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Matrix Norms

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MATRIX NORMS

by

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of the requirements for the degree

of

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in

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I-Hui C. Cheng
NOTATION

\[\mathbb{C}:\] The set of complex numbers.

\[\mathbb{C}_n: \] The set of all \(n\)-component vectors over the complex numbers.

\[\mathbb{C}^{n \times n}: \] The set of all \(n\times n\) matrices with complex entries.

\[\mathbb{O}_n^h: \] The unit sphere which is a set of all vectors \(x \in \mathbb{C}_n\) and \(h(x) = 1\), where \(h\) is some vector norm.

\[\langle x, y \rangle: \] The inner product of \(x\) and \(y\).

\[(a_{ij}): \] Matrix \(A\) where \(a_{ij}\) represents an entry of \(A\) in the \((i, j)\)th position.

\[A^*: \] The matrix obtained from \(A \in \mathbb{C}^{n \times n}\) by taking the transpose of the matrix whose elements are the complex conjugates of these of \(A\).

\[I: \] Identity matrix.

\[A^{-1}: \] Inverse of matrix \(A\).

\[\text{det}\ A: \] Determinant of matrix \(A\).

\[\text{tr}(A): \] Trace of matrix \(A\) defined as the sum of the diagonal elements of \(A\).

\[U: \] A unitary matrix.

\[\mathbf{e}_j: \] The unit vector with 1 in the \(j\)th position and 0 elsewhere.

\[\text{diag}\{a_1, a_2, \ldots, a_n\}: \] A diagonal matrix whose diagonal entries are \(a_{ij} = a_i\).
INTRODUCTION

In many situations it is very useful to have a single nonnegative real number to be, in some sense, the measure of the size of a vector or a matrix. As a matter of fact we do a similar thing with scalars, we let $|\lambda|$ represent the familiar absolute value or modulus of $\lambda$. For a vector $x \in \mathbb{C}^n$, one way of assigning magnitude is the usual definition of length,

$$\|x\| = \langle x, x \rangle^{1/2} = \{x_i^2\}^{1/2},$$

which is called the euclidean norm of $x$. In this case, length gives an overall estimate of the size of the elements of $x$. If $\|x\|$ is large, at least one of the elements in $x$ is large, and vice versa. There are many ways of defining norms for vectors and matrices. We will examine some of these in this paper.

In the first two sections, we will take an axiomatic approach to the formulation of measures of magnitude or norms for matrices and vectors.

In Section III, we will examine "Induced Matrix Norms", These norms are the ones which can be attained from a given matrix norm. In Theorem 3.3 we will work out in detail how one obtains the three most popular induced matrix norms $\|A\|_1$, $\|A\|_2$, and $\|A\|_\infty$ from the three most commonly used vector norms
\[ \|x\|_1, \|x\|_2, \text{ and } \|x\|_\infty. \] This will give us some insight on how matrix and vector norms are related.

After we have done all the introductory work, the last two sections will be devoted to two different areas of the applications of matrix norms.
PRELIMINARY RESULTS AND DEFINITIONS

The following results and definitions will be used in this paper, however no proof will be given.

Definition 0.1. The standard inner product of two vectors \( x, y \in \mathbb{C}^n \), is defined as
\[
\langle x, y \rangle = \sum_{i=1}^{n} x_i y_i^* + \overline{\sum_{i=1}^{n} x_i y_i}.
\]

Definition 0.2. If \( A \in \mathbb{C}^{n \times n} \), and if there exists a vector \( x \in \mathbb{C}^n \) such that \( Ax = \mu x \) where \( \mu \in \mathbb{C} \), then such a nonzero vector \( x \) is called an eigenvector of \( A \). The number \( \mu \) is called an eigenvalue of \( A \). Also \( x \) is said to be an eigenvector associated with the eigenvalue \( \mu \).

Definition 0.3. The matrix \( A \) is said to be Hermitian if and only if \( A^* = A \).

Definition 0.4. The matrix \( A \) is said to be normal if and only if \( A^* A = AA^* \).

Definition 0.5. The matrix \( U \) is unitary if and only if \( U^* U = I \).

Definition 0.6. Matrices \( A \) and \( B \) are said to be unitarily
similar if there is a unitary matrix $U$ such that $A = U^* B U$.

**Definition 0.7.** If $x, y \in \mathbb{C}_n$, the line segment joining $x$ and $y$ is defined to be the set of points in $\mathbb{C}_n$ which have the form $z = tx + (1 - t)y$ for some $t \in [0, 1]$.

**Definition 0.8.** A set of vectors $T \subseteq \mathbb{C}_n$ is said to be a convex set if and only if for each pair $x, y \in T$, the line segment joining $x$ and $y$ is also contained in $T$.

**Definition 0.9.** A real symmetric matrix $A \in \mathbb{C}_{n \times n}$ is said to be positive definite if and only if $x'Ax > 0$ for all non-zero vectors $x \in \mathbb{C}_n$.

**Theorem 0.1.** (Schwarz's Inequality) If $x, y \in \mathbb{C}_n$, then

$$
|x, y| = |\sum_j x_j y_j| \leq (\sum_j |x_j|^2)^{1/2} \cdot (\sum_j |y_j|^2)^{1/2}
$$

$$
= <x, x>^{1/2} \cdot <y, y>^{1/2}.
$$

**Theorem 0.2.** $A$ is normal if and only if there exists a set of orthonormal eigenvectors $\{x_1, x_2, \ldots, x_n\}$ with associated eigenvalues $\mu_1, \mu_2, \ldots, \mu_n$.

**Theorem 0.3.** $A$ is a definite matrix if and only if $A$ is real, symmetric and has nonnegative eigenvalues.
Theorem 0.4. A matrix $A \in \mathbb{C}_{n \times n}$ is unitarily similar to the diagonal matrix of its eigenvalues if and only if $A$ is normal.

Theorem 0.5. If two matrices $A$ and $B$ are similar then $A$ and $B$ have the same characteristic polynomials and hence the same eigenvalues.

Theorem 0.6. If $A$ is nonsingular, then the eigenvalues of $A^{-1}$ are the reciprocals of the eigenvalues of $A$.

Theorem 0.7. If $\mu_1, \mu_2, \ldots, \mu_n$ are the eigenvalues of $A$, then $\det A = \mu_1 \mu_2 \ldots \mu_n$.

Theorem 0.8. If $T$ is a closed and bounded set in a finite dimensional normed linear space and if $f$ is continuous on $T$, then $f(T)$ is a closed and bounded set.

Theorem 0.9. The eigenvalues of a Hermitian matrix are real.
A real-valued function defined on all n-square matrices A with complex elements is called a matrix norm, and usually written \( \|A\| \), if and only if it satisfies the following axioms:

1. \( \|A\| > 0 \) for \( A \neq 0 \) and \( \|A\| = 0 \) if and only if \( A = 0 \),
2. \( \|cA\| = |c| \|A\| \) for any complex number \( c \),
3. \( \|A + B\| \leq \|A\| + \|B\| \) (triangle inequality),
4. \( \|AB\| \leq \|A\| \|B\| \).

It is easily seen that the real-valued function \( \sum_{i,j} |a_{ij}| \) satisfies the above four axioms, and hence, is a matrix norm.

The following are some useful properties of matrix norm:

1. \( \|I\| \geq 1 \),
2. \( \|A^n\| \leq \|A\|^n \),
3. \( \|A^{-1}\| \geq 1/\|A\| \),
4. \( \|A - B\| \geq \|A\| - \|B\| \).

Proof: For any matrix norm \( \| \cdot \| \),

1. \( \|I\| \geq 1 \), by way of contradiction, we suppose \( 0 \leq \|I\| < 1 \) and \( A \neq 0 \), then \( \|AI\| \leq \|A\| \|I\| \), this implies \( \|A\| \leq \|A\| \|I\| \), which is a contradiction, hence \( \|I\| \geq 1 \).
2. \( \|A^n\| \leq \|A\|^n \), this follows directly from Axiom (iv).
3. \( \|A^{-1}\| \geq 1/\|A\| \), which follows easily from property (1).
(4) \( \| A - B \| \leq \| A - B \| \), which follows from the triangle inequality.

