Solutions of the Problem of Finding Confidence Intervals for the Two Normal Population with Unequal Variances

Ing-Haur Liu

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SOLUTIONS OF THE PROBLEM OF FINDING CONFIDENCE INTERVALS FOR THE TWO NORMAL POPULATION WITH UNEQUAL VARIANCES

by

Ing-Haur Liu

A report submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Applied Statistics

Plan B

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1970
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I would like to express my sincere appreciation to Professor Ronald V. Canfield for his guidance, ideas, and help in organizing this paper.

Thanks are also extended to Dr. Donald V. Sission and Professor Elwin G. Eastman of my Graduate Committee for their critical review of this paper and their helpful suggestions; and Dr. Rex L. Hurst, Head of the Department of Applied Statistics and Computer Science, for his advice and encouragement.

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Finally, to my wife, Patty, and to my daughter, Margaret, for their patience and support in fulfilling this paper, I extend a husband's and father's gratitude.

Ing-Haur Liu
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INTRODUCTION

Comparison of the means of two normal populations is a simpler problem when the variance (if unknown) are assumed to be equal than it is when they are not equal. The main concern of this report is the latter case also called the Behrens [4] - Fisher [5] problem.

In this report the solutions proposed by Behrens-Fisher, Scheffé [10], Welch [12], Banerjee [3] and Hájek [8], will be described and compared. To this problem Scheffé proposed a solution which has the advantage that no special table is necessary for its use, since the variate has an exact "Student's t" distribution. It may therefore be used for large sample sizes, but Welch's method will presumably give shorter intervals for very small sample sizes, where the loss of efficiency of Scheffé's statistic is greatest [9].

A Monte Carlo study was undertaken to compare these methods for large and small sample sizes. The results are reported in the Conclusion.

The main objective of this report is to present the efficient solutions to the reader who is familiar with statistical methods but not with theories.
THE BEHRENS-FISHER'S SOLUTION

Given two samples of $n_i$ ($i = 1,2$) units from two normal populations with $\bar{x}_i$ and $S_i^2$ ($i = 1,2$) as sample estimates of population means and variances, where $\mu_1$ and $\mu_2$ represent two population means, Fisher has given the relation

$$d = \frac{(\bar{x}_1 - \mu_1) - (\bar{x}_2 - \mu_2)}{\sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = t_1 \sin \theta - t_2 \cos \theta} \quad (2.1)$$

where $t_i = \frac{\bar{x}_i - \mu_i}{S_i/\sqrt{n_i}}$ ($i=1,2$) and $\tan \theta = \frac{S_1/\sqrt{n_1}}{S_2/\sqrt{n_2}}$.

A procedure originally due to Behrens based on $d$ may be used to test the hypothesis that $\delta (= \mu_1 - \mu_2)$ has the value zero. In 1938 Sukhatme [11] published critical values of the Behrens-Fisher test as defined in (2.1) for the 5 per cent level of significance for $v_1, v_2 = 6, 8, 12, 24$ and $\infty$ for $\theta = 0^\circ, 15^\circ, 30^\circ, 45^\circ, 60^\circ, 75^\circ, 90^\circ$ where

$$\tan \theta = \frac{S_1/\sqrt{n_1}}{S_2/\sqrt{n_2}}.$$ 

Further critical values for the 1 per cent level of significance for $v_1, v_2 = 6, 8, 12, 24$ and $\infty$ were published later. To calculate critical values of (2.1) Sukhatme assumed that for the given value of $\theta$, $t_1$ and $t_2$ were independently distributed as Student's t variate with $n_i - 1$ ($i=1,2$) d.f.
Critical values of the Behrens-Fisher test for small odd degrees of freedom $v_1, v_2 = 1, 3, 5$ and $7$ were published by Fisher and Healy [6].

Critical values of the Behrens-Fisher test have been tabulated for different values for where

$$\tan \theta = \frac{S_1/\sqrt{n_1}}{S_2/\sqrt{n_2}}.$$  

For $\theta = 0^\circ$, critical values of the Behrens-Fisher test are equal to critical values of Student's $t$ with $v_2$ d.f. Also for $\theta = 90^\circ$, critical values of the Behrens-Fisher test are exactly equal to critical values of Student's $t$ with $v_1$ d.f. For intermediate values of $\theta$, critical values of the Behrens-Fisher test for $v_1 = v_2 = v$ and $v = 6, 8, 12$ and $24$ are numerically less than tabulated critical values for $\theta = 0^\circ$ (or $90^\circ$, which are numerically equal in such cases) for the $5$ per cent and $1$ per cent levels of significance. For $v_1 \neq v_2$ and $v_1, v_2 = 6, 8, 12, 24$ and $\neq$ critical values of the Behrens-Fisher test for intermediate values of $\theta$ usually lie in between tabulated critical values for $\theta = 0^\circ$ and $\theta = 90^\circ$ and are occasionally less than both of them. For $v_1 = v_2 = 1$ critical values of the test for intermediate values $\theta$ are, however, numerically higher than corresponding critical values for $\theta = 0^\circ$ (or $90^\circ$) for the $10$ per cent, $5$ per cent, $2$ per cent and $1$ per cent significance levels.
The confidence interval of level $\alpha$ based on solution (2.1) is derived as follows:

\[ P \left( -d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} < \frac{L - \delta}{S_{\bar{x}_1 - \bar{x}_2}} < \frac{d_{\nu_1, \nu_2, l} - \alpha}{2} \right) \]

\[ = P \left( -d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2} < L - \delta < d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2} \right) \]

\[ = P \left( L - d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2} < \delta < L + d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2} \right) \]

\[ = 1 - \alpha \]

where $L = \bar{x}_1 - \bar{x}_2$

\[ \delta = \mu_1 - \mu_2 \]

\[ S_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} \]

$\alpha$ is the significance level.

Then the confidence interval for $\mu_1 - \mu_2$ is between

\[ (L - d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2}) \text{ and } (L + d_{\nu_1, \nu_2, l} - \frac{\alpha}{2} S_{\bar{x}_1 - \bar{x}_2}) \]
Table 1 is an example of the Behrens-Fisher's solution.

Table 1. Two samples randomly selected from two normal populations in which $\sigma_1^2 \neq \sigma_2^2$.

