Program for Missing Data in the Multivariate Normal Distribution

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PROGRAM FOR MISSING DATA IN THE MULTIVARIATE NORMAL DISTRIBUTION

by

Chi-Ping Lu

A report submitted in partial fulfillment of the requirements for the degree of

MASTER OF SCIENCE

in

Applied Statistics

Plan B

Approved:

UTAH STATE UNIVERSITY
Logan, Utah
1975
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Chi-Ping Lu
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CHAPTER I
INTRODUCTION

Missing data can often cause many problems in research work. Therefore for carrying out analysis, some procedure for obtaining estimates in the presence of missing data should be applied. Various theories and techniques have been developed for different types of problems.

Analysis of the Multivariate Normal Distribution with missing data is one of the areas studied. It has been discussed earlier by Wilkes (1932), Lord (1955), Edgett (1956) and Hartley (1958). They have established some basic concepts and an outline in the way of estimation.

In the last ten years, A. A. Afifi and R. M. Elasoff also have contributed some important techniques in estimating the parameters respective to mean, variance and covariance. R. R. Hocking, H. H. Oxpring and W. B. Smith are continuously improving it toward a more practical method of calculation. In their paper (1971), they gave the derivation of equations and a numerical example without explanation of the details.

The main purpose of this report is to evaluate the reliability and feasibility of this method by programming it. The procedure will be a general one available for large samples so that research workers can apply it conveniently in estimating the parameters when some observations are missing.
Evaluation of this particular method should consider the following properties:

1. The accuracy of the parameters.
2. The simplicity of the procedure; whether all algorithms needed are adequately described.
3. The generality and flexibility for arbitrary kinds of group classifications.
4. The computing costs.

The method we are discussing can be performed with iterations until all estimators converge within a criterion. Theoretically, it converges rapidly.

This study includes description of the whole set of theoretical algorithms and the application of an example. A computer program should be able to manipulate the total procedure rapidly and accurately.

Chapter three will provide an outline of the procedure, with each step followed by a numerical example. The construction of the program will also be illustrated.

Chapter four will give the results based on the research done.

Chapter five will bring out the general conclusions and some questionable points of this method.
2.1 Estimation of mean

The Maximum Likelihood method is a common way for finding point estimates of unknown parameters based on a sample.

Generally, the best statistics satisfy the criteria of "sufficiency", "efficiency", and "consistency" no matter what the distribution is. A maximum likelihood estimator for \( \theta \) (parameter) can be derived by means of differentiation.

Hocking and Smith have developed a maximum likelihood procedure for estimating parameters in the presence of missing data. This will be the principle concept of this paper. Hocking and Hartley completed a series of equations for the information matrix respective to the estimates of mean and variance/covariance successfully.

We considered the problem of estimating the parameters in multivariate normal population when some of the response vectors were incomplete. Assuming there were \( N \) observations taken from a \( P \)-variate normal population, which composed the multivariate normal distribution, denoted as \( N(\mu, \Sigma) \), the data could be divided into \( T \) groups based on the pattern of incompleteness, the \( t \)-th group containing \( N_t \) observations.

In the classical "Missing Data" problem, we dealt with the vectors as incomplete because not all elements of the \( P \)-vector were recorded. We modified the definition by allowing that an incomplete vector may consist of known
linear combinations of the original data vector.

The data for this problem can be described as follows:

\[ Y_{ti} \text{ for } i = 1, \ldots, n_t, \quad t = 1, \ldots, T \]

is \( q_t \)-variate normal which is \( N(u_t, \Sigma_t) \)

where \( u_t = D_t u \) \hspace{1cm} (1)

and \( \Sigma_t = D_t \Sigma D_t' \) \hspace{1cm} (2)

\( u \) and \( \Sigma \) are the objects of estimation. \( D_t \) is a matrix of zeros and ones indicating which observations are recorded, but in general \( D_t \) is comprised of known constants. The multivariate normal distribution function for each \( t \) is

\[
\frac{1}{(2\pi)^{n_t/2} |\Sigma_t|^{1/2}} e^{-1/2(Y_{ti} - u_t)' \Sigma_t^{-1} (Y_{ti} - u_t)}
\] \hspace{1cm} (3)

Let \( L_t \) denote the equation (3).

According to the definition of Maximum Likelihood function

\[ L = \prod_{t=1}^{T} L_t \]

Rewrite eq. (3) into

\[ \log L_t = C - 1/2 \log |\Sigma_t| - 1/2 \text{tr}[\Sigma_t^{-1} (Y_{ti} - u_t)(Y_{ti} - u_t)'] \]

\[ \log L = C - n_t/2 \log |\Sigma_t| - 1/2 \text{tr}(\Sigma_t^{-1} A_t) \] \hspace{1cm} (4)

The matrix \( A_t \) is given by

\[
A_t = \sum_{i=1}^{n_t} (Y_{ti} - u_t)(Y_{ti} - u_t)'
\]

\[
= \sum_{i=1}^{n_t} (Y_{ti} - \hat{u}_t)(Y_{ti} - \hat{u}_t)' + n_t(\hat{u}_t - u_t)(\hat{u}_t - u_t)'
\]

\[ = n_t(\hat{\Sigma}_t + H_t) \] \hspace{1cm} (5)
\( \hat{\Sigma}_t \) and \( H_t \) matrices will be defined by
\[
\hat{\Sigma}_t = \frac{1}{n_t} (Y_{ti} - \hat{u}_t)(Y_{ti} - \hat{u}_t)',
\]
\( H_t = (u_t - u_t)(u_t - u_t)' \)
where \( \hat{u}_t = \frac{1}{n_t} \sum_{i=1}^{n_t} Y_{ti} \)
\( \Sigma \) is a var/cov matrix shown as:
\[
\Sigma = \begin{pmatrix}
    \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} \\
    \sigma_{21} & \sigma_{22} & \sigma_{23} & \sigma_{24} \\
    \sigma_{31} & \sigma_{32} & \sigma_{33} & \sigma_{34} \\
    \sigma_{41} & \sigma_{42} & \sigma_{43} & \sigma_{44}
\end{pmatrix}
\]
For the convenience of calculation, convert the square matrix into a column vector whose elements are in the same order of the column of the \( \Sigma \) matrix, under the condition of \( (\sigma_{ij}, 1 \leq i \leq j = 1, \ldots, p) \)
\( p \) is the number of all variables.
The vector \( \sigma \) has a length of \( p(p+1)/2 \). The vector \( \sigma_t \) of length \( q_t(q_t+1)/2 \) represents the corresponding column array of the matrix \( \Sigma_t \). For instance,
\[
\Sigma_2 = \begin{pmatrix}
    \sigma_{11} & \sigma_{12} & \sigma_{13} \\
    \sigma_{21} & \sigma_{22} & \sigma_{23} \\
    \sigma_{31} & \sigma_{32} & \sigma_{33}
\end{pmatrix}
\]
\( \sigma = [\sigma_{11} \ \sigma_{12} \ \sigma_{22} \ \sigma_{13} \ \sigma_{23} \ \sigma_{33}]' \)
To relate \( \sigma_t \) with \( \sigma \), there exists a matrix \( C_t \), analogous to \( D_t \)
\[
\sigma_t = C_t \sigma
\]
To develop the likelihood equations, the following differentiations have to be done.
\[
\frac{\partial \log L_t}{\partial u_t} = \frac{\partial}{\partial u_t} \left( C - n_t/2 \log |\Sigma_t| - 1/2 \text{tr}(\Sigma_t^{-1} A_t) \right)
\]
and
\[
\frac{\partial \log L_t}{\partial u} = D_t \frac{\partial \log L_t}{\partial u_t} \tag{12}
\]

Hence
\[
\frac{\partial \log L}{\partial u} = \frac{\partial}{\partial u_t} \left( \frac{T}{2} \log L_t \right)
\]
\[
= \frac{T}{2} \sum_{t=1}^{T} D_t \frac{\partial \log L_t}{\partial u_t}
\]
\[
= - n_t \sum_{t=1}^{T} D_t \Sigma_t^{-1} (D_t u - \hat{u}_t) \tag{13}
\]

Here
\[
D_t u = u_t
\]

Differentiate \( \log L_t \) twice respective to \( u_t \), then
\[
\frac{\partial \log L_t}{\partial u_t \partial u_t} = - n_t \Sigma_t^{-1} \tag{14}
\]

We name the portion of the information matrix \( W_{ut} \) corresponding to \( u_t \):
\[
W_{ut} = n_t \Sigma_t^{-1} \tag{15}
\]

We may rewrite eq. (13) as follows:
\[
\frac{\partial \log L}{\partial u} = - \sum_{t=1}^{T} D_t' W_{ut} u_t + \sum_{t=1}^{T} D_t' W_{ut} \hat{u}_t
\]
\[
= - W_{u} u + \sum_{t=1}^{T} D_t' W_{ut} \hat{u}_t \tag{16}
\]
where
\[ W_u = \sum_{t=1}^{T} D_t W_{ut} D_t \quad (17) \]

If we differentiate log \( L \) twice with respect to \( u \), the total information matrix for \( u \) given \( W \) can be found.