A real-valued function satisfying axioms (1) through (iii), but not necessary (iv) will be called a generalized matrix norm. Thus a matrix norm is always a generalized matrix norm, but not conversely.

**Example 1.1.** The real-valued function \( \max_{i,j} |a_{ij}| \) is a generalized matrix norm which is not a matrix norm.

**Proof.** We verify that \( \max_{i,j} |a_{ij}| \) satisfies axioms (i), (ii), and (iii), but not (iv).

\[
(1) \quad \| A \| = \max_{i,j} |a_{ij}| \geq 0, \text{ and when } \| A \| = 0, \text{ that is, } \max_{i,j} |a_{ij}| = 0, \text{ then } a_{ij} = 0 \text{ for all } i, j, \text{ hence } A = 0. \quad \text{The converse is obvious.}
\]

\[
(ii) \quad \| cA \| = \max_{i,j} |ca_{ij}| = |c| \max_{i,j} |a_{ij}| = |c| \| A \| .
\]

\[
(iii) \quad \| A + B \| = \max_{i,j} |a_{ij} + b_{ij}| \leq \max_{i,j} (|a_{ij}| + |b_{ij}|)
\]

\[
\leq \max_{i,j} |a_{ij}| + \max_{i,j} |b_{ij}| = \| A \| + \| B \|.\]

\[
(iv) \quad \text{For Axiom (iv), } \| AB \| \leq \| A \| \| B \| , \text{ we will give a counterexample. Let}
\]

\[
A = \begin{bmatrix} 2 & 1 \\ 0 & -4 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -2 \\ 3 & 5 \end{bmatrix},
\]

then
AB = \begin{pmatrix} 5 & 1 \\ -12 & -22 \end{pmatrix}, \text{ and } ||AB|| = \max_{i,j} |c_{ij}| = 22,

but \( ||A|| = 4, ||B|| = 5 \), we have

\[ ||AB|| = 22 > ||A|| ||B|| = 20, \]

and this function does not satisfy axiom (iv), hence it is a generalized matrix norm which is not a matrix norm.

**Theorem 1.1.** A generalized matrix norm depends continuously on the elements of the matrix; that is, given \( \epsilon > 0 \) there exists a \( \delta > 0 \), such that whenever \( |a_{ij} - b_{ij}| < \delta \) for all pairs of \( i, j \), we have \( ||A - B|| < \epsilon \), where \( A = (a_{ij}), B = (b_{ij}) \).

**Proof.** First we let \( E_{ij} \) be the matrix having a one in the \( i, j \) position and zeros elsewhere. Then

\[ A - B = \sum_{i,j} (a_{ij} - b_{ij}) \cdot E_{ij}. \]

Now we define the number \( k = \max_{i,j} |E_{ij}| \) and \( k > 0 \) by Axiom (i),

\[ ||A - B|| = \sum_{i,j} |(a_{ij} - b_{ij})E_{ij}| \leq \sum_{i,j} |(a_{ij} - b_{ij})E_{ij}| \leq k \cdot \sum_{i,j} |a_{ij} - b_{ij}|. \]

For any \( \epsilon > 0 \), we define \( \delta = \epsilon/kn^2 \), and consider any pair of matrices \( A, B \), where \( A \) and \( B \) are of order \( n \), for which

\[ |a_{ij} - b_{ij}| < \delta \quad i, j = 1, 2, \ldots, n, \]

then

\[ ||A - B|| \leq k \cdot \sum_{i,j} |a_{ij} - b_{ij}| < k \cdot \sum_{i,j} \delta = kn^2 \delta = \epsilon. \]

Hence
\[ \|A\| - \|B\| \leq \|A - B\| < \varepsilon \]

by property (4) whenever \( |a_{ij} - b_{ij}| < \delta \).

The following theorem is also of fundamental importance, because with the help of this comparison theorem, we can reduce the study of properties of one generalized matrix norm to the study of another relatively simple norm, and which will be demonstrated in Theorem 1.5. But before introducing it, we give some definitions and a lemma which will be useful in proving the theorem.

**Definition 1.1.** Given a generalized matrix norm, \( \| \cdot \| \), on \( \mathbb{C}^{n \times n} \), we define a neighborhood \( N_\varepsilon(A) \) of \( A \in \mathbb{C}^{n \times n} \) to be

\[
N_\varepsilon(A) = \{ \alpha \in \mathbb{C}^{n \times n} \mid \| A - \alpha \| \leq \varepsilon \}.
\]

**Definition 1.2.** If \( L \) is a subset of \( \mathbb{C}^{n \times n} \), then \( A \) is a limit point of \( L \) if every \( N_\varepsilon(A) \) contains some point \( \alpha \in L \) with \( \alpha \neq A \).

**Lemma 1.2.** Let \( M \) be the set of all \( n \)-square matrices with \( \| \alpha \|_m = \max_{i,j} |a_{ij}| = 1 \), then \( M \) is closed.

**Proof.** To prove \( M \) is closed we show that every limit point of \( M \) is in \( M \). Let \( A \) be a limit point of \( M \), thus for any \( \varepsilon > 0 \), there exists an \( \alpha \in M \) such that
\[ \|A - \alpha\|_m \leq \varepsilon. \]

But
\[ \|A\|_m - \|\alpha\|_m \leq \|A - \alpha\|_m \leq \varepsilon \quad \text{and} \quad \|\alpha\|_m = 1, \]
so we have
\[ 1 - \varepsilon \leq \|A\|_m \leq 1 + \varepsilon \quad \text{for any} \quad \varepsilon > 0. \]
Consequently \( \|A\|_m = 1 \) and hence \( A \in M. \)

**Lemma 1.3.** Let \( M \) be the set defined in Lemma 1.2, and \( \| \cdot \| \)
a generalized matrix norm, then \( \| \cdot \| \) is continuous on \( M. \)

**Proof.** To prove \( \| \cdot \| \) is continuous at \( A \in M, \) we have to show that given \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that if
\[ \|A - X\|_m < \delta \quad \text{with} \quad X \in M, \]
we have \( \|A\| - \|X\| < \varepsilon. \) Theorem 1.1 states that a generalized matrix norm depends continuously on the elements of the matrix, that is, given \( \varepsilon > 0, \) there exists a \( \delta > 0 \) such that if
\[ |a_{ij} - x_{ij}| < \delta \quad \text{for all} \quad i, j, \]
then we have \( \|A\| - \|X\| < \varepsilon, \) but \( |a_{ij} - x_{ij}| < \delta, \) for all \( i, j, \) implies
\[ \max_{i,j} |a_{ij} - x_{ij}| < \delta, \] and this says \( \|A - X\|_m < \delta. \) Hence given any \( \varepsilon > 0, \) we were able to use the same \( \delta \) we found from the continuity of a generalized matrix norm to show the continuity of \( \| \cdot \|. \)

**Theorem 1.4.** Let \( \|A\| \) and \( N(A) \) be any two generalized matrix norms evaluated at \( A, \) then there exists positive numbers
\( r_1 \) and \( r_2 \), depending only on the choice of norms, such that

\[
\frac{1}{r_2} \leq \frac{|A|}{N(A)} \leq r_2
\]

for all \( A \in C_{n \times n} \) and \( A \neq 0 \).

**Proof.** Again we will use the matrices \( E_{ij} \) defined in Theorem 1.1 and define \( p_2 = \sum |E_{ij}|, \) and \( a = \max |a_{ij}|. \)

Since \( A = \sum a_{ij} E_{ij} \), we have

\[
|A| = \left\| \sum a_{ij} E_{ij} \right\| \leq \sum |a_{ij} E_{ij}| \leq \sum |a_{ij}| \left\| E_{ij} \right\| \leq a p_2,
\]

where \( p_2 \) is obviously independent of \( A \).

