A Non-Parametric Sequential Signed-Rank Test and Comparison with the Sequential T-Test

Kuei-Mei Even Sher
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A NON-PARAMETRIC SEQUENTIAL SIGNED-RANK TEST
AND COMPARISON WITH THE SEQUENTIAL T-TEST

by

Kuei-Mei Even Sher

A report submitted in partial fulfillment
of the requirements for the degree
of
MASTER OF SCIENCE
in
Applied Statistics
Plan B

Approved:

UTAH STATE UNIVERSITY
Logan, Utah

1971
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Finally, to my fiance, C. C. Chien, for his encouragement, I extend my appreciation.

Kuei-Mei Even Sher
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INTRODUCTION

Sequential analysis is a method of statistical inference whose characteristic feature is that the number of observations required by the procedure is not determined in advance of the experiment. The decision to terminate the experiment depends, at each stage, on the results of the observations previously made.

So far the general sequential t-test (without truncation) is a standard test for the mean of a normal distribution. A truncated sequential t-test has been developed by Suich and Iglewicz (1970). The difference between these two tests is that the former has a fixed critical value whenever the type I error ($\alpha$) and type II error ($\beta$) are given while the latter has different boundaries whenever the sample size changes. A non-parametric sequential signed-rank test was published in December (Miller, 1970). This test works for testing the mean of a symmetric distribution. It seems that the sequentila signed-rank test is more convenient and robust than other sequential tests, especially under a distribution with thicker tails.

The purpose of this paper therefore is to explain and compare the three sequential tests under three different distributions; that is, the normal distribution, the double exponential distribution (with thicker tails than the normal distribution), and the uniform distribution (with no tails). The results of these tests indicate that the non-parametric test may not be as powerful as the truncated t-test for some non-normal data. Examples will be given to explain the use of
the tests to make them easier to understand. Since the ASN (average sample number) and power are two important quantitative values for the sequential test, in section III, the comparison is therefore based on these two values. Monte Carlo methods will be used to investigate the power and stopping time distribution for each test.
DISCUSSION OF SEQUENTIAL TESTS IN GENERAL

Sequential t-Test

The benefits of sequential sampling have been applied to the t-test by Wald (1947), Rushton (1950) and others.

Let $x_1, x_2, \ldots, x_n$ be a sequence of independent normal distributed random variables with mean $\mu$ and variance $\sigma^2$. If we wish to test a hypothesis

$$H_0: \delta = 0$$

against

$$H_1: \delta = \delta_1 (\delta_1 > 0)$$

with risks of error $\alpha$ when $H_0$ is true and $\beta$ when $H_1$ is true where $\delta = \mu/\sigma$ and $\sigma$ is unknown, the general theory of sequential tests tells us to take observations sequentially and at each stage calculate the likelihood ratio $L$.

If

$$\beta/(1-\alpha) < L < (1-\beta)/\alpha$$

then we continue to take observations. If the right-hand inequality is broken we accept $H_1$, while if the left-hand inequality is broken we accept $H_0$. In the normal case, the likelihood function is

$$\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} \exp \left( - \frac{(x_i - \mu)^2}{2\sigma^2} \right) = \left( \frac{1}{\sqrt{2\pi}\sigma} \right)^n \exp \left( - \frac{(n-1)S^2 + n(x_\cdot - \mu)^2}{2\sigma^2} \right)$$

where $nx_\cdot = \Sigma x_i$ and $(n-1)S^2 = \Sigma_{1}^{n} (x_i - x_\cdot)^2$. 
Let \( t = \sqrt{nxs/s} \), the ratio \( t \) is unaltered by a transform \( X + aX \), that is, when \( X + aX \), \( \sigma \) becomes \( a\sigma \), \( s \) becomes \( as \), and \( au = a\delta \sigma \) so that any given value of \( \sigma \) can be changed into any other given value by a suitable choice of \( a \). Under these circumstances we can obtain a sequential test of \( H_0 \) against \( H_1 \) by considering only the distribution of \( t \). On \( H_1 \), \( t \) has the non-central \( t \) distribution with \( (n-1) \) degrees of freedom with parameter \( \delta_1 \), the probability density function (Johnson and Welch, 1940) being:

\[
\phi(t/\delta_1, n) = \frac{\Gamma(n) \exp\left[-\frac{1}{2} n(n-1) \delta_1^2/(n-1 + t^2)\right]}{2^{(n-2)/2} \Gamma((n-1)/2) \sqrt{\pi(n-1)}} \left(\frac{n-1}{n-1 + t^2}\right)^{n/2} \frac{H_{n-1}(-\delta_1 u)}{H_n(-\delta_1 u)} \tag{1}
\]

where

\[
u = \frac{t}{\sqrt{n/(n-1 + t^2)}} \tag{2}
\]

and

\[
H_n(x) = \int_0^\infty (z^n/n!) \exp\left[-\frac{1}{2} (z + x)^2\right] \, dz. \tag{3}
\]

On \( H_0 \), \( t \) has the probability density function \( \phi(t/\delta, n) \) or the central \( t \) distribution with \( (n-1) \) degrees of freedom, so that the likelihood ratio test criterion is

\[
L_n(t/\delta_1, \delta_1) = \phi(t/\delta_1, n) / \phi(t/\delta, n). \tag{4}
\]

In practice, it is easier to run the sequential procedure in terms of the logarithm of the likelihood ratio, i.e., to calculate

\[
\ln L_n(t/\delta, \delta_1) \text{ at each stage, and if}
\]

\[
\ln \delta/(1-\alpha) < \ln L_n(t/\delta, \delta_1) < \ln (1-\beta)/\alpha
\]

we take further observation, while if the right-hand inequality is
broken we accept $H_1$, and if the left-hand inequality is broken we accept $H_0$. Thus in practice the problem of carrying out the test reduces to the problem of evaluating $\ln L = \ln L_n(t/\delta, \delta_1)$ as a function of $t$, for each value of $n$.

In practice it is easier to work with $u$ than with $t$, since from [2] we find

$$u = \frac{\sum x_i}{\sqrt{(\sum x_i^2)}}$$

and $u$ can thus be calculated directly provided at each stage we keep a record of the cumulative sum $\sum x_i$ and the cumulative sum of squares $\sum x_i^2$. In terms of $u$, making the substitution from [1] into [4], after some reduction we find,

$$\ln L = G_n(\delta_1 u) - G_n(\delta u) - n(\delta_1^2 - \delta^2)/2$$

where

$$G_n(x) = x^2/2 - \ln Y_n(x)$$

and

$$Y_n(x) = \frac{H_n(-x)}{H_n(0)}.$$

By the proof of Rushton (1950), we obtain an approximation for

$$G_n(x) = x^2/4 + \sqrt{n} x (1 - 1/(4n) + x^2/(24n))$$

to make the procedure easier to calculate.

It is always an important matter, before carrying out a sequential test procedure, to obtain some estimate of the average sample number associated with it. In our case we use the formula given in Wald (1947) for the average sample number.
Let $N$ be an integer sufficiently large to allow the probability that $n > N$ to be neglected. Thus we shall assume that $n < N$. Then we can write

$$z_1 + z_2 + \cdots + z_n = (z_1 + z_2 + \cdots + z_n) + (z_{n+1} + z_{n+2} + \cdots + z_N) \quad [5]$$

where

$$z_\alpha = \ln \frac{f(x, \theta_1)}{f(x, \theta_0)}.$$

Taking expected values on both sides of [5], we obtain

$$N \ E(z) = E(z_1 + z_2 + \cdots + z_n) + E(z_{n+1} + z_{n+2} + \cdots + z_N). \quad [6]$$

Since, for $\alpha > n$, the random variable $z$ is distributed independently of $n$, the expected value of $z_{n+1} + z_{n+2} + \cdots + z_N$ is equal to the expected value of $(N-n)$ times the expected value of a single $z$, i.e.,

$$E(z_{n+1} + z_{n+2} + \cdots + z_N) = E(N-n) \ E(z)$$

$$= N \ E(z) - E(n) \ E(z) \quad [7]$$

from [6] and [7] it follows that

$$E(z_1 + z_2 + \cdots + z_n) = E(n) \ E(z) = 0.$$ 

Hence,

$$E(n) = \frac{E(z_1 + z_2 + \cdots + z_n)}{E(z)} \quad n \quad \text{if } E(z) \neq 0. \quad [8]$$

If $\theta$ is the true value of the parameter, then $E(n) = E_\theta(n)$ by the definition of the symbol $E_\theta(n)$.

If the excess of the probability ratio $P_{1m}/P_{0m}$ over the boundaries $A$ and $B$ ($A = (1-\beta)/\alpha$, $B = \beta/(1-\alpha)$) at the termination of the sequential
process is neglected, the random variable \(z_1 + z_2 + \cdots + z_n\) can take only the values \(\ln A\) and \(\ln B\) with the probabilities \(1-p(\theta)\) and \(p(\theta)\), respectively. Hence,

\[
E(z_1 + z_2 + \cdots + z_n) = p(\theta) \ln B + (1-p(\theta)) \ln A. \quad [9]
\]

From [8] and [9] we obtain the approximation formula,

\[
E_\theta(n) = p(\theta) \ln B + (1-p(\theta)) \ln A / E_\theta(z).
\]

In the normal case, i.e., when

\[
f(x, \theta) = \frac{1}{\sqrt{(2\pi)\sigma}} \exp \left(-\frac{(x-\theta)^2}{2\sigma^2}\right),
\]

we have

\[
z = \ln \frac{f(x, \theta_1)}{f(x, \theta_0)} = \frac{1}{2\sigma^2} (2(\theta_1 - \theta_0)x + \theta_0^2 - \theta_1^2).
\]

Hence

\[
E_\theta(z) = \frac{1}{2\sigma^2} (2(\theta_1 - \theta_0)x + \theta_0^2 - \theta_1^2)
\]

where

\[
p(\theta) = \frac{((1-\beta)/\alpha) h(\theta) - 1}{((1-\beta)/\alpha) h(\theta) - (\beta/(1-\alpha)) h(\theta)}
\]

\[
h(\theta) = (\theta_1 + \theta_0 - 2\theta)/(\theta_1 - \theta_0).
\]

**Truncated Sequential t-Test**

Although the test described above usually has a smaller ASN (Average Sample Number) than the equivalent fixed-sample procedure, there still remains the probability that an extremely large sample size
will be necessary to make a decision. To remedy this, Truncated Sequential t-Tests were proposed by Armitage and Schneiderman (1962), and Suich and Iglewicz (1970).

