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Development of a Discrete Mathematics Textbook and Guide for High School Teachers

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DEVELOPMENT OF A DISCRETE MATHEMATICS TEXTBOOK AND
GUIDE FOR HIGH SCHOOL TEACHERS

by

Rebecca A. Stokes

A project submitted in partial fulfillment
of the requirements for the degree
of
MASTER OF MATHEMATICS

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ABSTRACT

Development of a Discrete Mathematics Textbook and Guide for High School Teachers

by

Rebecca A. Stokes, Master of Mathematics
Utah State University, 2010

Major Advisor: David E. Brown
Department: Mathematics and Statistics

This project was to create the beginnings a textbook for teacher’s to supplement instruction of a discrete mathematics course at the high school level. The development of the text was guided by past and current efforts to place discrete mathematics in high school curriculum. A review of the literature and experiences of instructors were viewed and analyzed to guide the construction of the textbook. The text was written with the goal to give teachers information about the topics of discrete mathematics, extra resources, lesson ideas, and optional worksheets for students. Several lessons were created and one was implemented in a high school in Utah; the opinions of the students and teacher who participated were recorded and analyzed.

(133 pages)
ACKNOWLEDGMENTS

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I would like dedicate this project to my parents, Mark and Linda, for their encouragement to always seek to educate myself and others, also to my husband, Tyson, for his patience and support from the concept of this project to this milestone in its creation.

Rebecca A. Stokes
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CHAPTER 1
BACKGROUND

Over the past twenty years, discrete mathematics courses for high school students and the integration of discrete mathematics into secondary curriculum have become topics of interest (Dossey, 1990). Many state’s office of education, including Utah’s (Suddreth, n.d), have proposed its addition as an optional higher level course in which students may participate. Other school districts have used discrete math as the foundation for remedial mathematics courses (Picker, 1997), while others have offered a discrete mathematics course as an elective.

Because discrete mathematics has a direct bearing on computer science and engineering, it is an ideal choice for a high school mathematics course for some students, rather than or in conjunction with the common calculus courses. Discrete mathematics is unique to other mathematics courses in the set of topics covered and the approaches to solving real world problems (Dossey, 1991). Also discrete mathematics is often used in such content areas as the social sciences and political science (Roberts, 1976). Since the topics and strategies are different from those presented in the continuous mathematics courses seen in most high schools, it has been seen as an alternative for students not on the science track.

Of the six universities and two four-year colleges in the state of Utah, four universities and one college (Brigham Young University, Southern Utah University, University of Utah, Utah State University, and Westminster College) currently require a discrete mathematics course as part of their Mathematics-Education degrees (Brigham Young University n.d., Dixie State College n.d., Southern Utah University n.d., University of Utah n.d., Utah State University n.d., Utah Valley University n.d., Weber State University n.d, and Westminster College n.d.). While in the survey by Harold Bailey, he found that of the colleges that offered mathematics education majors, all required a discrete mathematics course (1995). In the same study Bailey also noted that of the 34 responses from high schools only seven taught discrete mathematics. Thus there is
a discrepancy between discrete mathematics being taught to preservice mathematics teachers and its limited implementation into the high school curriculum. Many sources (Dossey, 1990; Dossey, 1991; Hart, 1991, Rosenstain & DeBellis, 1997) claim that discrete mathematics is essential for the high school curriculum. Joseph G. Rosenstein and others at Rutgers University wishing to close the gap between the curriculum and discrete mathematics topics being taught, implemented in the 1990's a leadership program to train teachers with ideas and methods to implement discrete mathematics (1997). Many of the participants in this program have provided information and resources for educators; other individuals have taught discrete mathematics to high school students and even younger students.

The purpose of this project is to present the beginnings of a discrete mathematics textbook for educators. The text would allow teachers to revisit or learn for the first time the topics of discrete mathematics. The text would also include possible lesson plans and activities for teachers to implement into their classroom or to simply use the exercises and assessments. The end goal is to give teachers the resources to present discrete mathematics to high school students with research based methods.
CHAPTER 2
LITERATURE REVIEW

Goals and Methods of Review

Current research available in discrete mathematics coursework at the high school level is commonly assessed with respect to outcomes of students' skills and attitudes. The attitudes of students are indicated with a Likert scale in self reports or as expressed in interviews, and mathematical skills, as indicated by grades and achievement tests, needs to be examined. Thus to facilitate in the design of the current project, a review of the literature was conducted. The purpose of this literature review is to achieve the following objectives:

1. Analyze earlier studies and implementations of discrete mathematics curriculum in secondary education.
2. Examine research on the need for discrete mathematics in high schools.
3. Examine other texts implemented at the high school level.

A secondary source and primary source used in the review was *DIMACS Series in Discrete Mathematics and Theoretical Computer Science Volume 36: Discrete Mathematics in the Schools*. This volume is developed from papers presented at Rutgers University in 1992 and follow-ups of experiences of participants through 1997. The volume contains several outlooks on the influence of discrete mathematics in schools as well as several reports of action research, summaries of dissertations, and other studies. Several preliminary sources used in the review were Academic Search Premier, Digital Dissertations, ERIC, and PsychINFO accessed via USU's library services. The key words used with the preliminary sources are discrete math*, education, secondary education, and research. The search in preliminary sources also included specific topics in discrete mathematics such as graph theory, set theory, and combinatorics, using each as a key word in place of discrete math. Additional secondary sources were discovered in the reference sections of the articles, but few further primary sources on the subject of discrete
mathematics were found. The articles that are included in this sample followed the following inclusion criteria:

- Studies of secondary education (ages: 13-17)
- Studies where discrete mathematics was not a subtopic of a computer science courses
- Articles published after 1990, based on the National Council of Teachers of Mathematics addition of discrete mathematics topics as part of their standards in 1989 (NCTM, 1989)
- Articles that did not focus mainly on teaching practices rather than content

In all, seventeen articles and other resources in several areas of interest were found. They were analyzed and placed into three categories, discrete mathematics in the classroom, discrete mathematics curriculum in high schools, and other discrete mathematics texts.

Discrete Mathematics in the Classroom

Seven of the seventeen articles specifically discussed discrete mathematics in the classroom. These articles were systematically synthesized using a coding sheet with subcategories that were sorted into four categories: sample characteristics, conclusions, definition of discrete mathematics, and textbook and resources used. Based on the subcategories, the articles were analyzed to determine common features as well as disparities that influenced the authors' conclusions (see Appendix A and B).