Now consider the set \( M \) defined in Lemma 1.2, and let \( p_1 = \min_{B \in M} \|B\| \), such a \( p_1 \) will exist since \( M \) is closed and Lemma 1.3 provides the continuity of the generalized matrix norm \( \cdot \) on \( M \). Observe that \( p_1 > 0 \), and is also independent of \( A \).

For the given matrix \( A \), we may write \( A = a_{\mu\nu} B \) where \( |a_{\mu\nu}| = a \) and \( B \in M \). Then we have

\[
|A| = |a_{\mu\nu} B| = |a_{\mu\nu}| \|B\| = a \|B\| \geq a p_1,
\]

and combining the inequalities (1) and (2), we have

\[
ap_1 \leq |A| \leq a p_2.
\]

Similarly, we can find another two positive real numbers \( q_1 \) and \( q_2 \) depending only on the norm \( N \), such that

\[
a q_1 \leq N(A) \leq a q_2.
\]

With these two equalities, we have
The following theorem gives us an application of the above theorem.

**Theorem 1.5.** If \( A_1, A_2, A_3, \ldots \) is a sequence of matrices from \( \mathbb{C}^{nxn} \), then, for any generalized matrix norm, \( \| \cdot \| \), \( A_p \to A \), pointwise as \( p \to \infty \) if and only if \( \| A_p - A \| \to 0 \) as \( p \to \infty \).

**Proof.** Let \( N(A) = \sum \sum |a_{ij}| \). It is easy to verify that \( N(A) \) is a generalized matrix norm. Now suppose \( A_p \to A \) pointwise as \( p \to \infty \), we have

\[
\frac{a_{ij}}{p} \to \frac{a_{ij}}{p}, \text{ for every } i \text{ and } j,
\]

hence we have

\[
\sum \sum |a_{ij} - a_{ij}| \to 0, \text{ that is, } N(A_p - A) \to 0.
\]

Since \( \| \cdot \| \) is any generalized matrix norm, then by Theorem 1.4, there exists two real numbers \( r_1 \) and \( r_2 \), such that

\[
r_1 \leq \frac{\| A_p - A \|}{N(A_p - A)} \leq r_2.
\]

This implies

\[
N(A_p - A)r_1 \leq \| A_p - A \| \leq N(A_p - A)r_2.
\]

Hence as \( N(A_p - A) \to 0 \), we have

\[
\| A_p - A \| \to 0.
\]
Conversely, if \( \|A_p - A\| \to 0 \) as \( p \to \infty \), the proof is very similar to above, except we use Theorem 1.4 to find the existence of \( r_1 \) and \( r_2 \) such that

\[
\frac{N(A_p - A)}{\|A_p - A\|} \leq r_2,
\]

which implies

\[
r_1 \|A_p - A\| \leq N(A_p - A) \leq r_2 \|A_p - A\|,
\]

and as \( \|A_p - A\| \to 0 \) we have \( N(A_p - A) \to 0 \), this implies

\[
\sum_{i,j} |a_{ij} - p_{ij}| \to 0,
\]

hence \( a_{ij} \to p_{ij} \) for all \( i \) and \( j \). Therefore, \( A_p \to A \) as \( p \to \infty \), and the proof is complete.

Usually we define a matrix norm in term of the elements of the matrix, but if we let \( \mu_1, \mu_2, \ldots, \mu_n \) be the eigenvalues of \( A \in \mathbb{C}^{nxn} \), and \( \mu_A = \max_j |\mu_j| \), then we call this \( \mu_A \) the spectral radius of \( A \). The next example will show us that the spectral radius is not a matrix norm, but the following theorem will assure us the importance of the spectral radius in connection with the magnitude of matrix norms.

**Example 1.2.** Let \( A \) be the \( n \)-square matrix
to find the eigenvalues of $A$ we solve the equation $\det(A - \lambda I) = 0$,

$$A - \lambda I = \begin{bmatrix} -\lambda & 1 & 0 & 0 & \cdots & 0 \\ 0 & -\lambda & 1 & 0 & \cdots & 0 \\ 0 & 0 & -\lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & -\lambda \end{bmatrix}$$

$\det(A - \lambda I) = (-\lambda)^n = (-1)^n(\lambda)^n = 0$, this implies $\lambda_i = 0$ for all $i = 1, 2, \ldots, n$, therefore, $\mu_A = \max_i |\lambda_i| = 0$. But $A \neq 0$ and $\mu_A = 0$ do not imply $A = 0$, hence it does not satisfy the first axiom of matrix norms. Therefore, the spectral radius is not a matrix norm.

**Theorem 1.6.** If $A \in \mathbb{C}^{n \times n}$ and $\mu_A$ is the spectral radius of $A$, then for any matrix norm $\| \cdot \|$, we have

$$\|A\| \geq \mu_A.$$  

**Proof.** Let $\mu$ be an eigenvalue of $A$ with $\mu_A = |\mu|$, then there is a vector $x \neq 0$ such that $Ax = \mu x$. Define $A_x$ as the $n$-square matrix, where $x$ is the first column and all other columns are zero, that is, $A_x = [x \ 0 \ 0 \ \ldots \ 0]$. Then clearly

$$AA_x = \mu A_x,$$

but

$$|\mu| |A_x| = |\mu A_x| = \|AA_x\| \leq \|A\|A_x\|,$$

and $A_x \neq 0$ implies $\|A_x\| \neq 0$, hence we have

$$|\mu| = \mu_A \leq \|A\|.$$
The following are two of the most commonly used matrix norms:

(A) The euclidean, or Frobenius, norm
\[ \|A\| = \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2}. \]

(B) \[ M(A) = \max_{i,j} |a_{ij}|. \]

The following theorem gives us some properties of the euclidean norm.

**Theorem 1.7.** The euclidean norm is invariant under a unitary transformation, that is, if \( U \) is a unitary matrix, then
\[ \|A\| = \|AU\| = \|UA\|. \]

**Proof.** We first of all will prove that \( \|A\|^2 = \text{tr}(A^*A) \).

To prove this, we simply write out both sides in full,
\[ \|A\| = \left( \sum_{i,j} |a_{ij}|^2 \right)^{1/2}, \]
\[ \text{tr}(A^*A) = \sum_{i,j} \sum_{i,j} \overline{a_{ij}} a_{ij} = \sum_{i,j} |a_{ij}|^2, \]

hence \( \|A\|^2 = \text{tr}(A^*A) \). Now we have
\[ \|UA\|^2 = \text{tr}((UA)^*(UA)) = \text{tr}(A^*UA) = \text{tr}(A^*A) = \|A\|^2, \]
and using the same argument, we have \( \|AU\|^2 = \|A\|^2 \). Hence
\[ \|A\| = \|UA\| = \|AU\|, \]
and the proof is complete.
CHAPTER II

VECTORS NORMS

Just as for matrices, there are several useful measures of magnitude for the vectors of $\mathbb{C}^n$, in this section we shall use the word "norm" to include any nonnegative real-valued function of $x \in \mathbb{C}^n$, usually denoted $\|x\|$ or $h(x)$, that satisfies the following properties: Let $x, y \in \mathbb{C}^n$ then

1. $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$,
2. $\|kx\| = |k| \cdot \|x\|$ for any scalar $k$,
3. $\|x + y\| \leq \|x\| + \|y\|$.

Following is a list of some of the most commonly used vector norms:

(a) $\|x\|_1 = \sum_j |x_j|$
(b) $\|x\|_2 = \left( \sum_j |x_j|^2 \right)^{1/2}$
(c) $\|x\|_\infty = \max_j |x_j|$

There is also a class of vector norms which are a natural extension of the 1 and 2 norms, which includes the 1 and 2 norms, and gives motivation for calling $\max_j |x_j|$ the $\infty$-norm. These norms are called $p$-norms or Hölder norms. They are defined as follows:

$\|x\|_p = \left( \sum_1^n |x_1|^p \right)^{1/p}$. 
And the following theorem will give us a pretty good idea why \( \max |x_j| \) is called the \( \infty \)-norm, \( \|x\|_{\infty} \).