Let \( x_1, x_2, \ldots, x_n \) be a sequence of independent normally distributed random variables with mean \( \mu \) and variance \( \sigma \). We want to test the hypothesis

\[
H_0 : \delta = 0
\]

against

\[
H_1 : \delta = \delta_1 (\delta_1 > 0)
\]

where \( \delta = \mu / \sigma \) and \( \sigma \) is unknown.

Let

\[
t_k = \frac{x_k}{s_k} \quad k = 2, 3, 4, \ldots
\]

\[
x_k = \frac{1}{k} \sum_{i=1}^{k} x_i
\]

and

\[
s_k^2 = \frac{\sum (x_i - x_k)^2}{k-1}
\]

We will consider a continuation region of the form

\[
-c + dn < \frac{1}{\delta_1} \ln \left( \frac{L_1}{L_0} \right) < c-dn \quad [1]
\]

where \( c \) and \( d \) are positive and \( L_1 \) is the likelihood function of the sequence \( (t_1, t_2, \ldots, t_n) \) under \( H_1 \). Such boundaries lead to a maximum possible sample size at the intersection of the two lines \((-c + dn \) and \( c-dn)\), namely \((c/d)\). For known \( \sigma \), one can use the method
of Anderson (1960) to obtain c and d. The constants corresponding to some other value of \( \delta_1 \), say \( \delta'_1 \), are calculated (for \( \alpha = \beta = 0.05 \)) as follows:

\[
w = \delta'_1 / 0.2 \quad c = 19.905 / w \quad \text{and} \quad d = 0.03316 w.
\]

We make the assumption that Anderson's boundaries would yield reasonable results even when \( \sigma \) is unknown. This assumption makes it possible to use continuation region (1).

It is known that

\[
\ln \left( \frac{L_1}{L_0} \right) = \ln \left[ \frac{f(t_n; \delta_1)}{f(t_n; 0)} \right]
\]

where \( f(t_n; \delta) \) is the non-central t distribution with parameter \( \delta \). The above equation reduces to

\[
\ln \left( \frac{L_1}{L_0} \right) = -\frac{\delta_1^2 (n-u_n^2)}{2} + \ln H_{n-1}(-\delta_1 u) - \ln H_{n-1}(0)
\]

[2]

where

\[
u_n = \frac{nt_n}{\sqrt{(n - 1 + nt_n^2)}} = \frac{\sum_{i=1}^{n} x_i^2}{\sqrt{n} \sum_{i=1}^{n} x_i^2}.
\]

A further simplification can be achieved by noticing that \( u_n \) is a monotonic function of \( \ln \left( \frac{L_1}{L_0} \right) \). This makes it possible to convert continuation region (1) to the following equivalent form:

\[
\phi_L(u_n) < u_n < \phi_u(u_n)
\]

[3]

\( \phi_L(u_n) \) is obtained by solving

\[
-c + dn = \frac{1}{\delta_1} \ln \left( \frac{L_1}{L_0} \right)
\]

[4]
for \( u_n \). For this purpose, the following approximation is used.

\[
H_{n-1}(-\delta_n u_n) \approx \frac{s^{n-1}}{(n-1)!} \exp \left[ -\left( s - \delta_n u_n \right)^2/2 \right] 
\times \left[ \frac{(2\pi s^2) / (s^2 + n-1)}{(s^2 + n-1)} \right]^{\frac{2n}{2}} \left[ 1 - \frac{3(n-1)}{4(s^2 + n-1)^2} + \frac{5(n-1)^2}{6(s^2 + n-1)^3} \right] \tag{5}
\]

where

\[
s = \delta_n u_n + \frac{[\delta_n^2 u_n^2 + 4(n-1)]^{\frac{1}{2}}}{2}
\]

\[
H_{n-1}(0) = \frac{2^{n/2-1}}{(n-1)!} \frac{\Gamma(n/2)}{\Gamma(n/2)} \tag{6}
\]

Applying equations [2], [5], and [6] to equation [4] results in

\[
(-c + d_n) = (n-1) \ln s - ((n-2)/2) \ln 2 - \ln \Gamma(n/2) - \left( s - \delta_n u_n \right)^2/2
\]

\[
+ \frac{1}{2} \ln \left( \frac{(2\pi s^2) / (s^2 + n-1)}{(s^2 + n-1)} \right) + \ln \left[ 1 - \frac{3(n-1)}{4(s^2 + n-1)^2} \right]
\]

\[
+ \frac{5(n-1)^2}{6(s^2 + n-1)^3} \right] \frac{n\delta_n^2}{2} + \frac{\delta_n^2 u_n^2}{2}. \tag{7}
\]

The upper limits, \( \Phi_{u_n} \), are equivalently obtained by equating \( \delta_n (c-d_n) \) to the right-hand side of Equation [7].

For the case \( \alpha = \beta = 0.05 \), Table 1 gives values of \( \Phi_{u_n} \) and \( \phi_{u_n} \) for \( \delta_n = 1.0 \). Another table for \( \delta_n = 0.5 \) used in section III of this paper will be attached in Appendix I. Additional tables for \( \delta_n = 0.3 \) (0.1) 1.2., \( \alpha = \beta = 0.05; \alpha = \beta = 0.01 \) and the extension of \( \alpha \neq \beta \) are worked out by Suich (1970, unpublished).
Table 1. Truncated Sequential $t$-Test Table of boundary values with $\alpha = \beta = 0.05 \quad \delta_1 = \mu/\sigma = 1$

<table>
<thead>
<tr>
<th>$n$</th>
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<th>$\phi_u(U_{n})$</th>
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<td>3</td>
<td>-1.483</td>
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<td>20</td>
<td>1.898</td>
<td>2.139</td>
</tr>
<tr>
<td>21</td>
<td>1.979</td>
<td>2.156</td>
</tr>
<tr>
<td>22</td>
<td>2.058</td>
<td>2.173</td>
</tr>
<tr>
<td>23</td>
<td>2.134</td>
<td>2.190</td>
</tr>
</tbody>
</table>
The sampling procedure on each trail is:

a) accept \( H_0 \) if \( u_n \leq \Phi_L(u_n) \)
b) reject \( H_0 \) if \( u_n \geq \Phi(u_n) \)
c) continue otherwise.

If the maximum \( n \) given in Table 1 is reached and a decision has not been made, take another observation and accept \( H_0 \) if \( u_n \leq \Phi \) and reject otherwise. We will call this test procedure a truncated test.

Monte Carlo studies were run to estimate the average sample number for the Truncated Sequential t-Test. These methods appear in Appendix IV (A) and (B).

**Sequential Signed-Rank Test**

Let \( x_1, x_2, \ldots, x_n \) be a sequence of independent random variables, identically distributed according to the cdf \( F \). Assume that \( F \) has a density \( f \) which is symmetric about \( \theta \). The problem is to test the null hypothesis \( H_0 : \theta = 0 \) against the alternative \( H_1 : \theta \neq 0 \).

Let \( S_{+} \) be the sum of the positive ranks for the sample \( x_1, x_2, \ldots, x_n \). (Let \( R_{ni} \) be the rank of \( /x_i/ \) in \( /x_1/, \ldots, /x_n/ \); let \( x_i^+ = 1 \) if \( x_i > 0 \), 0 if \( x_i < 0 \). Then \( S_{+} = \sum_{i=1}^{n} x_i^+ R_{ni} \).

1. Continue sampling as long as
   
   \[
   |S_{+} - n(n+1)/4| < |z| \sqrt{\frac{n(n+1)(2n+1)}{24}}
   \]
   
   and
   
   \[ n < N \]

2. Stop sampling as soon as (i) or (ii) is violated.
   
   a) If (i) is violated, decide in favor of \( H_1 \).
   
   b) If (ii) is violated and not (i), decide in favor of \( H_0 \).
N and α are selected by the investigator; these determine $|z|_N^\alpha$ (see Table 2). The test thus consists of sampling until $SR^+_n$ drifts too many standard deviations $\sqrt{n(n+1)(2n+1)/24}$ away from its mean $n(n+1)/4$ under $H_0$ or until truncation point $N$ is reached.

In practice, $SR^+$ can be easily calculated for a sequence of observations from the following representation:

$$SR^+_n = \sum_{j=1}^{n} \sum_{i=1}^{j} (x_i + x_j)^+$$

$$= SR^+_{n-1} + \sum_{i=1}^{n} (x_i + x_n)^+$$

where

$$(x_i + x_j)^+ = \begin{cases} 1 & \text{if } x_i + x_j > 0 \\ 0 & \text{if } x_i + x_j < 0 \end{cases}$$

The Monte Carlo method is used to investigate ASN for Sequential Signed-Rank Test. The details were stated in Section III of this paper.

Table 2. Values of $|z|_N^\alpha$

<table>
<thead>
<tr>
<th>α</th>
<th>10</th>
<th>15</th>
<th>20</th>
<th>25</th>
<th>30</th>
<th>40</th>
<th>50</th>
<th>60</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.10</td>
<td>2.02</td>
<td>2.16</td>
<td>2.20</td>
<td>2.22</td>
<td>2.28</td>
<td>2.33</td>
<td>2.37</td>
<td>2.39</td>
</tr>
<tr>
<td>0.05</td>
<td>2.20</td>
<td>2.39</td>
<td>2.40</td>
<td>2.46</td>
<td>2.55</td>
<td>2.55</td>
<td>2.62</td>
<td>2.62</td>
</tr>
<tr>
<td>0.01</td>
<td>2.55</td>
<td>2.75</td>
<td>2.83</td>
<td>2.91</td>
<td>2.93</td>
<td>3.03</td>
<td>3.03</td>
<td>3.07</td>
</tr>
</tbody>
</table>

The table values were derived from the following relation:
Let \( Y_n = \frac{SR^+ - n(n+1)/4}{\sqrt{n(n+1)(2n+1)/24}} \)

and \(|Z|_N = \max_{1 \leq n \leq N} \{ |Y_n| \} \)

For the rejection boundary on \( SR^+_n - n(n+1)/4 \) defined by

\[ +z \sqrt{n(n+1)(2n+1)/24} \]

the test will decide in favor of \( H_1 \) if and only if \(|Z|_n > z\). Thus, \( z \) should be the upper \( \alpha \)-percential point of the distribution of \(|Z|_N\), i.e.,

\[ z = |z|_n^\alpha \text{ where } P( |Z|_n \geq |z|_n^\alpha ) = \alpha \].