<table>
<thead>
<tr>
<th></th>
<th>$X_1$</th>
<th>$X_2$</th>
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<tbody>
<tr>
<td>39</td>
<td>41</td>
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<td>30</td>
<td>19</td>
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<td>44</td>
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<td>15</td>
<td>53</td>
<td></td>
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<td>22</td>
<td>23</td>
<td></td>
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<td>30</td>
<td>52</td>
<td></td>
</tr>
<tr>
<td>27</td>
<td>69</td>
<td></td>
</tr>
</tbody>
</table>

Total 306 572

The 95 per cent confidence interval for $\mu_1 - \mu_2$ is as follows:

\[
n_1 = 9, \quad n_2 = 13
\]

\[
\bar{X}_1 = \frac{\sum X_{1i}}{n_1} = \frac{306}{9} = 34
\]

\[
\bar{X}_2 = \frac{\sum X_{2j}}{n_2} = \frac{572}{13} = 44
\]
\[ L = \overline{X}_1 - \overline{X}_2 = 34 - 44 = -12 \]

\[ S_1^2 = \frac{\sum (X_{1i} - \overline{X}_1)^2}{n_1 - 1} = \frac{1160}{8} = 145 \]

\[ S_2^2 = \frac{\sum (X_{2j} - \overline{X}_2)^2}{n_2 - 1} = \frac{3096}{12} = 258 \]

\[ S_{\overline{X}_1 - \overline{X}_2} = \sqrt{\frac{S_1^2}{n_1} + \frac{S_2^2}{n_2}} = \sqrt{\frac{145}{9} + \frac{258}{13}} = 5.99 \]

where \( i = 1, 2, \ldots, 9, \quad j = 1, 2, \ldots, 13 \)

\[ \tan \theta = \frac{\frac{S_1}{\sqrt{n_1}}}{\frac{S_2}{\sqrt{n_2}}} = \sqrt{\frac{145}{9}} = 0.81 \]

\( \theta = 39^\circ \)

From the Statistical Table of the distribution of 'd' with \( \theta = 39^\circ \), we obtain the value

\[ d_{8, 12, .975} = 2.22 \]

then

\[ L - d_{8, 12, .975} S_{\overline{X}_1 - \overline{X}_2} = -10 - 2.22 (5.99) \]

\[ = -23.3 \]
Thus the confidence interval for $\delta$ is between -23.3 and 3.3.
Scheffe's Solution

Given two samples of $n_i$ ($i=1,2$) units from two normal populations with means $\mu_1$ and $\mu_2$ also assumed $n_1 \leq n_2$, $\sigma_i^2$ ($i=1,2$) represents the variance of the populations and $\delta$, $L$ and $Q$ are defined as

$$\delta = \mu_1 - \mu_2$$
$$L = \bar{X}_1 - \bar{X}_2$$
$$Q = \frac{1}{n_1} \sum_{i=1}^{n_1} (u_i - \bar{u})^2$$

where

$$u_i = x_{1i} - \left( \frac{n_1}{n_2} \right)^{1/2} x_{2i}$$
$$\bar{u} = \frac{1}{n_1} \sum_{i=1}^{n_1} u_i$$

and

$$\sigma^2 = \sigma_1^2 + \left( \frac{n_1}{n_2} \right) \sigma_2^2$$

Scheffé indicated the interval estimation of the difference of means of two normal populations when the ratio of the variances of the populations is unknown as follows:

$$|\delta - L| \leq t_{n_1-1, 1-\frac{\alpha}{2}} \left\{ \frac{Q}{n_1(n_1-1)} \right\}^{1/2} \quad (2.2)$$

where $\alpha$ is the significance level.
Then

\[ p(-t_{n_1-1}, 1 - \frac{\alpha}{2}, t_{n_1-1}, 1 - \frac{\alpha}{2}) = 1 - \alpha \]

and their expected length is then

\[
E(\lambda) = 2t_{n_1-1}, 1 - \frac{\alpha}{2} \left[ n_1(n_1 - 1) \right]^{1/2} \sigma E \left( \frac{\sigma^2}{\sigma^2} \right)^{1/2}
\]

\[
= t_{n_1-1}, 1 - \frac{\alpha}{2} C_{n_1-1} \left( \frac{\sigma^2}{n_1} \right)^{1/2}
\]

where

\[
\frac{\bar{\sigma}^2}{\sigma^2} = \chi^2_{n_1-1} \quad C_k = 2k \quad E(\chi_k)
\]

\[
= \left( \frac{\sigma_k^2}{\bar{\sigma}^2} \right)^{1/2} \Gamma\left( \frac{1}{2} + \frac{1}{2} \right) / \Gamma\left( \frac{1}{2} k \right).
\]

For the case where the ratio of the variances, \( \theta = \frac{\sigma_1^2}{\sigma_2^2} \), is known, Scheffé also showed another solution as follows:

\[
|\delta - L| < t_{n_1 + n_2 - 2}, 1 - \frac{\alpha}{2} (n_1 + n_2 - 2)^{-1/2} \left( T_1 + T_2 / \theta \right)^{1/2}
\]

(2.3)

where

\[
T_1 = \sum_{i=1}^{n_1} (X_{1i} - \bar{X}_1)^2
\]

\[
T_2 = \sum_{j=1}^{n_2} (X_{2j} - \bar{X}_2)^2
\]
\[ \sigma^2 \frac{x_1 - x_2}{\bar{x}_1 - \bar{x}_2} = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \]

\[ \frac{(L - \delta)}{\bar{x}_1 - \bar{x}_2}, \frac{T_1}{\sigma_1^2}, \frac{T_2}{\sigma_2^2} \]

are mutually independently distributed, the first normally with zero mean and unit variance, and

\[ \frac{T_1}{\sigma_1^2} = \chi^2_{n_1 - 1}, \quad \frac{T_2}{\sigma_2^2} = \chi^2_{n_2 - 1} \]

where \( \chi^2_k \) is a generic notation for a random variable distribution according to the \( \chi^2 \) law with \( k \) d.f. The confidence intervals (2.3) are known to be highly efficient and their expected length to be

\[ E(L) = t_{n_1 + n_2 - 2, 1 - \frac{\alpha}{2}} C_{n_1 + n_2 - 2} \left[ \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2} \right]^{1/2} / n_1^{1/2}. \]

The ratio \( R \) of \( E(L) \) to \( E(L) \) is thus

\[ R = (t_{n_1 - 1, 1 - \frac{\alpha}{2}} C_{n_1 - 1}) / (t_{n_1 + n_2 - 2, 1 - \frac{\alpha}{2}} C_{n_1 + n_2 - 2}). \]

The following tables show the value of \( R \) when \( \alpha = .05 \) and .01.
### Table 2. Values of $R$ for $\alpha = .05$.