**Theorem 2.1.** We show that \( \|x\|_{\infty} = \max |x_j| = \lim_{p \to \infty} (\sum |x_j|^p)^{1/p}. \)

**Proof.** For any vector \( x \in \mathbb{C}^n \), suppose \( |x_m| = \max_j |x_j| \), then

\[
|x_m|^p \leq \sum_{j=1}^n |x_j|^p \leq n|x_m|^p
\]

for all \( p \) and

\[
|x_m| = (|x_m|^p)^{1/p} \leq (\sum_{j=1}^n |x_j|^p)^{1/p} \leq (n|x_m|^p)^{1/p} = n^{1/p}|x_m|,
\]

now taking the limit as \( p \to \infty \), we have

\[
limit_{p \to \infty} |x_m| \leq \lim_{p \to \infty} (\sum |x_j|^p)^{1/p} \leq \lim_{p \to \infty} n^{1/p}|x_m|,
\]

which leads to

\[
|x_m| \leq \lim_{p \to \infty} (\sum_{j=1}^n |x_j|^p)^{1/p} \leq |x_m|.
\]

Hence

\[
\lim_{p \to \infty} (\sum_{j=1}^n |x_j|^p)^{1/p} = \max_j |x_j| = \|x\|_{\infty}.
\]

Sometimes we will face the problem of finding the norm of a vector given in the form \( Ax \) where \( A \in \mathbb{C}^{n \times n}, x \in \mathbb{C}^n \).

Considering Axiom (iv) for a matrix norm, we might expect that there will be a matrix norm \( \| \cdot \| \) and vector norm \( h \) for which

\[
h(Ax) \leq |A| \cdot h(x).
\]

When this is the case we say the norms are compatible. To make the definition more precise: A matrix
norm is said to be compatible with a vector norm $h$ if and only if for all $x \in \mathbb{C}_n$ and $A \in \mathbb{C}_{n \times n}$

$$h(Ax) \leq \|A\|_2 h(x).$$

Following is an theorem to show that a matrix norm $M(A) = \max_{i,j} |a_{ij}|$ is compatible with the 1, 2, and $\infty$-norms which were listed above.

**Theorem 2.2.** $M(A)$ is compatible with the 1, 2, and $\infty$-norms.

**Proof.**

1. \[\|Ax\|_1 = \sum_{k=1}^{n} \|\sum_{i=1}^{n} a_{ik} x_k\|_{x_k} \leq \sum_{i,k} |a_{ik}| \|x_k\|_{x_k} \leq \sum_{i,k} M(A) \|x_k\|_{x_k} = M(A) \|x\|_1.\]

2. \[\|Ax\|_2^2 = \sum_{j,k} |a_{jk} x_k|^2 \leq \sum_{j,k} |a_{jk}|^2 (\sum_k |x_k|^2) = \|A\|_2^2 \|x\|_2^2,
\]

where $\| \cdot \|$ is the euclidean matrix norm, but

$$\|A\|_2^2 = \sum_{i,j} |a_{ij}|^2 \leq n^2 \max_{i,j} |a_{ij}|^2 = (n \max_{i,j} |a_{ij}|)^2 = M^2(A),$$

hence

$$\|Ax\|_2^2 \leq \|A\|_2^2 \|x\|_2^2 \leq M^2(A) \|x\|_2^2,$$

and we have
\[ \|Ax\|_2 \leq M(A)\|x\|_2. \]

(3) \[ \|Ax\|_\infty = \max_{j,k} |\sum_j a_{jk} x_k| \leq \max_{j,k} (\|a_{jk}\| \cdot \|x_k\|) \]
\[ \leq \max_{j,k} \left( \sum_j a_{jk} \right) \max_{j,k} (\|x_k\|) \leq \sum_j a_{jk} \|x\|_\infty \]
\[ \leq n \cdot \max_{j,k} (\max_{j,k} a_{jk}) \|x\|_\infty = M(A)\|x\|_\infty. \]

**Theorem 2.3.** If \( \| \cdot \| \) is the generalized matrix norm defined in Example 1.1, that is, \( \|A\| = \max_{i,j} |a_{ij}| \), then there is no vector norm compatible with \( \| \cdot \| \).

**Proof.** Let \( h \) be any vector norm and \( A \) be an \( n \)-square matrix \( A = (a_{ij}) \) where \( a_{ij} = 1 \) for all \( i, j \). Also let \( x \) be a vector such that \( x_j = 1 \) for all \( j \). Since \( Ax = nx \) and \( \|A\| = 1 \), we have
\[ h(Ax) = h(nx) = n \cdot h(x) > 1 \cdot h(x) = \|A\| \cdot h(x). \]
Therefore, there is no vector norm compatible with \( \|A\| = \max_{i,j} |a_{ij}|. \)
CHAPTER III
INDUCED MATRIX NORMS

Now we shall discuss a method in which we derive, from a given vector norm, a matrix norm that is compatible with the given vector norm. Since we must have \( h(Ax) \leq \| A \| h(x) \) where \( A \in \mathbb{C}^{n \times n} \) and \( x \in \mathbb{C}^n \), this suggests we define

\[
\| A \| = \sup_{x \neq 0} \frac{h(Ax)}{h(x)}.
\]

Later we will prove that for a fixed \( A \), this definition yields a well-defined nonnegative real number and that the resulting function, \( \| \cdot \| \), is a matrix norm. We will call it the matrix norm induced by the vector norm \( h \). In some books it is called the natural norm associated with the vector norm \( h \).

**Lemma 3.1.** For the function \( \| \cdot \| \) defined above we have:

1. \( \| A \| = \max_{h(x) = 1} h(Ax) \).
2. There is a vector \( x_0 \), depending on \( A \), such that \( h(x_0) = 1 \) and \( \| A \| = h(Ax_0) \).

**Proof.** First notice that if we replace the vector \( x \) by \( cx \), it does not affect the quotient \( h(Ax)/h(x) \). If we introduce \( z = x/h(x) \), we have \( h(z) = 1 \) and

\[
\frac{h(Az)}{h(z)} = h(Az),
\]

so we have \( \| A \| = \sup_{h(z) = 1} h(Az) \), or we say \( \| A \| = \sup_{h(z) = 1} h(Az) \), where \( z \in \overline{\Omega}_h \).
According to Theorem 1.1 we know that a vector norm depends continuously on the components of the vector argument. Furthermore, $\overline{O}_h$ is a closed and bounded set, hence attains its maximum. Therefore, both parts of the lemma follow immediately.

**Theorem 3.2.** If a vector $X_o$ exists such that the maximum is attained in the formula

$$\|A\| = \max_{h(z)=1} h(Az),$$

then $\|\cdot\|$ is a matrix norm, and it is compatible with the vector norm $h$.

**Proof.** We have to verify that $\|\cdot\|$ satisfies the axioms (i) through (iv) of matrix norm:

(i) It is clear that $\|A\| \geq 0$ and $A = 0$ implies $\|A\| = 0$ immediately. When $\|A\| = 0$ this implies that $h(Az) = 0$ for all $z$ with $h(z) = 1$, but $z \neq 0$ and $Az = 0$ for all $z \in C_n$, implies $A = 0$. Hence Axiom (i) is satisfied.

(ii) Let $c$ be any scalar, we have

$$\|cA\| = \max_{h(z)=1} h(cAz) = |c| \max_{h(z)=1} h(Az) = |c| \|A\|.$$ 

Thus Axiom (ii) is also satisfied.