Since in Section III we need \(|z|_n^\alpha\) values for \( N = 70, 80, \) and \( 90 \), a program was attached in appendix II to generate \(|z|_n^\alpha\) values by using the method described above. 4150 trials were run for \( N = 70 \), 4300 trials were run for \( N = 80 \), and 6400 trials run for \( N = 90 \) to generate the \(|z|_n^\alpha\) values as follow:

Table 3. Values of \(|z|_n^\alpha\)

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( N )</th>
<th>70</th>
<th>80</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.01</td>
<td>2.40</td>
<td>2.42</td>
<td>2.46</td>
<td></td>
</tr>
<tr>
<td>0.05</td>
<td>2.65</td>
<td>2.66</td>
<td>2.68</td>
<td></td>
</tr>
<tr>
<td>0.01</td>
<td>3.07</td>
<td>3.12</td>
<td>3.13</td>
<td></td>
</tr>
</tbody>
</table>
EXAMPLES FOR G.T., T.T., AND SR$^+$ TESTS

In this section, three examples are given to make the theory described above easier to understand. The symbols G.T., T.T., and SR$^+$ express the General sequential t-test, the Truncated sequential t-test, and Sequential Signed-Rank test. The numbers were used to generate random variable $x_i$ with normal distribution centered at 1, $N(1,1)$.

On G.T. test, the critical values are $\ln (1-\beta)/\alpha$ and $\ln \beta/(1-\alpha)$, that is, $\pm 2.94$ when $\alpha = \beta = 0.05$. On T.T. test, the boundary values come from Table 1 of this paper while on SR$^+$ test, the critical value comes from Table 2. By using the same calculation procedure expressed in this section, we may have any test on other distributions with a shift mean, such as the Uniform distribution and the Double Exponential distribution, etc. The programs used to generate random variables of these three distributions with a shift mean are attached in appendix III.
Table 4. An example for G. T. test with $X_i \sim N(1,1)$, $\alpha = \beta = 0.05$

<table>
<thead>
<tr>
<th>$n$</th>
<th>$X_i$</th>
<th>$\Sigma X_i$</th>
<th>$\Sigma X_i^2$</th>
<th>$\sqrt{\Sigma X_i^2}$</th>
<th>$U_n$</th>
<th>$G(\delta, u_n)$</th>
<th>$\frac{1}{2}n(\delta_1^2 - \delta_2^2)$</th>
<th>$\ln L$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.185</td>
<td>0.185</td>
<td>0.0342</td>
<td>0.185</td>
<td>1.000</td>
<td>1.042</td>
<td>0.5</td>
<td>0.542</td>
</tr>
<tr>
<td>2</td>
<td>2.082</td>
<td>2.267</td>
<td>4.3689</td>
<td>2.090</td>
<td>1.085</td>
<td>1.674</td>
<td>1.0</td>
<td>0.674</td>
</tr>
<tr>
<td>3</td>
<td>-0.123</td>
<td>2.144</td>
<td>4.3840</td>
<td>2.094</td>
<td>1.024</td>
<td>1.914</td>
<td>1.5</td>
<td>0.414</td>
</tr>
<tr>
<td>4</td>
<td>0.263</td>
<td>2.407</td>
<td>5.0757</td>
<td>2.253</td>
<td>1.068</td>
<td>2.292</td>
<td>2.0</td>
<td>0.292</td>
</tr>
<tr>
<td>5</td>
<td>-1.450</td>
<td>0.957</td>
<td>7.1782</td>
<td>2.679</td>
<td>0.357</td>
<td>0.763</td>
<td>2.5</td>
<td>-1.737</td>
</tr>
<tr>
<td>6</td>
<td>1.035</td>
<td>1.992</td>
<td>8.2494</td>
<td>2.872</td>
<td>0.694</td>
<td>1.756</td>
<td>3.0</td>
<td>-1.244</td>
</tr>
<tr>
<td>7</td>
<td>3.379</td>
<td>5.371</td>
<td>19.6670</td>
<td>4.435</td>
<td>1.211</td>
<td>3.484</td>
<td>3.5</td>
<td>-0.016</td>
</tr>
<tr>
<td>8</td>
<td>-0.019</td>
<td>5.352</td>
<td>19.6674</td>
<td>4.435</td>
<td>1.207</td>
<td>3.696</td>
<td>3.6</td>
<td>-0.304</td>
</tr>
<tr>
<td>9</td>
<td>0.352</td>
<td>5.704</td>
<td>19.7913</td>
<td>4.449</td>
<td>1.282</td>
<td>4.179</td>
<td>4.5</td>
<td>-0.321</td>
</tr>
<tr>
<td>10</td>
<td>-0.506</td>
<td>5.198</td>
<td>20.0473</td>
<td>4.477</td>
<td>1.161</td>
<td>3.937</td>
<td>5.0</td>
<td>-1.063</td>
</tr>
<tr>
<td>11</td>
<td>1.269</td>
<td>6.467</td>
<td>21.6577</td>
<td>4.654</td>
<td>1.390</td>
<td>5.023</td>
<td>5.5</td>
<td>-0.477</td>
</tr>
<tr>
<td>12</td>
<td>1.779</td>
<td>8.246</td>
<td>24.8225</td>
<td>4.982</td>
<td>1.655</td>
<td>6.353</td>
<td>6.0</td>
<td>0.353</td>
</tr>
<tr>
<td>13</td>
<td>0.741</td>
<td>8.987</td>
<td>25.3716</td>
<td>5.037</td>
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<td>7.171</td>
<td>6.5</td>
<td>0.671</td>
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<tr>
<td>14</td>
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<td>9.843</td>
<td>26.1043</td>
<td>5.109</td>
<td>1.927</td>
<td>8.090</td>
<td>7.0</td>
<td>1.090</td>
</tr>
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<td>11.036</td>
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<td>9.238</td>
<td>7.5</td>
<td>1.738</td>
</tr>
<tr>
<td>16</td>
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<td>10.592</td>
<td>27.7246</td>
<td>5.265</td>
<td>2.012</td>
<td>9.019</td>
<td>8.0</td>
<td>1.019</td>
</tr>
<tr>
<td>17</td>
<td>1.863</td>
<td>12.455</td>
<td>31.1954</td>
<td>5.585</td>
<td>2.230</td>
<td>10.414</td>
<td>8.5</td>
<td>1.914</td>
</tr>
<tr>
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<td>12.976</td>
<td>31.4668</td>
<td>5.610</td>
<td>2.313</td>
<td>11.136</td>
<td>9.0</td>
<td>2.136</td>
</tr>
<tr>
<td>19</td>
<td>1.801</td>
<td>14.777</td>
<td>34.7541</td>
<td>5.892</td>
<td>2.508</td>
<td>12.512</td>
<td>9.5</td>
<td>3.012*</td>
</tr>
</tbody>
</table>

*H_0 : \mu / \sigma = \delta = 0 against H_1 : \mu / \sigma = \delta_1 = 1, No truncation point.

When n=19, $\ln L = 3.012 > 2.94$, therefore reject the hypothesis H_0 : \mu / \sigma = 0.
Table 5. An example for T. T. test with $X_1 \sim N(1,1)$, $\alpha = \beta = 0.05$

<table>
<thead>
<tr>
<th>n</th>
<th>$X_i$</th>
<th>$\Sigma X_i$</th>
<th>$\Sigma X_i^2$</th>
<th>$\sqrt{\Sigma X_i^2}$</th>
<th>$U_n$</th>
<th>$\Phi_L(U_n)$</th>
<th>$\Phi_U(U_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.185</td>
<td>0.185</td>
<td>0.0342</td>
<td>0.185</td>
<td>1.000</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>2.082</td>
<td>2.267</td>
<td>4.3689</td>
<td>2.090</td>
<td>1.085</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>3</td>
<td>-0.123</td>
<td>2.144</td>
<td>4.3840</td>
<td>2.094</td>
<td>1.024</td>
<td>-1.483</td>
<td>2.235</td>
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<td>2.407</td>
<td>5.0757</td>
<td>2.253</td>
<td>1.068</td>
<td>-0.757</td>
<td>2.145</td>
</tr>
<tr>
<td>5</td>
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<td>0.957</td>
<td>7.1782</td>
<td>2.679</td>
<td>0.357</td>
<td>-0.313</td>
<td>2.089</td>
</tr>
<tr>
<td>6</td>
<td>1.035</td>
<td>1.992</td>
<td>8.2494</td>
<td>2.872</td>
<td>0.694</td>
<td>0.008</td>
<td>2.054</td>
</tr>
<tr>
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<td>5.371</td>
<td>19.6670</td>
<td>4.435</td>
<td>1.211</td>
<td>0.261</td>
<td>2.033</td>
</tr>
<tr>
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<td>1.207</td>
<td>0.472</td>
<td>2.022</td>
</tr>
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<td>5.704</td>
<td>19.7913</td>
<td>4.449</td>
<td>1.282</td>
<td>0.653</td>
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<td>5.198</td>
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<td>4.477</td>
<td>1.161</td>
<td>0.814</td>
<td>2.018</td>
</tr>
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<td>6.467</td>
<td>21.6577</td>
<td>4.654</td>
<td>1.390</td>
<td>0.959</td>
<td>2.022</td>
</tr>
<tr>
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<td>1.779</td>
<td>8.246</td>
<td>24.8225</td>
<td>4.982</td>
<td>1.655</td>
<td>1.091</td>
<td>2.029</td>
</tr>
<tr>
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<td>8.987</td>
<td>25.3716</td>
<td>5.037</td>
<td>1.784</td>
<td>1.241</td>
<td>2.038</td>
</tr>
<tr>
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<td>0.856</td>
<td>9.843</td>
<td>26.1043</td>
<td>5.109</td>
<td>1.927</td>
<td>1.328</td>
<td>2.050</td>
</tr>
<tr>
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<td>1.193</td>
<td>11.036</td>
<td>27.5275</td>
<td>5.247</td>
<td>2.103*</td>
<td>1.436</td>
<td>2.062*</td>
</tr>
</tbody>
</table>

* $H_0 : \nu/\sigma = 0.$ against $H_1 : \nu/\sigma = 1.$, Truncation point $n=24$

when $n=15$ $U_n = 2.103 > 2.062$ therefore, reject $H_0 : \nu/\sigma = 0.$
Table 6. An example for SR+ test with $X_i \sim N(1,1)$, $\alpha = \beta = 0.05$

