All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations

2012 Class. Quantum Grav. 29 077001

(http://iopscience.iop.org/0264-9381/29/7/077001)

IOP PUBLISHING CLASSICAL AND QUANTUM GRAVITY

Class. Quantum Grav. **29** (2012) 077001 (2pp) doi:10.1088/0264-9381/29/7/077001

NOTE

All homogeneous pure radiation spacetimes satisfy the Einstein–Maxwell equations

C G Torre

Department of Physics, Utah State University, Logan, UT 84322-4415, USA

Received 8 February 2012
Published 14 March 2012
Online at stacks.iop.org/CQG/29/077001

Abstract
It is shown that all homogeneous pure radiation solutions to the Einstein equations admit electromagnetic sources. This corrects an error in the literature.

PACS numbers: 04.20.−q, 04.20.Jb, 04.40.Nr

A spacetime (M, g) is said to be a homogeneous pure radiation solution of the Einstein equations if it admits a transitive group of isometries and if there exists a function Φ and a 1-form k such that the Einstein tensor G is of the form

$$G_{ab} = \Phi^2 k_a k_b, \quad \text{where} \quad g^{ab} k_a k_b = 0. \tag{1}$$

A classification of homogeneous pure radiation solutions is given in [1] based upon results of [2] and [3]. Such solutions are all pp waves, and they are either plane waves or are diffeomorphic to

$$g = \mathrm{d}x \otimes \mathrm{d}x + \mathrm{d}y \otimes \mathrm{d}y + \mathrm{d}u \odot \mathrm{d}v - 2\,\mathrm{e}^{2\rho x}\,\mathrm{d}u \otimes \mathrm{d}u, \qquad \rho = \text{const}. \tag{2}$$

The plane waves are known to arise from electromagnetic sources. It is asserted in [1, 3], based upon results of [4], that the metric (2) does not arise from an electromagnetic source. However, the metric (2) *does* arise from an electromagnetic source. In fact, infinitely many electromagnetic fields can serve as the source for the spacetime defined by (2).

For any function $f(u)$ the null 2-form

$$F = 2\rho\,\mathrm{e}^{\rho x}[\cos(\rho y + f(u))\,\mathrm{d}u \wedge \mathrm{d}x - \sin(\rho y + f(u))\,\mathrm{d}u \wedge \mathrm{d}y], \tag{3}$$

its Hodge dual $\tilde{F}$ and the metric (2) satisfy the source-free Maxwell equations $dF = 0 = d\tilde{F}$ and the Einstein equations:

$$G_{ab} = \tfrac{1}{2}\left(F_{ac}F_b{}^c + \tilde{F}_{ac}\tilde{F}_b{}^c\right) = 4\rho^2\,\mathrm{e}^{2\rho x}\nabla_a u \nabla_b u. \tag{4}$$

The electromagnetic field given in (3) is 'non-inheriting', that is, it is not invariant under the isometry group of (M, g). For example, F is not invariant under translations in y for any choice of $f(u)$.

Thus, granted theorem 12.6 of [1], all homogeneous pure radiation solutions of the Einstein equations arise from electromagnetic sources.

Acknowledgments

I thank Ian Anderson for discussions. These results were obtained using the *DifferentialGeometry* package in *Maple*.

References

[1] Stephani H, Kramer D, MacCallum M, Hoenselaers C and Herlt E 2003 *Exact Solutions of Einstein's Field Equations* 2nd edn (Cambridge: Cambridge University Press)
[2] Wils P 1989 *Class. Quantum. Grav.* **6** 1243
[3] Steele J 1990 *Class. Quantum. Grav.* **7** L81
[4] Sippel R and Goenner H 1986 *Gen. Rel. Grav.* **18** 1229