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Bayesian Estimate of System Reliability

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BAYESIAN ESTIMATE OF SYSTEM RELIABILITY

by

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Naresh Shah

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ABSTRACT

Bayesian Estimate of System Reliability

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A Bayesian estimate of reliability for each component in the system of n-components, each exponentially distributed, is developed which utilizes the basic notion of loss in estimation theory. Here we assume that each component is independently distributed. In reliability estimation, the loss associated with over-estimation is usually greater than the loss associated with under-estimation; and hence loss function can be a very useful tool. The prior distribution and loss function of reliability considered in this paper are flexible to be compatible with other situations in which reliability estimates are required. When the loss function is symmetric and no prior information is at hand, the resulting estimate is approximately the minimum variance unbiased estimate of reliability.

(39 pages)

INTRODUCTION

Reliability is the probability that a device will perform its purpose adequately for the period of time intended under the operating conditions encountered.

Generally, underestimation of reliability results in the unnecessary expense of redundancy or other measures to bring the reliability up to a desired level. Overestimation of reliability results in unwarranted confidence which may lead to total mission failure. In practice, the loss incurred by underestimation of reliability is usually less than the loss incurred when reliability is overestimated. For this reason, lower confidence bounds have been used as estimates of reliability. This approach neglects the basic notion of a loss function in decision theory (Lindgren, 1968).

Consider a system of n-components, subjected to an environmental life test. Due to time or budget limitations, it may be necessary to terminate testing after a limited number of failures or after a certain amount of time for a particular component. The engineer is required to establish the estimate of reliability for each component with this limited amount of data. A great deal of knowledge may be available through past experience of similar items. Bayesian theory permits the incorporation of this prior information into the reliability estimate and thus provides an attractive approach to overcoming the limited data problem.

This paper presents a solution of the estimation problem by using prior knowledge in Bayesian theory with a loss function. A loss function is described which permits weighting of loss to reflect any attitude toward overestimation.

The exponential model of reliability is used. Thus, the reliability $R(\theta,t)$ is given by

$$R(\theta,t) = e^{-\theta t} \tag{1}$$

where θ is the failure rate and t is the fixed mission time. It was observed that when loss function is symmetric and the prior distribution of failure rate is uniform (i.e., no prior knowledge), then the resulting reliability estimate is approximately the minimum variance unbiased estimate (Pugh, 1963).

To briefly restate the postulate of Bayes, assume that we know a certain conditional density function, $f(Z|\theta)$ and we desire to know $h(\theta|Z)$. We may write:

$$h(\theta|Z) = \frac{f(Z,\theta)}{f(Z)} = \frac{f(Z|\theta)g(\theta)}{\int_\theta f(Z|\theta)g(\theta)\,d\theta} \tag{2}$$

where the integration is performed over all θ, to give the marginal density of Z. The only unknown quantity in Equation (2) is a prior distribution of θ, $g(\theta)$. So, if we know prior distribution of θ, $g(\theta)$, then we obtained $h(\theta|Z)$, which is known as the posterior distribution.

We define our loss function as $\ell(\theta_\alpha,\theta)$, where θ_α is estimate for θ. In this case, Bayes posterior loss will be defined as

$$B(\theta_\alpha) = E[\ell(\theta_\alpha,\theta)]$$

$$= \int \ell(\theta_\alpha,\theta)h(\theta|Z)\, d\theta \tag{2a}$$

The Bayes principle calls for taking that value of θ_α which minimizes Bayes loss. In the case of a discrete distribution, the integration sign will be replaced by summation (Lindgren, 1968).

In the following pages, first we consider the case with single component and obtain the reliability estimation with Bayesian approach. Then we consider the system which consists of n-independent components; each component has been tested separately for its reliability estimate and then obtained the reliability estimates for the system.

ESTIMATE OF RELIABILITY WITH SINGLE COMPONENT

Introduction

Two cases are presented depending upon the manner in which the data are collected. Let T be accumulated test time and r the number of failures recorded. Case A: the test is terminated at the r^{th} failure; and Case B: the test is terminated after a preassigned number of hours (T) of test time. The number of failures, r, is recorded. For Case A, the quantity $2\theta T$ has the Chi-square distribution with 2r degrees of freedom; and for Case B, 2r is replaced by 2r + 2 in the following solution (Epstein and Sobel, 1953).

Consider the class of functions which are given by the usual confidence bound R_α of $R(\theta,t)$,

$$R_\alpha = R(\theta_\alpha,t) = e^{-\theta_\alpha t} \tag{3}$$

where

$$\theta_\alpha = \frac{\chi^2_{\nu,\alpha}}{2T}$$

and

$$F_{\chi^2_\nu}(2T\theta_\alpha) = 1 - \alpha.$$

The parameter ν has the value 2r in Case A.