<table>
<thead>
<tr>
<th>$n_2-1$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1-1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.15</td>
<td>1.20</td>
<td>1.23</td>
<td>1.25</td>
<td>1.28</td>
</tr>
<tr>
<td>10</td>
<td>1.05</td>
<td>1.07</td>
<td>1.09</td>
<td>1.03</td>
<td>1.11</td>
</tr>
<tr>
<td>20</td>
<td></td>
<td>1.03</td>
<td>1.03</td>
<td>1.01</td>
<td>1.05</td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td>1.01</td>
<td>1.02</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
<td></td>
<td></td>
<td></td>
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<td>1</td>
</tr>
</tbody>
</table>

### Table 3. Values of $R$ for $\alpha = .01$.

<table>
<thead>
<tr>
<th>$n_2-1$</th>
<th>5</th>
<th>10</th>
<th>20</th>
<th>40</th>
<th>$\infty$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$n_1-1$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>1.27</td>
<td>1.36</td>
<td>1.42</td>
<td>1.47</td>
<td>1.52</td>
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<tr>
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<td>1.16</td>
<td>1.20</td>
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<td>20</td>
<td></td>
<td>1.05</td>
<td>1.06</td>
<td>1.09</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td></td>
<td></td>
<td>1.02</td>
<td>1.04</td>
<td></td>
</tr>
<tr>
<td>$\infty$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
</tbody>
</table>
Tables 2 and 3 tell us that with \( n_1 > 10 \), and \( \alpha = .05 \) (0.01) the expected length of the confidence intervals (2.2) is at most 11 per cent (20%) longer than that of the optimum confidence intervals (2.3) available when the ratio \( e \) is known. While we may conclude from \( R \to 1 \) as \( n_1 \to \infty \), that solution (2.2) is asymptotically extremely efficient, we cannot conclude from Tables 2 and 3 that for small \( n_1 \) (2.2) is inefficient, since we do not know what the lengthening effect of the extra nuisance parameter in the Behrens-Fisher problem would be on "best" confidence intervals. Finally, by comparing solution (2.2) with solution (2.3), it has been possible to show that at least asymptotically confidence intervals (2.2) are very short.

The following is an example of Scheffe's solution:

From Table 1 we obtain

\[
\begin{align*}
n_1 &= 9, & n_2 &= 13 \\
\bar{X}_1 &= 34, & \bar{X}_2 &= 44 \\
L &= \bar{X}_1 - \bar{X}_2 = 34 - 44 = -10
\end{align*}
\]

then

\[
\sum_{i=1}^{n_1} u_i = \sum_{i=1}^{n_1} \left[ x_{1i} - \left( \frac{n_1}{n_2} \right)^{1/2} x_{2i} \right]
\]

\[= -31.02\]
\[ U = \sum_{i=1}^{n_1} \frac{U_i}{n_1} = \frac{-31.02}{9} = -3.45 \]

\[ Q = \sum_{i=1}^{n_1} (U_i - \bar{U}) = 1089.1704 \]

\[ \left[ \frac{Q}{n_1(n_1 - 1)} \right]^{1/2} = \left( \frac{1089.1704}{72} \right)^{1/2} \approx 3.88 \]

\[ L - t_{8, .975} \left[ \frac{Q}{n_1(n_1 - 1)} \right]^{1/2} = -10 - 8.95 \]

\[ = -18.95 \]

\[ L + t_{8, .975} \left[ \frac{Q}{n_1(n_1 - 1)} \right]^{1/2} = -10 + 8.95 \]

\[ = 1.05. \]

Thus the confidence interval for \( \delta \) is between -18.95 and -1.05.
WELCH'S SOLUTION

If $L$ is a normally distributed estimate of a population parameter $\delta$ with sampling variance $\lambda_1\sigma_1^2 + \lambda_2\sigma_2^2$, where $\lambda_1$ and $\lambda_2$ are known positive constants ($\lambda_1 = \frac{1}{n_1}$, $\lambda_2 = \frac{1}{n_2}$) and if $S_1^2$ and $S_2^2$ are estimates of $\sigma_1^2$ and $\sigma_2^2$, distributed in the standard fashion with $v_1$ and $v_2$ degrees of freedom respectively, and if $L$, $S_1^2$ and $S_2^2$ are all independent, then

$$V = \frac{(L - \delta)}{\sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}},$$

$$C = \frac{\lambda_1 S_1^2}{\lambda_1 S_1^2 + \lambda_2 S_2^2},$$

where

$$L = \bar{x}_1 - \bar{x}_2,$$

$$\delta = \mu_1 - \mu_2.$$  

Welch (1947) presented the solution as follows:

$$P[|V| \leq V(C, v_1, v_2, \frac{\alpha}{2})] = 1 - \alpha$$  \hspace{1cm} (2.4)

whatever the values of the unknown population variances may be

$\sigma_1^2$ and $\sigma_2^2$.

Critical values of Welch's solution for the two sample cases only have been calculated by Aspin [1] and will be given in the Appendix. They cover the range (i) $v_1$, ...


\( \nu_2 = 6, 8, 10, 15, 20 \) and \( \infty \) for \( \alpha = .05 \) and (ii) \( \nu_2 = 10, 12, 15, 20, 30 \) and \( \infty \) for \( \alpha = .01 \). Also further critical values for \( \alpha = .05 \) and \( \alpha = .01 \) are given in Welch, Trickelt and James [13].

For \( C = 0 \) critical values of Welch's solution are exactly equal to t-values of the Students t-values with \( \nu_2 \) degrees of freedom. Also for \( C = 1 \) critical values of Welch's solution are exactly equal to t-values of the Student's t-values with \( \nu_1 \) degrees of freedom. For inter­mediate values of \( C \), \( V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \) numerically lies in between \( V(0, \nu_1, \nu_2, \frac{\alpha}{2}) \) and \( V(1, \nu_1, \nu_2, \frac{\alpha}{2}) \).