Conclusions of the Authors

Of the seven articles related to discrete mathematics in the classroom, two of the studies reported that discrete mathematics coursework for students was manageable or there was an increased completion rate of students' work compared to other courses. Four of the eight studies reported that discrete mathematics course participation increased students'
interests and attitudes toward mathematics while one study found that it did not (Thompson, 1992). One author (Picker, 1997) found that coursework in discrete mathematics increased student motivation or attendance, and the other studies did not discuss motivation or attendance. Three studies acknowledged that discrete mathematics coursework increased students’ abilities in general mathematics or logic and one study noted the opposite (Thompson, 1992). Five of the articles discussed how discrete mathematics introduced students to current articles and research in mathematics or real-world applications using discrete mathematics. One study (Settergren, 1997) gave little information of students’ achievements and attitudes on mathematics.

Several of the subcategories were analyzed to determine any co-variation with authors’ conclusions.

**Definition of Discrete Mathematics**

While analyzing the studies it became apparent that the way the author defined or, more commonly, described the topics discrete mathematics had an impact on the outcome. Five of the studies described discrete mathematics in terms of the study of such topics as social decision making, game theory, graph theory, sequences, and discrete probability. Only one of the five studies that defined discrete mathematics with a common definition (Settergren, 1997) recorded no information on student achievement and attitudes. The other four studies with a common definition each concluded positive outcomes specifically with attitudes of students towards mathematics and in introducing students to real world problems and what mathematics is. One of the studies (Biehl, 1997) included several of the common topics of discrete mathematics, but
included other such topics as trigonometry and statistics. The conclusions of Biehl’s study include an increase in mathematical ability and attitudes towards mathematics of students. Finally, one study (Thompson, 1992) defined discrete mathematics in terms of pre-calculus topics and only included topics such as number theory, logic, and proof techniques that are not necessarily only seen in discrete mathematics. Thompson found that students’ attitudes towards mathematics decreased as well as their achievement. Thus it appears that common discrete mathematics topics tend to increase students’ perceptions of mathematics and their skills in other mathematics topics.

**Age or Academic Year of Students**

The question of when discrete mathematics is appropriate for students is necessary to guide at what age level the proposed text should be written. Only one study (Kowalski, 1997) introduced discrete mathematics to elementary age students. The instruction was on a lower level than the instruction in the other studies, and still it produced an increase in mathematical abilities of the students and introduced students to real world problems. Two studies were for students at the middle school level; of these one study (Settergren, 1997) gave no information on students’ achievements, but the study discussed the real world applications and current mathematically research to which students were introduced. The other study at the middle school level (Carney, 1997) reported a positive conclusion on the skills and perceptions of students on the discrete mathematics topics taught. Three studies taught discrete mathematics at the high school level. Of those three, two gave positive conclusions on mathematics specifically regarding either the attitudes or achievement of students involved in discrete mathematics. One (Picker, 1997) also included that the students who were enrolled in discrete mathematics had an increase in motivation and attendance than in lower level high school mathematics courses. Only one of the four studies performed at the high school level (Thompson, 1992), reported a decrease in the
attitudes of students and achievement in mathematics. Lastly, one study did not give an age group. The study was conducted at a science camp. In this study the students were able to learn more about mathematics and what mathematicians do, including an introduction to current research and real world problems. The conclusion from this study also included positive attitudes of students. From the above analysis it is apparent that discrete mathematics can be taught at any age or grade level and achieves positive outcomes for students in attitudes and mathematical success.

**Textbook Resources**

As this project is about textbooks, a review of the resources used in each of these studies was analyzed. There were three studies that did not give a specific textbook used to guide instruction, but instead used various sources. All three reported positive outcomes on students' skills, perceptions, or motivation. One study used the author's own text; the article concluded that the attitudes of students were engaged and motivated. Three studies used current research to guide instruction. Of the three, two expressed positive outcomes of student's achievement and opinions toward mathematics. The other study gave no report on student's abilities or perceptions. Of the seven studies, four included specific textbooks used by the instructor alone or with other sources. All of the studies except one (Thompson, 1992) expressed positive conclusions for students studying discrete mathematics. In general, using a variety of sources, current research and certain textbooks increased encouraging outcomes for students.

**Inclusion of Discrete Mathematics in High School Curriculum**

Of the original seventeen articles, seven articles were found that specifically discussed including discrete mathematics into high school curriculum. Two of the articles were studies, while the other five were commentaries about discrete mathematics being added to the
curriculum. The two studies were reviewed and analyzed. A narrative for each article and the four commentaries are summarized below. An analysis of the articles is also given.

Studies

In Harold Bailey’s dissertation, *Discrete Mathematics in Undergraduate and High School Mathematics Curriculum* (1995), he studied the place of discrete mathematics at both the collegiate and secondary levels. This review will focus on discrete mathematics in high schools. Bailey sent a questionnaire to 50 high school’s with 34 responding. He chose the high schools based on a random sample from a previous study (p. 59). Approximately 20% of the high schools from Bailey’s study reported teaching discrete mathematics (p. 79). In general, those that reported teaching discrete mathematics taught it to only 11th and 12th graders (p. 85). Those opposed to teaching discrete mathematics at the high school level argued mainly toward budgetary reasons and perceptions of discrete mathematics. Thus courses in discrete mathematics are low on the priority list due to other factors in high schools (pp. 85-86).

Olgamary Rivera-Marrero also examined discrete mathematics in high schools in her 2007 dissertation, *The Place of Discrete Mathematics in the School Curriculum: An Analysis of Preservice Teachers’ Perceptions of the Integration of Discrete Mathematics into Secondary Level Courses*. Rivera-Marrero chose four preservice teachers and performed an in-depth case study for each teacher to determine their view of mathematics in high schools. The participants were chosen based on accessibility and requirements of previous experience with discrete mathematics (p. 25). The students expressed views that demonstrated an understanding of the common topics of discrete mathematics with each discussing the applications that can be used especially at the high school level (p. 73-74). Discrete mathematics was lauded by the pre-service teachers for the opportunity to teach students to be problem solvers (p. 77) The curiosity and
motivation of students were discussed by the preservice teachers, who claimed that discrete mathematics problems are “challenging, interesting, and enjoyable to students and keep them involved in mathematics” (p. 75). In conclusion, the four preservice teachers also advocated that discrete mathematics should be included in the general high school coursework (p.82).