Loss Function

Let R be the true reliability and

$$L_1(\theta_\alpha,\theta) = \frac{1}{\theta}\left(\frac{R_\alpha}{R} - 1\right)^2 \tag{4}$$

$$L_2(\theta_\alpha,\theta) = \frac{2\gamma}{\theta}\left(\frac{R_\alpha}{R} - 1\right) \tag{5}$$

If underestimation has occurred, then $R_\alpha \leq R$ which implies $e^{-\theta_\alpha t} \leq e^{-\theta t}$ or $\theta_\alpha \geq \theta$; and in this case, loss function is given by

$$L(\theta_\alpha,\theta) = L_1(\theta_\alpha,\theta)$$

$$= \frac{1}{\theta}\left(\frac{R_\alpha}{R} - 1\right)^2 \tag{6}$$

If overestimation has occurred, then $R_\alpha > R$ which implies $e^{-\theta_\alpha t} > e^{-\theta t}$ or $\theta_\alpha < \theta$; and in this case the loss function is given by

$$L(\theta_\alpha, \theta) = L_1(\theta_\alpha, \theta) + L_2(\theta_\alpha,\theta)$$

$$= \frac{1}{\theta}\left(\frac{R_\alpha}{R} - 1\right)^2 + \frac{2\gamma}{\theta}\left(\frac{R_\alpha}{R} - 1\right) \tag{7}$$

The parameter γ is seen to control the bias for overestimation in the loss. If $\gamma = 0$, the loss function is symmetric. Figure 1 shows the graph of the loss function for various values of γ, when the true reliability is 0.9. For example, if the reliability is underestimated by an amount 0.625, the loss is seen to be 0.048. If, however $\gamma = 1$,

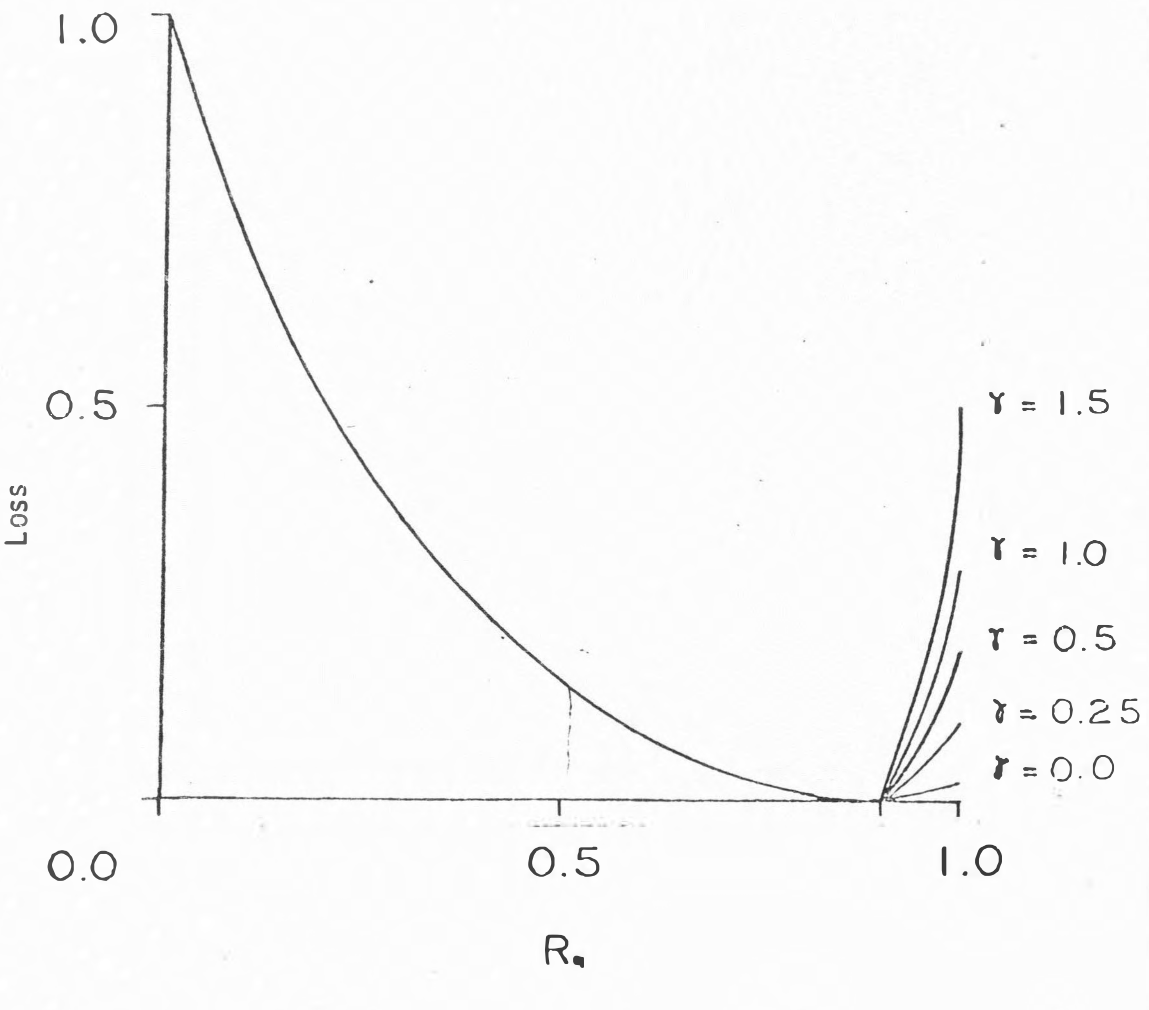

Figure 1. Loss function.

overestimation by the same amount results in a loss of 0.187. When $\gamma = 1.5$, the same amount of overestimation shows a loss of 0.275. Thus a larger γ indicates greater loss for overestimation as compared with underestimation.