The confidence interval of level \( \alpha \) based on solution (2.4) derived as follows:

\[
P[-V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \leq \frac{L - \delta}{\sqrt{\frac{\lambda_1 S_1^2}{\nu_1} + \frac{\lambda_2 S_2^2}{\nu_2}}} \leq V(C, \nu_1, \nu_2, \frac{\alpha}{2})]
\]

\[
= P[-V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \leq \frac{\delta}{\sqrt{\frac{\lambda_1 S_1^2}{\nu_1} + \frac{\lambda_2 S_2^2}{\nu_2}}} \leq L - \delta]
\]

\[
= P[L - V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \leq \frac{\delta}{\sqrt{\frac{\lambda_1 S_1^2}{\nu_1} + \frac{\lambda_2 S_2^2}{\nu_2}}} \leq L + V(C, \nu_1, \nu_2, \frac{\alpha}{2})]
\]

\[
= 1 - \alpha.
\]
Then the confidence interval for $\mu_1 - \mu_2$ is between

$$[L - V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}]$$

and

$$[L + V(C, \nu_1, \nu_2, \frac{\alpha}{2}) \sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}]$$

Example of Welch's solution.

From Table 1 we obtain

$$n_1 = 9, \quad n_2 = 13$$

$$\bar{x}_1 = 34, \quad \bar{x}_2 = 44$$

$$L = \bar{x}_1 - \bar{x}_2 = 34 - 44 = -10$$

$$S_1^2 = 145, \quad S_2^2 = 258$$

Let $\lambda_i = \frac{1}{n_i}$

$$C = \frac{\lambda_i S_i^2}{\lambda_1 S_1^2 + \lambda_2 S_2^2} = \frac{145}{9} + \frac{258}{13} = \frac{16.11}{35.96} = 0.45.$$ 

From the Statistical table [7],

$$V(C, \nu_1, \nu_2, \frac{\alpha}{2}) = V (.45, 8, 12, .025) = 2.07$$

then

$$L - V(c, \nu_1, \nu_2, \frac{\alpha}{2}) \sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}$$

$$= -10 - 2.07 (5.99)$$

$$= -10 - 12.4 = -22.4$$

$$L + V(c, \nu_1, \nu_2, \frac{\alpha}{2}) \sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}$$

$$= -10 + 12.4 = 2.4.$$ 

Thus the confidence interval for $\delta$ is between -22.4 and 2.4.
BANERJEE'S SOLUTION

Given two samples of \( n_i \) \((i=1,2)\) units from two normal populations with means \( \mu_1 \) and \( \mu_2 \) and variances \( \sigma_1^2 \) and \( \sigma_2^2 \), \( \bar{X}_i \) and \( S_i^2 \) \((i=1,2)\) are sample estimates of population means and variances respectively. \( c_i \) \((i=1,2)\) are known coefficients, for example, chosen to be +1 and -1 for a confidence interval on \( \mu_1 - \mu_2 \), and \( \alpha \) is a pre-assigned probability level (between 0 and 1). Banerjee has shown the following relationship holds

\[
P\left\{ \sum_{i=1}^{2} c_i (\bar{X}_i - \mu_i) \right\}^2 \leq \frac{\sum_{i=1}^{2} \frac{t_i^2 c_i^2 S_i^2}{n_i}}{1 - \alpha} \geq 1 - \alpha \quad (2.5)
\]

where \( t_i \) \((i=1,2)\) are so chosen that

\[
\frac{1}{\sqrt{\nu_i}} \frac{1}{\beta(\frac{v_i}{2}, \frac{1}{2})} \int_{-t_i}^{t_i} \left( 1 + \frac{t_i^2}{\nu_i} \right)^{-\frac{\nu_i + 1}{2}} dt = 1 - \alpha
\]

Namely, \( t_i \) \((i=1,2)\) are \( t \) - values of Student's \( t \) table of \( \nu_i \) \((i=1,2)\) d.f. Corresponding to confidence coefficient \( \alpha \).

Banerjee [2] has also shown that non-central confidence interval with confidence coefficient not less than any pre-assigned probability level is possible.

\[
P \left\{ -T_1 \leq \sum_{i=1}^{2} c_i (\bar{X}_i - \mu_i) \leq T_2 \right\} > 1 - \alpha \quad (i=1,2)
\]
where
\[ T_1 = \sqrt{\frac{t_{11}^2 C_1^2 S_1^2}{n_1} + \frac{t_{21}^2 C_2^2 S_2^2}{n_2}} } \]

\[ T_2 = \sqrt{\frac{t_{12}^2 C_1^2 S_1^2}{n_1} + \frac{t_{22}^2 C_2^2 S_2^2}{n_2}} \]

and \( t_{ij} \) (\( i, j = 1, 2 \)) have been so determined that

\[ \int_{-t_{11}}^{t_{12}} f\left(\frac{t}{\nu_1}\right) dt = \varepsilon_{11} \]

\[ \int_{0}^{t_{12}} f\left(\frac{t}{\nu_1}\right) dt = \varepsilon_{12} \]

\[ \int_{-t_{21}}^{t_{22}} f\left(\frac{t}{\nu_2}\right) dt = \varepsilon_{21} \]

\[ \int_{0}^{t_{22}} f\left(\frac{t}{\nu_2}\right) dt = \varepsilon_{22} \]

where
\[ t_{ij} > 0 \quad (i, j = 1, 2) \]

and
\[ \varepsilon_{11} + \varepsilon_{12} = \varepsilon_{21} + \varepsilon_{22} = 1 - \alpha. \]

However, confidence intervals considered in this report are central by nature. Therefore, the confidence interval for \( C_1 \mu_1 + C_2 \mu_2 \) from solution (2.5) can be derived as follows:

\[ P\left[ \sum_{i=1}^{2} C_i (X_i - \mu_i) \right] \leq \frac{2}{\chi^2_{1}} \frac{t_i^2 C_i^2 S_i^2}{n_i} \]
\[
\begin{align*}
&= P\left[ -\sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}} \leq \frac{2}{\sum_{i=1}^2} C_i (\bar{X}_1 - \mu_i) \right. \\
&\quad \left. \leq \sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}} \right] \\
&= P\left[ \frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \bar{X}_i - \sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}} \leq \frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \mu_i \leq \frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \bar{X}_i + \sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}} \right] \\
&= 1 - \alpha.
\end{align*}
\]

Then the confidence interval on \( \frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \mu_i \) is between
\[
\frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \bar{X}_i - \sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}} \quad \text{and} \quad \frac{2}{\sum_{i=1}^2} \sum_{i=1}^2 C_i \bar{X}_i + \sqrt{\frac{2}{\sum_{i=1}^2} \frac{t_i^2 C_i^2 S_i^2}{n_i}}.
\]

The following is an example of Banerjee's solution.