<table>
<thead>
<tr>
<th>n</th>
<th>$X_i$</th>
<th>SR+</th>
<th>$n(n+1)/4$</th>
<th>$\sqrt{n(n+1)(2n+1)/24}$</th>
<th>$\bar{Y}_n$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.185</td>
<td>1</td>
<td>0.5</td>
<td>0.500</td>
<td>1.000</td>
</tr>
<tr>
<td>2</td>
<td>2.082</td>
<td>3</td>
<td>1.5</td>
<td>1.118</td>
<td>1.342</td>
</tr>
<tr>
<td>3</td>
<td>-0.123</td>
<td>5</td>
<td>3.0</td>
<td>1.871</td>
<td>1.069</td>
</tr>
<tr>
<td>4</td>
<td>0.263</td>
<td>9</td>
<td>5.0</td>
<td>2.739</td>
<td>1.460</td>
</tr>
<tr>
<td>5</td>
<td>-1.450</td>
<td>10</td>
<td>7.5</td>
<td>3.708</td>
<td>0.674</td>
</tr>
<tr>
<td>6</td>
<td>1.035</td>
<td>15</td>
<td>10.5</td>
<td>4.770</td>
<td>0.943</td>
</tr>
<tr>
<td>7</td>
<td>3.379</td>
<td>22</td>
<td>14.0</td>
<td>5.916</td>
<td>1.352</td>
</tr>
<tr>
<td>8</td>
<td>-0.019</td>
<td>27</td>
<td>18.0</td>
<td>7.141</td>
<td>1.260</td>
</tr>
<tr>
<td>9</td>
<td>0.352</td>
<td>35</td>
<td>22.5</td>
<td>8.441</td>
<td>1.481</td>
</tr>
<tr>
<td>10</td>
<td>-0.506</td>
<td>38</td>
<td>27.5</td>
<td>9.811</td>
<td>1.070</td>
</tr>
<tr>
<td>11</td>
<td>1.269</td>
<td>48</td>
<td>33.0</td>
<td>11.247</td>
<td>1.334</td>
</tr>
<tr>
<td>12</td>
<td>1.779</td>
<td>60</td>
<td>39.0</td>
<td>12.748</td>
<td>1.647</td>
</tr>
<tr>
<td>13</td>
<td>0.741</td>
<td>72</td>
<td>45.5</td>
<td>14.309</td>
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</tr>
<tr>
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<td>0.856</td>
<td>85</td>
<td>52.5</td>
<td>15.930</td>
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<tr>
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<td>1.193</td>
<td>99</td>
<td>60.0</td>
<td>17.607</td>
<td>2.215</td>
</tr>
<tr>
<td>16</td>
<td>-0.444</td>
<td>107</td>
<td>68.0</td>
<td>19.339</td>
<td>2.017</td>
</tr>
<tr>
<td>17</td>
<td>1.863</td>
<td>124</td>
<td>76.5</td>
<td>21.125</td>
<td>2.249</td>
</tr>
<tr>
<td>18</td>
<td>0.521</td>
<td>141</td>
<td>85.5</td>
<td>22.962</td>
<td>2.417*</td>
</tr>
</tbody>
</table>

*$H_0 : \delta = 0$ against $H_1 : \delta = 1$, Truncation point $n=21$

when $n=18$ \( \bar{Y}_n = 2.417 > 2.41 \), therefore reject the hypothesis $H_0 : \delta = 0$. 

18
It is of interest now to compare ASN and Power of these three tests. It is also of interest to compare these tests for a variety of distributions. The t-tests are optimal for the normal distribution. Therefore, this distribution will be used. The double exponential distribution has thicker tails than the normal and according to Miller (1970), should have better power for the Signed-Rank test. This distribution is also easily generated on the computer and so will be used. A distribution with thin tails does not favor the Signed-Rank test. For this case, the uniform distribution will be used.

The symbol N (1,1) used in this section means under normal distribution with mean 1 and standard deviation 1; U(0.289,0.289) means under uniform distribution with mean 0.289 and standard deviation 0.289; Exp (1.414,1.414) means under double exponential distribution with mean 1.414 and standard deviation 1.414, etc..

Normal Distribution

Since the values of \( \alpha \) and \( \beta \) are required to decide the boundary values on G.T. test and T.T. test, we may have \( \alpha = \beta = 0.05 \) in the following comparisons without any doubt. But on SR\(^+\) test, the \( |z|^n \) values were generated only under specified values of \( \alpha \). Therefore, in the following comparisons we have to decide the truncation number \( N \) first by evaluating \( \beta = 0.05 \) on SR\(^+\) test. In this part, programs on appendix V were run for N (1,1) and N (0.5,1). The results were shown on Table 7 and Table 8.
Table 7. Values for evaluating $\beta = \alpha = 0.05$ under $N(1,1)$

<table>
<thead>
<tr>
<th>N</th>
<th>20</th>
<th>21</th>
<th>22</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>6.2%</td>
<td>5.4%</td>
<td>3.6%</td>
<td>2.0%</td>
</tr>
</tbody>
</table>

Table 8. Values for evaluating $\beta = \alpha = 0.05$ under $N(0.5,1)$

<table>
<thead>
<tr>
<th>N</th>
<th>60</th>
<th>70</th>
<th>75</th>
<th>80</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>10.0%</td>
<td>6.8%</td>
<td>5.2%</td>
<td>4.0%</td>
</tr>
</tbody>
</table>

Therefore, we may have $N = 21$ and $N = 75$ as the truncation points for SR$^+$ test to $N(1,1)$ and $N(0.5,1)$ respectively under $\alpha = \beta = 0.05$.

Now by using the programs on appendix III, appendix IV, and appendix V, 500 sets of samples were used; the average sample number and the standard deviation were thus calculated. The numbers of samples for each set depends on the truncation number $N$. Because the G.T. test has no truncation point, Wald's formula (W's) was used to evaluate ASN. This ASN was also estimated in the Monte Carlo simulation (M.C.) for the G.T. test. As shown on appendix III, the writer uses $N = 45$ for $\frac{\mu}{\sigma} = 0.5$. The results were tabled as follows:
Table 9. Comparison of ASN for G.T., T.T., and SR$^+$ tests under normal distribution $N(1,1)$

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T.</th>
<th>T.T.</th>
<th>SR$^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>W's</td>
<td>M.C.</td>
<td>M.C.</td>
<td>M.C.</td>
</tr>
<tr>
<td>ASN</td>
<td>5.292</td>
<td>8.462</td>
<td>9.388</td>
</tr>
</tbody>
</table>

standard deviation of $n$ | 4.412 | 2.987 | 3.850 |

Table 10. Comparison of ASN for G.T., T.T., and SR$^+$ tests under normal distribution $N(0.5,1)$

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T.</th>
<th>T.T.</th>
<th>SR$^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td>W's</td>
<td>M.C.</td>
<td>M.C.</td>
<td>M.C.</td>
</tr>
<tr>
<td>ASN</td>
<td>21.168</td>
<td>25.880</td>
<td>25.598</td>
</tr>
</tbody>
</table>

standard deviation of $n$ | 15.941 | 11.694 | 18.339 |

Uniform Distribution

Same as the normal distribution, we need some tests to find $N$ with $\beta = \alpha = 0.05$. Programs on appendix V were run for this purpose. The results were tabled as follows:
Table 11. Values for evaluating $\beta = \alpha = 0.05$ under $U(0.289,0.289)$

<table>
<thead>
<tr>
<th>$N$</th>
<th>15</th>
<th>19</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>19.8%</td>
<td>8.0%</td>
<td>4.8%</td>
<td>2.4%</td>
</tr>
</tbody>
</table>

Therefore, the truncation points are $N = 20$ for $U(0.289,0.289)$ and $N = 80$ for $U(0.145,0.289)$.

The ASN's and standard deviations of $n$ for three tests under Uniform distribution were tabled as follows:

Table 12. Values for evaluating $\beta = \alpha = 0.05$ under $U(0.145,0.289)$

<table>
<thead>
<tr>
<th>$N$</th>
<th>70</th>
<th>80</th>
<th>85</th>
<th>90</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>8.4%</td>
<td>5.0%</td>
<td>4.0%</td>
<td>3.2%</td>
</tr>
</tbody>
</table>

Table 13. Comparison of ASN for G.T., T.T., and SR tests under uniform distribution $U(0.289,0.289)$

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T. $W's$</th>
<th>M.C.</th>
<th>T.T. M.C.</th>
<th>SR$^+$ M.C.</th>
</tr>
</thead>
</table>

Table 13. continued

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T.</th>
<th>T.T.</th>
<th>SR$^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W's M.C.</td>
<td>M.C.</td>
<td>M.C.</td>
</tr>
</tbody>
</table>

| ASN         | 5.292 | 8.752 | 8.706 | 12.206 |
| standard deviation of n | 4.570 | 2.967 | 3.681 |

Table 14. Comparison of ASN for G.T., T.T., and SR$^+$ tests under uniform distribution U(0.145,0.289)

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T.</th>
<th>T.T.</th>
<th>SR$^+$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>W's M.C.</td>
<td>M.C.</td>
<td>M.C.</td>
</tr>
</tbody>
</table>

| standard deviation of n | 17.144 | 14.605 | 24.158 |

Double Exponential Distribution

Programs on appendix V were run for finding N with $\beta = \alpha = 0.05$ under double exponential distribution. Here the double exponential distribution is expressed as:

$$f(x) = \frac{1}{2} e^{-\frac{|x|}{2}} \quad -\infty < x < \infty$$
Table 15. Values for evaluating $\beta = \alpha = 0.05$ under $\text{Exp}(1.414, 1.414)$

<table>
<thead>
<tr>
<th>N</th>
<th>15</th>
<th>18</th>
<th>19</th>
<th>20</th>
<th>25</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>13.6%</td>
<td>6.8%</td>
<td>5.2%</td>
<td>4.4%</td>
<td>0.8%</td>
</tr>
</tbody>
</table>

Table 16. Values for evaluating $\beta = \alpha = 0.05$ under $\text{Exp}(0.707, 1.414)$

<table>
<thead>
<tr>
<th>N</th>
<th>50</th>
<th>55</th>
<th>56</th>
<th>60</th>
<th>70</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>7.2%</td>
<td>5.4%</td>
<td>5.0%</td>
<td>3.2%</td>
<td>1.8%</td>
</tr>
</tbody>
</table>

Therefore, the truncation points are $N = 19$ for $\text{Exp}(1.414, 1.414)$ and $N = 56$ for $\text{Exp}(0.707, 1.414)$.