Prior Distribution

The selection of a prior distribution for reliability, or for failure rate r, allows the practitioner to use information which he has gathered through experience or history of similar items. Since this information is usually subjective, a significant criticism of Bayesian methods is that it allows the practitioner to inject his desires rather than his experience into the solution.

The uniform prior distribution means no prior information. So, it is desirable to choose a prior distribution which indicates a general trend toward the previous experience, and which in general does not have a small variance as compared with the uniform distribution since this could significantly bias the estimated value of reliability.

A prior distribution of reliability for fixed mission time t is intuitively appealing on these grounds in the Beta distribution.

The density function for Beta distribution is:

$$f(R) = \frac{1}{\beta(P,q)} R^{P-1}(1-R)^{q-1} \tag{8}$$

for $0 \leq R \leq 1$, and P,q is the parameters of the Beta distribution. If we select P is greater than q, then a trend toward higher values of reliability is indicated. The variance of the Beta distribution is:

$$\sigma^2 = \frac{Pq}{(P+q)^2(P+q+1)}$$ (For derivation see Appendix A.)

This variance decreases with increasing P and q. More accurate prior information gives a smaller variance of the prior distribution. The uniform distribution on (0,1) is seen to be the special case of the Beta with $P = q = 1$. A change in the mission time generally requires a change in the values of P and q for the prior on R. For the exponential case an increase in t causes a decrease in reliability and thus the prior should reflect this effect by showing a trend toward lower reliabilities as t increases.

It is difficult to determine this type of functional relationship. For this reason it is convenient to use a prior distribution on the failure rate instead of R to avoid dependence on t. The following allows one to incorporate the desirable features of (8) into the prior distribution of θ without determining the functional realtion of t in (8). The prior distribution of θ should be the same no matter what mission time is contemplated; thus in determining the prior for θ it suffices to first determine the prior for R for some convenient fixed t using (8) and then use standard transformation techniques to determine the prior for θ. Let $g(\theta)$ be the prior on θ and $t = t_0$ in (8), then

$$g(\theta) = f(R)\left|\frac{dR}{d\theta}\right| \tag{9}$$

and $R = e^{-\theta t}$

Therefore by transformation $R = e^{-\theta t_0}$.

$$g(\theta) = \frac{1}{\beta(P,q)} \; (e^{-\theta t_o})^{P-1} \; (1-e^{-\theta t_o})^{q-1} \; [\frac{d}{d\theta} \; e^{-\theta t_o}]$$

$$= \frac{1}{\beta(P,q)} \; e^{-P\theta t_o} \; (1-e^{-\theta t_o})^{q-1} \cdot t_o \tag{10}$$

The uniform prior on θ is obtained when $P = 0$ and $q = 1$.

Posterior Distribution

The posterior distribution for θ is derived using the prior density (1) of failure rate. First we consider Case A for which the test is terminated at the time of r^{th} failure. Here $2\theta T$ has the Chi-square distribution with 2r degrees of freedom. The condition density for T given value of θ is

$$f(T|\theta) = \frac{\theta^r}{\Gamma(r)} \; T^{r-1} \; e^{-T\theta} \tag{11}$$

By definition, the posterior distribution $h(\theta|T)$ of (Lindgren, 1968) is

$$h(\theta|T) = \frac{f(T|\theta)g(\theta)}{\int_\theta f(T|\theta)g(\theta)\,d\theta}$$

$$= \frac{\frac{\theta^r\,T^{r-1}\,e^{-T\theta}}{\Gamma(r)} \cdot \frac{t_o}{\beta(P,q)} \; e^{-P\theta t_o} \; (1-e^{-\theta t_o})^{q-1}}{\int_\theta \frac{\theta^r\,T^{r-1}\,e^{-T\theta}}{\Gamma(r)} \; \frac{t_o}{\beta(P,q)} \; e^{-P\theta t_o} \; (1-e^{-\theta t_o})^{q-1} \; d\theta}$$

$$= K\,\theta^r e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \tag{12}$$

where

$$K = \int_\theta \theta^r e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \, d\theta.$$

Here K is the normalizing constant, t mission time, and t_o is the time used to determine the prior distribution $g(\theta)$. The posterior distribution for Case B is similar with putting r + 1 for r in Equation (11).

Reliability Estimation

The reliability is estimated by deriving the expression for Bayes loss as a function of the estimate θ_α of the parameter θ. Let $B(\theta_\alpha)$ is the Bayes loss. Then

$$B(\theta_\alpha) = \int_0^\infty L(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta$$

$$= \int_0^{\theta_\alpha} L_1(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta + \int_{\theta_\alpha}^\infty L_1(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta$$

$$+ \int_{\theta_\alpha}^\infty L_2(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta$$

$$= \int_0^\infty L_1(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta + \int_{\theta_\alpha}^\infty L_2(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta \tag{13}$$

The Bayes estimate for reliability is obtained by the value of α which minimizes (13). To find the value of θ_α which minimizes (13), differentiate (13) with respect to θ_α and make equal to zero and solve it.