(iii) Since we know that

$$h((A + B)x) = h(Ax + Bx) \leq h(Ax) + h(Bx);$$

hence by definition

$$\|A + B\| = \max_{h(z)=1} h((A + B)z) \leq \max_{h(z)=1} (h(Az) + h(Bz)).$$
\[ \text{Thus we have Axiom (iii).} \]

(iv) If we let \( z = x/h(x) \) and \( h(z) = 1 \), we have

\[ h(Ax) = h(h(x)Az) = h(x)h(Az) \leq h(x) \text{Max}_{h(z)=1} h(Az) \]

This gives us that the matrix norm \( \| \cdot \| \) is compatible with the vector norm \( h \). With this fact and again the existence of \( x_0 \) such that \( \| AB \| = h(ABx_0) \) and \( h(x_0) = 1 \), we have

\[ \| AB \| = h(ABx_0) = h(A(Bx_0)) \leq \| A \| h(Bx_0) \]

This gives us that the matrix norm \( \| \cdot \| \) is compatible with the vector norm \( h \). With this fact and again the existence of \( x_0 \) such that \( \| AB \| = h(ABx_0) \) and \( h(x_0) = 1 \), we have

\[ \| AB \| = h(ABx_0) = h(A(Bx_0)) \leq \| A \| h(Bx_0) \]

Thus Axiom (iv) is satisfied and hence the proof is complete.

The following theorem gives us some important and useful matrix norms, namely those which induced by 1, 2, and \( \infty \) vector norms.

**Theorem 3.3.** The induced matrix norms associated with the 1, 2, and \( \infty \) vector norms are:

(a) \( \| A \|_1 = \text{Max}_j \sum_{i=1}^n |a_{ij}| \) (the maximum absolute column sum),

(b) \( \| A \|_2 = \text{maximum eigenvalue of } A^*A \)^{1/2},

(c) \( \| A \|_\infty = \text{Max}_i \sum_{j=1}^n |a_{ij}| \) (the maximum absolute row sum).

**Proof.** (a) If \( \| z \|_1 = 1 \), we have \( \sum_{j=1}^n |z_j| = 1 \). But

\[ \| Az \|_1 = \sum_{j=1}^n |a_{ij}z_j| \leq \sum_{i,j}^n |a_{ij}| |z_j| \]
\[ = \sum_j (|z_j| \sum_i |a_{ij}|) \leq (\max_j \sum_i |a_{ij}|) \sum_j |z_j| \]

= \max_j \sum_i |a_{ij}|.

Hence it follows from Lemma 3.1 that

\[ \|A\|_1 = \max \|Az\|_1 \leq \max \sum_j |a_{ij}|. \tag{1} \]

Now suppose that we attain the maximum of \( \sum_i |a_{ij}| \) at \( j = k \), and choose the vector \( z \) such that \( z_k = 1, z_i = 0 \) (\( i \neq k \)).

For this vector \( z \) we have

\[ \|Az\|_1 = \sum_i |a_{ik}| = \max \sum_j |a_{ij}|. \]

But

\[ \|A\|_1 = \max \|Az\|_1 \geq \|Az\|_1 = \sum_j |a_{ij}|, \tag{2} \]

combining the inequalities (1) and (2) we have

\[ \|A\|_1 = \max \sum_j |a_{ij}|. \]

(b) The induced matrix norm associated with 2 norm is

\[ \|A\|_2 = \{\text{maximum eigenvalue of } A^*A\}^{1/2}. \]

To prove this, we first observe that \( A^*A \) is Hermitian and positive definite, since \( (A^*A)^* = A^*A \) and

\[ x^* (A^*A)x = (Ax)^* (Ax) = \langle Ax, Ax \rangle \geq 0 \]

for all \( x \in \mathbb{C}^n \). It then follows from Theorem 0.3 and 0.9 that the eigenvalues of \( A^*A \) are real and nonnegative.

Since \( A^*A \) is Hermitian, thus a normal matrix, there exists a set of orthonormal right eigenvectors \( x_1, x_2, \ldots, x_n \) with associated eigenvalues \( \lambda_1, \lambda_2, \ldots, \lambda_n \). Now for any \( x \) with \( \|x\|_2 = 1 \) we write
Then
\[ x = \sum_{j=1}^{n} \xi_j x_j, \]
and hence
\[ (Ax)^* (Ax) = \sum_{j=1}^{n} \lambda_j \xi_j^* \xi_j. \]
Applying the fact that
\[ x_i x_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}, \]
we obtain
\[ \|Ax\|_2 = \left( \sum_{j=1}^{n} \lambda_j \right)^{1/2} \]
and since \( \|x\|_2 = 1 \) and \( x = \sum_{j=1}^{n} \xi_j x_j \), we have
\[ \|x\|_2 = \langle x, x \rangle = \sum_{i=1}^{n} \sum_{j=1}^{n} \xi_i \xi_j^* \xi_i \xi_j = \sum_{j=1}^{n} \lambda_j = 1 \]
by using the fact that \( \xi_j \)s are a set of orthonormal vectors.

Now \( \lambda_j \geq 0 \) for \( j = 1, 2, \ldots, n \), and it then follows that if
\[ \lambda_A = \max_j \lambda_j, \]
then
\[ \|Ax\|_2 = \left( \sum_{j=1}^{n} \lambda_j \right)^{1/2} \leq \left( \sum_{j=1}^{n} \lambda_A \right)^{1/2} = \lambda_A^{1/2}. \]
Hence
\[ \max_{\|x\|_2 = 1} \|Ax\|_2 \leq \lambda_A^{1/2}. \]
But if we let \( \lambda_A = \lambda_k \) and \( x = x_k \), then we have
\[ \|Ax\|_2 = \left( \sum_{j=1}^{n} \lambda_j \right)^{1/2} = \lambda_k^{1/2} = \lambda_A^{1/2} \]
since \( \xi_j = 0 \) for \( j \neq k \), and \( \xi_j = 1 \) for \( j = k \). Thus for \( x = x_k \)
we obtain the maximum of \( \|Ax\|_2 = \lambda_A^{1/2} \) with \( \|x\|_2 = 1 \). Hence
\[ \|A\|_2 = \text{Max} \|Ax\|_2 = \lambda_A^{1/2}. \]

(c) Now we consider the \( \infty \) norm. If \( \|z\|_\infty = 1 \), then \( \text{Max} |z_j| = 1 \).

thus we have

\[
\|Az\|_\infty = \text{Max} \sum_{i=1}^{n} a_{ij} |z_j| \leq \text{Max} \sum_{i=1}^{n} |a_{ij}| |z_j| \leq \text{Max} \left( \sum_{i=1}^{n} |a_{ij}| (\text{Max} |z_j|) \right) \\
\leq \text{Max} \sum_{i=1}^{n} |a_{ij}|.
\]

Hence

\[
\|Az\|_\infty = \text{Max} \frac{\|Az\|_\infty}{\|z\|_\infty} \leq \text{Max} \sum_{i,j} |a_{ij}|.
\]

Again suppose the value of \( i \) which gives the maximum sum of the right is \( i = k \). We can construct a vector \( z \) with \( z_j = +1 \) if \( a_{kj} \geq 0 \) and \( z_j = -1 \) if \( a_{kj} < 0 \). For this vector \( z \)

\[
\|Az\|_\infty = \sum_{j=1}^{n} |a_{kj}|.
\]

With the exact same argument as above (a), we may conclude that

\[
\|Az\|_\infty = \text{Max} \sum_{i,j} |a_{ij}|.
\]

Therefore, our proof is complete.

\[ \|A\|_2 \] is known as the spectral norm, and should be distinguished from the spectral radius \( \mu_A \). As an example shows:

**Example 3.1.** Let

\[ A = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \]

with \( k \) real, then

\[ A^* A = \begin{bmatrix} 1 & k \\ k & 1+k^2 \end{bmatrix} \]
and it is easy to see that \( \mu_A = 1 \) and \( \|A\|_2 = \left\{ 1 + \frac{1}{2} k^2 + \frac{1}{2} k(4 + k^2)^{1/2} \right\}^{1/2} \). Hence spectral radius is usually different from the spectral norm.