The ASNs and standard deviations of $n$ for three tests were tabled as follows:

Table 17. Comparison of ASN for G.T., T.T., and SR$^+$ tests under double exponential distribution $\text{Exp}(1.414, 1.414)$

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T. W's</th>
<th>G.T. M.C.</th>
<th>T.T. M.C.</th>
<th>SR$^+$ M.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>W's</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>M.C.</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Table 17. continued

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T. W's M.C.</th>
<th>T.T. M.C.</th>
<th>SR⁺ M.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASN</td>
<td>5.292 7.930</td>
<td>7.744</td>
<td>10.802</td>
</tr>
<tr>
<td>standard deviation of n</td>
<td>4.664</td>
<td>2.603</td>
<td>3.358</td>
</tr>
</tbody>
</table>

Table 18. Comparison of ASN for G.T., T.T., and SR⁺ tests under double exponential distribution Exp (0.707,1.414)

<table>
<thead>
<tr>
<th>Test Method</th>
<th>G.T. W's M.C.</th>
<th>T.T. M.C.</th>
<th>SR⁺ M.C.</th>
</tr>
</thead>
<tbody>
<tr>
<td>standard deviation of n</td>
<td>16.562</td>
<td>11.799</td>
<td>13.348</td>
</tr>
</tbody>
</table>

We have ASNs on G.T. test and T.T. test with a specified value $\alpha = \beta = 0.05$. It shows that the ASNs are almost the same. Therefore, on the comparison of power, we just need some trials to get the ASN of SR⁺ test and have these ASNs within $0.5$ to the ASN of G.T. test or T.T. test. The table for Comparison of power under three distributions are:
Table 19. Table for comparison of power with $\frac{\mu}{\sigma} = 1$

<table>
<thead>
<tr>
<th>Dis. Test</th>
<th>$N(1,1)$</th>
<th>$U(0.289,0.289)$</th>
<th>$Exp(1.414,1.414)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASN</td>
<td>8.462</td>
<td>8.752</td>
<td>7.930</td>
</tr>
<tr>
<td>power</td>
<td>97.4%</td>
<td>97.6%</td>
<td>96.2%</td>
</tr>
</tbody>
</table>

Table 20. Table for comparison of power with $\frac{\mu}{\sigma} = 0.5$

<table>
<thead>
<tr>
<th>Dis. Test</th>
<th>$N(0.5,1)$</th>
<th>$U(0.145,0.289)$</th>
<th>$Exp(0.707,1.414)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>ASN</td>
<td>25.880</td>
<td>26.512</td>
<td>25.154</td>
</tr>
<tr>
<td>power</td>
<td>96.4%</td>
<td>94.6%</td>
<td>96.0%</td>
</tr>
</tbody>
</table>

Since there was no published work on the robustness of the T.T. test, it seems reasonable to test whether the T.T. test works good for the double exponential distribution and the uniform distribution. The simulated values of type I error ($\alpha$) and the type II error ($\beta$) were listed below:
Table 21. Comparison of α and β under double exponential distribution.

<table>
<thead>
<tr>
<th>Test</th>
<th>SR⁺</th>
<th>T.T.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_0: \mu/\sigma = 0$</td>
<td>$H_1: \mu/\sigma = 0.5$</td>
</tr>
<tr>
<td>$H$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \mu/\sigma = 1$</td>
<td>4.4%</td>
<td>6.6%</td>
</tr>
<tr>
<td>$H_1: \mu/\sigma = 0$</td>
<td>5.2%</td>
<td>5.0%</td>
</tr>
</tbody>
</table>

Table 22. Comparison of α and β under uniform distribution

<table>
<thead>
<tr>
<th>Test</th>
<th>SR⁺</th>
<th>T.T.</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H_0: \mu/\sigma = 0$</td>
<td>$H_1: \mu/\sigma = 0.5$</td>
</tr>
<tr>
<td>$H$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$H_0: \mu/\sigma = 1$</td>
<td>4.2%</td>
<td>6.4%</td>
</tr>
<tr>
<td>$H_1: \mu/\sigma = 0$</td>
<td>4.8%</td>
<td>5.0%</td>
</tr>
</tbody>
</table>

According to the binomial distribution with probability of success $p=0.05$, 95% of the values of the random variable fall within 3.2% and 6.8%. Therefore it may be noted from tables 21 and 22 that all of the Monte Carlo estimates of α and β fall within this range. Thus a test of the hypothesis $H_0: \alpha = \beta = 0.05$ would be accepted in all cases. However, it may be noted that consistently the β for the T.T. test is smaller indicating the T.T. test may be more powerful than the SR⁺ test on the dis-
tributions considered. Since these distributions were picked to represent distributions with marked differences from the normal, the results may be generalized beyond the two distributions considered. The extent of this generalization cannot be inferred from this limited study.
CONCLUSION

As a usual way, the smaller the ASN the better the test method. Although the SR$^+$ test applies to any symmetric distribution while the T.T. test is used for testing normal distribution, the result shows that under the two symmetric distributions (the uniform distribution and the double exponential distribution chosen in this paper) the T.T. test appears to be better than SR$^+$ test. Since these distributions are very different, it indicates that the t-tests may be very robust. This area could be further studied.

According to the tables on section III of this paper, the truncated sequential t-test seems to be best for the hypothesis and alternatives considered, not only because of the smaller ASN but of smaller standard deviation of $n$. Another benefit of the T.T. test is that we can find the truncation point by the procedure described on section I whereas the G.T. test needs many trials to decide how large the sample size should be. The result is that the G.T. test needs a much larger sample size than T.T. test.

It should be noted that using Wald's formula the estimated ASNs remain the same values in every distribution. It seems that the Wald's formula is not very sensitive. It may be seen that the calculated W's values were consistently smaller than those estimated from the Monte Carlo method. The consistency of the Monte Carlo method tends to show that these estimates are much closer to the true values.

It's a trend for ASN getting large and the power getting large, too. According to the tables 19 and 20, power of SR$^+$ test is smaller than G.T.
test and T.T. test. As a conclusion we may say that the T.T. test is more convenient and economic than the other two sequential tests in spite of the fact that the SR$^+$ test is designed to any symmetric distribution.
SELECTED BIBLIOGRAPHY


## APPENDIX I

Table of Boundary Values with $\alpha = \beta = 0.05$ and $\delta_l = 0.5$

<table>
<thead>
<tr>
<th>n</th>
<th>$\phi_L(U_n)$</th>
<th>$\phi_U(U_n)$</th>
<th>n</th>
<th>$\phi_L(U_n)$</th>
<th>$\phi_U(U_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>7</td>
<td>-2.585</td>
<td>3.100</td>
<td>52</td>
<td>1.241</td>
<td>2.200</td>
</tr>
<tr>
<td>8</td>
<td>-2.222</td>
<td>2.898</td>
<td>54</td>
<td>1.306</td>
<td>2.203</td>
</tr>
<tr>
<td>10</td>
<td>-1.687</td>
<td>2.727</td>
<td>56</td>
<td>1.368</td>
<td>2.208</td>
</tr>
<tr>
<td>12</td>
<td>-1.300</td>
<td>2.604</td>
<td>58</td>
<td>1.429</td>
<td>2.213</td>
</tr>
<tr>
<td>14</td>
<td>-0.997</td>
<td>2.512</td>
<td>60</td>
<td>1.489</td>
<td>2.218</td>
</tr>
<tr>
<td>16</td>
<td>-0.749</td>
<td>2.442</td>
<td>62</td>
<td>1.546</td>
<td>2.224</td>
</tr>
<tr>
<td>18</td>
<td>-0.538</td>
<td>2.387</td>
<td>64</td>
<td>1.602</td>
<td>2.230</td>
</tr>
<tr>
<td>20</td>
<td>-0.355</td>
<td>2.343</td>
<td>66</td>
<td>1.657</td>
<td>2.237</td>
</tr>
<tr>
<td>22</td>
<td>-0.192</td>
<td>2.308</td>
<td>68</td>
<td>1.711</td>
<td>2.244</td>
</tr>
<tr>
<td>24</td>
<td>-0.045</td>
<td>2.280</td>
<td>70</td>
<td>1.763</td>
<td>2.251</td>
</tr>
<tr>
<td>26</td>
<td>0.088</td>
<td>2.258</td>
<td>72</td>
<td>1.814</td>
<td>2.258</td>
</tr>
<tr>
<td>28</td>
<td>0.211</td>
<td>2.240</td>
<td>74</td>
<td>1.864</td>
<td>2.266</td>
</tr>
<tr>
<td>30</td>
<td>0.325</td>
<td>2.226</td>
<td>76</td>
<td>1.913</td>
<td>2.273</td>
</tr>
<tr>
<td>32</td>
<td>0.431</td>
<td>2.215</td>
<td>78</td>
<td>1.961</td>
<td>2.281</td>
</tr>
<tr>
<td>34</td>
<td>0.531</td>
<td>2.206</td>
<td>80</td>
<td>2.009</td>
<td>2.289</td>
</tr>
<tr>
<td>36</td>
<td>0.625</td>
<td>2.200</td>
<td>82</td>
<td>2.055</td>
<td>2.298</td>
</tr>
<tr>
<td>38</td>
<td>0.715</td>
<td>2.196</td>
<td>84</td>
<td>2.100</td>
<td>2.306</td>
</tr>
<tr>
<td>40</td>
<td>0.800</td>
<td>2.193</td>
<td>86</td>
<td>2.145</td>
<td>2.315</td>
</tr>
<tr>
<td>42</td>
<td>0.881</td>
<td>2.191</td>
<td>88</td>
<td>2.189</td>
<td>2.323</td>
</tr>
<tr>
<td>44</td>
<td>0.958</td>
<td>2.191</td>
<td>90</td>
<td>2.232</td>
<td>2.322</td>
</tr>
<tr>
<td>46</td>
<td>1.033</td>
<td>2.192</td>
<td>92</td>
<td>2.275</td>
<td>2.341</td>
</tr>
<tr>
<td>48</td>
<td>1.105</td>
<td>2.194</td>
<td>94</td>
<td>2.316</td>
<td>2.349</td>
</tr>
<tr>
<td>50</td>
<td>1.174</td>
<td>2.196</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
APPENDIX II

CC****THIS PROGRAM WAS USED TO GENERATE THE VALUES OF \( \alpha \), WITH N=80. THE LIMITATION OF THE TOTAL AREA AVAILABLE OF THE COMPUTER MAKES THIS PROGRAM GENERATE ABOUT 400 VALUES IN EACH RUN. FOR N=70 AND N=90, WE MAY JUST CHANGE THE NUMBER 80 IN THIS PROGRAM TO 70 AND 90.