$$\frac{\partial}{\partial\theta_\alpha} B(\theta_\alpha) = B'(\theta_\alpha)$$

$$= \int_0^\infty \frac{\partial}{\partial\theta_\alpha} L_1(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta \quad + \int_{\theta_\alpha}^\infty \frac{\partial}{\partial\theta_\alpha} L_2\,(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta$$

$$- L_2(\theta_\alpha,\theta_\alpha)\, h(\theta_\alpha|T) \tag{14}$$

Now

$$L_2(\theta_\alpha,\theta_\alpha) = 0, \text{ so}$$

$$B'(\theta_\alpha) = \int_0^\infty \frac{\partial}{\partial\theta_\alpha} L_1(\theta_\alpha,\theta)\, h(\theta|T)\, d\theta \quad + \int_{\theta_\alpha}^\infty \frac{\partial}{\partial\theta_\alpha} L_2(\theta_\alpha,\theta)\, h(\theta|T) d\theta$$

Equating this equal to zero and after simplification (details are in Appendix B), we get

$$R_\alpha = (1 - \frac{t}{T})^r \; \frac{\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} A_i - \gamma \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} A_i\, \alpha_i^*}{\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} [1 - \frac{t-(P+i)t_0}{T-t}]^{-r}} \tag{15}$$

where

$$A_i = [1 - \frac{t - (P+i)t_o}{T}]^{-r}$$

$$\alpha = 1 - F_{\chi^2_{2r}}(2T\theta_\alpha)$$

and

$$\alpha_i^* = 1 - F_{\chi^2_{2r}}\{2T\theta_\alpha[1 - \frac{t - (P+i)t_o}{T}]\} \tag{16}$$

By selecting level α in such a way that above equality holds, the R_α is the estimate of reliability. Biometrika Tables for Statisticians will be helpful for evaluation of α_i^*.

When unbiased estimate is desired (i.e., Y = 0 and uniform prior on R for all mission times) then (15) reduces to

$$R_\alpha = (1 - \frac{t}{T})^r \tag{17}$$

and this estimate is exactly the minimum variance unbiased estimate of reliability (Pugh, 1963). (For details, see Appendix C.)

In most of the cases T >> t, and then approximately

$$\alpha_i^* \doteq \alpha - \frac{[(P+i)t_o - t]\ (T\theta_\alpha)^r\ e^{-T\theta_\alpha}}{T\Gamma(r)} \tag{18}$$

(For details see Appendix D.) This approximation may be useful in solving R_α.

Also from (16) or (18), if T is sufficiently large, then $\alpha_i^* \doteq \alpha$ for all i and then Equation (15) simplifies to

$$R_\alpha = (1 - \gamma\alpha)(1 - \frac{t}{T})^r \frac{\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i}[1 - \frac{t-(P+i)t_o}{T}]^{-r}}{\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i}[1 - \frac{t-(P+i)t_o}{T-t}]^{-r}}$$

$$= (1 - \gamma\alpha)(1 - \frac{t}{T})^r \text{ for large T.} \qquad (19)$$

The solution for Case B where the life test is terminated after preassigned accumulated time T is similar to the above. The only difference is r in Equation (15) is replaced by r + 1.

SYSTEM RELIABILITY

Introduction

In this chapter we extend our results to the case of a system. Let the system consist of n-independent components in series. Components in series means that the system will fail if any one component fails. Independent components means failure of one component has no relation to the failure of any other components. For each component, the exponential model of reliability is used. Thus the reliability is

$$R_i(\theta_i, t) = e^{-\theta_i t} \qquad i = 1,2,\ldots,n \tag{20}$$

where θ_i is the failure rate for i^{th} component and t is the mission time. For simplicity we assume t is the same for all components. Two cases are presented. Case A, the test is terminated at the r_i^{th} failure for i^{th} component and Case B, the test is terminated after a pre-assigned number of hours (T_i) of test time for i^{th} component. Now, the quantity $2\theta_i T$ has the Chi-square distribution with $2r_i$ degrees of freedom for Case A. The result for Case B is obtained by substituting r+1 in r for Case A.

Let R_{α_i} be the lower confidence bound of reliability for i^{th} component such that $P(R_{\alpha_i} < R_i) \geq 1 - \alpha_i$ and $\hat{R}$ be

$$\hat{R} = R_{\alpha_i} \cdot R_{\alpha_2} \ldots \cdot R_{\alpha_n} \tag{21}$$

Now as in the previous section,

$$R_{\alpha_i} = R(\theta_{\alpha_i},t) = e^{-\theta_{\alpha_i} t} \tag{22}$$

where $2T\theta_{\alpha_i}$ has Chi-square distribution with V_i degrees of freedom, and

$$F_{\chi^2_{V_i}}(2T\theta_{\alpha_i}) = 1 - \alpha_i \tag{23}$$

The parameter V_i has the value $2r_i$ for Case A and $2r_i+2$ for Case B.

Let $\theta_\alpha = \theta_{\alpha_1} + \theta_{\alpha_2} + \ldots + \theta_{\alpha_n}$, then

$$\hat{R} = R(\theta_\alpha,t) = \prod_{i=1}^{n} R_{\alpha_i}$$

$$= e^{-t \sum_{i=1}^{n} \theta_{\alpha_i}}$$

$$= e^{-t\theta_\alpha} \tag{24}$$

Let R_i be the true reliability for i^{th} component and R be the true reliability for system. Then

$$R = R_1 \cdot R_2 \cdot \ \ldots\ldots \cdot R_n$$

$$= e^{-t \sum_{i=1}^{n} \theta_i} = e^{-t\theta} \text{ where } \theta = \sum_{i=1}^{n} \theta_i \tag{25}$$