From Table 1 we obtain,
\[
\begin{align*}
n_1 &= 9, \quad n_2 = 13. \\
\bar{X}_1 &= 34 \quad \bar{X}_2 = 44 \\
L &= \bar{X}_1 - \bar{X}_2 = 34 - 44 = -10 \\
S_1^2 &= 145, \quad S_2^2 = 258
\end{align*}
\]

From the Student's t table,
\[
t_1 = t_{8, .975} = 2.306
\]
\[ t_2 = t_{12}, \, .975 = 2.179 \]

Then

\[
\sqrt{\frac{t^2_{12} \sum_{i=1}^{n_1} C_i^2 S_i^2}{n_1}} = \sqrt{\frac{t^2_{12} C_1^2 S_1^2}{n_1}} + \frac{t^2_{2} C_2^2 S_2^2}{n_2}
\]

Let

\[ C_1 = 1, \, C_2 = -1 \]

\[ = \sqrt{16.11(2.306)^2} + 19.85(2.179)^2 = 13.4 \]

\[ L - \sqrt{\frac{t^2_{12} \sum_{i=1}^{n_1} C_i^2 S_i^2}{n_1}} = -10 - 13.4 = -23.4 \]

\[ L + \sqrt{\frac{t^2_{12} \sum_{i=1}^{n_1} C_i^2 S_i^2}{n_1}} = -10 + 13.4 = 3.4 \]

Thus the confidence interval for \( \mu_1 - \mu_2 \) is between -23.4 and 3.4.
HAJEK'S SOLUTION

Let $X$ be a normally distributed $(\mu, \sigma^2)$ random variable, and let $S^2$ be an estimate for $\sigma^2$ with the structure

$$S^2 = \sigma^2 \sum_{j=1}^{K} \frac{\lambda_j}{m_j} \chi_j^2 (m_j)$$

(2.6)

$$\lambda_j \geq 0, \quad \sum_{j=1}^{K} \lambda_j = 1$$

Where $\lambda_j$ are unknown constants, and the random variables $\chi_j^2 (m_j)$ have Chi-square distribution with $m_j$ d.f. and are independent of each other and of $X$; we take the arbitrary limits $t' < 0 < t''$. Under these conditions the probability $P$ of the event

$$t' \leq \frac{X - \mu}{S} \leq t'', \quad t' < 0 < t''$$

(2.7)

lies within the limits $P_v \leq P \leq P_m$, where $P_m$ and $P_v$ are the probability of the event (2.7) under the condition that $(X - \mu)/S$ has Student's distribution with $m$ and $v$ degrees of freedom respectively, where $m = m_1 + m_2 + \cdots + m_k$ and $v$ is any integer

$$v \leq \min_{1 \leq j \leq k} \frac{m_j}{\lambda_j}$$

for example

$$v = \min_{1 \leq j \leq k} m_j.$$
This result can be used to get a confidence interval for two population means as follows:

Let \( X = \bar{X}_1 - \bar{X}_2 \) \( \mu = \mu_1 - \mu_2 \),

\[
\frac{S^2}{X_1 - X_2} = \frac{S^2}{n_1} + \frac{S^2}{n_2}
\]

Since

\[
\frac{S^2}{n_1} + \frac{S^2}{n_2} = \sigma_1^2 \frac{\chi^2(m_1)}{m_1} + \sigma_2^2 \frac{\chi^2(m_2)}{m_2}
\]

\[
= \frac{2}{\sum_{j=1}^{\lambda} \sigma_j^2} \frac{\chi^2(m_j)}{m_j} = \sigma^2 \frac{2}{\sum_{j=1}^{\lambda} \sigma_j^2} \frac{\chi^2(m_j)}{m_j}
\]

\[
= \sigma^2 \frac{2}{\sum_{j=1}^{\lambda} \frac{\lambda_j^2 \chi^2(m_j)}{m_j}}
\]

where

\( m_j = n_j - 1, \quad \lambda_j = \frac{\sigma_j^2}{\sigma^2} \) \( (j=1,2) \)

and

\( \sigma^2 = \sigma_1^2 + \sigma_2^2 + \ldots + \sigma_j^2 \)

therefore \( \frac{S^2}{X_1 - X_2} \) has the structure (2.6).

Let

\[
t = \frac{(X_1 - X_2) - (\mu_1 - \mu_2)}{S^2 \frac{1}{X_1 - X_2}} \quad \text{and} \quad v = \min_{1 \leq j \leq k} m_j .
\]

Thus the confidence interval of \( \alpha \) level based on this theorem is derived as follows: Let \( t' = -t_v, \quad 1 - \frac{\alpha}{2} \),

\( t' = t_v, \quad 1 - \frac{\alpha}{2} \),
then
\[
P \left[ \frac{(\bar{X}_1 - \bar{X}_2) - (\mu_1 - \mu_2)}{\frac{S}{\sqrt{\bar{X}_1 - \bar{X}_2}}} < t_\nu, 1 - \frac{\alpha}{2} \right] \]

\[
= P \left[ (\bar{X}_1 - \bar{X}_2) - t_\nu, 1 - \frac{\alpha}{2} \frac{S}{\sqrt{\bar{X}_1 - \bar{X}_2}} < \mu_1 - \mu_2 \right] \]

\[
< (\bar{X}_1 - \bar{X}_2) + t_\nu, 1 - \frac{\alpha}{2} \frac{S}{\sqrt{\bar{X}_1 - \bar{X}_2}} \]

\[
\geq 1 - \alpha. \]