However the following theorem gives us the special case, where the spectral radius and spectral norm are the same:

**Theorem 3.4.** If \( A \) is a normal matrix, then \( \|A\|_2 = \mu_A \).

**Proof.** Since \( A \) is normal, it follows from Theorem 0.4 that \( A \) is unitarily similar to the diagonal matrix of its eigenvalues, that is, \( A = U^* D U \) where \( U = \text{diag}\{\mu_1, \mu_2, \ldots, \mu_n\} \) where \( \mu_i \)'s are the eigenvalues of \( A \). According to definition,

\[
\|A\|_2 = \lambda_A = \left\{ \text{maximum eigenvalue of } A^* A \right\}^{1/2},
\]

\[
A^* A = (U^* DU)^* (U^* DU) = U^* D^* D U,
\]

but this means \( A^* A \) is similar to \( D^* D \) and it follows from Theorem 0.5 that they have the same eigenvalues, and the eigenvalues of \( D^* D = (\mu_1 \overline{\mu}_1, \mu_2 \overline{\mu}_2, \ldots, \mu_n \overline{\mu}_n) = (|\mu_1|^2, |\mu_2|^2, \ldots, |\mu_n|^2) \). Hence

\[
\|A\|_2 = \lambda_A = \left\{ \text{Max}_j \ |\mu_j|^2 \right\}^{1/2} = \text{Max}_j \ |\mu_j| = \mu_A.
\]

Therefore, if \( A \) is normal, the spectral norm of \( A \) is equal to the spectral radius of \( A \). Furthermore, this is the case where the equality in Theorem 1.6 holds.

Up to now, we notice that the 1, 2, and \( \infty \) norms discussed
above depended only on the absolute values of the elements of the vector arguments, and such norms are called the absolute vector norms. In the following we shall discuss some properties concerning these kind of norms. The first is called the monotonic property. We will denote by $|x|$ the vector whose elements are the absolute values of the elements of $x$, and by $|x| \leq |y|$, for any $x, y \in \mathbb{C}_n$, we will mean $|x_j| \leq |y_j|$ for all $j = 1, 2, \ldots, n$. A vector norm $h$ will be called monotonic if and only if $|x| \leq |y|$ implies $h(x) \leq h(y)$. The second property we will discuss concerns the induced matrix norm. When a matrix norm is evaluated at a diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$, it would be very convenient if $|D| = \text{Max} \; |d_j|$. Here we will show that $\| \cdot \|$ has this property if and only if it is a norm induced by an absolute vector norm.

**Theorem 3.5.** If $h$ is a vector norm and $\| \cdot \|$ denotes the matrix norm induced by $h$, then the following conditions are equivalent:

(i) $h$ is absolute;

(ii) $h$ is monotonic;

(iii) $\|D\| = \text{Max} \; |d_j|$ for any diagonal matrix $D = \text{diag}(d_1, d_2, \ldots, d_n)$.

**Proof.** (i) implies (ii). Suppose $h$ is an absolute vector norm and we write $h(x) = H(x_1, x_2, \ldots, x_n)$ where $H$ is a function
of absolute values of $x_1, x_2, \ldots, x_n$. Now suppose we consider $H$ as a function of $x_1$, that is, leave $x_2, \ldots, x_n$ fixed, and let $H(x_1, x_2, \ldots, x_n) = f(x)$. By way of contradiction, suppose there exist nonnegative numbers $p$ and $q$ for which $p < q$ and $f(p) > f(q)$. But since $f(x) = f(|x|)$, we have $f(-q) = f(q)$.

Let $B_h$ be the set consisting of all vectors $x$ such that $h(x) \leq f(q)$, that is, $B_h = \{x \mid h(x) \leq f(q) \text{ for } x \in \mathbb{R}^n \}$. Now let $x, y \in B_h$, then $h(tx + (1 - t)y) \leq th(x) + (1 - t)h(y) \leq tf(q) + (1 - t)f(q)$ and therefore $B_h$ is convex. Let $y_1$ and $y_2$ be the vectors

$$y_1 = \begin{pmatrix} -q \\ x_2 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad y_2 = \begin{pmatrix} q \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

which belong to the set $B_h$, since $h(y_1) = h(y_2) = f(q)$, and consider the vector

$$z = \begin{pmatrix} p \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$z$ belongs to the line segment joining $y_1$ and $y_2$ obviously, and since $B_h$ is a convex set, it follows that $h(z) \leq f(q)$. But $h(z) = H(p, x_2, \ldots, x_n) = f(p) > f(q)$, this is a contradiction.

Each component would be considered in a similar fashion and hence when $h$ is an absolute vector norm, $H$ is a nondecreasing function, and consequently $h$ is monotonic.
(ii) implies (iii). We first suppose \( D \neq 0 \) and define
\[
d = \max_j |d_j|.
\]
Since
\[
Dx = \begin{bmatrix}
d_1x_1 \\
d_2x_2 \\
\vdots \\
d_nx_n
\end{bmatrix}
\]
and
\[
dx = \begin{bmatrix}
dx_1 \\
dx_2 \\
\vdots \\
dx_n
\end{bmatrix},
\]
clearly \(|Dx| \leq |dx|\) and since \( h \) is monotonic, this implies that
\[
h(Dx) \leq h(dx) = dh(x).
\]
But \( \| \cdot \| \) is the induced matrix norm by \( h \), thus
\[
\|D\| = \sup_{x \neq 0} \frac{h(dx)}{h(x)} \leq \sup_{x \neq 0} \frac{dh(x)}{h(x)} = d.
\]
Now suppose we attain \( d \) at \( d = |e_m| \), then
\[
\frac{h(De_m)}{h(e_m)} = \frac{|d_m h(e_m)|}{h(e_m)} = \frac{|d_m| h(e_m)}{h(e_m)} = |d_m| = d.
\]
This means we attain the maximum in the inequality
\[
\|D\| = \sup_{x \neq 0} \frac{h(Dx)}{h(x)} \leq d,
\]
at \( x = e_m \), and hence \( \|D\| = d \). Therefore when \( h \) is monotonic then
\[
\|D\| = d = \max_j |d_j|
\]
for any diagonal matrix \( D = \text{diag}(d_1, \ldots, d_n) \).

(iii) implies (i). For any vector \( x \) we can construct a diagonal matrix \( D \) in such a way that \( d_j = 1 \) if \( x_j = 0 \) and \( d_j = |x_j|/x_j \) if \( x_j \neq 0 \), so we have \( |d_j| = 1, j = 1, 2, \ldots, n \) and \( Dx = |x| \).

By (iii) we know that \( \|D\| = \|D^{-1}\| = \max_j |d_j| = 1 \), hence we have
\[ h(|x|) = h(Dx) \leq \|D\| h(x) = h(x), \]
and since \( x = D^{-1} |x| \)
\[ h(x) = h(D^{-1} |x|) \leq \|D^{-1}\| h(|x|) = h(|x|). \]
Combining these two inequalities we have
\[ h(x) = h(|x|), \]
and this simply means that \( h \) is an absolute vector norm, and our proof is complete.

**Theorem 3.6.** If \( \| \cdot \| \) is a matrix norm induced by an absolute vector norm \( h \), and \( |A| \) is the matrix with elements \( |a_{ij}| \), then \( \|A\| \leq \||A|\|. \)

**Proof.** Let \( x \) be any vector with \( h(|x|) = 1 \). Clearly we have
\[ A|x| \leq |A|x| = |Ax|, \]
and since \( h \) is absolute, by the previous theorem we have
\[ h(A|x|) \leq h(|Ax|), \]
but since \( \| \cdot \| \) is an induced matrix norm by \( h \), we have
\[ \|A\| = \operatorname{Max}_{h(x)=1} h(A|x|) \leq \operatorname{Max}_{h(x)=1} h(|A|x|) = \||A|\|, \]
hence \( \|A\| \leq \||A|\|. \)
In this section we will be examining a function defined on all square matrices \( A \) which is nonnegative and is zero if and only if \( \det A = 0 \). This function will give us a measure of departure of a matrix from singularity. Such a measure, called the lower bound of matrix \( A \), can be obtained from a definition similar to that of the induced matrix norm. We define the lower bound of \( A \) with respect to the vector norm \( h \).