Let

$$L_{i1}(\theta_{\alpha_i},\theta_i) = \frac{1}{\theta_i}\left(\frac{R_{\alpha_i}}{R_i} - 1\right)^2$$

$$L_{i2}(\theta_{\alpha_i},\theta_i) = \frac{\gamma_i^2}{\theta_i}\left(\frac{R_{\alpha_i}}{R_i} - 1\right)$$

(The parameter γ_i will control the bias for overestimation of i^{th} component in the system loss.)

and

$$L_i(\theta_{\alpha_i},\theta_i) = \begin{cases} L_{i1}(\theta_{\alpha_i},\theta_i) \text{ if } \theta_{\alpha_i} \geq \theta_i \\ \\ L_{i1}(\theta_{\alpha_i},\theta_i) + L_{2i}(\theta_{\alpha_i},\theta_i) \text{ if } \theta_{\alpha_i} < \theta_i \end{cases}$$

Then loss function for system is given by

$$L_s(\theta_\alpha,\theta) = \sum_{i=1}^{n} L_i(\theta_{\alpha_i},\theta_i)\, a_i \text{ where } a_i \text{ is the weight} \tag{26}$$

attached to i^{th} component.

Prior Distribution

A prior distribution of system reliability is the product of all prior distributions for each component. As previously shown, the Beta distribution is used for the prior distribution for each component.

The prior distribution for i^{th} component is

$$f(R_i) = \frac{1}{\beta(P_i,q_i)} R_i^{P_i-1} (1 - R_i)^{q_i-1} \text{ for } 0 < R_i < 1$$

for $i = 1,2,\ldots,n$ and P_i, q_i is the parameter of Beta distribution.

The prior distribution of θ_i for fixed $t = t_o$ will be

$$g(\theta_i) = f(R_i)\left[\frac{dR_i}{d\theta_i}\right]$$

$$= \frac{t_o}{\beta(P_i,q_i)}\, e^{-P_i\theta_i t_o}\,(1-e^{-\theta_i t_o})^{q-1}$$

and hence prior distribution for system will be

$$g(\theta_1,\theta_2,\ldots,\theta_n) = t_o^{\,n} \prod_{i=1}^{n} \frac{1}{\beta(P_i,q_i)}\, e^{-P_i\theta_i t_o}\,(1-e^{-\theta_i t_o})^{q_i-1}$$

$$= \frac{t_o^{\,n}}{\prod_{i=1}^{n} \beta(P_i,q_i)}\, e^{-t_o\sum_{i=1}^{n} P_i\theta_i} \prod_{i=1}^{n} (1-e^{-\theta_i t_o})^{q_i-1} \qquad (27)$$

Posterior Distribution

For i^{th} component and Case A, as shown previously by Equation (11),

$$f(T_i|\theta_i) = \frac{\theta_i^{r_i}}{\Gamma(r_i)}\, T_i^{\,r_i-1}\, e^{-T_i\theta_i}$$

and hence

$$f(\underline{T}|\theta_1,\theta_2,\ldots,\theta_n) = f(T_1|\theta_1)\, f(T_2|\theta_2) \;\ldots.\; f(T_n|\theta_n)$$

$$= \frac{\theta_1^{r_1}\,\theta_2^{r_2}\;\ldots.\;\theta_n^{r_n}}{\prod_{i=1}^{n}\Gamma(r_i)} \prod_{i=1}^{n} T_i^{\,r_i-1}\, e^{-T_i\theta_i} \qquad (28)$$

From Equation (12), the posterior distribution of $h(\theta_i|T)$ is

$$h(\theta_i|T_i) = K_1\theta_i^{r_i} e^{-(P_i t_o+T_i)\theta_i} (1-e^{-\theta_i t_o})^{q_i-1}$$

and hence

$$h(\theta_1,\theta_2,\ldots,\theta_n|\underline{T}) = H(\theta_1|T_1)\cdot h(\theta_2|T_2)\cdot\ldots\ldots\cdot h(\theta_n|T_n)$$

$$= \frac{g(\theta_1,\theta_2,\ldots,\theta_n)\cdot f(\underline{T}|\theta_1,\theta_2,\ldots,\theta_n)}{\int_\Omega g(\theta_1,\theta_2,\ldots,\theta_n)\, f(\underline{T}|\theta_1,\theta_2,\ldots,\theta_n)\, d\theta_1 d\theta_2,\ldots,d\theta_n}$$

$$h(\theta_1,\theta_2,\ldots,\theta_n\, |\underline{T}) = K \prod_{i=1}^{n} [\theta_i^{r_i} e^{-(P_i t_o+T)\theta_i} (1-e^{-\theta_i t_o})^{q_i-1}]$$

$$= K\theta_1^{r_1}\cdot\theta_2^{r_2},\ldots,\theta_n^{r_n} e^{-\sum_{i=1}^{n} (P_i t_o+T_i)\theta_i} \prod_{i=1}^{n} (1-e^{-\theta_i t_o})^{q_i-1} \qquad (29)$$

Estimation of Reliability

Now the Bayes loss for the system will be $B(\theta_\alpha)$. Then

$$B(\theta_\alpha) = E[L_s(\theta_\alpha,\theta)]$$

$$= \int \ldots\ldots \int \sum_{i=1}^{n} a_i\, L_i(\theta_{\alpha_i},\theta_i) h(\theta_1\ldots.\theta_n|\underline{T}) d\theta_1 d\theta_2 \ldots d\theta_n$$

$$= \sum_{i=1}^{n} a_i \int \ldots\ldots \int L_i(\theta_{\alpha_i},\theta_i) h(\theta_1,\ldots,\theta_n|\underline{T}) d\theta_1 d\theta_2\ldots d\theta_n \qquad (30)$$