Commentaries

In 1991, NCTM published a yearbook, based on discrete mathematics throughout the mathematics curriculum from elementary school to high school. Three of the commentaries come from this volume including John Dossey’s article “Discrete Mathematics: The Math of Our Times” (1991). In this commentary, Dossey claims the case for including discrete mathematics into the high school curriculum has been made by information from NCTM, NFS, the Conference Board of the Mathematical Sciences, other authors and the growth in articles on the topics, mentions of discrete mathematics in textbooks, and teacher’s knowledge of discrete mathematics (p. 4). Dossey also explains the efforts by the NCTM to determine an outline for how to implement discrete mathematics topics and into classrooms (pp. 5-6). To finalize his argument, Dossey asserts that discrete mathematics allows the students to investigate unique problems that are not easily viewed in the parlance of algebra or geometry. Discrete mathematics allows students to create models, scrutinize special cases and find solutions using easily understood and simple problems (p 8).

On the other hand, Anthony Gardiner wrote “A Cautionary Note” also in the NCTM yearbook. In the commentary, Gardiner begins by expressing that discrete mathematics is not “new”, but it seems the topic of current interest. He suggests that just because it is of current interest, discrete mathematics is not necessarily the mathematics the students need in the changing world (p. 11). A few of the arguments that Gardiner asserts against pushing discrete mathematics into high schools are as follows. First, discrete mathematics in its “natural state” is
beyond the abilities and capacities of most high school students and their teachers (p.12). Gardiner also claims that discrete mathematics requires students to consider every aspect of a problem, which is ideal, but the students are being “helped along” by teachers to solve the problem. This action by teacher does not allow this process and problems to be the learning experience they need to be (p. 12). Another quam that Gardiner expresses is that for high school students to think in the realm of discrete mathematics is “unnatural” (p. 12). Lastly Gardiner states that students will not understand the underlying principles of algorithms that are being taught. The algorithms are given and asked to be taken as valid, though students are not taught the process of finding the algorithm; this is more important to the student as the algorithm in time may be obsolete (p. 13). Gardiner does concede that if discrete mathematics is to be included in the curriculum it should focus on only a few topics and the processes of counting that leads to the addition and multiplication rules. Furthermore, students must be adequately prepared for the techniques used in discrete mathematics (p. 13). In conclusion, Gardiner does not deny that discrete mathematics has its place because of computers; but if it is taught at the high school level it must be done with care (p. 17).

In the same volume Eric Hart provides another view on the subject with “Discrete Mathematics: An Exciting and Necessary Addition” (1991). Hart states that many topics commonly defined as discrete mathematics are already seen in secondary curriculum but not identified as such. Due to the business and industry applications, other aspects and problems in discrete mathematics should be introduced at the secondary level (p. 67). Hart also notes that most reports in the late 1980’s and early 1990’s indicate the need for discrete mathematics to be integrated into the secondary curriculum (p. 74). Specifically he notes that discrete mathematics should be included into the curriculum due to the following arguments. First, mathematics is not a dead study and students should be introduced to current research. Also, teaching discrete
mathematics is a conduit to teaching problem solving and modeling with many applications including social science and management science. Finally, discrete mathematics enhances and works well with the traditional areas of the curriculum (pp. 74-75). Hart concludes with the notion that discrete mathematics can be easily integrated into the current curriculum or taught as a separate course (p. 77).

Kenneth Boggart also commented about the inclusion of discrete mathematics in secondary schools in “The Roles of Finite and Discrete Mathematics in College and High School Mathematics” (1991). Boggart claims that many finite math topics can be introduced in the second year of algebra and have many applications (p. 82). With respect to discrete mathematics, Boggart asserts that many college bound students would be better prepared for college by taking a one-year course in discrete mathematics than AP calculus. His assertion is based on a view that discrete mathematics focuses more on students’ critical thinking skills (p. 83). Boggart also states that a discrete mathematics course would be well placed after the second year of algebra for all students; he believes discrete mathematics would be an essential mathematics course for students who are not college-bound (p. 86).

In the DIMACS series on discrete mathematics in schools, Joseph Malkevitch discussed views of mathematics and how discrete mathematics differs from common opinions of mathematics in “Discrete Mathematics and Public Perceptions of Mathematics” (1997). Malkevitch begins by contrasting the common algebra problems used to assess understanding in high school mathematics with a list of discrete mathematics problems. He comments that the contrast in the lists are highlighted with what is required to solve the problems, the manipulation of symbols and one correct answer, for algebra prompts, verses the prompts from discrete mathematics which require critical thinking and a focus on the process rather than the answer (pp. 89-92). Malkevitch gives that the goals of mathematics include the developing critical thinking
skills, comprehending “spatial concepts” and preparing students for their future in society and job opportunities; he continues in the next paragraph that “discrete mathematics lends itself to achieving some of the goals for mathematics education more effectively than what is currently taught” (p. 92). Discussing what mathematics is needed in schools, Malkevitch gives several topics many of which, though not all, require an understanding of discrete mathematics and which would show students that mathematics is included in many real world applications (pp. 94-95).

To conclude, Malkevitch proposes that the mathematics taught in schools should change the perceptions of mathematics to the non-mathematical community; discrete mathematics is a testing ground to change these views for the better (p. 96).

Conclusion

Arguments for the inclusion of discrete mathematics into the curriculum vary depending on author, but most authors seek to include discrete mathematics due to the NCTM (1989) recommendations. Most commentators and preservice teachers see the inclusion of discrete mathematics in the curriculum due to the motivational and interesting applications involving discrete mathematics in many areas not just the hard sciences. They also argue that discrete mathematics is different from the mathematics most students see in geometry or in algebra and that it fosters critical thinking. Another author and school administrators (or department heads) argue that it is not feasible due to content difficulty, school structure, and funding. Thus the placement of discrete mathematics has many arguments for and some against its implementation into the high school curriculum. Due to the fact that most educational decisions are made at the state and local level, it will be for state offices and boards of education to determine if discrete mathematics will be put into practice in the schools.
The final three articles in this literature review suggest resources and texts for teaching discrete mathematics at the high school level while one particularly promotes a specific text.

Fred S. Roberts gives applications for discrete mathematics and resources in his article included in the DIMACS publication (1997). Roberts gives a list of several sources (42 in all) for applications that are relevant to discrete mathematics. He includes sample applications in three areas: the traveling salesman problem, graph coloring, and Eulerian chains (walks). Roberts concludes his list with the comments that part of the reason discrete mathematics should be taught is the richness of applications and helping students to see that mathematics is applicable in real world situations (pp. 115-116). In general, Roberts’s list of resources comes from textbooks and other articles giving instructors many places to find and bring the topics of interest to their classrooms.

