem 133

Theorem 137. $A \wedge B$ finite $\longrightarrow A \times B$ is finite

Cardinal Numbers

The last section of the report is an outline of the development of the arithmetic of the cardinal numbers and the necessary theorems are included making it possible to prove Cantor's theorem on the greatest cardinal number. A typical method of introducing the cardinal number of a set is define cardinal numbers as equivalence classes of equipollent sets (sets that have the same power) but according to Suppes it can't be proven that the appropriate equivalence classes exist. Consequently he introduces a new axiom containing a new primitive notion, that of the cardinal number of a set (in symbols: $K(A)$). Thus the axiom is given that associates to each set A, an object $K(A)$ such that to two equipollent sets there is associated the same cardinal number.

<u>Axiom for Cardinal Numbers:</u>

$$K(A) = K(B) \longleftrightarrow A \simeq B$$

Obviously without an axiom of infinity, the existance of infinite cardinals is not known to exist so all the theorems in this section deal with finite cardinals, but it is known that some of these theorems would carry-over to infinite cardinals.

The English lower case letters 'a', 'b', 'c' with and without subscript and superscript will be used to denote cardinal numbers.

<u>Definition 55.</u> a is a cardinal number $\longleftrightarrow$ there is a set A such that $K(A) = a$.

<u>Theorem 138.</u> $(\exists A)(\exists B)$ such that 1) $A \cap B = 0$; 2) $K(A) = a$; 3) $K(B) = b$.

Proof: By definition there exist sets A' and B' such that $K(A') = a$ and $K(B') = b$ but by theorem 102 there exist sets A and B such that $A \cap B = 0$, $A \simeq A'$ and $B \simeq B'$ so by the axiom for cardinals $K(A) = a$ and $K(B) = b$.

The next theorem is the usual justifying theorem for an operator, in this case for the addition of cardinal numbers.

<u>Theorem 139.</u> $(E!c)(\exists A)(\exists B)$ such that i) $A \cap B = 0$; ii) $K(A) = a$; iii) $K(B) = b$; iv) $K(A \cup B) = c$

Proof: Parts i), ii) and iii) follow from theorem 138 and the existance of c is trivial but its uniqueness is more interesting. That is, it is needed that c be independent of the particular sets A and B. Suppose there are sets A' and B' and a cardinal number c' such that

$$A' \cap B' = 0 \quad - - - - - - \quad (1)$$

and

$$K(A') = a \quad - - - - - - \quad (2)$$

and $K(B') = b$ - - - - - - - (3)

and $K(A' \cup B') = c'$ - - - - - - - (4)

Now from (2) and (3) and the axiom for cardinals $A' \approx A$ and $B' \approx B$

consequently $A' \cup B' \approx A \cup B$

So $K(A' \cup B') = K(A \cup B)$ by the axiom for the cardinal numbers.

Definition 56. $(a + b = c) \longleftrightarrow (\exists A)(\exists B)$ such that i) $A \cap B = 0$; ii) $K(A) = a$; iii) $K(B) = b$; iv) $K(A \cup B) = c$

The commutativity and associativity of cardinal numbers follows from the commutativity and associativity of sets respectively.

Theorem 139. $a + b = b + a$

Theorem 140. $(a + b) + c = a + (b + c)$

The following definition of the cardinal numbers 0, 1 and 2 are done in the obvious manner. There is no new notation introduced to distinguish between the empty set and the cardinal number zero as it should be obvious when in context.

Definition 57. $0 = K(0)$

$1 = K(\{0\})$

$2 = K(\{0, \{0\}\})$

Theorem 141. $2 = \{0, \{0\}\} \longrightarrow PA \approx 2^A$

Proof: Let $B \in PA$. Then there exists a function $g_B \in 2^A$ such that

$$g_B(x) = \begin{cases} 0 & \text{if } x \in B \\ \{0\} & \text{if } x \in A - B \end{cases}$$

Then to each B there corresponds a unique g_B, and for each $h \in 2^A$ there exists a unique $B \in PA$ such that $h = g_B$ which proves $PA \approx 2^A$.

The additive identity is established in the next theorem and its proof follows from theorem 24.

Theorem 142. $a + 0 = a$

The justifying theorem for multiplication is proven similiar to that for addition only, of course, using the corresponding Cartesian product theorem (theorem 98) in place of the theorem for union.