The estimator of \( u \) can be obtained by setting eq. (16) to zero and multiplying by \( W_u^{-1} \) under the condition that \( \Sigma \) is known.

\[ \hat{u} = W_u^{-1} \sum_{t=1}^{T} D_t W_{ut} \hat{u}_t \quad (18) \]

2.2 Estimation of variance/covariance

Develop the likelihood equations for \( \sigma \).

Let \( \sigma_{tij} \) denote the elements of \( \Sigma \), and

\[ \Sigma_{tij} = \frac{\partial \Sigma_t}{\partial \sigma_{tij}} \quad (19) \]

thus, \( \Sigma_{tij} \) has a one in positions \((i, j)\) and \((j, i)\), and zeros elsewhere. Since we knew

\[ \frac{\partial \log |\Sigma_t|}{\partial \sigma_{tij}} = \text{tr} (\Sigma_t^{-1} \Sigma_{tij}) \quad (20) \]

and

\[ \frac{\partial \Sigma_t^{-1}}{\partial \sigma_{tij}} = - \Sigma_t^{-1} \Sigma_{tij} \Sigma_t^{-1} \quad (21) \]

easily solve

\[ \frac{\partial \log L_t}{\partial \sigma_{tij}} = \frac{\partial}{\partial \sigma_{tij}} \left( C - n_t/2 \log |\Sigma_t| - 1/2 \text{tr}(\Sigma_t^{-1} A_t) \right) \quad (22) \]

by placing eq. (19), (20), and (21) into (22).

\[ \frac{\partial \log L_t}{\partial \sigma_{tij}} = -n_t/2 \text{tr}(\Sigma_t^{-1} \Sigma_{tij}) + 1/2 \text{tr}(\Sigma_t^{-1} \Sigma_{tij} \Sigma_t^{-1} A_t) \quad (23) \]
as a likelihood estimate equation of variance.

Concerning the estimate equation of covariance, eq.(23) have to be differentiated once more, easily yielding

\[
\frac{\partial \log L_t}{\partial \sigma_{\text{tr}s}} = \frac{n_t}{2} \text{tr}(E^{-1}_t E^{-1}_t \Sigma_{\text{tr}s} E^{-1}_t \Sigma_{\text{tr}s} E^{-1}_t \Sigma_{tij} E^{-1}_t A_t) - \frac{1}{2} \text{tr}(E^{-1}_t E^{-1}_t \Sigma_{\text{tr}s} E^{-1}_t A_t) - \frac{1}{2} \text{tr}(E^{-1}_t E^{-1}_t \Sigma_{tij} E^{-1}_t E^{-1}_t A_t)
\]

(24)

because \( E(A_t) = n_t E_t \)

The expected value of (24) is

\[
- \frac{n_t}{2} \text{tr}(E^{-1}_t E_{tij} E^{-1}_t E_{\text{tr}s})
\]

(25)

Logically, we can obtain the likelihood equation of variance by setting eq.(23) to zero, but in fact, this equation are too complicated and difficult to solve. We try another way to simplify it.

We define \( \sigma \) vector from \( E \) matrix as

\[
\sigma = (\sigma_{11} \sigma_{12} \ldots \sigma_{1p} \ldots \sigma_{pp})
\]

the order of the elements is in the column order of \( E \); same is \( \sigma_{ij} \) of \( E^{-1} \) displayed as:

\[
\sigma^{-1} = (\sigma_{11} \sigma_{12} \ldots \sigma_{1p} \ldots \sigma_{pp})
\]

(26)

By the definition of \( \sigma \) and \( \sigma^{-1} \), we can define the following
The following relationship exists,

\[
\begin{align*}
\frac{\partial \sigma_{ij}}{\partial \sigma_{uv}} &= \begin{cases} 
-\sigma_{iu} \sigma_{jv} & \text{if } u = v \\
-(\sigma_{iu} \sigma_{jv} + \sigma_{iv} \sigma_{ju}) & \text{if } u \neq v
\end{cases}
\end{align*}
\]  

(27)

Thus the information matrix defined as \( W_{ot} \) can be found, and it is a square matrix of dimension \( q_t (q_t+1)/2 \).

\[
W_{ot} = n_t (C_t^t U C_t)^{-1}
\]

(28)

Where \( U \) is a square matrix of dimension \( p(p+1)/2 \), whose components are the products of covariances defined in eq. (27).

Displaying \( U \) matrix more explicitly, the element in row \((u, v)\) and column \((i, j)\) for \( 1 \leq u \leq v = 1, \ldots , p \), \( 1 \leq i \leq j = 1, \ldots , p \) is expressed as

\[
U_{(u, v)ij} = \sigma_{iu} \sigma_{jv} + \sigma_{iv} \sigma_{ju}
\]

(29)

The symmetrical \( U \) matrix is

\[
\begin{bmatrix}
\mu_{11,11} & \mu_{11,12} & \mu_{11,22} & \mu_{11,13} & \mu_{11,23} & \mu_{11,33} & \mu_{11,14} & \cdots & \mu_{11,44} \\
\mu_{12,11} & \mu_{12,12} & \mu_{12,22} & \mu_{12,13} & \mu_{12,23} & \mu_{12,33} & \mu_{12,14} & \cdots & \mu_{12,44} \\
\mu_{22,11} & \mu_{22,12} & \mu_{22,22} & \mu_{22,13} & \mu_{22,23} & \mu_{22,33} & \mu_{22,14} & \cdots & \mu_{22,44} \\
\mu_{13,11} & \mu_{13,12} & \mu_{13,22} & \mu_{13,13} & \mu_{13,23} & \mu_{13,33} & \mu_{13,14} & \cdots & \mu_{13,44} \\
\mu_{23,11} & \mu_{23,12} & \mu_{23,22} & \mu_{23,13} & \mu_{23,23} & \mu_{23,33} & \mu_{23,14} & \cdots & \mu_{23,44} \\
\mu_{33,11} & \mu_{33,12} & \mu_{33,22} & \mu_{33,13} & \mu_{33,23} & \mu_{33,33} & \mu_{33,14} & \cdots & \mu_{33,44} \\
\mu_{14,11} & \mu_{14,12} & \mu_{14,22} & \mu_{14,13} & \mu_{14,23} & \mu_{14,33} & \mu_{14,14} & \cdots & \mu_{14,44} \\
\mu_{24,11} & \mu_{24,12} & \mu_{24,22} & \mu_{24,13} & \mu_{24,23} & \mu_{24,33} & \mu_{24,14} & \cdots & \mu_{24,44} \\
\mu_{34,11} & \mu_{34,12} & \mu_{34,22} & \mu_{34,13} & \mu_{34,23} & \mu_{34,33} & \mu_{34,14} & \cdots & \mu_{34,44}
\end{bmatrix}
\]

Moreover, the information matrix for likelihood \( L_t \), combined by \( W_{ut} \) and \( W_{ot} \), is block diagonal.
\[ W_t = \begin{pmatrix} W_{ut} & 0 \\ 0 & W_{\sigma t} \end{pmatrix} \] (30)

Since we already know

\[ E_t = \sum_{(all \ rs)} \sigma_{trs} E_{trs} \] (31)

therefore

\[ \text{tr} \left( E_t^{-1} E_{tij} \right) = \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} E_t \right) \]

\[ = \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} \left( \sum_{(all \ rs)} \sigma_{trs} E_{trs} \right) \right) \]

\[ = \sum_{(all \ rs)} \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} E_{trs} \right) \sigma_{trs} \] (32)

Recalling (23) the estimation equation of covariance is

\[- n_t/2 \text{tr} \left( E_t^{-1} E_{tij} \right) + 1/2 \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} A_t \right)\]

Compare (32) to the first term on the right-side of eq. (23), it obviously indicating this term is the \((i, j)\)th component of \(W_{\sigma t} \cdot \sigma_t\).

Further, we note that the second term on the right-side of eq. (23) happened to be

\[ 1/2 \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} A_t \right) = 1/2 \text{tr} \left( E_t^{-1} E_{tij} E_t^{-1} n_t (\hat{E}_t + H_t) \right) \]

\[ = W_{\sigma t} (\hat{\sigma}_t + h_t) \] (33)

where \(\hat{\sigma}_t\) and \(h_t\) are in vector forms of \(\hat{E}_t\) and \(H_t\). Therefore

\[ \frac{\partial \log L_t}{\partial \sigma} = - W_{\sigma t} (\sigma_t - (\hat{\sigma}_t + h_t)) \] (34)

Similar to \(u\), the likelihood equation of \(\sigma\) is given as
\[
\frac{\partial \log L_t}{\partial \sigma} = C_t' \frac{\partial \log L_t}{\partial \sigma_t}
\]

Since \( \sigma_t = C_t \sigma \)

\[
\frac{\partial \log L}{\partial \sigma} = \sum_{t=1}^{T} \frac{\partial \log L_t}{\partial \sigma} = \sum_{t=1}^{T} \frac{C_t' W_{\sigma t}}{C_t' W_{\sigma t} (\hat{\sigma_t} + h_t)}
\]

Where \( W_\sigma \) is the total information matrix for \( \sigma \) shown as

\[
W_\sigma = \sum_{t=1}^{T} C_t' W_{\sigma t} C_t
\]

The two likelihood estimation equations of \( W_u \) - eq.(16) and \( W_\sigma \) - eq.(35) basically consisted of an identical structure.