Then the limits of the confidence interval for \(\mu_1 - \mu_2\) is between
\[
(\bar{X}_1 - \bar{X}_2) - t_\nu, 1 - \frac{\alpha}{2} \frac{S}{\sqrt{\bar{X}_1 - \bar{X}_2}} \]
and
\[
(\bar{X}_1 - \bar{X}_2) + t_\nu, 1 - \frac{\alpha}{2} \frac{S}{\sqrt{\bar{X}_1 - \bar{X}_2}} \]

The theorem can also be used to test the null Hypothesis \(\mu_1 = \mu_2\) versus \(\mu_1 \neq \mu_2\) in the following manner:
Let \(t^*\) be the upper critical value for the test.
Then \(P(|t| < t^*) = 1 - \alpha.\)

*Figure 1.* Graphical illustration of the acceptance and rejection regions for \(P(|t| < t^*) = 1 - \alpha.\)
From the theorem

$$P(|t| < t_m, 1 - \frac{\alpha}{2}) \leq 1 - \alpha.$$

Therefore

$$t_m, 1 - \frac{\alpha}{2} < t^*$$

Figure 2. Graphical illustration of the acceptance and rejection regions for $P(|t| < t_m - 1 - \frac{\alpha}{2}) \leq 1 - \alpha$.

also from theorem

$$P(|t| < t_V, 1 - \frac{\alpha}{2}) \leq 1 - \alpha.$$

Thus

$$t_m, 1 - \frac{\alpha}{2} < t^* < t_V, 1 - \frac{\alpha}{2}.$$
The acceptance region \((-t^*, t^*)\) contains the interval 
\((-t_m, 1 - \frac{\alpha}{2}), (t_m, 1 - \frac{\alpha}{2})\). The rejection region
\((-\infty, -t^*)\) and 
\((t^*, \infty)\) contains the region
\((-\infty, -t_v, 1 - \frac{\alpha}{2})\) and 
\((t_v, 1 - \frac{\alpha}{2}, \infty)\).

If the statistic falls between \(\pm t_v, 1 - \frac{\alpha}{2}\) and
\(\pm t_m, 1 - \frac{\alpha}{2}\), most cases this even has a small probability
of occurring.

The following is an example of Hájek's solution.

From Table 1 we obtain

\[ n_1 = 9, \quad n_2 = 13 \]
\[ \bar{x}_1 = 34, \quad \bar{x}_2 = 44 \]
\[ L = \bar{x}_1 - \bar{x}_2 = 34 - 44 = -10 \]
\[ s_1^2 = 145, \quad s_2^2 = 258 \]

\[ s_{\bar{x}_1 - \bar{x}_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = 5.99. \]
From the Student's t table.

\[ t_{\nu}, \, 1 - \frac{\alpha}{2} = t_8, \, .975 = 2.306 \]

then

\[
L - t_8, \, .975 \frac{S}{\bar{x}_1} - \bar{x}_2 = -10 - 2.306 (5.99) = -10 - 13.81 = -23.81
\]

\[
L + t_8, \, .975 \frac{S}{\bar{x}_1} - \bar{x}_2 = -10 + 13.81 = 3.81
\]

thus the confidence interval for \( \mu_1 - \mu_2 \) is between -23.81 and 3.81.
CONCLUSION

The comparison of the critical values among these solutions would not be strictly valid for the comparison of confidence interval between two population means, because some solutions calculate critical values restricting to sub-sets having observed values \( \frac{S_1}{\sqrt{n_1}} \) whereas in others the critical values refer to unrestricted variation of the four sample estimates \( \bar{X}_1, \bar{X}_2, S_1^2 \) and \( S_2^2 \).

It is useful to compare these methods according to their ease of application. Scheffé's statistic has a restriction between two sample sizes and its calculation is quite tedious; Hájek's and Banerjee's statistics have easy calculations, but all three statistics have the advantage that no special tables are necessary for their use since the variates have an exact "Student's t" distribution. Behrens-Fisher's and Welch's statistics require special tables.

A Monte Carlo study was conducted to compare these methods according to the average width of the confidence intervals, the variance, and the number of times the confidence interval actually covered the true value of the parameter. The following tables present the results of this study:
In Table 4, samples of size 9 and 13 were drawn for two normal populations. The sample size of 9 was drawn from a normal population whose mean equals zero and whose variance equals two, whereas the sample size of 13 was drawn from a normal population whose mean equals zero and whose variance equals one. The sampling was repeated 200 times and the average width, variance and the number of times the interval covered zero. Table 5 was derived similarly, except the sample sizes were 101 and 101.

Kendall and Stuart (1961) noted that Welch's statistics will presumably give shorter intervals for very small sample sizes.
sizes, where the loss of efficiency of Scheff's statistic is greatest. From the above tables it appears that Welch's statistic gives the shortest intervals not only for small sample sizes but also for large sample sizes.
BIBLIOGRAPHY


12. Welch, B. L. "The significance of the difference between two means when the population variances are unequal," Biometrika, 29, 350-62. 1937.

APPENDIXES
### Table 6. 5 per cent points of the distribution of 'd'.

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<th>( n_1 = 12 )</th>
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### Appendix B

#### Upper \( \frac{1}{2} \) per cent critical values of \( v = (y - \eta)/\sqrt{\frac{\lambda_1 s^2_1}{\lambda_1 s^2_1 + \lambda_2 s^2_2}} \) (i.e., upper \( \frac{1}{2} \) per cent critical values of \(|v|\)*

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*Explanation following Table 10, page 41.*
Table 8. Upper $\frac{1}{2}$ per cent critical values of $v = (y - \eta) / \sqrt{\frac{1}{\lambda_1 S_1^2} + \frac{1}{\lambda_2 S_2^2}}$ (i.e., upper 5 per cent critical values of $|v|$)*

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*Note: $\lambda_1$ and $\lambda_2$ are the eigenvalues of the matrix $S$. $S_1^2$ and $S_2^2$ are the variances of the two independent normal distributions.
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*Explanation following Table 10, page 41.
Table 9. Value of \( v = \frac{y - \eta}{\sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}} \) exceeded with probability \( \frac{\alpha}{2} = 0.01^* \) (or of \(|v|\) exceeded with probability \( \alpha = 0.02 \)).