DIMENSION Z(400, 80), ZN(80)
DO 2 M=1,400
DO 2 J=1,80
SX=0.
DO 1 T=1,12
1 SX=SX+P(49173)
7(M,J)=SX-6.0
2 CONTINUE
DO 10 M=1,400
DO 10 J=1,80
ZN(J)=0.
10 CONTINUE
SR=0.
DO 8 J=1,80
DO 6 K=1, J
IF(Z(M,K)+Z(M,J)) 5, 5, 0
5 SR=SR+0.
GO TO 6
6 CONTINUE
W=SR-J*(J+1)/4.
ABSP=ABS(W)
V=J*(J+1)*(2*J+1)/24.
QV=SORT(V)
ZN(J)=7N(J)+ABSP/QV
8 CONTINUE
CC****TO HAVE THE MAXIMUM VALUE ON ZN(80)
DO 12 N=1,80
DO 12 NN=N,80
IF(ZN(N)) LE. ZN(NN)) GO TO 12
TEMP=ZN(NN)
ZN(NN)=ZN(N)
ZN(N)=TEMP
12 CONTINUE
WRITE(6,200) ZN(80)
200 FORMAT(1H, 10.2)
10 CONTINUE
STOP
END
APPENDIX III(A)

CC****THIS PROGRAM WAS USED TO GENERATE THE AVERAGE SAMPLING NUMBER AND POWER FOR GENERAL ORDER METRIC WITH \( n(1,1) \) UNDER THE HYPOTHESIS \( H_0: \theta = 0 \) AGAINST \( H_1: \theta \neq 0 \).

CC****IF WE WANT TO HAVE SAMPLES COME FROM UNIFORM DISTRIBUTION AND DOUBLE EXPONENTIAL DISTRIBUTION, WE MAY ONLY CHANGE THIS PROGRAM FROM LINE 6 TO LINE 7 BY:

\[
Z(M,J) = \text{PN}(62417) + 0.289 \quad \text{UNIFORM OR}
\]

\[
X_1 = \text{RN}(62417)
X_2 = \text{RN}(62417)
Y_1 = -\text{ALOG}(X_1)
Y_2 = -\text{ALOG}(X_2)
Z(M,J) = (Y_1 - Y_2) + 1.414 \quad \text{DOUBLE EXPONENTIAL}
\]

DIMENSION Z(500,45)
DO 2 M=1,500
DO 2 J=1,45
SX=0.
DO 1 I=1,12
1 SX=SX+\text{RN}(63491)
Z(M,J)=SX-6.*1.
2 CONTINUE
DO 3 M=1,500
SUM=0.
SQ=0.
DO 4 J=1,45
SUM=SUM+Z(M,J)
SQ=SQ+Z(M,J)**2
SQT=SQR(T(SQ))
UU=SUM/SQT
U=UU**1.
F=J
A=(U**2)/4
B=SQR(T(F))
C=(U**2)/(24*J)
D=1/(4*J)+C
GU=A+B*U*D
GL=GU-J/2.
N=J
IF(GL.GT.2.94) GO TO 800
IF(GL.LT.-2.94) GO TO 800
4 CONTINUE
800 WRITE(6,200) N
200 FORMAT(1H,'REJECT H0 AT N = ' I20)
GO TO 3
900 WRITE(6,300) N
200 FORMAT(1H,'ACCEPT H0 AT N = ' I20)
3 CONTINUE
STOP
END
APPENDIX III (A)

**THIS PROGRAM WAS USED TO GENERATE THE AVERAGE SAMPLE NUMBER AND POWER FOR G. T. TEST WITH H0: \( M = 0 \) UNDER THE HYPOTHESIS H0: \( M/\sigma = 0 \) AGAINST H1: \( M/\sigma = 0.5 \).**

```
DIMENSION Z(250,145)
DO 3 M=1,250
DO 3 J=1,145
SX=0.
DO I=1,12
1 SX=SX+RN(56247)
Z(M,J)=SX-6.+0.5
3 CONTINUE
DO 4 M=1,250
SUM=0.
SQ=0.
DO J=1,145
SUM=SUM+Z(M,J)
SQ=SQ+Z(M,J)**2
SORT=SQR(SQ)
UU=SUM/SORT
U=UU**0.5
A=(U**2)/4
B=SQR(F)
C=(U**2)/(124*J)
D=1-1/(4*J)+C
GU=A+B*U*D
GL=GU-(J**0.25)/2.
N=J
IF(GL.GT.2.94) GO TO 900
IF(GL.LT.-2.94) GO TO 700
2 CONTINUE
800 WRITE(6,200) N
200 FORMAT(1H,'REJECT H. AT N= ' I20)
GO TO 4
900 WRITE(6,300) N
300 FORMAT(1H,'ACCEPT H. AT N= ' I20)
4 CONTINUE
STOP
END
```
APPENDIX IV (A)

This program was used to generate the average sample number and power for T.T. test with EXP(1.414, 1.414) under the hypothesis H₀: \( \mu / \sigma = 0 \) against H₁: \( \mu / \sigma = 1 \).

DIMENSION Z(500, 24)
DO 3 M=1, 500
DO 3 J=1, 24
X1=RN(58617)
X2=RN(58617)
Y1=-ALOG(X1)
Y2=-ALOG(X2)
Z(M, J)=(Y1-Y2) + 1.414
3 CONTINUE

DO 4 M=1, 500
SUM=0.
SQ=0.
DO 2 J=1, 24
SUM=SUM+Z(M, J)
SQ=SQ+Z(M, J)**2
SQT=SQR(T(SQ))
U=SUM/SQT
IF(J.GT.3) GO TO 5
IF(U.LT.2.235 AND U.GT.-1.483) GO TO 2
N=J
IF(U.GT.2.235) GO TO 800
GO TO 900
5 IF(J.EQ.4) GO TO 54
IF(J.EQ.5) GO TO 55
IF(J.EQ.6) GO TO 56
IF(J.EQ.7) GO TO 57
IF(J.EQ.8) GO TO 58
IF(J.EQ.9) GO TO 59
IF(J.EQ.10) GO TO 60
IF(J.EQ.11) GO TO 61
IF(J.EQ.12) GO TO 62
IF(J.EQ.13) GO TO 63
IF(J.EQ.14) GO TO 64
IF(J.EQ.15) GO TO 65
IF(J.EQ.16) GO TO 66
IF(J.EQ.17) GO TO 67
IF(J.EQ.18) GO TO 68
IF(J.EQ.19) GO TO 69
IF(J.EQ.20) GO TO 70
IF(J.EQ.21) GO TO 71
IF(J.EQ.22) GO TO 72
IF(J.EQ.23) GO TO 73
IF(J.EQ.24) GO TO 74
54 IF(U.LT.2.145 AND U.GT.-0.757) GO TO 2
N=J
IF(U.GT.2.145) GO TO 800
GO TO 900
55 IF(U.LT.2.086.AND.U.GT.-0.313) GO TO 2
N=J
IF(U.GT.2.086) GO TO 800
GO TO 900
56 IF(U.LT.2.054.AND.U.GT.0.08) GO TO 2
N=J
IF(U.GT.2.054) GO TO 800
GO TO 900
57 IF(U.LT.2.033.AND.U.GT.0.261) GO TO 2
N=J
IF(U.GT.2.033) GO TO 800
GO TO 900
58 IF(U.LT.2.022.AND.U.GT.0.472) GO TO 2
N=J
IF(U.GT.2.022) GO TO 800
GO TO 900
59 IF(U.LT.2.017.AND.U.GT.0.683) GO TO 2
N=J
IF(U.GT.2.017) GO TO 800
GO TO 900
60 IF(U.LT.2.018.AND.U.GT.0.214) GO TO 2
N=J
IF(U.GT.2.018) GO TO 800
GO TO 900
61 IF(U.LT.2.022.AND.U.GT.0.956) GO TO 2
N=J
IF(U.GT.2.022) GO TO 800
GO TO 900
62 IF(U.LT.2.029.AND.U.GT.1.051) GO TO 2
N=J
IF(U.GT.2.029) GO TO 800
GO TO 900
63 IF(U.LT.2.038.AND.U.GT.1.214) GO TO 2
N=J
IF(U.GT.2.038) GO TO 800
GO TO 900
64 IF(U.LT.2.050.AND.U.GT.1.329) GO TO 2
N=J
IF(U.GT.2.050) GO TO 800
GO TO 900
65 IF(U.LT.2.062.AND.U.GT.1.436) GO TO 2
N=J
IF(U.GT.2.062) GO TO 800
GO TO 900
66 IF(U.LT.2.076.AND.U.GT.1.537) GO TO 2
N=J
IF(U.GT.2.076) GO TO 800
GO TO 900
67 IF(U.LT.2.091.AND.U.GT.1.634) GO TO 2
N=J
IF(U.GT.2.091) GO TO 800
GO TO 900
68 IF(U.LT.2.106.AND.U.GT.1.725) GO TO 2
N=J
IF(U.GT.2.106) GO TO 800
GO TO 900
69 IF(U.LT.2.122.AND.U.GT.1.813) GO TO 2
N=J
IF(U.GT.2.122) GO TO 800
GO TO 900
70 IF(U.LT.2.139.AND.U.GT.1.895) GO TO 2
N=J
IF(U.GT.2.139) GO TO 800
GO TO 900
71 IF(U.LT.2.156.AND.U.GT.1.979) GO TO 2
N=J
IF(U.GT.2.156) GO TO 800
GO TO 900
72 IF(U.LT.2.173.AND.U.GT.2.058) GO TO 2
N=J
IF(U.GT.2.173) GO TO 800
GO TO 900
73 IF(U.LT.2.190.AND.U.GT.2.134) GO TO 2
N=J
IF(U.GT.2.190) GO TO 800
GO TO 900
74 IF(U.LT.2.208) GO TO 800
GO TO 900
800 WRITE(6,200) N
200 FORMAT(1H,'ACCEPT H, AT N= ',120)
GO TO 4
800 WRITE(6,300) N
300 FORMAT(1H,'REJECT H, AT N= ',120)
GO TO 4
2 CONTINUE
4 CONTINUE
STOP
END
APPENDIX IV (B)

CC**** THIS PROGRAM WAS USED TO GENERATE THE AVERAGE SAMPLE NUMBER AND POWER FOR T.T. TEST WITH EXP(0.7,07,1.414)
UNDER THE HYPOTHESIS H0: \( \mu/\sigma = 0 \) AGAINST H1: \( \mu/\sigma = 0.5 \).