We minimize this Bayes loss by taking the partial derivative with respect to θ_{α_i} for i = 1,2,...,n and we obtain the estimation for R_{α_i} for each i.

$$\frac{\partial B(\theta_\alpha)}{\partial \theta_{\alpha_i}} = B_i'(\theta_\alpha)$$

$$= \frac{\partial}{\partial \theta_{\alpha_i}} [\sum_{i=1}^{n} a_i \int \cdots\cdots \int L_i(\theta_{\alpha_i}, \theta_i) h(\theta_1 \ldots\ldots \theta_n | \underline{T}) d\theta_1 \ldots d\theta_n]$$

$$= \int a_i L_i(\theta_{\alpha_i}, \theta_i) h(\theta_i | T_i)\ d\theta_i \prod_{\substack{j=1 \\ j \neq i}}^{n} \int h(\theta_j | T_j)\ d\theta_j$$

$$= a_i \int L_i\ (\theta_{\alpha_i}, \theta_i) h(\theta_i | T_i)\ d\theta_i$$

$$B_i'(\theta_\alpha) = a_i \int_0^\infty L_{i1}(\theta_{\alpha_i}, \theta_i) h(\theta_i | T_i)\ d\theta_i + \int_{\theta_{\alpha_i}}^\infty a_i L_{i2}(\theta_{\alpha_i} \theta_i) h(\theta_i | T_i) d\theta_i$$

$$B_i'(\theta_\alpha) = a_i \int_0^\infty \frac{\partial}{\partial \theta_{\alpha_i}} L_{i1} h(\theta_i | T_i) d\theta_i + a_i \int_{\theta_{\alpha_i}}^\infty \frac{\partial}{\partial \theta_{\alpha_i}} L_{i2} h(\theta_i | T_i) d\theta_i\ -$$

$$- L_{i2}(\theta_{\alpha_i}, \theta_{\alpha_i}) h(\theta_i | T_i)$$

By solving $B_i'(\theta_\alpha) = 0$, we obtained:

$$R_{\alpha_i} = (1-\frac{t}{T_i})^{r_i} \frac{\sum_{j=0}^{q_i-1} (-1)^j \binom{q_i-1}{j} A_{ij} - \gamma_i \sum_{j=0}^{q_i-1} (-1)^j \binom{q_i-1}{j} A_{ij} \quad \alpha^*_{ij}}{\sum_{j=0}^{q_i-1} (-1)^j \binom{q_i-1}{j} \left[1 - \frac{t-(P_i+j)t_o}{T_i - t}\right]^{-r_i}}$$

where

$$A_{ij} = \left[1 - \frac{t-(P_i+j)t_o}{T_i}\right]^{-r_i}$$

$$\alpha^*_{ij} = 1 - F_{x^2_{2r_i}} \left\{ \left[1 - \frac{t-(P_i+j)t_o}{T_i}\right] 2T_i \theta_{\alpha_i} \right\}$$

$$\alpha_i = 1 - F_{x^2_{2r_i}} (2T_i \theta_{\alpha_i})$$

So our estimation for $\hat{R}$ will be

$\hat{R} = \prod_{i=1}^{n} R_{\alpha_i}$ where R_{α_i} has above estimated value.

Here we note that whatever weight we attached to a particular component loss in the combined losses for a system does not enter into the estimation of component reliability. This is because of the independence of the components of the system and the particular loss function used. Since the loss for the system is the sum of component losses, by independence the minimum loss occurs when each term is minimum.

Remarks

In system reliability estimation, we may consider some other loss functions. The following loss function is a particularly appealing one.

$$L_s = \frac{1}{\theta_1 \cdots \theta_n} \left(\frac{R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n}}{R_1 R_2 \cdots R_n} - 1 \right)^2 + \frac{2\gamma}{\theta_1 \theta_2 \cdots \theta_n} \left(\frac{R_{\alpha_1} R_{\alpha_2} \cdots R_{\alpha_n}}{R_1 R_2 \cdots R_n} - 1 \right)$$

With the above loss function, I tried to evaluate R_α, but final result is too complicated and requires advanced contour integration.

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APPENDIXES

Appendix A

Variance of Beta Distribution

The calculation of variance for Beta distribution is as under:

$$f(R) = \frac{1}{\beta(P,q)} R^{P-1} (1-R)^{q-1}, \; 0 < R < 1$$

$$E(R) = \int_0^1 R\, f(R)\, dR$$

$$= \int_0^1 \frac{R^P (1-R)^{q-1}}{\beta(P,q)} dR$$

$$= \frac{\beta(P+1,q)}{\beta(P,q)} = \frac{P}{P+q}$$

where

$$\beta(P,q) = \int_0^1 x^{P-1}(1-x)^{q-1}\, dx = \frac{\Gamma(P)\Gamma(q)}{\Gamma(P+q)} \text{ and } \Gamma(P) = (P-1)!$$

and

$$E(R^2) = \int_0^1 R^2 f(R)\, dR$$

$$= \frac{\beta(P+2,q)}{\beta(P,q)} = \frac{P(P+1)}{(P+1)(P+q+1)}$$

$$\text{Variance} = E(R^2) - [E(R)]^2$$

$$= \frac{P(P+1)}{(P+q)(P+q+1)} - \left(\frac{P}{P+q}\right)^2 = \frac{Pq}{(P+q)^2(P+q+1)}$$