Nancy Crisler, Patience Fisher, and Gary Froelich created a textbook on discrete mathematics and describe their efforts in, “A Discrete Mathematics Textbook for High Schools” (1997). Using a connection with COMAP, the Consortium for Mathematics and its Applications, and the authors’ view that implementing discrete mathematics into high schools is impeded by the lack of a proper text, the three authors set out to create just such a text. Using the NCTM report and standards written by John Dossey (1990) the authors began to organize the topics and themes to be included in the text (pp. 324-325). The text includes chapters on Election Theory; Fair Division; Matrix Operations and Applications; Graphs and Their Applications; More Graphs, Subgraphs, and Trees; Counting and Probability; Matrices Revisited; and Recursion (pp.326-330). The author’s based the order of the chapters on their own experience teaching discrete mathematics (pp. 325). They tested the textbook on their own students and in teacher forums.
The authors received mixed reviews of the textbook from those who viewed it; nevertheless their textbook was published in 1994 under the title *Discrete Mathematics Through Applications* (pp. 324-326).

Also included in the *DIMACS* volume is a review by Deborah S. Franblau and Janice C. Kowalski on information for teachers in the section “Recommended Resource for Teaching Discrete Mathematics” (1997). Franblau and Kowalski give a list of resources based on their experience in the Discrete Mathematics Leadership Program at Rutgers (p. 373). They have organized their resource list by type and give an appropriate age level for each item (p. 374-375). Franblau and Kowalski list only four texts, over a dozen sources for student activities, a dozen trade books and literature for students, texts for teacher’s references, videos, software, and internet sites (pp. 375-403). The four texts the authors recommend are *Mathematics a Human Endeavor*, *For All Practical Purposes*, *Excursions in Modern Mathematics*, and *Discrete Mathematics through Applications* written by Crisler, Fisher, and Froelich (p. 375). For each text Franblau and Kowalski give comments from publishers, topics in the text, intended use, and some comments. The other sources are summarized similarly.

Thus the resources for teaching discrete mathematics exist, but whether the common mathematics instructor knows of their existence and can find access to them is a different question the literature did not discover. As suggested by Franblau and Kowalski the instructor is the best judge of what material is appropriate and would work in their classrooms (1997); thus the resources should be analyzed by instructors and implemented at their discretion.
CHAPTER 3

PROJECT DESCRIPTION

From my coursework as a preservice teacher, I learned many disciplines in mathematics that I did not know existed as a high school student. From experiences in my undergraduate courses, I felt students at the high school level could be introduced to the material I was studying and develop an understanding in such topics. In a capstone class and a graph theory class, I developed ideas and lessons to implement topics such as graph theory, game theory, and number theory for high school students. This project grew from those ideas and lessons and focused solely on implementing discrete mathematics content at the high school level.

Goals of the Textbook

The textbook in development will present discrete topics to students and teachers at the high school level. The text will be a reference book for teachers with accompanying lesson plans and a few student handouts and task sheets. The education level of the text is intended specifically for students who have passed geometry and the second year of algebra, though the materials can be arranged and modified to work in any classroom. In conjunction with the findings in the literature review, the textbook is intended as a guide for teachers and contains optional resources to aide instructors.

The focus of the topics in the text will be on discrete mathematics and its applications in areas such as combinatorics, graph theory, set theory, number theory, sequences, and social decision making. Also, the text aims to present students with new ways to view mathematics and their own thinking. Besides the coursework in the above topics the text will attempt to achieve the following goals through the suggested lessons it contains.

- Teach students the philosophy of mathematics.
• Give students a historical perspective of mathematics and incorporate historical
documents into student learning.
• Train students in the nature and techniques of proofs.
• Introduce students to the language of mathematics and have them use it continually, with
a goal of fluency.
• Present to students techniques to model real life problems.
• Increase student appreciation for mathematics and logic.
• Introduce students to open problems in mathematics research.
• Give students and teachers the opportunity to direct their own learning and focus on areas
of interest as well as current research in mathematics.

The design and layout of the textbook is to give instructors a brief tutorial on the topic, an
optional lesson plan, and assessment ideas. Each lesson will also include prerequisite topics and
topics that can be considered as extensions of the lessons. At the end of each lesson are several
resources used to create the lesson and technology that can be implemented to aid discussion.

Table of Contents of the Textbook

Below is a proposed table of contents for the textbook. The chapters, or main topics, are
given with sub topics that would be included in the chapter. An explanation to the instructor is
given as well.

This text is designed specifically for teachers; included are definitions, theorems, lesson
and activity ideas that can be implemented in teaching discrete math. The format of the
text is such that topics need not be addressed in the order they are presented, and hence
they are not numbered below. Overlap of topics is expected and encouraged to enhance
connections.
The instructor is encouraged to create their own progression of lessons based on students’ interest and needs. Optional task sheets for some lessons are provided.

Table of Contents

- What is Discrete Mathematics?
- Sets and Notation
  * Axioms of Set Theory: Naïve and Axiomatic Set Theory, Set Notation, Subsets, Operations on Sets, and Cardinality (Size of Sets)
- Logic
  * Statements, Notation, Deduction, Induction, and Contrapositives and Indirect Proofs
- Counting
  * Combinations, Permutations, Principle of Inclusion Exclusion, and Multi-sets
- Functions
  * Injection, Surjection, Bijection, and Relationships to Counting Problems
- Sequences
  * Fibonacci, Notation, Recursion, and Generating Functions
- Graph Theory
  * Notation, Definitions, Subgraphs, Trees, Regular graphs, Coloring Problems, Directed graphs, Interval graphs, and Open Problems.
- Tournaments
  * Notation, Transitive tournaments, Cycles, Kings
- Matrices and Linear Algebra
  * Introduction, Basic Operations, Applications, and Connections to Discrete Math Topics
- Design Theory
  * Game Theory, Election Theory, Latin Squares
- History and Discrete Math
  * Historical Context of Early Problems and Mathematicians of Interest
- Applications
  * An array of applications of select topics covered in the text.
- Resources
  * A brief list of other materials on the topics covered in the text, including other texts, journals, software, and other resources.
Most of the topics presented are standard for several of the more common textbooks on discrete mathematics. To aid the goal as a teacher's guide, the last three chapters were included to assist the teacher in the endeavor to increase content knowledge, teach mathematics from a historical perspective, and encourage real world examples.