We denote this lower bound of \( A \) by \( \text{glb}_h(A) \).

\[
\text{glb}_h(A) = \inf_{x \neq 0} \frac{h(Ax)}{h(x)}
\]

Using exactly the same kind of proof as in Lemma 3.1, we can obtain the following results:

**Lemma 4.1.** (i) \( \text{glb}_h(A) = \min_{h(x) = 1} h(Ax) \). (ii) There is a vector \( y_0 \), depending on \( A \), such that \( h(y_0) = 1 \) and \( \text{glb}_h(A) = h(Ay_0) \).

We will use the following theorems to justify our claim that this lower bound is a measure of departure from singularity.

**Theorem 4.2.** If \( \| \cdot \| \) is the matrix norm induced by the vector norm \( h \), then
glb_h(A) = \frac{1}{A^{-1}} \quad \text{if } \det A \neq 0 \\
0 \quad \text{if } \det A = 0

\textbf{Proof.} If } \det A \neq 0, \text{ define the vector } y \text{ in terms of } x \\
\text{by } y = Ax. \text{ For any } x \neq 0, \text{ we have } y \neq 0, \text{ since } A \text{ is nonsingular and}
\begin{align*}
glb_h(A) &= \inf h(Ax) = \inf h(y) = \left(\sup \frac{h(A^{-1}y)}{h(y)}\right)^{-1} \\
&= \frac{1}{\|A^{-1}\|}.
\end{align*}

But if } \det A = 0, \text{ then there exists a nonzero vector } z \text{ such} \\
\text{that } Az = 0 \text{ and } h(z) = 1, \text{ if not, we can always let } z_0 = z/h(z), \\
\text{then } Az_0 = 0 \text{ and } h(z_0) = 1. \text{ By Lemma 4.1, we have}
\begin{align*}
glb_h(A) &= \min h(Ax) = h(Az) = 0 \\
h(x) &= 1
\end{align*}
\text{and this minimum is attained when } x = z.

\textbf{Corollary 4.3.} If } A, B \in \mathbb{C}_{n \times n}, \text{ then } \text{glb}_h(AB) \geq \text{glb}_h(A) \cdot \text{glb}_h(B).

\textbf{Proof.} The proof follows immediately from the fact that}
\begin{align*}
\| (AB)^{-1} \| &= \| B^{-1} A^{-1} \| \leq \| B^{-1} \| \| A^{-1} \|.
\end{align*}

\textbf{Corollary 4.4.} If } h \text{ is an absolute norm and } D = \{d_1, d_2, \\
\ldots, d_n\}, \text{ then } \text{glb}_h(D) = \min \max |d_j| .

\textbf{Proof.} If } D \text{ is singular, then } d_j \neq 0 \text{ for all } j \text{ and}
\begin{align*}
\text{glb}_h(D) &= \frac{1}{\|D^{-1}\|} = \frac{1}{\max_j (1/|d_j|)} = \min_j |d_j| ,
\end{align*}
\text{by Theorem 3.5. Now if } D \text{ is singular, then } \text{glb}_h(D) = 0 \text{ by}
\text{Theorem 4.2, and clearly } \min_j |d_j| = 0.
Theorem 4.5. Let \( \mu_1, \mu_2, \ldots, \mu_n \) be the eigenvalues of \( A \), then for any vector norm \( h \), \( \text{glbh}(A) \leq \text{Min}_j |\mu_j| \). Equality holds if \( A \) is normal and \( h \) is the euclidean norm.

Proof. (1) If \( A \) is nonsingular, that is, \( \det A \neq 0 \), then by Theorem 4.2, we have

\[
\text{glbh}(A) = \frac{1}{\|A^{-1}\|}.
\]

But it follows from Theorem 1.6 that if \( \lambda_{-1} \) is the spectral radius of \( A^{-1} \), then for any matrix norm

\[
\|A^{-1}\| \geq \lambda_{-1}^{-1}.
\]

Hence we have

\[
\text{glbh}(A) = \frac{1}{\|A^{-1}\|} \leq \frac{1}{\lambda_{-1}} \leq \frac{1}{\lambda_j}
\]

for \( j = 1, 2, \ldots, n \), where \( \lambda_j \)'s are the eigenvalues of \( A^{-1} \).

By Theorem 0.6 we have that \( 1/\lambda_j \), \( j = 1, 2, \ldots, n \), are the eigenvalues of \( A \). Therefore, we have

\[
\text{glbh}(A) \leq 1/\lambda_j = \mu_j
\]

for \( j = 1, 2, \ldots, n \), where \( \mu_j \)'s are the eigenvalues of \( A \).

Consequently, \( \text{glbh}(A) \leq \text{Min}_j |\mu_j| \). If \( A \) is singular, \( \text{glbh}(A) = 0 \) and hence \( \text{glbh}(A) \leq \text{Min}_j |\mu_j| \) holds obviously.

(2) If \( A \) is normal, and \( h \) is the euclidean vector norm, then it follows from Theorem 4.2 that \( \text{glbh}(A) = 1/\|A^{-1}\| \), and from Theorem 3.4 that if \( A \) is normal and \( h \) is the euclidean norm, then the spectral radius and the spectral norm are the same, that is, \( \|A^{-1}\| = \lambda_{-1} \) where \( \lambda_{-1} \) is the spectral radius of \( A^{-1} \).
Hence
\[ \text{glb}_h(A) = \frac{1}{\|A^{-1}\|} = \frac{1}{\lambda_{\text{max}}}. \]

Again using Theorem 0.6 we have
\[ \frac{1}{\lambda_{\text{max}}} = \text{Min} |\mu_j| \]
for \( j = 1, 2, \ldots, n \), where \( \mu_j \)'s are the eigenvalues of \( A \).
Therefore, we have \( \text{glb}_h(A) = \text{Min} |\mu_j| \), and proof is complete.

By combining Theorem 4.5 with Theorem 1.6, it is easy to conclude that \( \text{glb}_h(A) \leq |\mu_j| \leq \|A\| \), where \( h \) is any vector norm and \( \| \cdot \| \) any matrix norm. This defines an annular region of the complex plane within which the eigenvalues of \( A \) must all lie.

With the help of Theorem 4.2 and 4.5 we can now explain the reason why the lower bound of \( A \) is a measure of departure from singularity. First, let \( \{A_i\} \) be a sequence of \( n \)-square matrices and suppose this sequence of matrices approaches a singular matrix, by this we simply mean that \( \det A_i \to 0 \) as \( i \to \infty \). But Theorem 0.7 states that the \( \det A \) is the product of the eigenvalues of \( A \), hence, as \( \det A_i \to 0 \) at least one of the eigenvalues of \( A \) approaches zero, or \( \text{Min} |\mu_j| \to 0 \) where \( \mu_j \)'s are the eigenvalues of \( A \). It follows from Theorem 4.5 that \( \text{glb}_h(A) \leq \text{Min} |\mu_j| \); hence we have \( \text{glb}_h(A) \to 0 \). Thus Theorem 4.2 actually defines a measure of departure of a matrix from singularity.
CHAPTER V

THE FIELD OF VALUES

If A is n-square matrix over the complex numbers, the eigenvalues of A form a set of n points in the complex plane. Some of the most useful ideas concerning the distribution of these points can be developed from the concept of the "field of values" of A. Furthermore, all the eigenvalues of A lie in the field of values of A, which will be shown later. We define the field of values, $F(A)$, of $A \in \mathbb{C}^{n \times n}$ to be the set of all numbers

$$x^* Ax = \langle x, Ax \rangle = \sum_{i,j} a_{ij} x_i^* x_j$$

where $x \in \mathbb{C}^n$ and $h(x) = 1$. Throughout this section $h$ will denote the Euclidean vector norm.