Appendix B

Solution of Bayes Loss Equation

Details of calculations for arriving at Equation (15) are as follows:

From (13)

$$B(\theta_\alpha) = \int_0^\infty L_1(\theta_\alpha,\theta)h(\theta|T)\,d\theta + \int_{\theta_\alpha}^\infty L_2(\theta_\alpha,\theta)h(\theta|T)\,d\theta$$

$$= \int_0^\infty \frac{1}{\theta}\left(\frac{R_\alpha}{R}-1\right)^2 \cdot K\,\theta^r\,e^{-\theta(Pt_o+T)}(1-e^{-\theta t_o})^{q-1}\,d\theta$$

$$+ \int_{\theta_\alpha}^\infty \frac{2\gamma}{\theta}\left(\frac{R_\alpha}{R}-1\right)\cdot K\,\theta^r\,e^{-\theta(Pt_o+T)}(1-e^{-\theta t_o})^{q-1}\,d\theta$$

Therefore

$$\frac{\partial}{\partial\theta_\alpha}B(\theta_\alpha) = \frac{\partial}{\partial\theta_\alpha}\left[\int_0^\infty \frac{1}{\theta}\left(\frac{R_\alpha}{R}-1\right)^2 \cdot K\,\theta^r\,e^{-\theta(Pt_o+T)}(1-e^{-\theta t_o})^{q-1}\,d\theta\right]$$

$$+ \frac{\partial}{\partial\theta_\alpha}\left[\int_{\theta_\alpha}^\infty \frac{2\gamma}{\theta}\left(\frac{R_\alpha}{R}-1\right) K\theta^r\,e^{-\theta(Pt_o+T)}(1-e^{-\theta t_o})^{q-1}\,d\theta\right]$$

$$B'(\theta_\alpha) = K \int_0^\infty \theta^r e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{\theta} \frac{\partial}{\partial\theta_\alpha} [(\frac{R_\alpha}{R} - 1)^2] \, d\theta$$

$$+ 2\gamma K \int_{\theta_\alpha}^\infty \theta^r e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{\theta} \frac{\partial}{\partial\theta_\alpha} (\frac{R_\alpha}{R} - 1) \, d\theta$$

$$= K \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \cdot 2(\frac{R_\alpha}{R} - 1) \frac{1}{R} R_\alpha (-t) \, d\theta$$

$$+ 2\gamma K \int_{\theta_\alpha}^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{\alpha}{R} (-t) \, d\theta$$

Taking this derivative expression equal to zero, we get

$$0 = 2K R_\alpha(-t) \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{R} (\frac{R_\alpha}{R} - 1) \, d\theta$$

$$+ 2K\gamma R_\alpha(-t) \int_{\theta_\alpha}^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{R} \, d\theta$$

By cancelling common factor $2K R_\alpha(-t)$ and substituting

$R = e^{-\theta t}$, we get

$$0 = \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{e^{-\theta t}} \frac{R_\alpha}{e^{-\theta t}} \, d\theta$$

$$- \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{e^{-\theta t}} \, d\theta \; +$$

$$+ \gamma \int_{\theta_\alpha}^\infty \theta^{r-1} e^{-\theta(Pt_o+T)} (1-e^{-\theta t_o})^{q-1} \frac{1}{e^{-\theta t}} \, d\theta$$

Therefore,

$$R_\alpha \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T-2t)} (1-e^{-\theta t_o})^{q-1} \, d\theta \; =$$

$$= \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T-t)} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

$$- \gamma \int_{\theta_\alpha}^\infty \theta^{r-1} e^{-\theta(Pt_o+T-t)} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

or $\qquad R_\alpha = \dfrac{A - B}{C}$

where

$$A = \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T-t)} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

$$B = \gamma \int_{\theta_\alpha}^\infty \theta^{r-1} e^{-\theta(Pt_o+T-t)} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

and

$$C = \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T-2t)} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

Now we evaluate the above integrals. First,

$$A = \int_0^\infty \theta^{r-1} e^{-(Pt_o+T-t)\theta} (1-e^{-\theta t_o})^{q-1} \, d\theta$$

$$= \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+t-t)} \left[\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} e^{-i\theta t_o}\right] d\theta$$

$$= \sum_{i=0}^{q-1} (-1)^8 \binom{q-1}{i} \int_0^\infty \theta^{r-1} e^{-\theta(Pt_o+T-t+it_o)} \, d\theta$$

Put $\frac{Z}{2} = \theta(Pt_o+T-t+it_o)$, hence,

$$\frac{dZ}{2(Pt_o+T-t+it_o)} = d\theta$$

Therefore,

$$A = \sum_{i=0}^{q-1} (-1)^{8}\binom{q-1}{i} \int_0^{\infty} \left[\frac{Z}{2(Pt_o+T-t+it_o)}\right]^{r-1} e^{-Z/2} \frac{1}{2(Pt_o+T-t+it_o)}\, dZ$$

$$= \sum_{i=0}^{q-1} (-1)^{i}\binom{q-1}{i} \frac{1}{(Pt_o+T-t+it_o)^{r}} \int_0^{\infty} \left(\frac{Z}{2}\right)^{r-1} e^{-Z/2}\cdot \frac{1}{2}\, dZ$$

$$= \Gamma(r) \sum_{i=0}^{q-1} (-1)^{i}\binom{q-1}{i} [(P+i)t_o+T-t]^{-r}$$

$$= \Gamma(r)\, T^{-r} \sum_{i=0}^{q-1} (-1)^{i}\binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T}\right]^{-r}$$