Methods Used in and Description of the Lessons

For this master's project, five lessons or units were developed covering several topics. The lessons were guided by my study of mathematics education and the methods presented in Teaching Mathematics in Secondary and Middle School: An Interactive Approach by James S. Cangelosi (2003). Other sources included my time in discrete mathematics courses, such as combinatorics and graph theory, several texts on discrete mathematics, articles on discrete mathematics problems, articles with activities in discrete mathematics classrooms, the internet, and the sharing of ideas with colleagues.

The main text used to determine the methods of lesson plans presented is the aforementioned text by James S. Cangelosi (2003). In his book, he discusses how research-based methods for teaching do not focus on repeated algorithms or "drills", and that computation should not be the end goal for mathematics curriculum (2003, pg. 142). In connection with research-based principles, the lessons of the text seek that students construct concepts and that algorithms are tools and not the primary focus in mathematics.

Specifically the lessons are designed to be permuted by the discretion of the instructor as suggested by Cangelosi (2003, pg. 131):

Because mathematics is widely misunderstood to be a linear sequence of skills to be mastered one at a time in a fixed order, some people think teaching mathematics is a matter of following a
prescribed curriculum guide or mathematics text book. In reality, there are three reasons you must creatively develop curricula to succeed as a mathematics teacher.

The three reasons Cangelosi cites include the responsibility of a teacher to organize materials and lessons to help his student develop the state and district competencies; not all linear presentations of mathematics are the best way to teach all students, though for some topics other information is prerequisite for increased understanding; and the order and method of teaching has an impact on how your students understand and view mathematics (2003). Thus each lesson or series of lessons in the text can be introduced at any time.

Another aspect that influenced the formatting of the lessons and structure of the text is given in an excerpt from the National Council of Teachers of Mathematics. In the excerpt, the description of a perfect classroom and school for teaching mathematics is described. It includes the following ideas: technology in the classroom is an essential part of the learning atmosphere, that curriculum is powerfully mathematical, students are speaking and writing mathematics, concepts are being derived with understanding, and students are taught to be true problem-solvers seeking answers and moving with ease through the unexpected turns in problems (2000, pg. 3). These aspects inspired the tools for teaching in the lessons and the desired goals of the textbook.

The lessons included in this project each contain three or four sections. At the beginning of the lesson there is a set of goals or objectives that the students should achieve during the lesson. The goals are to give the teacher a guide and a glimpse of the topics to be covered in the lesson. After the goals, there are a set of introductory notes for the teacher. The introduction is general information that an instructor can use to learn or review the material to be taught. The introduction utilizes more notation and assumes the instructor has mathematical maturity for self-education. Following the introduction are lesson plan and activity ideas. This section includes information on how many days the lesson would take, any prerequisite material, and notes on
implementation. Lessons utilize different groupings of students, classroom discussion, and activities to construct concepts, discover relationships, and apply the material to real world settings. Some of the lessons include task sheets or handouts to use while implementing the lesson given. All of the material is intended as a guide and can be modified, ignored, and embellished as to the needs of the instructor and students. Finally, several exercises are included for each lesson. The exercises vary in intensity and degree of difficulty. Some exercises are specifically intended for homework or an assessment. All of the prompts and exercises are suggestions based on the previous lesson. The instructor is encouraged to determine what is appropriate for the students and to change any prompt in order reflect the classroom experience.

In appendices C, D, E, F, and G, the five lessons are presented to the reader. The lessons included cover set theory, introduction to graph theory, sequences and functions, generating functions, and voting theory. These five lessons given are to represent several of the possible topics that can be introduced. The lessons on sequences and generating functions were chosen to demonstrate how two sections can build upon each other. The lessons presented exhibit real world examples and applications of topics, each encourages students to write and communicate mathematics, and several exhibit current research and topics in mathematics.
CHAPTER 4
LESSON EXECUTION
Students and Classroom Characteristics

On January 28th 2010, the lesson on set theory (appendix C) was presented to a college preparation class at Lehi High School. The class consisted of twenty-six 11th and 12th graders. The students had “passed” the second year of algebra with a C or D letter grade. The students were deemed by faculty not ready for a pre-calculus course. Thus the students were encouraged to take this course to prepare for college mathematics entering in at the pre-calculus level. As the teacher of the course, Ms. Merissa Cunningham, stated that the course was basically a repeat of Algebra 2 with a little more room to cover other topics (personal communication Sept. 3, 2009). Using this “room” in the course schedule, arrangements were made for the execution of the lesson at a convenient time for the class.

Ms. Cunningham’s classroom is usually conducted with a bell ringer exercise first. Then a brief discussion of the homework is given to the class. Afterward Ms. Cunningham delivers a lesson using various methods including lecturing, group work, and activities. The class is usually given ample time in class to do homework and ask questions of the instructor. The class contains some technology, mostly for presentation purposes. The technology for student-use is limited only to calculators. Whether Ms. Cunningham’s class has access to personal computers on a somewhat regular basis, is unknown. Lehi High School is on an A-B schedule. The days of the week are divided into “A” days and “B” days. This allows for longer periods since there are only four periods in a day. Thus Ms. Cunningham has about an hour and twenty minutes for the period to perform all classroom procedures and tasks.

Before the set theory lesson was executed in the classroom, the students had learned basic operations of sets, union and intersection. Students did not have a strong understanding that
sets are mathematically objects or that there existed more operations among sets besides union and intersection. The lesson was used by Ms. Cunningham as a review for a midterm that the students would be taking the next class period. Basic set operations were included in the midterm.

Method of Lesson Execution and Data Collection

The lesson at Lehi High School was taught by the creator of the text with Ms. Cunningham aiding in classroom management and group monitoring. The class was given its normal bell ringer activity, but with the content on operations between sets. After a discussion from Ms. Cunningham and a brief explanation of this project and some of the topics of discrete mathematics by the author, the class was given the task sheet from the set theory lesson. The students were grouped into groups of threes and fours based upon the position of their sits. While students worked in groups the author and Ms. Cunningham walked among the groups asking questions and observing the discussions among the groups. The lesson continued by transitioning between the groups and the class discussions. Technology was used in the lesson to show concepts and diagrams as given on websites. Due to time restrictions, approximately one-third of the lesson material was covered during the time in the classroom. The material in the lesson that was taught included constructing the concept of a set, discussion of mathematical objects with circular definitions, binary operations of sets, the complement of a set, the empty set and the universe, notation, subsets, and the construction of Venn diagrams.

