Introduction

The current research builds off previous research [4] exploring the dynamics of a magnetic dipole and a systematic search for periodic bouncing modes. Here we are concerned with the stability of these periodic modes. Using chaotic indicators and implementing our own simplistic method, we characterize the long-term stability of nearly all modes found in the previous research.

Chaotic Indicators

The characterization of dynamical systems and their long-term behavior is the study of chaos theory. Many contributions [2, 3, 4, 5, 6, 7] have been made to the study of dynamical systems and there are many ways in which to characterize their behavior. The canonical method is to use Lyapunov exponents (LEs). Lyapunov exponents calculate the exponential growth or decay of neighboring orbits over time with each time step’s deformation added to the array of values and after a proscribed amount of time, the values in the array were divided by the number of steps taken. With the largest Lyapunov exponent characterizing chaos, and for every state tested, all maximum LEs were positive. However, using MapPhysics software [14] to visualize, the long-term behavior of states tested indicated stability. This is how I came to the conclusion the non-smooth nature of this magnetic dipole system might not work with LE calculations. To ensure the implementation of the Lyapunov spectrum calculations were correct, I computed it for several known systems, including the Henon map, the Lorenz attractor, and the Rossler attractor, all of which agreed with established values for each [15].

In order to compute a spectrum of LEs, I followed closely the implementation found in Alligood [13]. Beginning with a $6 \times 6$ identity matrix, representing the unit hypersphere in Fig. 2, one integration time step was taken using a Dormand-Prince adaptive step size numerical integration algorithm. After the integration step was taken, the Jacobian for the newly computed state variables was found and the initially unit sphere was then deformed in different directions. The logarithm of the amount of deformation in each axis was stored and added to an array, and the new distorted sphere was then orthonormalized using a QR decomposition. This process was repeated for a proscribed amount of time with each time step’s deformation added to the array of values and a proscribed amount of time, the values in the array were divided by the number of steps taken. With the largest Lyapunov exponent characterizing chaos and for every state tested, all maximum LEs were positive. However, using MapPhysics software [14] to visualize, the long-term behavior of states tested indicated stability. This is how I came to the conclusion the non-smooth nature of this magnetic dipole system might not work with LE calculations. To ensure the implementation of the Lyapunov spectrum calculations were correct, I computed it for several known systems, including the Henon map, the Lorenz attractor, and the Rossler attractor, all of which agreed with established values for each [15].

The computation of the SALI and GALI were very similar to the Lyapunov spectrum. The only issue arose as with the Lyapunov spectrum, that the non-smooth nature of the magnetic dipole system was not workable with LE calculations. To ensure the implementation of Lyapunov spectrum calculations were correct, I computed it for several known systems, including the Henon map, the Lorenz attractor, and the Rossler attractor, all of which agreed with established values for each [15].

Chaotic Indicators, cont.

As the Lyapunov exponent measures the growth or decay of axes of an ellipsoid to capture the chaotic nature of dynamical systems, the SALI and GALI measure the alignment of originally orthogonal deviations vectors to determine chaos.

Fig. 1: The Small Alignment Index is a quick indicator of chaos in dynamical systems. Two distinct vectors will align with one another in a chaotic system whereas they will still be pointing away from each other if the orbit is stable. Image credit [9].

Defining the SALI at time $t$ to be $SALI(t) = \min (\|\hat{w}_1(t) + \hat{w}_2(t)\|, \|\hat{w}_1(t) - \hat{w}_2(t)\|)$ we can see in the figure above the alignment of two initially orthogonal vectors will begin to align if there is a degree of chaos in the dynamical system. The SALI tends to give a more efficient means of measuring chaos as some threshold value can be established, below which you can determine if an orbit is chaotic and the SALI falls towards zero exponentially quickly for chaotic orbits.

Fig. 2: The Lyapunov spectrum implementation.

Lyapunov Spectrum Implementation

Because of the discontinuous nature of the magnetic dipole system, the LEs, SALI, and GALI were not able to properly provide a reliable indication of chaos in the modes tested. The underlying assumption about the chaotic nature these systems are able to characterize relies on them being smooth, something known as Oseledec’s Theorem. The bounding criterion imposed for the system at hand is therefore inherently invalid to this assumption and can’t use these chaotic indicators. To determine stability, then, I looked for a simple condition as the system was integrated: whether or not the free dipole returned to it’s initial value after $\pi$ or $\pi$ (depending on whether the dipole is in phase or out of phase) after one period of motion. The criterion for whether the dipole returned to its original state was within a tolerance of $10^{-2}$ and integration was for up to 500 periods, after which if the bouncing dipole hadn’t aligned within that tolerance was deemed chaotic. After determining the long-term behavior of each unique periodic state, I then found the percentage for each mode (those families of states with the same $m$, $n$, $p$, $q$, and $r$ values) which was stable and whether the final state in each mode (the one with the highest value of $E$ and $p_q$) was stable. If the final state in a given mode was stable, I characterized the entire mode as completely stable whether or not 100% of the states in the mode were stable.

The following tables summarize the data by the number of bozos for periodic states and whether a mode is in-phase or out of-phase. I used a threshold value of 50% of states in a given mode as stable to determine whether a mode was partially stable. This seemed like the best cutoff value for this condition given the data.

Table 1: A select few entries of periodic states characterized by their number of bounces and all modes characterized by either in-phase or out-of-phase.

<table>
<thead>
<tr>
<th>State</th>
<th>In-phase</th>
<th>Out-of-phase</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.0%</td>
<td>0.0%</td>
</tr>
<tr>
<td>2</td>
<td>31.3</td>
<td>68.7%</td>
</tr>
<tr>
<td>3</td>
<td>12.5</td>
<td>87.5%</td>
</tr>
<tr>
<td>4</td>
<td>16.7</td>
<td>83.3%</td>
</tr>
<tr>
<td>5</td>
<td>17.9</td>
<td>82.1%</td>
</tr>
<tr>
<td>6</td>
<td>12.5</td>
<td>87.5%</td>
</tr>
<tr>
<td>7</td>
<td>4.6</td>
<td>95.4%</td>
</tr>
<tr>
<td>8</td>
<td>5.7</td>
<td>94.3%</td>
</tr>
</tbody>
</table>

References

[11] Tab. 1: A select few entries of periodic states characterized by their number of bounces and all modes characterized by either in-phase or out-of-phase.

Results

Chaotic Indicators

Chaotic Indicators

The computation of the SALI and GALI were very similar to the Lyapunov spectrum. The only difference is every step, the $b \times m$ matrix that took the spot of the identity matrix, whose columns represented orthogonal deviation vectors, was normalized and the value of the SALI/GALI was computed. When the SALI fell below a threshold value of $10^{-2}$ the orbit was taken to be chaotic. The same issue arose as with the Lyapunov spectrum, that the non-smooth nature of the magnetic dipole equations meant long-term stable orbits were characterized as chaotic. To ensure my implementation was working, I checked it against results from [9] and found it to be working for systems in that publication.

To overcome the difficulties with the particular dynamical system at hand, I devised a more simplistic approach to determining the chaotic nature of the orbits which I describe in the Results section.