It was assumed that all elements of \( u \) are estimable. It didn't follow that all elements of \( \sigma \) are estimable too, hence in eq.(35) the vector \( \sigma \) should be interpreted as the vector of estimable parameters from \( \Sigma \).

In terms of finding the solutions for \( u \) and \( \sigma \) based on equation (16) and (35), there are several ways to do, but the simplest one is setting them to zero.

Therefore the two likelihood estimation equations for \( u \) and \( \sigma \) are

\[
W_u \cdot u = \sum_{t=1}^{T} D_t' W_{ut} \hat{u_t}
\]

\[
W_\sigma \cdot \sigma = \sum_{t=1}^{T} C_t' W_{\sigma t} (\hat{\sigma_t} + h_t)
\]

Eventually
\[ u = W_u^{-1} \sum_{t=1}^{T} D_t W_{ut} \hat{u}_t \]  
(39)

\[ \sigma = W_\sigma^{-1} \sum_{t=1}^{T} C_t W_{\sigma t} (\hat{\sigma}_t + h_t) \]  
(40)

The estimation equations for \( u \) and \( \sigma \) are shown in eq. (39) and (40). The estimation of \( u \), \( \sigma \) and \( h_t \) started by estimating \( W_{ut} \), \( W_{\sigma t} \) and \( h_t \) with the initial estimators of \( u \) and \( \sigma \) as we assumed previously. After the first time, we repeat the whole process by replacing the initial estimators with the resulting estimates from last iteration. Empirically, after a period of iterations, the modified \( u \) and \( \sigma \) will rapidly converge on both of the simulated and actual data.
3.1 General description

For carrying out the theory, a numerical example was applied to follow the outlined sequence. Every individual step will be explained in detail with actual data so that the whole method can be observed clearly.

The example was the Iris setosa of Multiple Measurements from a taxonomic problem, it was assumed $n = 50$ observation taken on a 4 - variate normal population, which were divided into 4 groups, according to the pattern of incompleteness. Of the 50 records 31 were complete and 19 were incomplete in different ways.

The original data of 4 groups were shown in appendix 1.

The 4 groups were tabulated as:

P: Presence
M: Missing

<table>
<thead>
<tr>
<th>Group</th>
<th>$X_1$</th>
<th>$X_2$</th>
<th>$X_3$</th>
<th>$X_4$</th>
<th>$n_i$: Obs. in $i$th grn.</th>
<th>$q_i$: No of vat.</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>$n_1 = 31$</td>
<td>$q_1 = 4$</td>
</tr>
<tr>
<td>2</td>
<td>P</td>
<td>P</td>
<td>P</td>
<td>M</td>
<td>$n_2 = 8$</td>
<td>$q_2 = 3$</td>
</tr>
<tr>
<td>3</td>
<td>P</td>
<td>P</td>
<td>M</td>
<td>M</td>
<td>$n_3 = 6$</td>
<td>$q_3 = 2$</td>
</tr>
<tr>
<td>4</td>
<td>M</td>
<td>M</td>
<td>P</td>
<td>M</td>
<td>$n_4 = 5$</td>
<td>$q_4 = 1$</td>
</tr>
<tr>
<td>Total T = 4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td>N = 50</td>
<td></td>
</tr>
</tbody>
</table>
3.2 Procedure

The whole procedure was iterated by means of replacing the estimators of $u$, $\varepsilon$ and $H_t$ until they converged to constants.

The stepwise sequence in each iteration was following:

1. The $u^{(\ell)}$ ($\ell$ indicated the number of iteration) would be one of the objectives of estimation. Besides, it also needed an initial value for starting. The best recommended way was using $\hat{u}_1$ as initial estimator of $u$, denoted as $u^{(0)}$ (0 represented the initial estimator) which came from complete data group. $D_t$ was designed with 0 and 1 indicating the elements of complete data group. Thus $u^{(\ell)}_t = D_t u^{(\ell)}$ and $H^{(\ell)}_t = (u^{(\ell)}_t - \hat{u}^{(\ell)}_t)(u^{(\ell)}_t - \hat{u}^{(\ell)}_t)^\top$ could be obtained, where $\hat{u}_t = 1/n_t \sum_{i=1}^{n_t} y_{ti}$ kept as constants in every iteration. In our case,

$$\hat{u}_1 = u^{(0)} = 1/31 \sum_{i=1}^{31} y_{ti} = \begin{bmatrix} 4.99 \\ 3.43 \\ 1.45 \\ 0.24 \end{bmatrix}, \quad \hat{u}_2 = 1/8 \sum_{i=1}^{8} y_{ti} = \begin{bmatrix} 4.90 \\ 3.35 \\ 1.37 \end{bmatrix}$$

$$\hat{u}_3 = 1/6 \sum_{i=1}^{6} y_{ti} = \begin{bmatrix} 5.12 \\ 3.63 \end{bmatrix}, \quad \hat{u}_4 = 1/5 \sum_{i=1}^{5} y_{ti} = \begin{bmatrix} 1.52 \end{bmatrix}$$

$D_t$ could be defined as:

$$D_1 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad D_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$D_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad D_4 = \begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$$

Therefore $u^{(0)}_t = D_t u^{(0)}$ and $H^{(0)}_t$ were able to be computed.
\[ H_1^{(0)} = (u_1 - D_t u_1^{(0)}) (u_1 - D_t u_1^{(0)})' = 0 \]

\[ H_2^{(0)} = \begin{bmatrix} 0.0081 & 0.0072 & 0.0072 \\ 0.0064 & 0.0064 & 0.0064 \\ 0.0064 & 0.0064 & 0.0064 \end{bmatrix} \]

\[ H_3^{(0)} = \begin{bmatrix} 0.0169 & 0.0260 \\ 0.0400 \\ 0.0400 \end{bmatrix} \quad \quad H_4^{(0)} = [0.0049] \]

2. The \( \sigma \) vector was another objective of estimation whose components were in the column order of \( \Sigma \) matrix. Its best initial estimator used \( \hat{\Sigma} \) as \( \Sigma^{(0)} \), where \( \hat{\Sigma} \) was the covariance matrix of group 1 which was computed by complete data.

\[ \hat{\Sigma}_t = 1/n_t \sum_{i=1}^{n_t} (Y_{ti} - \hat{u}_t)(Y_{ti} - \hat{u}_t)' \]

where \( \hat{u}_t \) came from step 1. All of \( \hat{\Sigma}_t \) would be constants in every iteration.

\[ \hat{\Sigma}_1 = 1/31 \sum_{i=1}^{31} (Y_{ti} - \hat{u}_1)(Y_{ti} - \hat{u}_1)' \]

\[ = 1/31 \sum_{i=1}^{31} (Y_{ti} - u_1^{(0)})(Y_{ti} - u_1^{(0)})' \]

\[ \Sigma^{(0)} = \begin{bmatrix} 0.158 & 0.140 & 0.015 & 0.010 \\ 0.185 & 0.013 & 0.008 \\ 0.026 & 0.004 & 0.012 \end{bmatrix} \]

\[ \hat{\Sigma}_2 = \begin{bmatrix} 0.070 & 0.021 & 0.027 \\ 0.053 & -0.009 & 0.024 \end{bmatrix} \]

\[ \hat{\Sigma}_3 = \begin{bmatrix} 0.018 \\ 0.026 \\ 0.096 \end{bmatrix} \quad \quad \hat{\Sigma}_4 = [0.018] \]
3. \( W_{ct} = n_t (C_t U C_t')^{-1} \)

\( n_t \) came from step 1. \( U \) was a matrix in rank \( p(p+1)/2 = 10 \).