The next class day, the students completed a survey on their views of the lesson (see appendix H). Only 23 of the 26 students who participated in the lesson completed the survey. The discrepancy between the two numbers was likely due to absences from class or students choosing not to complete the survey. The surveys and the task sheets used by the students were returned to the author for analysis. The responses of the students were recorded for the first eight
prompts on the survey (appendix 1). Simple statistics were viewed on the data from the surveys. Comments from the author and Ms. Cunningham were recorded to guide feedback on the lesson for modifications and future implementation (personal communication, January 28, 2010).
CHAPTER 5

RESULTS OF THE LESSON EXECUTION

Student Survey Data Analysis

With only a relatively small number of students surveyed on the lesson, the author chose to only view simple statistics such as average and standard deviation for some prompts, basic percentages for others, and qualitative analysis for the final prompts. The task sheets were viewed and analyzed, but as most of the work was in monitored groups or as a large class, the responses did not deviate a great deal from the general discussion in class when the lesson was presented.

After the student surveys were recorded, the average and standard deviation were calculated for each of the first six responses. Likewise frequency analysis was performed to determine if a general trend persisted. The frequency distributions are given in appendix J. A table is given with the average and standard deviations for each aspect of the lesson to which students responded.

<table>
<thead>
<tr>
<th></th>
<th>Engaging</th>
<th>Informative</th>
<th>Clear Instruction</th>
<th>Level of Difficulty</th>
<th>Format</th>
<th>Activities</th>
</tr>
</thead>
<tbody>
<tr>
<td>Average</td>
<td>2.826</td>
<td>3.739</td>
<td>3.087</td>
<td>3.478</td>
<td>3.261</td>
<td>3.045</td>
</tr>
<tr>
<td>SD</td>
<td>1.302</td>
<td>0.964</td>
<td>1.083</td>
<td>1.163</td>
<td>1.251</td>
<td>1.495</td>
</tr>
</tbody>
</table>

Of special note is the “level of difficulty” prompt. Several students relabeled the numbering on their survey on the prompt. Instead of “1” representing “poor” and “5” representing “excellent”, students indicated that “1” represented “easy” and “5” represented “hard”. Since less than half of the students made these notes on their surveys and others did not,
the data may not represent the author’s intent or may be a mixture of data for two different prompts.

Students were also invited to respond to prompts about discrete mathematics; specifically if they would take a discrete mathematics course it were offered and if they believed discrete mathematics is more applicable for real-life problems than algebra. Of the 23 students, six (26%) responded that they would take a discrete mathematics class, and 17 (74%) said they would not. For the prompt on discrete mathematics versus algebra in real-life problems ten (43%) of the students responded “yes”, 12 (52) responded “no”, and one did not answer.

Lastly, the students were invited to leave comments on the lesson. Specifically the students were asked to note what they liked the most and least as well as what they learned about sets. Several comments are given below.

- “You didn’t have to solve equations.”
- “I liked that we could work in groups”
- “It was boring”
- “I didn’t understand what was going on.”
- “I didn’t like all the drawing.”
- “I didn’t like the packet, the fact that I couldn’t just do the assignment and turn it in without the notes.”
- “The examples were fun.”
- “It’s more real life problems and a little bit easier to figure out.”

Other general comments were on the author’s teaching style as opposed to their normal instructors. Though an instructor’s teaching style is crucial in the task of learning, the lessons will be implemented by many instructors with their own classroom management and style of presentation. Thus only the comments specifically about the lesson were included.
Instructor and Author Comments and Feedback

After the lesson was given Ms. Cunningham and the author discussed the lesson and its execution. Ms. Cunningham made note that as an instructor she would like a filled out task sheet to aid in her teaching. Though some prompts she understood the intentions of the author, others would have left her confused. Another thought Ms. Cunningham expressed was that all instructions from the task sheet should be read aloud and then discussed. Thus students would have a general idea before they worked as a group or individual on what was expected of them. Ms. Cunningham expressed that she would modify what she saw in class to fit her own needs and understanding as well as those of her students. Other suggestions included using web-quests for students to explore the concepts outside of class and utilizing technology more (personal communication, January 28, 2010).

The author also made notes soon after giving the lesson. Similar to Ms. Cunningham, the author felt that the instructions though clear contained phrases and words unfamiliar to the students. For example, the task sheet asks students to create a Venn diagram containing three sets, where no one set is a subset of the others. Though the students had learned what a subset was and had seen a Venn diagram for two sets without proper containment, most students only drew a Venn diagram very similar the one in the task sheet. Similarly during discussion of this prompt and the activities involving the prompt, the author noted that students demonstrated difficulty in identifying regions. When asked about the drawing students expressed a belief that regions were the same as the set representations. Another example of a mis understanding of language was the use of the logical phrase “if and only if”. When discussing the definition of a set, the students did not understand the meaning of the phrase. This made it difficult to complete several of the tasks.
The author noticed that despite the use of unfamiliar mathematical language when the prompts were clearly communicated, the students were able to complete the tasks. The tasks often presented a challenge and when a discussion took place between the author and the students, the students were able to express their thinking and then apply it to the prompt. Thus the prompts in the task sheet were manageable, but required the students to do more than follow an algorithm.

With respect to classroom instruction and management, the author noted that the students worked best as a class or in groups. Discussion was better as a class for the development of definitions and looking at operations on sets, but the activities produced more thinking and creativity when performed in groups. The author felt at a disadvantage in the classroom, as she did not know the standard procedures of Ms. Cunningham or anything about students’ previous experiences. The classroom management was more difficult because of these deficiencies.

For the lesson itself, the time limit in the classroom was a hindrance. Several of the topics could not be taught after teaching all of the prerequisite material in the limited time given. The author and Ms. Cunningham felt that three days of 50 or more minutes would be sufficient to present the information. Lastly, the author felt the motivation within the lesson itself was weak. The students were somewhat attentive at Lehi High School, because it was their homework for the day and part of the review for their midterm.

Conclusion

From the feedback given from the students and their instructor, as well as the author’s observations, the data described above gave formative feedback for restructuring the set theory lesson and other lessons. The areas of needed improvement and advancement would include
lecture structure, directions for instructor, and monitoring of difficulty level of prompts and tasks assigned.

The student’s feedback expresses the need for improvement in engagement and motivation specifically within activities. The students found the lesson informative despite the need for greater motivation within the lesson. From the comments and the prompt results for discrete mathematics in real-life situations, the students demonstrate that the content is seen as useful as algebra. Most students indicated that they would not consider taking a discrete mathematics if it were offered, but most students are seniors and will only take the math they need in college. The level of difficult for students like those taught at Lehi High School is unsure and must be gauged by the instructor.