DIMENSION M(250,95)
DO 3 M=1,250
DO 3 J=1,95
X1=RN(51647)
X2=RN(51647)
Y1=-ALOG(X1)
Y2=-ALOG(X2)
Z(M,J)=(Y1-Y2)*0.707
2 CONTINUE
DO 4 M=1,250
SUM=0.
SQ=0.
DO 2 J=1,95
SUM=SUM+Z(M,J)
SQ=SQ+Z(M,J)*Z(M,J)
SQT=SQRT(SQ)
U=SUM/SQT
IF(J.GT.7) GO TO 5
IF(U.LT.3.100 .AND. U.GT.-2.586) GO TO 2
M=J
IF(U.GT.3.100) GO TO 800
IF(U.LT.-2.585) GO TO 800
5 IF(J.EQ.8) GO TO 52
IF(J.EQ.9) GO TO 50
IF(J.EQ.10) GO TO 50
IF(J.EQ.11) GO TO 50
IF(J.EQ.12) GO TO 42
IF(J.EQ.13) GO TO 63
IF(J.EQ.14) GO TO 64
IF(J.EQ.15) GO TO 65
IF(J.EQ.16) GO TO 66
IF(J.EQ.17) GO TO 67
IF(J.EQ.18) GO TO 68
IF(J.EQ.19) GO TO 69
IF(J.EQ.20) GO TO 70
IF(J.EQ.21) GO TO 71
IF(J.EQ.22) GO TO 72
IF(J.EQ.23) GO TO 73
IF(J.EQ.24) GO TO 74
IF(J.EQ.25) GO TO 75
IF(J.EQ.26) GO TO 76
IF(J.EQ.27) GO TO 77
IF(J.EQ.28) GO TO 78
IF(J.EQ.29) GO TO 79
IF (J.EQ.30) GO TO 40
IF (J.EQ.31) GO TO 41
IF (J.EQ.32) GO TO 42
IF (J.EQ.33) GO TO 43
IF (J.EQ.34) GO TO 44
IF (J.EQ.35) GO TO 45
IF (J.EQ.36) GO TO 46
IF (J.EQ.37) GO TO 47
IF (J.EQ.38) GO TO 48
IF (J.EQ.39) GO TO 49
IF (J.EQ.40) GO TO 50
IF (J.EQ.41) GO TO 51
IF (J.EQ.42) GO TO 52
IF (J.EQ.43) GO TO 53
IF (J.EQ.44) GO TO 54
IF (J.EQ.45) GO TO 55
IF (J.EQ.46) GO TO 56
IF (J.EQ.47) GO TO 57
IF (J.EQ.48) GO TO 58
IF (J.EQ.49) GO TO 59
IF (J.EQ.50) GO TO 60
IF (J.EQ.51) GO TO 61
IF (J.EQ.52) GO TO 62
IF (J.EQ.53) GO TO 63
IF (J.EQ.54) GO TO 64
IF (J.EQ.55) GO TO 65
IF (J.EQ.56) GO TO 66
IF (J.EQ.57) GO TO 67
IF (J.EQ.58) GO TO 68
IF (J.EQ.59) GO TO 69
IF (J.EQ.60) GO TO 70
IF (J.EQ.61) GO TO 71
IF (J.EQ.62) GO TO 72
IF (J.EQ.63) GO TO 73
IF (J.EQ.64) GO TO 74
IF (J.EQ.65) GO TO 75
IF (J.EQ.66) GO TO 76
IF (J.EQ.67) GO TO 77
IF (J.EQ.68) GO TO 78
IF (J.EQ.69) GO TO 79
IF (J.EQ.70) GO TO 80
IF (J.EQ.71) GO TO 81
IF (J.EQ.72) GO TO 82
IF (J.EQ.73) GO TO 83
IF (J.EQ.74) GO TO 84
IF (J.EQ.75) GO TO 85
IF (J.EQ.76) GO TO 86
IF (J.EQ.77) GO TO 87
IF (J.EQ.78) GO TO 88
IF (J.EQ.79) GO TO 89
IF(J.EQ.901) GO TO 130
IF(J.EQ.91) GO TO 131
IF(J.EQ.92) GO TO 132
IF(J.EQ.93) GO TO 133
IF(J.EQ.94) GO TO 134
IF(J.EQ.95) GO TO 135
IF(J.EQ.96) GO TO 136
IF(J.EQ.97) GO TO 137
IF(J.EQ.98) GO TO 138
IF(J.EQ.99) GO TO 139
IF(J.EQ.100) GO TO 140
IF(J.EQ.101) GO TO 141
IF(J.EQ.102) GO TO 142
IF(J.EQ.103) GO TO 143
IF(J.EQ.104) GO TO 144
IF(J.EQ.105) GO TO 145