Similarly, integral (C) will be

$$C = \Gamma(r)\, (T-t)^{-r} \sum_{i=0}^{q-1} (-1)^{i}\binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T-t}\right]^{-r}$$

Now

$$B = \int_{\theta_\alpha}^{\infty} \theta^{r-1} e^{-\theta(Pt_o+T-t)} (1-e^{-\theta t_o})^{q-1}\, d\theta$$

$$= \gamma \sum_{i=0}^{q-1} (-1)^{i}\binom{q-1}{i} \int_{\theta_\alpha}^{\infty} \theta^{r-1} e^{-\theta(Pt_o+T-t+t_o i)}\, d\theta$$

Put $Z/2 = \theta(Pt_o+T-t+t_o i)$

$$B = \gamma \sum_{i=0}^{q-1} \frac{(-1)^i \binom{q-1}{i}}{[(P+i)t_o+T-t]^r} \int_{2\theta_\alpha T[1-\frac{t-(P+i)t_o}{T}]}^{\infty} (Z/2)^{r-1}\, e^{-Z/2} \cdot \frac{1}{2} \cdot dZ$$

$$= \Gamma T^{-r}\, \Gamma(r) \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T}\right]^{-r} \cdot \alpha_i^*$$

where

$$\alpha_i^* = \frac{1}{\Gamma(r)} \int_{2\theta_\alpha T(1-\frac{t-(P+i)t_o}{T})}^{\infty} (Z/2)^{r-1}\, e^{-Z/2}\, d(Z/2) = 1 - F_{\chi^2_{2r}}\left[2\theta_\alpha T\left(1 - \frac{t(P+i)t_o}{T}\right)\right]$$

Putting the value of A, B, and C in R, we get

$$R_\alpha = \frac{\Gamma(r)\, T^{-r} \left\{ \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T}\right]^{-r} - \gamma \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T}\right]^{-r} \alpha_i^* \right\}}{\Gamma(r)(T-t)^{-r} \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T-t}\right]^{-r}}$$

$$= \left(1 - \frac{t}{T}\right)^r \frac{\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} A_i - \gamma \sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} A_i\, \alpha_i^*}{\sum_{i=0}^{q-1} (-1)^i \binom{q-1}{i} \left[1 - \frac{t-(P+i)t_o}{T-t}\right]^{-r}}$$

where

$$A_i = \left[1 - \frac{t-(P+i)t_o}{T}\right]^{-r}$$

This R_α is the form of Equation (15)

Appendix C

Unbiased Estimation

From Equation (15)

$$R = (1 - \frac{t}{T})^r \frac{\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} A_i - \gamma \sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} A_i \quad \alpha_i^*}{\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} [1 - \frac{t-(P+i)t_o}{T-t}]^{-r}}$$

Now, when unbiased estimate is desired, i.e., $\gamma = 0$, and uniform prior on R for all t, $P = q = 1$ and $t_o = t$. So $q - 1 = 0$ gives $i = 0$ and $t - (P+i)t_o = 0$. Hence

$$A_i = [1 - \frac{t-(P+i)t_o}{T}]^{-r} = 1$$

and hence

$$\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} A_i = 1.$$

Similarly

$$\sum_{i=1}^{q-1} (-1)^i \binom{q-1}{i} [1 - \frac{t-(P+i)t_o}{\quad}]^{-r} = 1.$$

and

$$R_\alpha = (1 - \frac{t}{T})^r$$

which is the form of Equation (17).

Appendix D

Solution for Large T

From Equation (16),

$$\alpha_i^* = 1 - F_{\chi^2_{2r}}\{2T\theta_\alpha [1 - \frac{t-(P+i)t_o}{T}]\}$$

Now $2T\theta_\alpha = \chi^2_{2r,\alpha}$, then for large T, by Taylor's series expansion,

$$F_{\chi^2_{2r}}\{[1 - \frac{t-(P+i)t_o}{T}] 2T\theta_\alpha\} = F(2\theta_\alpha T) + 2T [\frac{t-(P+i)t_o}{T}] F'(2T\theta_\alpha) + O(\frac{1}{T})$$

$$\doteq 1 - \alpha + 2T\theta_\alpha [\frac{t-(P+i)t_o}{T}] f(2T\theta_\alpha)$$

$$\doteq 1 - \alpha + [\frac{t-(P+i)t_o}{T}] 2T\theta_\alpha \frac{1}{2^r \Gamma(r)} (2T\theta_\alpha)^{r-1} e^{-T\theta_\alpha}$$

$$\doteq 1 - \alpha + [\frac{t-(P+i)t_o}{T}] \frac{(T\theta_\alpha)^r}{\Gamma(r)} e^{-T\theta_\alpha}$$

which gives

$$\alpha_i^* = -\frac{[t-(P+i)t_o]}{T\Gamma(r)} (T\theta_\alpha)^r e^{-T\theta_\alpha}$$

which is the same as Equation (18).

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