The elements of \( U^{(k)} \) in row \((u, v)\), column \((i, j)\) = \( \sigma_{iu} \sigma_{jv} + \sigma_{iv} \sigma_{ju} \)

where \( u < v = 1, \ldots, p \), \( i < j = 1, \ldots, p \)

In the first iteration the value of \( \sigma_{ij} \) came from step 2, otherwise came from step 6. \( C_t \) was matrix related \( \sigma \) to \( \sigma_t \), satisfying \( \sigma_t = C_t \sigma \), which designed with 0 and 1 indicating the elements of \( \sigma \) presented in \( \sigma_t \). In our case, \( C_1 = I(10 \times 10) \). The length of \( C_t \) was \( q_t(q_t+1)/2 \).

\[ C_2(6 \times 10) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ C_3(3 \times 10) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ C_4(1 \times 10) = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix} \]

Therefore

\[ \sigma_1(10 \times 1) = C_1(10 \times 10) \times \sigma(10 \times 1) \]

\[ = [\sigma_{11} \ \sigma_{12} \ \sigma_{22} \ \sigma_{13} \ \sigma_{23} \ \sigma_{33} \ \sigma_{14} \ \sigma_{24} \ \sigma_{34} \ \sigma_{44}]' \]

\[ \sigma_2(6 \times 1) = C_2(6 \times 10) \times \sigma(10 \times 1) \]

\[ = [\sigma_{11} \ \sigma_{12} \ \sigma_{22} \ \sigma_{13} \ \sigma_{23} \ \sigma_{33}]' \]

\[ \sigma_3(3 \times 1) = C_3(3 \times 10) \times \sigma(10 \times 1) \]

\[ = [\sigma_{11} \ \sigma_{12} \ \sigma_{22}]' \]
\[ \sigma_{4(1\times1)} = C_{4(1\times10)} \times \sigma_{(10\times1)} \]
\[ = [\sigma_{33}]^t \]

4. \[ W_{\sigma} = \sum_{t=1}^{T} C_t W_{\sigma t} C_t \]
   \(C_t\) and \(W_{\sigma t}\) came from step 3.

5. \[ \sigma(e) = W_{\sigma}^{-1} \sum_{t=1}^{T} C_t W_{\sigma t} (\sigma_t + h_t) \]
   \(W_{\sigma t}\) came from step 3.
   \(\sigma_t\) came from step 2.
   \(h_t\) came from step 1.

6. \[ \Sigma_t(e) = D_t \Sigma_t(e) D_t^t \]
   \(\Sigma_t(e)\) came from step 5.
   \(D_t\) came from step 1.

7. \[ W_{ut} = \eta_t \Sigma_t(e)^{-1} \]
   \(\Sigma_t(e)\) came from step 6.

8. \[ W_u = \sum_{t=1}^{T} D_t^t W_{ut}(\sigma) D_t \]
   \(W_{ut}(\sigma)\) came from step 7.

9. \[ u(e) = W_u^{-1} \sum_{t=1}^{T} D_t^t W_{ut}(\sigma) u_t \]
   \(W_u^{-1}\) came from step 8.
\( \hat{u}_t \) came from step 1.

\( \mathbf{W}_t^{(z)} \) came from step 7.

1. \( u_t^{(z)} = D_t u_t^{(z)} \)

\( D_t \) came from step 1.

\( u_t^{(z)} \) came from step 9.

11. \( H_t = (u_t - \hat{u}_t^{(z)})(u_t - \hat{u}_t^{(z)})' \)

\( \hat{u}_t \) came from step 1.

\( u_t^{(z)} \) came from step 10.

The new \( u_t^{(z)} \) was obtained from step 9, \( \sigma_t^{(z)} \) from step 5, and \( h_t^{(z)} \) from step 11.

From the second iteration place the new \( \varepsilon \) into step 3, and complete the whole procedure up to step 11, then \( u_t^{(z)} \), \( \sigma_t^{(z)} \), and \( h_t^{(z)} \) are able to be produced from every iteration. Repeating the whole procedure until \( u_t^{(z)} \), \( h_t^{(z)} \), and \( \sigma_t^{(z)} \) vectors converged within a given criteria, the eventual estimates of mean, variance and covariance were found.

### 3.3 Construction of the program

For the convenience of program manipulation, some modifications have been made.

1. \( D_t \) and \( C_t \) matrices have been shown with different ranks among 4 groups. In the program, all \( D_t \) and \( C_t (t = 1, \ldots, 4) \) were modified into square and symmetrical matrices. All \( D_t \) were \((4 \times 4)\) matrices.
wherever the data missed filled with 0. So did in $C_t(10 \times 10)$. The $C_t$ and $D_t$ matrices were shown in the beginning of the output.

2. For the rank conformity of the initial vector $u_t$ to $D_t$, kept $u_t$ in size $(4 \times 1)$ and filled with 0 when observation missed. $h_t$ was $(10 \times 1)$ conformed to the rank of $C_t$.

3. The output vector of $h$ and $\sigma$ were in the following format:

\[
h = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} & h_{22} & h_{23} & h_{24} & h_{33} & h_{34} & h_{44} \end{bmatrix}^T
\]

\[
\sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} & \sigma_{14} & \sigma_{22} & \sigma_{23} & \sigma_{24} & \sigma_{33} & \sigma_{34} & \sigma_{44} \end{bmatrix}^T
\]

They were in different order from them shown before because of rearrangement.

4. Since the initial value of $\sigma$ and $h$ vector not in the array of column's order but of the row's order of $\Sigma$ and $H$ matrix. Therefore the related matrices $C_t$ and $D_t$ also were rearranged. However they still satisfied the condition of $\sigma_t = C_t \sigma$ and $u_t = D_t u$.

Above all, those improvements didn't have any influence on the result of estimators.

The basic structure of the program followed the outline shown above.

In the very beginning, we started with calculating

\[ W_{\sigma t} = n_t (C_t U C_t')^{-1} \] where the elements of $U$ came from initial value of $\Sigma$.

Reorganized those steps agreeing to the program sequence:

STEP 1. $E_t = D_t' \Sigma D_t$

STEP 2. $W_{ut} = n_t \Sigma_t^{-1}$
STEP 3. \( W_u = \sum_{t=1}^{T} D_t W\hat{u}_t D_t \)

STEP 4. new \( u^{(k)} = W_u^{-1} \sum_{t=1}^{T} D_t W\hat{u}_t \)

STEP 5. \( u^{(k)}_t = D_t u^{(k)} \)

STEP 6. new \( H^{(k)}_t = (u_t - u^{(k)}_t)(u_t - u^{(k)}_t)' \)

STEP 7. \( W_{\sigma_t} = n_t (C_t U C_t')^{-1} \)

The elements of \( U \) came from the input \( \Sigma^{(0)} \) in the first iteration.

STEP 8. \( W_{\sigma} = \sum_{t=1}^{T} C_t W_{\sigma_t} C_t \)

STEP 9. new \( \sigma^{(k)} = W_{\sigma}^{-1} \sum_{t=1}^{T} C_t W_{\sigma_t} (\hat{\sigma}_t + h_t) \)

\( h_t \) came from step 6.

Therefore \( u^{(k)} \), \( H^{(k)}_t \), and \( \sigma^{(k)}_t \) could be procured from the \( k \)th iteration, the \( k+1 \)th iteration started from placing the \( \Sigma^{(k)} \) to step 1.

\( C_t \), \( D_t \), \( n_t \) and \( \hat{\sigma}_t \) were a part of input, keeping as constants all the time.

There were some matrix calculation involved, the IBM scientific subroutine could be applied.

1. Transpose a matrix .... subroutine WTRA.
2. Subtract two matrices .... subroutine MSUB.
3. Products of two matrices .... subroutine GMPRD.
4. Multiple a matrix by a scalar subroutine SMPY.

5. Inverse a matrix (generalized inverse) subroutine DMTSQ supplied by Dr. Hurst.

6. Computing the U matrix subroutine UMAT.
CHAPTER IV
RESULTS

The conclusions were based on the program output. The \( u \) and \( \sigma \) vector converged very fast requiring only 4 iterations to obtain 4 - decimal accuracy. The estimates of \( u \) and \( \sigma \) were shown as:

\[
\begin{bmatrix}
4.9964 \\
3.4451 \\
1.4446 \\
0.2396
\end{bmatrix}
\begin{bmatrix}
\Sigma^{(4)}
\end{bmatrix}
= 
\begin{bmatrix}
0.1239 & 0.1055 & 0.0158 & 0.0078 \\
0.1518 & 0.0083 & 0.0059 & 0.009 \\
0.0261 & 0.0041 & 0.0019 & 0.0013
\end{bmatrix}
\]

In our case, since we had such a large ratio of complete records (\( p = 31/50 \)) the estimates of group 1 were good for exploiting as initial estimates.

The estimates of \( u \) and \( \Sigma \) were initiated as the estimates of group 1, and modified by the additional information of incomplete data, consequently, the portion of complete data was playing an important role as to the reliability of the estimates.

The parameters of complete data \( N = 50 \) should be as following:

\[
\begin{bmatrix}
5.0041 \\
3.4265 \\
1.4633 \\
0.2469
\end{bmatrix}
\begin{bmatrix}
\Sigma
\end{bmatrix}
= 
\begin{bmatrix}
0.1266 & 0.1168 & 0.0168 & 0.0106 \\
0.1466 & 0.0120 & 0.0095 & 0.0061 \\
0.0307 & 0.0061 & 0.0059 & 0.0037 \\
0.0091 & 0.0059 & 0.0037 & 0.0013
\end{bmatrix}
\]

Being statistics, the estimates of \( u \) and \( \Sigma \) satisfied the property of "unbiaseness" and "consistency" by comparing them with the parameters. The fact was that although there were 3 incomplete groups, none of them contained too much information. Evidence seems to indicate that accuracy increases with a smaller proportion of missing data, and also with smaller variates.
CHAPTER V
CONCLUSIONS

According to the properties of this method, the following conclusions might be made.