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*Explanation following Table 10, page 41.*
Table 10. Value of $v = \frac{Y - \eta}{\sqrt{\lambda_1 S_1^2 + \lambda_2 S_2^2}}$ exceeded with probability $\frac{\alpha}{2} = 0.05^*$
(or of $|v|$ exceeded with probability $\alpha = 0.10$).

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* $y$ is normally distributed about $\eta$ with variance $\lambda_1 s_1^2 + \lambda_2 s_2^2$, and $s_1^2$ and $s_2^2$ are independent estimates of $\sigma_1^2$ and $\sigma_2^2$, based on $\nu_1$ and $\nu_2$ degrees of freedom, respectively. $\lambda_1$ and $\lambda_2$ are known constants.

In the problem of comparing the means of samples taken from two normal populations, put $y = (\bar{X}_1 - \bar{X}_2)$, $\nu_1 = (n_1 - 1)$, $\nu_2 = (n_2 - 1)$, $\lambda_1 = 1/n_1$, $\lambda_2 = 1/n_2$ where $n_1$ and $n_2$ are the sample sizes.
Appendix C

DIMENSION XI(9), X2(13), U(9), X(4400)
WRITE(6,200)
LL=4400
IR=69597
DO 300 IK=1,4400
  Y=0.
  DO 301 JK=1,12
    Y=Y+RN(IR)
    X(IK)=(Y-6.)
  300 CONTINUE
IK=0
N1=0
N2=0
N3=0
N4=0
N5=0
SRFD=0.
SSFD=0.
SWHD=0.
SRAND=0.
SHAJD=0.
SSRFD=0.
SSSF=0.
SSWHD=0.
SSRAND=0.
SSHAJD=0.
DO 99 K=1,200
DO 90 I=1,9
  IK=IK+1
80 XI(I)=X(IK)*2.
  DO 91 J=1,13
    IK=IK+1
81 X2(J)=X(IK)
  SMX1=0.
  SMX2=0.
  DO 15 I=1,9
15 SMX1=SMX1+XI(I)
  DO 20 J=1,13
20 SMX2=SMX2+X2(J)
  FN1=9.
  FN2=13.
  X1MN=SMX1/FN1
  X2MN=SMX2/FN2
  SMSX1=0.
  SMSX2=0.
  DO 25 I=1,9
25 SMSX1=SMSX1+(XI(I)-X1MN)**2
DO 30 J = 1, 13
30 SMSX2 = SMSX2 + (X2(J) - X2MNJ ** 2)
VRX1 = SMSX1 / (FN1 - 1.)
VRX2 = SMSX2 / (FN2 - 1.)
S12 = SQRT(VRX1 / FN1 + VRX2 / FN2)
SMU = 0.
DO 35 I = 1, 9
U(I) = X1(I) - SQRT(FN1 / FN2) * X2(I)
35 SMU = SMU + U(I)
UBAR = SMU / FN1
Q = 0.
DO 40 I = 1, 9
40 Q = Q + (U(I) - UBAR) ** 2
QA = SQRT(Q / 72.1)
C = (VRX1 / FN1) / (S12 ** 2)
B = SQRT((2.306 ** 2 * VRX1 / FN1 + 2.179 ** 2 * VRX2 / FN2)
TAN = SQRT((VRX1 ** FN2) / (VRX2 ** FN1))
IF (TAN .GE. 0. AND. TAN .LT. 0.27) GO TO 51
IF (TAN .GE. 0.27 .AND. TAN .LT. 0.58) GO TO 52
IF (TAN .GE. 0.58 .AND. TAN .LT. 1.0) GO TO 53
IF (TAN .GE. 1.0 .AND. TAN .LT. 1.74) GO TO 54
IF (TAN .GE. 1.74 .AND. TAN .LT. 3.74) GO TO 55
IF (TAN .EQ. 3.74) GO TO 56
IF (TAN .GT. 3.74) GO TO 57
51 D = 2.179 + (2.184 - 2.179) * TAN / 0.27
GO TO 60
52 D = 2.184 + (2.202 - 2.184) * (TAN - 0.27) / 0.31
GO TO 60
53 D = 2.202 + (2.229 - 2.202) * (TAN - 0.58) / 0.42
GO TO 60
54 D = 2.229 + (2.263 - 2.229) * (TAN - 1.0) / 0.74
GO TO 60
55 D = 2.263 + (2.293 - 2.263) * (TAN - 1.74) / 2.
GO TO 60
56 D = 2.293
GO TO 60
57 D = 2.306
GO TO 60
60 IF (C .GE. 0. .AND. C .LT. 0.1) GO TO 61
IF (C .GE. 0.1 .AND. C .LT. 0.2) GO TO 62
IF (C .GE. 0.2 .AND. C .LT. 0.3) GO TO 63
IF (C .GE. 0.3 .AND. C .LT. 0.4) GO TO 64
IF (C .GE. 0.4 .AND. C .LT. 0.5) GO TO 65
IF (C .GE. 0.5 .AND. C .LT. 0.6) GO TO 66
IF (C .GE. 0.6 .AND. C .LT. 0.7) GO TO 67
IF (C .GE. 0.7 .AND. C .LT. 0.8) GO TO 68
IF (C .GE. 0.8 .AND. C .LT. 0.9) GO TO 69
IF (C .GE. 0.9 .AND. C .LT. 1.0) GO TO 70
V = 2.31
GO TO 71
61 V = 2.18 - (2.18 - 2.14) * C / 0.1
GO TO 71
62 V = 2.14 - (2.14 - 2.11) * (C - 0.1) / 0.1
GO TO 71
63 \( V = 2.11 - (2.11 - 2.08) \times (C - 0.2) / 0.1 \)
GO TO 71
64 \( V = 2.08 - (2.08 - 2.07) \times (C - 0.3) / 0.1 \)
GO TO 71
65 \( V = 2.07 \)
GO TO 71
66 \( V = 2.07 - (2.07 - 2.10) \times (C - 0.5) / 0.1 \)
GO TO 71
67 \( V = 2.10 - (2.10 - 2.15) \times (C - 0.6) / 0.1 \)
GO TO 71
68 \( V = 2.15 - (2.15 - 2.20) \times (C - 0.7) / 0.1 \)
GO TO 71
69 \( V = 2.20 - (2.20 - 2.25) \times (C - 0.8) / 0.1 \)
GO TO 71
70 \( V = 2.25 - (2.25 - 2.31) \times (C - 0.9) / 0.1 \)
GO TO 71
71 BFBD = 2.0*S12
SFD = 2.*2.306*QA
WHD = 2.*VS12
BAND = 2.*R
HAJD = 2.*2.306*S12
FL = XMN - X2MN
IF (ABS(FL) . LE. D*S12) N1 = N1 + 1
IF (ABS(FL) . LE. 2.306*QA) N2 = N2 + 1
IF (ABS(FL) . LE. VS12) N3 = N3 + 1
IF (ABS(FL) . LE. R) N4 = N4 + 1
IF (ABS(FL) . LE. 2.306*S12) N5 = N5 + 1
SBFD = SBFD + BFBD
SSFD = SSFD + SFD
SWHD = SWHD + WHD
SBAND = SBAND + BAND
SHAJD = SHAJD + HAJD
SSBFD = SSBFD + BFBD**2
SSSFD = SSSFD + SFD**2
SSWHD = SSWHD + WHD**2
SSBAND = SSBAND + BAND**2
SSHAJD = SSSHAJD + HAJD**2
CONTINUE
ABFD = SBFD/200.
ASFD = SSFD/200.
AWHD = SWHD/200.
ABAND = SBAND/200.
AHJAD = SHAJD/200.
VRBFD = (SSBFD - 200. * ABFD**2)/199.
VRSFD = (SSSFD - 200. * ASFD**2)/199.
VRWHD = (SSWHD - 200. * AWHD**2)/199.
VRBAND = (SSBAND - 200. * ABAND**2)/199.
VRHAJD = (SSHAJD - 200. * AHJAD**2)/199.
WRITE(6, 202) ABFD, ASFD, AWHD, ABAND, AHJAD
202 FORMAT(1X, 'MEAN', 5F10.5)
WRITE(6, 203) VRBFD, VRSFD, VRWHD, VRBAND, VRHAJD
203 FORMAT(1X, 'VAR', 1X, 5F10.5)
WRITE(6,201) N1,N2,N3,N4,N5
201 FORMAT(1X,'N',3X,5I10)
STOP
END