The feedback from Ms. Cunningham and the author, reemphasize the focus for the textbook on preparing the instructor to teach the content. The textbook should focus on preparing the instructor to teach discrete mathematics and to engage students in problem solving. Also, the textbook should include other resources the instructor could use to supplement their teaching. The suggestions made by the students, Ms. Cunningham, and the author will be integrated fully into the complete textbook.

The execution of lessons suggested in the textbook would be different based on each instructor. If the instructor uses research based methods of teaching mathematics and implementing classroom management, the material will likely be taught with greater success than that done at Lehi High School by the author. The textbook will need to be implemented in many classrooms upon its completion and publication to determine its true quality and use.
REFERENCES


APPENDICES
# APPENDIX A

## ARTICLE SUMMARY OF LITERATURE REVIEW

### Discrete Mathematics in the Classroom

<table>
<thead>
<tr>
<th>Author(s)</th>
<th>Conclusions of author(s)</th>
<th>Definition of discrete mathematics</th>
<th>Sample characteristics age (academic year)</th>
<th>Sample characteristics geographic region</th>
<th>Textbook and Resources Used</th>
</tr>
</thead>
<tbody>
<tr>
<td>Biehl, L.C.</td>
<td>DM task make it easier for cooperative learning tasks. DM coursework improves grades from earlier math courses. DM coursework improves work completion rate. DM tasks help students have positive perceptions of math in their lives. DM increase awareness of math ability and intrinsic motivation. DM course allowed students to communicate ideas with mathematicians at university level.</td>
<td>Includes extraneous topics such as trigonometry</td>
<td>High school</td>
<td>Delaware East Coast</td>
<td>No textbook, used real-world examples and various resources.</td>
</tr>
<tr>
<td>Carney, P.</td>
<td>DM activities increased enthusiasm for review materials. DM activities increase general math skills such as skills with fractions and estimation. DM activities increase successful cooperative learning. DM activities increase attitudes of students about their math abilities. DM course allowed students to learn from current mathematicians and their research.</td>
<td>Commonly defined</td>
<td>Middle school, specifically 7th and 8th graders</td>
<td>Maryland East Coast</td>
<td>No specific text, various resources.</td>
</tr>
<tr>
<td>Casey, N.</td>
<td>Graph theory activities introduced students to open problems. Graph theory activities, taught logic and the difference between proof and counter example. Graph theory introduced students to math without &quot;numbers&quot;. Students were engaged and enthusiastic. Students learn what mathematicians do, by reviewing and applying information from mathematicians.</td>
<td>Commonly defined</td>
<td>No info, presumably high school</td>
<td>Idaho Mountain West</td>
<td>Author's illustrated texts, current research and articles from mathematicians</td>
</tr>
<tr>
<td>Author(s)</td>
<td>Conclusions of author(s)</td>
<td>Definition of discrete mathematics</td>
<td>Sample characteristics age (academic year)</td>
<td>Sample characteristics geographic region</td>
<td>Textbook and Resources Used</td>
</tr>
<tr>
<td>----------------</td>
<td>------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>------------------------------------</td>
<td>---------------------------------------------</td>
<td>------------------------------------------</td>
<td>-------------------------------------------------------------------------------------------</td>
</tr>
<tr>
<td>Kowalski, J.C.</td>
<td>Introduction of Fibonacci numbers to elementary students to review adding and counting. Students become aware of what a mathematician is and what mathematics is. Students learn and understand the process of mathematical modeling. Students make “real world” connections to mathematics.</td>
<td>Commonly Defined</td>
<td>Elementary School (4th graders)</td>
<td>Rhode Island East Coast</td>
<td>Fascinating Fibonacci, Excursion in Modern Mathematics</td>
</tr>
<tr>
<td>Picker, S.H.</td>
<td>DM tasks foster cooperative learning. DM coursework increases attendance in low attending students. DM coursework increases students continuation in mathematics courses. DM see math as more than arithmetic and see math more like a mathematician.</td>
<td>Commonly defined</td>
<td>High school</td>
<td>New York East Coast</td>
<td>No specific text, various sources including college textbooks, e.g. A Primer of Discrete Mathematics</td>
</tr>
<tr>
<td>Settergren, R. J.</td>
<td>Game theory lessons gave students a chance to review current mathematics and articles. Students’ attitudes and achievements not discussed.</td>
<td>Commonly defined</td>
<td>Middle School, 8th graders</td>
<td>California West Coast</td>
<td>Current Research For All Practical Purposes</td>
</tr>
<tr>
<td>Thompson, D. R.</td>
<td>Students exposed to the new precalculus dm curriculum found the coursework manageable. Students exposed to new curriculum did not have an increased interest in mathematics over the study. The new curriculum did not promote or increase proof logic.</td>
<td>Other: defines with precalculus</td>
<td>High school</td>
<td>Illinois Midwest</td>
<td>Precalculus and Discrete Mathematics (A text in development at the University of Chicago)</td>
</tr>
</tbody>
</table>
# APPENDIX B