1. The accuracy of the estimates seemed dependent on some factors.

   A. It was inevitable using a initial estimate of $u$ and $\Sigma$, the precision and reliability of eventual estimates were highly affected by the characteristics of this value. Consequently, a larger portion of complete data produced a better estimates of $u$ and $\Sigma$. In other words, the initial estimates have to supply sufficient informations of the variables.

   B. For the sake that the initial estimates came from complete data, this method definitely was not available for those problems which didn't include any complete data, nor the amount of these data not enough to make $\hat{\Sigma}_1$ to be a non-singular and full rank matrix. For this point, we concerned ourselves with about the "estimability condition" of this method, which meant that $n_t > p$ and $n_t > q_t$ were necessary for any group. Once we failed to satisfy that all $n_t > q_t$, it never yielded the convergent estimates, even in the case of $n_t = p$, or $n_t = q_t$.

   C. This method is available for large sample size, but also works for small sample sizes as long as the variable was taken from the normal distribution and met the condition of B.

Sample size also affected the accuracy of the estimates, because a small sample size didn't have as much as information to work as did
in large sample size. Therefore, selecting moderate amount of sample size will produce better estimates.

2. The whole procedure could be performed in one program, it was convenient for users by simply placing the initial values for mean, variance/covariance, \( H_t, C_t, D_t \) matrices.

3. The possible combinations of missing data types are \( 2^p - 1 \) (\( p \) is the number of variables), for large \( p \), the group numbers of \( C_t \) and \( D_t \) becoming tremendous values so that they are too large to compute. For instance, if \( p = 10 \), the maximum group combination could be \( 2^{10} - 1 = 1023 \). It becomes a heavy load for user to prepare the input data, even though it possibly could be done. This method provided absolute flexibility in arbitrary kinds of group classifications, but with small variables it worked more efficiently.

4. The costs of computer time are also high both of storage for the various data and C. P. U. processing time for iteration. Recalling the example of 10 variables and assuming the iteration times were 10, how much it would cost? However, in our example, it cost $3.50 for 4 variables in 4 group with 4 iterations.

For further study, it is suggested that under the conditions of variables \( p < 10 \), and the classification \( T < 10 \), this method will be a convenient and powerful one for estimation.


Appendix 1

The Table of 4 Groups of Multiple Measurements in Taxonomic Problem

X: Indicates the data missed

$q_t$: The number of variable in $t^{th}$ group

$n_t$: The number of observations in $t^{th}$ group

Group 1. $q_1 = 4, \ n_1 = 31$

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<th>$x_3$</th>
<th>$x_4$</th>
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Group 3. $q_3 = 2, n_3 = 6$

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Group 4. $q_4 = 1, n_4 = 5$

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Appendix 2

Computer Program

The list of all subroutines and main program referred for missing data in the multivariate normal distribution.
SUBROUTINE MTRA

PURPOSE
TRANSPOSE A MATRIX

USAGE
CALL MTRA(A,M,N,M,MS)

DESCRIPTION OF PARAMETERS
A = NAME OF MATRIX TO BE TRANSPOSED
R = NAME OF OUTPUT MATRIX
N = NUMBER OF ROWS OF A AND COLUMNS OF R
M = NUMBER OF COLUMNS OF A AND ROWS OF R
MS = DIGIT NUMBER FOR STORAGE MODE OF MATRIX A (AND R)
   0 = GENERAL
   1 = SYMMETRIC
   2 = DIAGONAL

REMARKS
MATRIX R CANNOT BE IN THE SAME LOCATION AS MATRIX A

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
MCPY

METHOD
TRANSPOSE N BY M MATRIX A TO FORM M BY N MATRIX R BY MOVING EACH ROW OF A INTO THE CORRESPONDING COLUMN OF R. IF MATRIX A IS SYMMETRIC OR DIAGONAL, MATRIX R IS THE SAME AS A.

SUBROUTINE MTRA(A,M,N,M,MS)
DIMENSION A(100),R(100)

IF MS IS 1 OR 2, COPY A

IF(MS) 10,20,10
10 CALL MCPY(A,R,N,N,MS)
RETURN

TRANSPUSE GENERAL MATRIX

20 IR=0
DO 30 I=1,N
   J=1
   DO 30 J=1,M
   1=I+J
   IR=IR+1
30   R(IR)=A(I,J)
RETURN
END
SUBROUTINE MSUB

PURPOSE
SUBTRACT TWO MATRICES ELEMENT BY ELEMENT TO FORM RESULTANT MATRIX

USAGE
CALL MSUB(A, B, R, N, M, MSA, MSB)

DESCRIPTION OF PARAMETERS
A = NAME OF INPUT MATRIX
B = NAME OF INPUT MATRIX
R = NAME OF OUTPUT MATRIX
N = NUMBER OF ROWS IN A & B
M = NUMBER OF COLUMNS IN A & B
MSA = ONE DIGIT NUMBER FOR STORAGE MODE OF MATRIX A
0 = GENERAL
1 = SYMMETRIC
2 = DIAGONAL
MSB = SAME AS MSA EXCEPT FOR MATRIX B

REMARKS
NONE

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

METHOD
STRUCTURE OF OUTPUT MATRIX IS FIRST DETERMINED. SUBTRACTION
OF MATRIX B ELEMENTS FROM CORRESPONDING MATRIX A ELEMENTS
IS THEN PERFORMED.
THE FOLLOWING TABLE SHOWS THE STORAGE MODE OF THE OUTPUT
MATRIX FOR ALL COMBINATIONS OF INPUT MATRICES

<table>
<thead>
<tr>
<th>A</th>
<th>B</th>
<th>R</th>
</tr>
</thead>
<tbody>
<tr>
<td>GENERAL</td>
<td>GENERAL</td>
<td>GENERAL</td>
</tr>
<tr>
<td>GENERAL</td>
<td>SYMMETRIC</td>
<td>GENERAL</td>
</tr>
<tr>
<td>GENERAL</td>
<td>SYMMETRIC</td>
<td>SYMMETRIC</td>
</tr>
<tr>
<td>SYMMETRIC</td>
<td>DIAGONAL</td>
<td>SYMMETRIC</td>
</tr>
<tr>
<td>SYMMETRIC</td>
<td>SYMMETRIC</td>
<td>SYMMETRIC</td>
</tr>
<tr>
<td>DIAGONAL</td>
<td>GENERAL</td>
<td>GENERAL</td>
</tr>
<tr>
<td>DIAGONAL</td>
<td>SYMMETRIC</td>
<td>SYMMETRIC</td>
</tr>
<tr>
<td>DIAGONAL</td>
<td>DIAGONAL</td>
<td>DIAGONAL</td>
</tr>
</tbody>
</table>

SUBROUTINE MSUB(A, B, R, N, M, MSA, MSB)
DIMENSION A(100), B(100), R(100)

DETERMINE STORAGE MODE OF OUTPUT MATRIX

IF(MSA-MSB) 7,5,7
5 CALL LOC(N, M, N, M, MSA)
GO TO 100
7 MTEST=MSA*MSB
MSN=0
IF(MTEST) 20,20,10
10 MSX=1
20 IF(MTEST=2) 35,35,30
30 MSX=2

LOCATE ELEMENTS AND PERFORM SUBTRACTION

35 DO 90 J=1,N
   90 DO I=1,N
      CALL LOC(I,J,1JR,N,M,MSH)
      IF(IJR) 40,90,40
   40 CALL LOC(I,J,1JA,N,M,MSA)
      AEL=0.0
      IF(IJA) 50,60,50
   50 AEL=A(IJA)
   60 CALL LOC(I,J,1JH,N,M,MSB)
      HEL=0.0
      IF(IJH) 70,60,70
   70 HEL=H(IJH)
   80 R(IJR)=AEL-HEL
   90 CONTINUE
      RETURN

SUBTRACT MATRICES FOR OTHER CASES

100 DO 110 I=1,NM
   110 RETURN
      END
SUBROUTINE LOC

PURPOSE

COMPUTE A VECTOR SUBSCRIPT FOR AN ELEMENT IN A MATRIX OF
SPECIFIED STORAGE MODE

USAGE

CALL LOC (I,J,IR,N,M,MS)

DESCRIPTION OF PARAMETERS

I = ROW NUMBER OF ELEMENT
J = COLUMN NUMBER OF ELEMENT
IR = RESULTANT VECTOR OF ROWS IN MATRIX
M = NUMBER OF COLUMNS IN MATRIX
MS = ONE DIGIT NUMBER FOR STORAGE MODE OF MATRIX

0 = GENERAL
1 = SYMMETRIC
2 = DIAGONAL

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED

NONE

SUBROUTINE LOC(I,J,IR,N,M,MS)