50 IF(U.LE.2.808,.AND.,U.GT.-2.222) GO TO 2
N=J
IF(U.GT.2.298) GO TO 800
GO TO 900

50 IF(U.LE.2.813,.AND.,U.GT.-1.655) GO TO 2
N=J
IF(U.GT.2.313) GO TO 800
GO TO 900

60 IF(U.LE.2.727,.AND.,U.GT.-1.687) GO TO 2
N=J
IF(U.GT.2.727) GO TO 800
GO TO 900

61 IF(U.LE.2.666,.AND.,U.GT.-1.494) GO TO 2
N=J
IF(U.GT.2.666) GO TO 800
GO TO 900

62 IF(U.LE.2.604,.AND.,U.GT.-1.333) GO TO 2
N=J
IF(U.GT.2.604) GO TO 800
GO TO 900

63 IF(U.LE.2.508,.AND.,U.GT.-1.143) GO TO 2
N=J
IF(U.GT.2.508) GO TO 800
GO TO 900

64 IF(U.LE.2.512,.AND.,U.GT.-0.997) GO TO 2
N=J
IF(U.GT.2.512) GO TO 800
GO TO 900

65 IF(U.LE.2.477,.AND.,U.GT.-0.873) GO TO 2
N=J
IF(U.GT.2.477) GO TO 800
GO TO 900
66 IF(U.LT.2.442 AND U.GT.-0.740) GO TO 7
   N=J
   IF(U.GT.-0.442) GO TO 900
   GO TO 900
67 IF(U.LT.2.415 AND U.GT.-0.646) GO TO 2
   N=J
   IF(U.GT.2.415) GO TO 900
   GO TO 900
68 IF(U.LT.2.387 AND U.GT.-0.536) GO TO 2
   N=J
   IF(U.GT.2.387) GO TO 900
   GO TO 900
69 IF(U.LT.2.365 AND U.GT.-0.447) GO TO 2
   N=J
   IF(U.GT.2.365) GO TO 900
   GO TO 900
70 IF(U.LT.2.343 AND U.GT.-0.355) GO TO 2
   N=J
   IF(U.GT.2.343) GO TO 900
   GO TO 900
71 IF(U.LT.2.326 AND U.GT.-0.274) GO TO 2
   N=J
   IF(U.GT.2.326) GO TO 900
   GO TO 900
72 IF(U.LT.2.308 AND U.GT.-0.197) GO TO 2
   N=J
   IF(U.GT.2.308) GO TO 900
   GO TO 900
73 IF(U.LT.2.294 AND U.GT.-0.124) GO TO 2
   N=J
   IF(U.GT.2.294) GO TO 900
   GO TO 900
74 IF(U.LT.2.280 AND U.GT.-0.045) GO TO 2
   N=J
   IF(U.GT.2.280) GO TO 900
   GO TO 900
75 IF(U.LT.2.269 AND U.GT.0.022) GO TO 2
   N=J
   IF(U.GT.2.269) GO TO 900
   GO TO 900
76 IF(U.LT.2.259 AND U.GT.0.063) GO TO 2
   N=J
   IF(U.GT.2.259) GO TO 900
   GO TO 900
77 IF(U.LT.2.249 AND U.GT.0.158) GO TO 2
   N=J
   IF(U.GT.2.249) GO TO 900
   GO TO 900
78 IF(U.LT.2.240 AND U.GT.0.211) GO TO 2
N=J
IF(U.GT.2.241) GO TO 800
GO TO 900
79 IF(U.LT.2.233.AND.U.GT.0.264) GO TO 2
N=J
IF(U.GT.2.233) GO TO 400
GO TO 900
80 IF(U.LT.2.226.AND.U.GT.0.325) GO TO 2
N=J
IF(U.GT.2.226) GO TO 400
GO TO 900
81 IF(U.LT.2.221.AND.U.GT.0.371) GO TO 2
N=J
IF(U.GT.2.221) GO TO 400
GO TO 900
82 IF(U.LT.2.215.AND.U.GT.0.421) GO TO 2
N=J
IF(U.GT.2.215) GO TO 400
GO TO 900
83 IF(U.LT.2.211.AND.U.GT.0.481) GO TO 2
N=J
IF(U.GT.2.211) GO TO 400
GO TO 900
84 IF(U.LT.2.206.AND.U.GT.0.531) GO TO 2
N=J
IF(U.GT.2.206) GO TO 400
GO TO 900
85 IF(U.LT.2.203.AND.U.GT.0.572) GO TO 2
N=J
IF(U.GT.2.203) GO TO 400
GO TO 900
86 IF(U.LT.2.200.AND.U.GT.0.625) GO TO 2
N=J
IF(U.GT.2.200) GO TO 400
GO TO 900
87 IF(U.LT.2.194.AND.U.GT.0.670) GO TO 2
N=J
IF(U.GT.2.194) GO TO 400
GO TO 900
88 IF(U.LT.2.186.AND.U.GT.0.714) GO TO 2
N=J
IF(U.GT.2.186) GO TO 400
GO TO 900
89 IF(U.LT.2.185.AND.U.GT.0.780) GO TO 2
N=J
IF(U.GT.2.185) GO TO 400
GO TO 900
90 IF(U.LT.2.193.AND.U.GT.0.800) GO TO 2
N=J
IF(U.GT.2.193) GO TO 400
GO TO 300
01 IF(U.LT.2.192.AND.U.GT.0.641) GO TO 2
  N=J
  IF(U.GT.2.192) GO TO 300
  GO TO 900
02 IF(U.LT.2.191.AND.U.GT.0.831) GO TO 2
  N=J
  IF(U.GT.2.191) GO TO 300
  GO TO 900
03 IF(U.LT.2.191.AND.U.GT.0.220) GO TO 2
  N=J
  IF(U.GT.2.191) GO TO 300
  GO TO 900
04 IF(U.LT.2.191.AND.U.GT.0.55) GO TO 2
  N=J
  IF(U.GT.2.191) GO TO 300
  GO TO 900
05 IF(U.LT.2.192.AND.U.GT.1.031) GO TO 2
  N=J
  IF(U.GT.2.192) GO TO 300
  GO TO 900
06 IF(U.LT.2.193.AND.U.GT.1.040) GO TO 2
  N=J
  IF(U.GT.2.193) GO TO 300
  GO TO 900
07 IF(U.LT.2.194.AND.U.GT.1.105) GO TO 2
  N=J
  IF(U.GT.2.194) GO TO 300
  GO TO 900
08 IF(U.LT.2.195.AND.U.GT.1.140) GO TO 2
  N=J
  IF(U.GT.2.195) GO TO 300
  GO TO 900
09 IF(U.LT.2.196.AND.U.GT.1.174) GO TO 2
  N=J
  IF(U.GT.2.196) GO TO 300
  GO TO 900
10 IF(U.LT.2.198.AND.U.GT.1.203) GO TO 2
  N=J
  IF(U.GT.2.198) GO TO 300
  GO TO 900
11 IF(U.LT.2.200.AND.U.GT.1.241) GO TO 2
  N=J
  IF(U.GT.2.200) GO TO 300
  GO TO 900
12 IF(U.LT.2.202.AND.U.GT.1.274) GO TO 2
N=J
IF(U.GT.2.202) GO TO 900
GO TO 900
104 IF(U.LT.2.203.AND.U.GT.1.306) GO TO 2
N=J
IF(U.GT.2.203) GO TO 900
GO TO 900
105 IF(U.LT.2.206.AND.U.GT.1.337) GO TO 2
N=J
IF(U.GT.2.206) GO TO 900
GO TO 900
106 IF(U.LT.2.206.AND.U.GT.1.368) GO TO 2
N=J
IF(U.GT.2.206) GO TO 900
GO TO 900
107 IF(U.LT.2.211.AND.U.GT.1.398) GO TO 2
N=J
IF(U.GT.2.211) GO TO 900
GO TO 900
108 IF(U.LT.2.213.AND.U.GT.1.429) GO TO 2
N=J
IF(U.GT.2.213) GO TO 900
GO TO 900
109 IF(U.LT.2.216.AND.U.GT.1.459) GO TO 2
N=J
IF(U.GT.2.216) GO TO 900
GO TO 900
110 IF(U.LT.2.218.AND.U.GT.1.489) GO TO 2
N=J
IF(U.GT.2.218) GO TO 900
GO TO 900
111 IF(U.LT.2.221.AND.U.GT.1.518) GO TO 2
N=J
IF(U.GT.2.221) GO TO 900
GO TO 900
112 IF(U.LT.2.224.AND.U.GT.1.548) GO TO 2
N=J
IF(U.GT.2.224) GO TO 900
GO TO 900
113 IF(U.LT.2.227.AND.U.GT.1.574) GO TO 2
N=J
IF(U.GT.2.227) GO TO 900
GO TO 900
114 IF(U.LT.2.230.AND.U.GT.1.602) GO TO 2
N=J
IF(U.GT.2.230) GO TO 900
GO TO 900
115 IF(U.LT.2.234.AND.U.GT.1.634) GO TO 2
N=J
IF(U.GT.2.234) GO TO 900
GO TO 900
116 IF(U.LT.2.237.AND.U.GT.1.457) GO TO 2
N=J
IF(U.GT.2.237) GO TO 900
GO TO 900
117 IF(U.LT.2.241.AND.U.GT.1.684) GO TO 2
N=J
IF(U.GT.2.241) GO TO 900
GO TO 900
118 IF(U.LT.2.244.AND.U.GT.1.711) GO TO 2
N=J
IF(U.GT.2.244) GO TO 900
GO TO 900
119 IF(U.LT.2.248.AND.U.GT.1.737) GO TO 2
N=J
IF(U.GT.2.248) GO TO 900
GO TO 900
120 IF(U.LT.2.251.AND.U.GT.1.763) GO TO 2
N=J
IF(U.GT.2.251) GO TO 900
GO TO 900
121 IF(U.LT.2.255.AND.U.GT.1.788) GO TO 2
N=J
IF(U.GT.2.255) GO TO 900
GO TO 900
122 IF(U.LT.2.258.AND.U.GT.1.814) GO TO 2
N=J
IF(U.GT.2.258) GO TO 900
GO TO 900
123 IF(U.LT.2.262.AND.U.GT.1.839) GO TO 2
N=J
IF(U.GT.2.262) GO TO 900
GO TO 900
124 IF(U.LT.2.266.AND.U.GT.1.864) GO TO 2
N=J
IF(U.GT.2.266) GO TO 900
GO TO 900
125 IF(U.LT.2.270.AND.U.GT.1.899) GO TO 2
N=J
IF(U.GT.2.270) GO TO 900
GO TO 900
126 IF(U.LT.2.273.AND.U.GT.1.913) GO TO 2
N=J
IF(U.GT.2.273) GO TO 900
GO TO 900
127 IF(U.LT.2.277.AND.U.GT.1.937) GO TO 2
N=J
IF(U.GT.2.277) GO TO 900
128 IF (U.LT.2.281.AND.U.GT.1.961) GO TO 2
    N=J
    IF (U.GT.2.281) GO TO 900
    GO TO 900
129 IF (U.LT.2.285.AND.U.GT.1.985) GO TO 2
    N=J
    IF (U.GT.2.285) GO TO 900
    GO TO 900
130 IF (U.LT.2.289.AND.U.GT.2.009) GO TO 2
    N=J
    IF (U.GT.2.289) GO TO 900
    GO TO 900
131 IF (U.LT.2.294.AND.U.GT.2.032) GO TO 2
    N=J
    IF (U.GT.2.294) GO TO 900
    GO TO 900
132 IF (U.LT.2.298.AND.U.GT.2.055) GO TO 2
    N=J
    IF (U.GT.2.298) GO TO 900
    GO TO 900
133 IF (U.LT.2.302.AND.U.GT.2.078) GO TO 2
    N=J
    IF (U.GT.2.302) GO TO 900
    GO TO 900
134 IF (U.LT.2.306.AND.U.GT.2.100) GO TO 2
    N=J
    IF (U.GT.2.306) GO TO 900
    GO TO 900
135 IF (U.LT.2.311.AND.U.GT.2.123) GO TO 2
    N=J
    IF (U.GT.2.311) GO TO 900
    GO TO 900
136 IF (U.LT.2.315.AND.U.GT.2.145) GO TO 2
    N=J
    IF (U.GT.2.315) GO TO 900
    GO TO 900
137 IF (U.LT.2.319.AND.U.GT.2.167) GO TO 2
    N=J
    IF (U.GT.2.319) GO TO 900
    GO TO 900
138 IF (U.LT.2.323.AND.U.GT.2.189) GO TO 2
    N=J
    IF (U.GT.2.323) GO TO 900
    GO TO 900
139 IF (U.LT.2.328.AND.U.GT.2.211) GO TO 2
    N=J
    IF (U.GT.2.328) GO TO 900
    GO TO 900
140 IF(U.LT.2.332.AND.U.GT.2.232) GO TO 2
N=J
IF(U.GT.2.332) GO TO 800
GO TO 900
141 IF(U.LT.2.337.AND.U.GT.2.254) GO TO 2
N=J
IF(U.GT.2.337) GO TO 800
GO TO 900
142 IF(U.LT.2.341.AND.U.GT.2.275) GO TO 2
N=J
IF(U.GT.2.341) GO TO 800
GO TO 900
143 IF(U.LT.2.345.AND.U.GT.2.294) GO TO 2
N=J
IF(U.GT.2.345) GO TO 800
GO TO 900
144 IF(U.LT.2.349.AND.U.GT.2.314) GO TO 2
N=J
IF(U.GT.2.349) GO TO 800
GO TO 900
900 WRITE(6,200) N
200 FORMAT(14,'ACCEPT U AT N= ',I20)
GO TO 4
800 WRITE(6,300) N
300 FORMAT(14,'REJECT U AT N= ',I20)
GO TO 4
2 CONTINUE
4 CONTINUE
STOP
END
APPENDIX V (A)

THS PROGRAM WAS USED TO GENERATE THE AVERAGE SAMPLE NUMBER AND POWER FOR SEQUENTIAL STONG-PANK TEST WITH \( \mu \), 0.289, 0.289) UNDER THE HYPOTHESIS \( H_0: \mu = 0 \) AGAINST \( H_1: \mu = 1 \).

DIFIMENSION Z(500,20)
DO 2 M=1,500
DO 2 J=1,20
Z(M,J)=RM(34289)-0.211
2 CONTINUE
DO 10 M=1,500
SR=0.
DO 9 J=1,20
DO 6 K=1,J
IF(Z(M,K)+Z(M,J)) 5,5,2
5 SR=SR+0.
GO TO 6
9 SR=SR+1.
6 CONTINUE
W=SR-J*(J+1)/4.
ABS=ABS(W)
V=J*(J+1)*(2*J+1)/24.
QV=SQR(W)
IF(ABS.LT.2.40) GO TO 8
N=J
WRITE(6,300) N
300 FORMAT(1H1,'REJECT H_0 AT N = ',I20)
GO TO 10
8 CONTINUE
N=20
WRITE(6,301) N
301 FORMAT(1H1,'ACCEPT H_0 AT N = ',I20)
10 CONTINUE
STOP
END
APPENDIX V (R)

THE ts PROGRAM WAS USED TO GENERATE THE AVERAGE SAMPLE NUMBER AND POWER FOR SEQUENTIAL SIGNED-RANK TEST WITH U(0.145, 0.286) UNDER THE HYPOTHESIS H_0 : μ/σ = 0 AGAINST H_1 : μ/σ = 0.5.

DIMENSION Z(500,80)
DO 2 M=1,500
DO 2 J=1,80
Z(M,J)=RN(37163)-0.355
2 CONTINUE
DO 10 M=1,500
SR=0.
DO 8 J=1,80
DO 6 K=1,J
IF(Z(M,K)+Z(M,J)) 5,5,0
5 SR=SR+1.
GO TO 6
6 CONTINUE
W=SR-J*(J+1)/4.
ABSR=ABS(W)
V=J*(J+1)*(2*J+1)/24.
QV=SORT(V)
ZN=ABS/W
IF(ZN .LT. 2.66) GO TO 9
N=J
WRITE(6,300) N
300 FORMAT(1H, 'REJECT H_0 AT N= ', I20)
GO TO 10
9 CONTINUE
N=80
WRITE(6,400) N
400 FORMAT(1H, 'ACCEPT H_0 AT N= ', I20)
10 CONTINUE
STOP
END