Since $F(A)$ is the image of $\mathbb{C}_h$ under a continuous function, it follows from Theorem 0.8 that $F(A)$ is a closed and bounded set in the complex plane. Furthermore, it can be easily proved that each eigenvalue $\mu_j$ of A is in $F(A)$. Suppose $\mu_j$, $j = 1, 2, \ldots, n$, are eigenvalues of A, and $x_j$ is the corresponding right eigenvector of A such that $h(x_j) = 1$, we then have $Ax_j = \mu_j x_j$.

But $x_j^* Ax_j = x_j^* \mu_j x_j = \mu_j x_j^* x_j$, and $x_j^* x_j = h(x_j) = 1$, hence we have $\mu_j = x_j^* Ax_j \in F(A)$.

Theorem 5.1. The field of values of a matrix A is invariant
under a unitary similarity transformation, that is, if U is unitary, then \( F(A) = F(UA^*U^*) \).

**Proof.** If \( z \in F(UA^*U^*) \), there is an \( x \in \mathbb{C}^n \) with \( h(x) = 1 \) and \( z = x^* UAU^* x = y^*Ay \) where \( y = U^*x \). But we proved that the euclidean vector norm is invariant under unitary transformation, Theorem 1.7, hence \( h(y) = 1 \) and so \( z \in F(A) \). Hence \( F(UA^*U^*) \subseteq F(A) \). With the same type of argument we can get the inclusion going the other way, and so \( F(UA^*U^*) = F(A) \).

Next we want to characterize the field of values of a normal matrix geometrically. To accomplish this task, we first will introduce the following definition and theorem:

**Definition 5.1.** Let \( Z_k \) be a set of \( k \) distinct vectors in \( \mathbb{C}^n \). The smallest convex set \( H(Z_k) \subseteq \mathbb{C}^n \) with \( Z_k \subseteq H(Z_k) \) is called the convex hull of \( Z_k \). By "smallest" we mean that if \( Z_k \subseteq T \subseteq \mathbb{C}^n \) and \( T \) is convex, then \( H(Z_k) \subseteq T \).

**Theorem 5.2.** If \( z_1, z_2, \ldots, z_k \) are the numbers of \( Z_k \), then \( z \in H(Z_k) \) if there are numbers \( \alpha_i \geq 0, i = 1, 2, \ldots, k \), for which

\[
\sum_{i=1}^{k} \alpha_i = 1 \quad \text{and} \quad z = \sum_{i=1}^{k} \alpha_i z_i.
\]

**Proof.** Let \( T_k \) be the set of vectors of the form \( \sum_{i=1}^{k} \alpha_i z_i \).
with $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$. Let $x, y$ be any two elements of $T_k$

then $x = \sum_{i=1}^{k} \alpha_i z_i$ and $y = \sum_{i=1}^{k} \beta_i z_i$ with $\alpha_i, \beta_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = \sum_{i=1}^{k} \beta_i = 1$. The line segment joining $x$ and $y$ is $z = tx + (1 - t)y$, $0 \leq t \leq 1$. To prove that $T_k$ is convex, we will show that

$z \in T_k$:

$$z = tx + (1 - t)y = t \sum_{i=1}^{k} \alpha_i z_i + (1 - t) \sum_{i=1}^{k} \beta_i z_i$$

$$= \sum_{i=1}^{k} (t \alpha_i + (1 - t) \beta_i) z_i.$$  

But $t \alpha_i + (1 - t) \beta_i \geq 0$ for all $i = 1, 2, \ldots, k$, and

$$\sum_{i=1}^{k} (t \alpha_i + (1 - t) \beta_i) = t \sum_{i=1}^{k} \alpha_i + (1 - t) \sum_{i=1}^{k} \beta_i = t + (1 - t) = 1,$$

hence $z \in T_k$. Therefore, $T_k$ is a convex set, and $Z_k \subseteq T_k$.

To finish the proof, we need to show that $T_k = H(Z_k)$. Let $T$ be any convex set in $C_n$ such that $Z_k \subseteq T$, then we will show, by induction on $k$, that $T_k \subseteq T$. When $k = 2$, let $x \in T_2$, then

$$x = a_1 z_1 + a_2 z_2 = a_1 z_1 + (1 - a_1) z_2 \in T$$

where $a_1 + a_2 = 1$ and $a_1, a_2 \geq 0$. Hence $T_2 \subseteq T$. It remains to show that $T_{k-1} \subseteq T$ and $z_k \in T$ imply $T_k \subseteq T$. Choose an arbitrary $x \in T_k$, then

$$x = a_1 z_1 + a_2 z_2 + \ldots + a_k z_k$$

$$= (1 - \frac{a_k}{1}) \left( \sum_{i=1}^{k-1} \frac{1}{1 - a_k} \right) \sum_{i=1}^{k} \alpha_i z_i + \frac{a_k}{1} z_k$$
\[ \sum_{i=1}^{k} \alpha_i = 1. \]
Since
\[ \frac{1}{1 - \alpha_k} \sum_{i=1}^{k-1} \alpha_i = \frac{1}{1 - \alpha_k} (1 - \alpha_k) = 1, \]
it follows that
\[ \frac{1}{1 - \alpha_k} \sum_{i=1}^{k-1} \alpha_i z_i \in T, k-1 \subseteq T. \]
Also since \( z_k \in T \) and \( T \) is convex, \( x \in T \) follows immediately and hence the proof is complete.

The geometric characterization of the field of values of a normal matrix is well-explained in the following two theorems:

**Theorem 5.3.** The field of values of a normal matrix coincides with the convex hull of the eigenvalues.

**Proof.** If \( A \) is a normal matrix with eigenvalues \( \mu_1, \mu_2, \ldots, \mu_n \), then by Theorem 0.4 there is a unitary matrix \( U \) such that \( A = UDU^* \) where \( D = \text{diag} \{ \mu_1, \mu_2, \ldots, \mu_n \} \). According to Theorem 5.1, we have \( F(A) = F(UAU^*) = F(D) \). By definition, \( z \in F(A) \) if and only if there is an \( x \) with \( h(x) = 1 \) and
\[ z = x^* D x = \sum_{j=1}^{n} |x_j|^2 \mu_j. \]
Since \( h^2(x) = \sum_{j} |x_j|^2 = 1 \), it follows from Theorem 5.2 that \( z \) is in the convex hull of the eigenvalues. Hence \( F(A) \subseteq H(Z_n) \).

Now if \( Z_n \) is the set of eigenvalues of \( A \). Since each eigenvalue is in \( F(A) \) and \( F(A) \) is obviously a convex set when \( A \) is normal,
we may conclude from the definition of convex hull that
\[ F(A) \supseteq H(Z_n) \]. This completes the proof.

**Theorem 5.4.** The field of values of the matrix \( A \) is an interval of the real line if and only if \( A \) is Hermitian.

**Proof.** First if \( A \) is Hermitian, that is, \( A^* = A \), then \( A^* A = AA^* \) and hence \( A \) is normal. By the preceding theorem, \( F(A) \) is the convex hull of the eigenvalues of \( A \), but Theorem 0.9 states that the eigenvalues of \( A \) are all real, and so their convex hull is obviously an interval of the real line. Now, conversely, if \( A \in C_{\text{n} \times \text{n}} \), we write
\[
B = \frac{1}{2} (A + A^*) \quad \text{and} \quad C = -\frac{1}{2} i(A - A^*),
\]
obviously \( B = B^*, C = C^*, \) and \( A = B + iC \). Then
\[
x^* Ax = x^* (B + iC) = x^* Bx + i x^* Cx
\]
where \( x^* Bx \) and \( x^* Cx \) are all real. If \( F(A) \) consists only of real numbers, which implies \( x^* Cx = 0 \) for every \( x \in C_n \), then \( C = 0 \). But \( C = 0 \) implies \( A = A^* \). Therefore, \( A \) is Hermitian and the proof is complete.
REFERENCES


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