IX=I
JX=J

IF(MS-1) 10,20,30
10 IX=IX*(JX-1)+IX
GO TO 36

20 IF(IX-JX) 22,24,24
22 IX=IX+(JX*JX-JX)/2
GO TO 36

24 IX=JX+(IX*IX-IX)/2
GO TO 36

30 IX=0

IF(IX-JX) 30,32,36
32 IX=IX
36 IX=IX
RETURN
END
SUBROUTINE SMPY

PURPOSE
MULTIPLY EACH ELEMENT OF A MATRIX BY A SCALAR TO FORM A RESULTANT MATRIX

USAGE
CALL SMPY(A,C,R,N,M,MS)

DESCRIPTION OF PARAMETERS
A - NAME OF INPUT MATRIX
C - SCALAR
R - NAME OF OUTPUT MATRIX
N - NUMBER OF ROWS IN MATRICES A AND R
M - NUMBER OF COLUMNS IN MATRICES A AND R
MS - ONE DIGIT NUMBER FOR STORAGE MODE OF MATRIX A (AND R)
 0 - GENERAL
 1 - SYMMETRIC
 2 - DIAGONAL

REMARKS
NONE

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
LUC

METHOD
SCALAR IS MULTIPLIED BY EACH ELEMENT OF MATRIX

SUBROUTINE SMPY(A,C,R,N,M,MS)
DIMENSION A(100),R(100)

COMPUTE VECTOR LENGTH, IT

IF (N-10) 2*10,10
10 CALL LUC(10,10,10,10,10,0)
GO TO 100
2 CALL LUC(4,4,4,4,0,0)
100 DU 1=1,14

MULTIPLY BY SCALAR

1 R(I)=A(I)*C
RETURN
END
SUBROUTINE GMFDU

PURPOSE
MULTIPLY TWO GENERAL MATRICES TO FORM A RESULTANT GENERAL
MATRIX

USAGE
CALL GMFDU(A,B,R,N,M,L)

DESCRIPTION OF PARAMETERS
A = NAME OF FIRST INPUT MATRIX
B = NAME OF SECOND INPUT MATRIX
R = NAME OF OUTPUT MATRIX
N = NUMBER OF ROWS IN A
M = NUMBER OF COLUMNS IN A AND ROWS IN B
L = NUMBER OF COLUMNS IN B

REMARKS
ALL MATRICES MUST BE STORED AS GENERAL MATRICES
MATRIX R CANNOT BE IN THE SAME LOCATION AS MATRIX A
MATRIX R CANNOT BE IN THE SAME LOCATION AS MATRIX B
NUMBER OF COLUMNS OF MATRIX A MUST BE EQUAL TO NUMBER OF ROWS
OF MATRIX B

SUBROUTINES AND FUNCTION SUBPROGRAMS REQUIRED
NONE

METHOD
THE M BY L MATRIX B IS PREMULTIPLIED BY THE N BY M MATRIX A
AND THE RESULT IS STORED IN THE N BY L MATRIX R.

SUBROUTINE GMFDU(A,B,R,N,M,L)
DIMENSION A(100),B(100),R(100)

IR=0
IK=M
DO 10 K=1,L
IK=IK+M
DO 10 J=1,N
IR=IR+1
JL=J+N
IK=IK
R(IR)=0
DO 10 I=1,N
JL=JL+1
IK=IK
R(IR)=R(IR)+A(IR)*B(IJ)
10 CONTINUE
RETURN
END
SUBROUTINE MCPY

PURPOSE
COPY ENTIRE MATRIX

USAGE
CALL MCPY

DESCRIPTION OF PARAMETERS
A = NAME OF INPUT MATRIX
H = NAME OF OUTPUT MATRIX
N = NUMBER OF ROWS IN A OR H
M = NUMBER OF COLUMNS IN A OR H
MS = ONE DIGIT NUMBER FOR STORAGE RULE OF MATRIX A (AND H)

0 = GENERAL
1 = SYMMETRIC
2 = DIAGONAL

SUBROUTINE REQUIRED LOC

SUBROUTINE MCPY(A,H,N,M,MS)
DIMENSION A(100),H(100)
CALL LOC(N,M,1,N,M,MS)
DO 1 I=1,N
   1 R(I)=A(I)
RETURN
END
SUBROUTINE UMTO(A,N1,N2,N3,N4,DET,TEST,N1)

* ASYMMETRIC MATRIX INVERSION ROUTINE * INVERTS PORTION BETWEEN N1 AND N2, RIGHT HAND SIDES START AT N1 AND GO FOR NY COLUMNS.
* INVERSE WILL REPLACE ORIGINAL MATRIX AND SOLUTIONS WILL REPLACE
* RIGHT HAND SIDES , THE DETERMINANT IS ALSO COMPUTED * TEST IS A CRITICAL 
* TO BE USED IN OBTAINING A G- INVERSE, TEST IS A SUITABLE SMALL CONSTANT 
* ANY DIAGONAL ELEMENT SMALLER THAN TEST WILL BE SET TO ZERO.
* THE NUMBER OF RIGHT HAND SIDES MAY BE ZERO * THE SIZE ON THE ARRAY IN THE MAIN PROGRAM IS 'N1'.

DIMENSION A(N1,N1)
NK=N3*N4-1
DET=1.0
GO 608 L=N1,N2
IF (ABS(A(L,L)) .GE. TEST) GO TO 601
A(L,L)=0.0
DET=DET*A(L,L)
A(L,L)=1.0/A(L,L)
601 CONTINUE
GO 605 I=N1,N2
IF (I.EQ.L) GO TO 605
A(I,L)=A(I,L)*A(L,L)
 GO 603 J=N1,N2
IF (J.EQ.L) GO TO 603
A(I,J)=A(I,J)*A(I,L)*A(L,J)
 603 CONTINUE
IF (N4.EQ.0) GO 10 605
GO 604 J=N3,NK
A(I,J)=A(I,J)*A(I*L)*A(L,J)
604 CONTINUE
GO 605 J=N1,N2
IF (J.EQ.L) GO TO 605
A(L,J)=A(L,J)*A(L,L)*A(L,J)
605 CONTINUE
IF (N4.EQ.0) GO 10 608
GO 607 J=N3,NK
A(L,J)=A(L,J)*A(L,L)*A(L,J)
607 CONTINUE
IF (N4.EQ.0) GO 10 610
GO 609 I=N1,N2
GO 609 J=N3,NK
A(I,J)=A(I,J)
609 CONTINUE
610 RETURN
END
SUBROUTINE UMAT

PURPOSE
CALCULATE THE U MATRIX

DESCRIPTION OF PARAMETERS
A = NAME OF INPUT MATRIX
B = NAME OF OUTPUT MATRIX

USAGE
CALL UMAT

SUBROUTINE UMAT(A,B)
INTEGER A,L
DIMENSION A(4,4),B(10,10)
DO 1 I=1,4
DO 2 J=1,4
RI=(I-1)*4+J
1 IF(RI=5) 11,2,12
12 RL=RI+1
GC TO 11
13 RI=RI-3
GC TO 11
14 IF(RI=10) 2,12,15
15 IF(RI=12) 10,16,17
16 RI=RI-3
GC TO 11
17 IF(RI=16) 2,10,2
18 RI=10
11 DO 3 K=1,4
20 DO 4 L=1,4
CI=(K-1)*4+L
22 IF(CI=5) 21,4,22
23 CI=CI+1
24 GC TO 21
25 IF(CI=10) 4,4,25
26 CI=CI+3
GC TO 21
27 IF(CI=16) 4,8,4
28 CI=10
21 B(RI,CI)=A(K,L)+A(L,J)+A(K,J)*A(L,I)
4 CONTINUE
3 CONTINUE
2 CONTINUE
1 CONTINUE
RETURN
END
**DIMENSION A(4,4),AA(4,4),H(10,10),U(10,10),C(10,10),M(10,10),
Y(4,4),Q(4,4),W(4,4),X(4,4),N(4,4),P(4,4),K(4,4),L(4,4),O(4,10),
#T(4,4),TU(4,4),TA(4,4),AH(4,4),LY(4,4),TY(4,4),
#R(10,10),H(10,10),F(10,10),U(10,10),M(10,10),
#Q(10,10),F(10,10),PP(10,10),QW(4,4),HK(10,10),S(10,10),X(10,10),
#D(4,4),Y(4,4),H(10,10),Q(4,4),T(4,4),Y(4,4),
#C(4,10,10),MT(10,10),VV(4,10,10),TA(10,10),Y(10,10),M(4,10,10),
#V(4,10,10),YY(4)
READ(5,10) ((AA(I,J),J=1,4),I=1,4)
READ(5,52) (YY(J),J=1,10),I=1,10)
READ(5,70) ((TV(I,J),J=1,10),I=1,10)
READ(5,14) ((CM(I),I=1,4),I=1,4)
READ(5,16) ((TV(M),M=1,4),M=1,4)
CONTINUE
WHITE(6,123)
DO 122 I=1,4
122 WRITE(6,111) (AA(I,J),J=1,4),I=1,4)
WHITE(6,127)
DO 130 J=1,10
130 WRITE(6,62) (HT(I,J),I=1,4)
WHITE(6,131)
DO 132 J=1,10
132 WRITE(6,62) (TV(I,J),I=1,4)
DO 100 N=1,7
IF (N=1) 11,11,99
CONTINUE
CALL GMF(UT,AA,AT,4,4,4)
CALL GMPH(DT,UT,4,4,4)
CALL UMSW (T,1,0,0,DELT,TE,1,11)
C** STEP 1.
C SIG(T)=U(T)*SIG(DT)*DT
C T= SIG(T)
99 DO 80 I=1,4
80 YK=Y(I,1)
DO 81 I=1,4
81 YK=Y(J,1)
DO 81 J=1,4
YK=Y(J,1)
DI=Y(J,1)
t=U(J,1)
CONTINUE
C** STEP 2.
C W(UT)=INV OF SIG(T)
C TT=W(UT)
CALL DMPY(IY,IT,IT,4,4,0)