<u>Theorem 143.</u> $(E!c)(\exists A)(\exists B)$ such that: i) $K(A) = a$; ii) $K(B) = b$; iii) $K(A \times B) = c$

<u>Definition 58.</u> $ab = c \longleftrightarrow (\exists A)(\exists B)$ such that: i) $K(A) = a$; ii) $K(B) = b$; iii) $K(A \times B) = c$

Again the commutativity and associativity of cardinal multiplication follows from the commutativity and associativity of the Cartesian product.

<u>Theorem 144.</u> $ab = ba$

<u>Theorem 145.</u> $(ab)c = a(bc)$

<u>Theorem 146.</u> $a \cdot 1 = a$

<u>Theorem 147.</u> $a(b + c) = ab + ac$

Proof: Let A, B and C be pairwise disjoint sets such that $K(A) = a$, $K(B) = b$ and $K(C) = c$. Then since

$$B \cap C = 0 \rightarrow (A \times B) \cap (A \times C) = 0$$

and

$$A \times (B \cup C) = (A \times B) \cup (A \times C)$$

the theorem follows immediately.

The justifying theorem for cardinal exponentiation follows from theorem 103.

<u>Theorem 148.</u> $(E!c)(\exists A)(\exists B)$ such that: i) $K(A) = a$; ii) $K(B) = b$; iii) $K(A^B) = c$

<u>Definition 59.</u> $a^b = c \longleftrightarrow (\exists A)(\exists B)$ such that: i) $K(A) = a$; ii) $K(B) = b$; iii) $K(A^B) = c$.

Some of the more obvious theorems are given below for cardinal exponentiation.

<u>Theorem 149.</u> $a^{b+c} = a^b a^c$

Proof follows from theorem 104.

<u>Theorem 150.</u> $(ab)^c = a^c b^c$

Proof follows from theorem 105.

<u>Theorem 151.</u> $(a^b)^c = a^{bc}$

Proof follows from theorem 106.

<u>Theorem 152.</u> $a^1 = a$

<u>Theorem 153.</u> $a^0 = 1$

<u>Definition 60.</u> $a \leq b \leftrightarrow (\exists A)(\exists B)$ such that:

i) $K(A) = a$; ii) $K(B) = b$; iii) $A \precsim B$.

From theorems 107, 108, 109, and 110 it follows that i) $a \leq a$;

ii) $(a \leq b \wedge b \leq c) \rightarrow a \leq c$; iii) $(a \leq b \wedge b \leq a) \rightarrow a = b$.

<u>Definition 61.</u> $a < b \leftrightarrow a \leq b \wedge a \neq b$

<u>Theorem 154.</u> $a < b \leftrightarrow (\exists A)(\exists B)$ such that: i) $K(A) = a$;

ii) $K(B) = b$; iii) $A \precnsim B$.

Proof: $\leftarrow$ Given i), ii) and iii) suppose $b \leq a$. Then $B \succsim A$ contradicting iii).

Proof: $\rightarrow$ Given $a < b$

$$a < b \rightarrow a \leq b \wedge a \neq b$$

$$\rightarrow (\exists A)(\exists B)(K(A) = a \wedge K(B) = b \wedge A \precsim B) \wedge a \neq b$$

$$\rightarrow \text{i)} \wedge \text{ii)} \wedge A \precsim B \wedge a \neq b$$

$$\rightarrow \text{i)} \wedge \text{ii)} \wedge A \precsim B \wedge (a < b \vee b < a)$$

but $\qquad b < a$ contradicts $A \precsim B$

so $\qquad \rightarrow \text{i)} \wedge \text{ii)} \wedge A \precsim B \wedge a < b$

$$\text{i)} \wedge \text{ii)} \wedge \text{iii)}.$$

<u>Theorem 155.</u> $a < 2^a$

Proof: Let A be such that $K(A) = a$

then $\qquad 2^a = K(\{0, \{0\}\}^A)$ by theorem 141

but theorem 114 says $\qquad A \precnsim PA$

so $\Lambda \prec \{0, \{0\}\}^{\Lambda}$ by theorem 115

thus $a < 2^a$ by theorem 154.

With theorem 155 it is now possible to prove the well-known Cantor theorem that there exists no greatest cardinal number.

Theorem 156. $(\forall a)(\exists b)(a < b)$

Proof: Suppose there is a greatest cardinal number, call it a. Then by the definition of a cardinal number, it is known that there is a set Λ such that $K(\Lambda) = a$. However, if Λ exists, then $P\Lambda$ exists and by the axiom for cardinals there is a cardinal number b such that $K(P\Lambda) = b$. But theorem 114 says $\Lambda \prec P\Lambda$. So $a < b$ by theorem 154 thus contradicting the assumption that a is the largest cardinal number.

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