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DIMENSION X1(101), X2(101), U(101), X(40400)
WRITE(6, 200)
LL=40400
IR=45679
DO 300 IK=1,40400
  Y=0.
  DO 301 JK=1,12
    Y=Y+RN(IR)
    X(IK)=(Y-6.)
  301 CONTINUE
IK=0
N1=0
N2=0
N3=0
N4=0
N5=0
SBFD=0.
SSFD=0.
SWHD=0.
SBAND=0.
SHAJD=0.
SSBFD=0.
SSFD=0.
SSWHD=0.
SSBAND=0.
SSHAJD=0.
DO 99 K=1,200
  DO 90 I=1,101
    IK=IK+1
    90 X1(I)=X(IK)*2.
    DO 81 J=1,101
      IK=IK+1
      81 X2(J)=X(IK)
    SMX1=0.
    SMX2=0.
    DO 15 I=1,101
      15 SMX1=SMX1+X1(I)
    DO 20 J=1,101
      20 SMX2=SMX2+X2(J)
    FN1=101.
    FN2=101.
    X1MN=SMX1/FN1
    X2MN=SMX2/FN2
    SMSX1=0.
    SMSX2=0.
    DO 25 I=1,101
      25 SMSX1=SMSX1+(X1(I)-X1MN)**2
    DO 30 J=1,101
      30 SMSX2=SMSX2+(X2(J)-X2MN)**2
    VRX1=SMSX1/(FN1-1.)
    VRX2=SMSX2/(FN2-1.)
    S12=SQR(T( VRX1/FN1+VRX2/FN2)
SMU=0.
DO 35 I=1,101
U(I)=X1(I)-SQRT(FN1/FN2)*X2(I)
35 SMU=SMU+U(I)
UBAR=SMU/FN1
Q=0.
DO 40 I=1,101
40 Q=Q+(U(I)-UBAR)**2
QA=SQRT(O/10100.)
B=SQRT(1.984**2*VRX1/FN1+1.984**2*VRX2/FN2)
D=1.96
V=1.96
BF=2.*D*S12
SFD=2.*1.984*QA
WH=2.*V*S12
BAND=2.*B
HAJD=2.*1.984*S12
FL=X1MN-X2MN
IF(ABS(FL).LE.D*S12) N1=N1+1
IF(ABS(FL).LE.1.984*QA) N2=N2+1
IF(ABS(FL).LE.V*S12) N3=N3+1
IF(ABS(FL).LE.B) N4=N4+1
IF(ABS(FL).LE.1.984*S12) N5=N5+1
SBFD=SFD+BF
SSFD=SSFD+SFD
SWHD=SWHD+WH
SBAND=SBAND+BAND
SHAJD=SHAJD+HAJD
SSBFD=SSBFD+BFD**2
SSSFD=SSSFD+SFD**2
SSWH=SSWH+WH
SSBAND=SSBAND+BAND**2
SSHAJD=SSHAJD+HAJD**2
99 CONTINUE
ABFD=SBFD/200.
ASFD=SSFD/200.
AWHD=SWHD/200.
ABAND=SBAND/200.
AHAJD=SHAJD/200.
VRBFD=(SSBFD-200.*ARFD**2)/199.
VRSFD=(SSSFD-200.*ASFD**2)/199.
VRWH=(SSWH-200.*AWHD**2)/199.
VRBAND=(SSBAND-200.*ABAND**2)/199.
VRAJD=(SSHAJD-200.*AHAJD**2)/199.
WRITE(6,202) ABFD,ASFD,AWHD,ABAND,AHAJD
202 FORMAT(1X,'MEAN',5F10.5)
WRITE(6,203) VRBFD,VRSFD,VRWH,VRBAND,VRAJD
203 FORMAT(1X,'VAR',5F10.5)
WRITE(6,201) N1,N2,N3,N4,N5
201 FORMAT(1X,'N',5I10)
STOP
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