C***********************************************************************
C**  STEP 3.
C  YU=U(T)*M(U)*U(T)
C  M(U)=SUM(U,U)*M(U)*U(T)
C  M=M(U)
C  CALL DMPHU(UIL,IT,TA,4,4,0)
C  CALL DMPHU(IA,DL,TD,4,4,0)
C  CALL DMPHU(1AP,Y,E,4,4,1)
DO 82 II=1,4
DO 82 JJ=1,4
YT(II,II,1)=YT(II,II,1)
TI(II,II,JJ)=TI(II,II,JJ)
82 CONTINUE
80 CONTINUE

C***********************************************************************
C**  STEP 4.
C  WI=M(U)*M(U)
C  WI=SUM(U,U)*(U(U)*M(U))*U(T)
C  M=M(U)
C  DO 84 II=1,4
C  DO 84 JJ=1,4
C  WI(II,II,1)=WI(II,II,1)+YT(II,II,1)
C  DC 85 JJ=1,4
C  WI(II,II,1)=M(II,II,1)+YT(II,II,1)
C  DC 85 I=1,4
C  CONTINUE
C  CALL UM1SW(WM,II,II,1,0,0,0,0,0,DET,1,E3T,4)
C  CALL DMPRU(WM,HI,H4,4,4,1)
C  WRITE(6,54)
C  DC 87 I=1,4
C  WRITE(6,77) WI(II,II)

C***********************************************************************
C**  STEP 5.
C  UI(U)=U(T)*U
C  M=M(U)
C  DO 90 I=1,4
C  NI=1,4
C  DO 90 JJ=1,4
C  YR(II,II,1)=YT(II,II,1)
C  DI(II,II,JJ)=DI(II,II,JJ)
C  CONTINUE

C***********************************************************************
C**  STEP 6.
C  M(I)=(E(U)-U)*THAN(E(U)-U)
C  M=M(U)
C  CALL DMPRU(DM,NT,RA,4,4,1)
C  CALL MSUB(YM,MC,NB,4,1,0)
C  CALL MTRM(MH,NC,NU,4,1,0)
C  CALL DMPHU(MH,NC,NU,4,1,0)
DO 93 M=1,4
C  MK(M,1)=MU(M,1)
DO 94 M=2,4
C  MK(M+3,1)=MU(2,M)
DO 95 M=3,4

40 CALL DMPY(IY,IT,IT,4,4,0)

82 CONTINUE

80 CONTINUE

84 CONTINUE

85 CONTINUE

78 WRITE(6,77) WI(II,II)

91 CONTINUE

93 CONTINUE

94 MK(M+3,1)=MU(2,M)
DO 95 M=3,4
**FILE 9**

```plaintext
95 M(K,S+M,L)=MU(3,L)
M(K,10,J)=MU(4,J)
DO 96 J=1,10
96 M(K,1,J)=AK(S+M,L)
90 CONTINUE
WHITE(6,J) N=1
WHITE(6,79)
DO 61 I=1,10
61 WRITE(6,62) (M(I,J,J),J=1,10)

C******************************************************************************
C** STEP 7.
C G(SIGT)=NT*( INV UF (CT*UU*CT))
G=K(SIGT)
WHITE (6,71) N
11 CALL UMAT(MAXBU)
DO 51 I=1,4
Y=YY(I)
DO 2 J=1,10
DO 2 J=1,10
C(I,J,J)=Y(I,J,J)
2 CALL GMPKU ('CPRM=U10*10*10)
CALL GMPKU ('U'*H*10*10*10)
TE=0,5=1.0/(10*10)
CALL UIMSG('R'=10*0,0,DET,TEST,10)
CALL SMPY('YP=10*10*10')
DO 55 J=1,10
DO 55 K=1,10
55 G(J,J,K)=f(UJK)
51 CONTINUE

C******************************************************************************
C** STEP 8.
C G= SUM UF K(SIGT)
G=K
DO 152 I=1,10
DO 152 J=1,10
152 G(I,J)=0
DO 60 I=1,4
DO 60 J=1,10
DO 60 K=1,10
60 G(UJK)=G(J,K)+G(I,J,K)

C******************************************************************************
C** STEP 9.
C G = INV UF K(SIGT)
G=U(UJK)
G=SUM UF (C(T)*=K(SIGT)*K(SIGT +HT))
G=U(UJK)
C S=VAR AND COV MATRIX
DO 100 I=1,4
DO 7 J=1,10
DO 7 L=1,10
7 C(J,J,L)=CC(1,J,J,L)
DO 8 J=1,10
DO 8 L=1,10
8 CONTINUE
DO 72 I=1,10
J=1
72 M(I,J)=HM(I,J)+V(I,J)
DO 20 I=1,10
DO 20 J=1,10
20 M(I,J)=G(I,J)
```

**FILE 9**

```plaintext
95 M(K,S+M,L)=MU(3,L)
M(K,10,J)=MU(4,J)
DO 96 J=1,10
96 M(K,1,J)=AK(S+M,L)
90 CONTINUE
WHITE(6,J) N=1
WHITE(6,79)
DO 61 I=1,10
61 WRITE(6,62) (M(I,J,J),J=1,10)

C******************************************************************************
C** STEP 7.
C G(SIGT)=NT*( INV UF (CT*UU*CT))
G=K(SIGT)
WHITE (6,71) N
11 CALL UMAT(MAXBU)
DO 51 I=1,4
Y=YY(I)
DO 2 J=1,10
DO 2 J=1,10
C(I,J,J)=Y(I,J,J)
2 CALL GMPKU ('CPRM=U10*10*10)
CALL GMPKU ('U'*H*10*10*10)
TE=0,5=1.0/(10*10)
CALL UIMSG('R'=10*0,0,DET,TEST,10)
CALL SMPY('YP=10*10*10')
DO 55 J=1,10
DO 55 K=1,10
55 G(J,J,K)=f(UJK)
51 CONTINUE

C******************************************************************************
C** STEP 8.
C G= SUM UF K(SIGT)
G=K
DO 152 I=1,10
DO 152 J=1,10
152 G(I,J)=0
DO 60 I=1,4
DO 60 J=1,10
DO 60 K=1,10
60 G(UJK)=G(J,K)+G(I,J,K)

C******************************************************************************
C** STEP 9.
C G = INV UF K(SIGT)
G=U(UJK)
G=SUM UF (C(T)*=K(SIGT)*K(SIGT +HT))
G=U(UJK)
C S=VAR AND COV MATRIX
DO 100 I=1,4
DO 7 J=1,10
DO 7 L=1,10
7 C(J,J,L)=CC(1,J,J,L)
DO 8 J=1,10
DO 8 L=1,10
8 CONTINUE
DO 72 I=1,10
J=1
72 M(I,J)=HM(I,J)+V(I,J)
DO 20 I=1,10
DO 20 J=1,10
20 M(I,J)=G(I,J)
```
CALL GMPRU(L,X,Y,10,10,10)
CALL GMPRU(P,P,H,10,10,10)
DO 73 K=1,10
L=1
73 Q(K,K,L)=PP(K,L)
100 CONTINUE
DO 150 I=1,10
156 RR(I,1)=0
DO 75 J=1,10
75 RR(J,1)=RR(J,1)+Q(J,J,1)
75 CONTINUE
TEST=0.5*1.0/(10*10)
CALL UMTSW(W=1.0,0.0,0.0,DETA,TEST,10)
CALL GMPRU(W,RR,10,10,10)
WRITE(6,/) N
WRITE(6,53)
DO 76 I=1,10
76 WRITE(6,77) S(I,1)
DO 3 I=1,4
3 AA(I,1)=S(I,1)
DO 4 I=2,4
4 AA(I,2)=S(I+3,1)
AA(3,3)=S(8,1)
AA(3,4)=S(9,1)
AA(4,3)=S(4,1)
AA(4,4)=S(10,1)
1000 CONTINUE
C

*** ***
10 FORMAT(16F5.3)
14 FORMAT(3612/2012)
16 FORMAT(16F5.3)
52 FORMAT(15,3I3,/*5X*,17I2,/*5X*,2012)
53 FORMAT(' VAR/COV VECTOR IS')
59 FORMAT(' MEAN VECTOR IS')
62 FORMAT(4F16.6)
70 FORMAT(G9.4,/849.4/4F9.4)
71 FORMAT(' THE '12p' ITERATION')
77 FORMAT(F12.0)
79 FORMAT('H(1)',12X,'H(2)',12X,'H(3)',12X,'H(4)')
111 FORMAT(4F12.0)
112 FORMAT(4(4I3,6X))
114 FORMAT(' THE U(T) MATRICES ARE: /*5X*,U(1)',14X,'U(2)',14X
       #',U(3)',14X,'U(4)')
116 FORMAT(' THE ESTIMATED MEAN VECTOR FOR U(1) IS /*5X*,U(1)',
           '12X,'U(2)',12X,'U(3)',12X,'U(4)')
117 FORMAT(' U(',12p') MATRIX IS')
123 FORMAT(' THE INITIAL VAR/COV MATRIX IS')
124 FORMAT(4F15.3)
126 FORMAT(' U(',12p') MATRIX IS')
127 FORMAT(' THE INITIAL M(T) VECTORS ARE: /*5X*,H(1)',12X,'H(2)',
          '12X,'H(3)',12X,'H(4)')
131 FORMAT(' THE SIG(1) VECTORS ARE: /*5X*,S(1)',12X,'S(2)',12X,
          'S(3)',12X,'S(4)')
135 FORMAT(' THE PROGRAM OUTPUT IS FOLLOWING: */)
777 FORMAT(1013)
STOP
END
Appendix 3

Program Output

This output included the initial estimators of mean vector, var/cov matrix, $C_t$, $D_t$, $H_t$ matrices. Records of estimates contained 6 iterations.
The program output is following:

The D(1) matrices are:

<table>
<thead>
<tr>
<th></th>
<th>D(1)</th>
<th>D(2)</th>
<th>D(3)</th>
<th>D(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1 0 0 0</td>
<td>1 0 0 0</td>
<td>1 0 0 0</td>
<td>0 0 0 0</td>
</tr>
<tr>
<td>0 1 0 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0 1 0</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0 0 0 1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The estimated mean vector for u(t) is:

<table>
<thead>
<tr>
<th></th>
<th>u(1)</th>
<th>u(2)</th>
<th>u(3)</th>
<th>u(4)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>4.990</td>
<td>4.990</td>
<td>2.120</td>
<td>0.000</td>
</tr>
<tr>
<td>3.430</td>
<td></td>
<td></td>
<td>3.630</td>
<td>0.000</td>
</tr>
<tr>
<td>1.430</td>
<td></td>
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THE INITIAL VAR/COV MATRIX IS:

\[
\begin{array}{cccc}
0.158000 & 0.140000 & 0.015000 & 0.010000 \\
0.140000 & 0.185000 & 0.013000 & 0.008000 \\
0.015000 & 0.013000 & 0.026000 & 0.004000 \\
0.010000 & 0.008000 & 0.004000 & 0.012000 \\
\end{array}
\]

THE INITIAL H(1) VECTORS ARE:

\[
\begin{array}{cccc}
H(1) & H(2) & H(3) & H(4) \\
0.000000 & 0.000100 & 0.016000 & 0.000000 \\
0.000000 & 0.007700 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000640 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000640 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
\end{array}
\]

THE SIG(1) VECTORS ARE:

\[
\begin{array}{cccc}
S(1) & S(2) & S(3) & S(4) \\
0.153000 & 0.070000 & 0.018000 & 0.000000 \\
0.136000 & 0.021000 & 0.026000 & 0.000000 \\
0.014000 & 0.027000 & 0.000000 & 0.000000 \\
0.009000 & 0.000000 & 0.000000 & 0.000000 \\
0.179000 & 0.053000 & 0.096000 & 0.000000 \\
0.013000 & -0.009000 & 0.000000 & 0.000000 \\
0.008000 & 0.000000 & 0.000000 & 0.000000 \\
0.026000 & 0.024000 & 0.000000 & 0.000000 \\
0.004000 & 0.000000 & 0.000000 & 0.000000 \\
0.011000 & 0.000000 & 0.000000 & 0.000000 \\
\end{array}
\]

1st VAR/COV VECTOR IS:

\[
\begin{array}{c}
0.123911 \\
0.105525 \\
0.016572 \\
0.007077 \\
0.151949 \\
0.008775 \\
0.005707 \\
0.026335 \\
0.004262 \\
0.010940 \\
\end{array}
\]

1st MEAN VECTOR IS:

\[
\begin{array}{c}
4.996597 \\
3.445231 \\
1.444255 \\
0.239588 \\
\end{array}
\]

1st H(1) VECTORS:

\[
\begin{array}{cccc}
H(1) & H(2) & H(3) & H(4) \\
0.000044 & 0.009331 & 0.012284 & 0.000000 \\
0.000100 & 0.007149 & 0.002891 & 0.000000 \\
-0.000035 & 0.007218 & 0.000000 & 0.000000 \\
-0.000003 & 0.000000 & 0.000000 & 0.000000 \\
0.000232 & 0.004964 & 0.194134 & 0.000000 \\
-0.000090 & 0.007116 & 0.000000 & 0.000000 \\
-0.000000 & 0.000000 & 0.000000 & 0.000000 \\
0.000028 & 0.005884 & 0.000000 & 0.000000 \\
0.000002 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
\end{array}
\]
THE 2 ITERATION

VARIANCE VECTOR IS

\[
\begin{pmatrix}
0.123455 \\
0.0105676 \\
0.015783 \\
0.007908 \\
0.015127 \\
0.008542 \\
0.005776 \\
0.026130 \\
0.004109 \\
0.010456
\end{pmatrix}
\]

MEAN VECTOR IS

\[
\begin{pmatrix}
4.996341 \\
3.445045 \\
1.440445 \\
0.239002
\end{pmatrix}
\]

\[
\begin{array}{cccc}
M(1) & M(2) & M(3) & M(4) \\
0.000041 & 0.007291 & 0.013279 & 0.000000 \\
0.000046 & 0.004166 & 0.022379 & 0.000000 \\
0.000034 & 0.007195 & 0.000000 & 0.000000 \\
0.000023 & 0.009042 & 0.034172 & 0.000000 \\
0.000031 & 0.000000 & 0.000000 & 0.000000 \\
0.000006 & 0.000000 & 0.000000 & 0.000000 \\
0.000029 & 0.0000572 & 0.000000 & 0.000000 \\
0.000002 & 0.000000 & 0.000000 & 0.000000 \\
0.000000 & 0.000000 & 0.000000 & 0.000000 \\
\end{array}
\]

THE 3 ITERATION

VARIANCE VECTOR IS

\[
\begin{pmatrix}
0.123461 \\
0.105380 \\
0.015883 \\
0.007866 \\
0.151829 \\
0.008321 \\
0.005797 \\
0.026154 \\
0.004119 \\
0.010463
\end{pmatrix}
\]

MEAN VECTOR IS

\[
\begin{pmatrix}
4.996417 \\
3.445108 \\
1.444055 \\
0.239002
\end{pmatrix}
\]
3

<table>
<thead>
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<th>H(2)</th>
<th>H(3)</th>
<th>H(4)</th>
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<td>0.009296</td>
<td>0.015273</td>
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<td>0.000097</td>
<td>0.009176</td>
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<td>-0.000034</td>
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**THE 4 ITERATION**

4

**VAR/COV VECTOR IS**

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</tr>
<tr>
<td>0.010463</td>
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4

**MEAN VECTOR IS**

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**THE 5 ITERATION**

5

**VAR/COV VECTOR IS**

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### 5. Mean Vector IS

1.3.996414
3.445106
1.444654
0.239603

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### 6. Iteration

**The 6 Iteration**

**Var/Cov Vector IS**

| 0.123561 |
| 0.105579 |
| 0.015372 |
| 0.007376 |
| 0.151329 |
| 0.005349 |
| 0.004366 |
| 0.026132 |
| 0.004115 |
| 0.01063 |

**Mean Vector IS**

1.3.996414
3.445106
1.444654
0.239603

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</table>
VITA

Chi-Ping Lu

Candidate for the Degree of

Master of Science

Report: The Program of Incomplete Data in Multivariate Normal Distribution

Major Field: Applied Statistics

Biographical Information:

Personal Data: Born at Pei-Ping, China, May 5, 1949, daughter of Hsueh-Lee and Win-Jin Lu.

Education: Graduated from Provincial Taipei Second Girl High School, Taiwan in 1967; received a Bachelor of Science degree from National Chen-Chi University, majoring in Statistics, Taiwan in 1971; completed requirements for the Master of Science degree in Applied Statistics at Utah State University in 1975.