## SUMMARY DATA OF THE LITERATURE REVIEW

### Discrete Mathematics in the Classroom

<table>
<thead>
<tr>
<th>Category (n=7)</th>
<th>Frequency</th>
<th>Percentage</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Conclusion(s) of author(s)</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. Coursework for students is manageable/ increased completion rate</td>
<td>2</td>
<td>28.5%</td>
</tr>
<tr>
<td>b. Coursework increases students interest in and attitudes toward mathematics and what it is</td>
<td>4</td>
<td>57%</td>
</tr>
<tr>
<td>c. Coursework does not increase in students interest in and attitudes toward mathematics</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>d. Coursework increases student motivation/attendance</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>e. Coursework increases students math/logic abilities</td>
<td>3</td>
<td>43%</td>
</tr>
<tr>
<td>f. Coursework does not increase students math/logic abilities</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>g. Coursework introduced students to real world applications and current research in mathematics</td>
<td>5</td>
<td>71.5%</td>
</tr>
<tr>
<td><strong>Definition of discrete math</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. Commonly defined</td>
<td>5</td>
<td>71.5%</td>
</tr>
<tr>
<td>b. Including extraneous topics</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>c. Other</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td><strong>Age/Academic year</strong></td>
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<td></td>
</tr>
<tr>
<td>a. Elementary School</td>
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<td>14%</td>
</tr>
<tr>
<td>b. Middle School</td>
<td>2</td>
<td>28.5%</td>
</tr>
<tr>
<td>c. High School</td>
<td>3</td>
<td>43%</td>
</tr>
<tr>
<td>d. No information</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td><strong>Geographical Region</strong></td>
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<td></td>
</tr>
<tr>
<td>a. East Coast</td>
<td>4</td>
<td>57%</td>
</tr>
<tr>
<td>b. Midwest</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>c. Mountain West</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>d. West Coast</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td><strong>Textbook and Resources Used</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>a. No specific textbook</td>
<td>3</td>
<td>43%</td>
</tr>
<tr>
<td>b. Author’s Textbook</td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>c. Current research articles and real world examples</td>
<td>3</td>
<td>43%</td>
</tr>
<tr>
<td>d. <em>Fascinating Fibonacci</em></td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>e. <em>Excursions in Modern Mathematics</em></td>
<td>1</td>
<td>14%</td>
</tr>
<tr>
<td>f. <em>Primer of Discrete Mathematics</em></td>
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</tr>
<tr>
<td>g. <em>For All Practical Purposes</em></td>
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<td>14%</td>
</tr>
<tr>
<td>h. <em>Precalculus and Discrete Mathematics</em></td>
<td>1</td>
<td>14%</td>
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</table>
APPENDIX C

SETS AND NAÏVE SET THEORY

Goals:
• Students construct the concept of a set and its elements
• Students are introduced to set notation and appreciate its utility in mathematics.
• Students understand operations between sets.
• Students are introduced to naïve set theory and its paradoxes.
• Students appreciate the use of sets to group mathematical concepts and objects.

Introduction:

A set is a collection of objects. This is not a definition of a set, as it is very circular. A set is not a well defined object, similar to points and lines in geometry. Something that should be pointed out to students is that the circular description of sets is not a definition. Below are some of the properties of a set that helps us to create the concept.

A set is also an object that can belong to another set. We denote a set by using the curly braces with a list of the elements contained in the braces; the elements are separated by commas. We use a capital letter to name the set; for example
\[ A = 0, 1, \sqrt{2}, e, \pi \] \[ B = \{ \text{apple}, \text{orange} \}. \] Or we can define a set by describing the characteristics of the elements in the set, for example
\[ T = x : x \text{ is a student at East High School}. \]

Consider the following set and its elements,
\[ \{ 0, 1, \}, \{ 1 \}. \] The elements of the set are as follows: the empty set, the set containing 1, a comma, and the numeral 1.

Another way of representing sets is pictorially, often using a Venn diagram. This picture usually involves one or more sets and the relationship that the sets have to each other. An example of a Venn diagram (below) which represents sets \( A \) and \( B \), their union and intersection (both of these terms which we will discuss below).
The objects that belong to the set are called the **elements** of the set. The notation to denote that \( x \) is an element of the set \( S \) is often stated \( x \) is in \( S \). Moreover, the string of symbols \( x \in S \) means that "\( x \) is contained in \( S \)" and \( y \notin S \) means that \( y \) is not an element of the set \( S \) or \( y \) is not contained in the set \( S \).

The elements of a set are distinct and can be any object, a number, person, property, etc. Thus the use of sets in mathematics is vast. By definition, \( A \) is a set if and only if there is an \( x \) that is an element of \( A \) or \( A \) is the empty set. Note that \( \pi, \pi, \pi, \pi = \pi \), that is repeats of an element does not affect the properties of the set. Order in a set is inconsequential unlike a sequence or a tuple; therefore \( m, a, t, h = a, h, m, t \).

As an aside, an \( n \)-tuple is an ordering of \( n \) elements. An example of a 5-tuple is student, exam 1 score, exam 2 score, exam 3 score, final exam score. This can represent a student's exam scores over the term. It can be noted that only the second entry represents the student's score on exam 1. Also note that the numbers or elements in the 5-tuple need not be distinct.

There exists a unique set that contains no elements; it is called the **empty set** and is denoted by \( \emptyset \). In contrast we will call \( U \) the **universal set**. \( U \) is the set from which all others are a subset and thus contains all elements of interest in the context.

As with other mathematical objects there are operations we can perform with sets. The operations are given below with their name and definition.

- **Union of two sets**, \( A \cup B := \{ x : x \in A \text{ or } x \in B \} \)
- **Intersection of two sets**, \( A \cap B := \{ x : x \in A \text{ and } x \in B \} \)
- **Cartesian product of two sets**, \( A \times B := \{ x, y : x \in A \text{ and } y \in B \} \)
- **Set theoretic difference between \( A \) and \( B \)**, \( A \setminus B := \{ x : x \in A \text{ and } x \notin B \} \)
- **Complement of a set**, \( \bar{A} := \{ x : x \notin A \text{ and } x \in U \} \).

Notice, that each of these operations is binary. With some proving the union, intersection, and Cartesian product can be applied to more than two sets in a recursive manner similar to
multiplying 3 or more numbers to each other. Thus we would define \( A \cup B \cup C = A \cup B \cup C \).

Also note that each operation also yields a set.

Sets also have relationships to each other. The set \( B \) is a subset of the set \( A \) if each element of \( B \) is also an element of \( A \), and is denoted \( B \subseteq A \); in set notation it can be written that \( B \) is a subset of \( A \) if and only if for \( x \in B \), \( x \in A \). The empty set is a subset of any other set. There is also the notion of a proper subset, denoted \( D \subset C \). \( D \) is a proper subset of \( C \) if and only if \( D \subseteq C \) and \( D \neq C \).

Two sets are equal if and only they contain the exact same elements, that is \( B \subseteq A \) and \( A \subseteq B \) if and only if \( A = B \).

At times we may also wish to describe the size of a set, or in other words, the number of elements that belong to a set. This is sometimes called the cardinality of the set and is denoted by \( |S| \). For example if we take the set \( A = \{0, 1, \sqrt{2}, e, \pi\} \), we have \( |A| = 5 \). Note that the cardinality of a set is always a nonnegative integer.

Like all mathematical topics, there are axioms to guide set theory. There are two types of set theory, naïve and axiomatic. This lesson will introduce naïve set theory; as an extension to this lesson you can choose to explore the axioms of axiomatic set theory. In naïve set theory, the only axioms are the properties of union, intersection, complement, equality, subsets and that for some object \( x \) and set \( A \), either \( x \in A \) or \( x \notin A \), not both.

Though the axioms of naïve set theory seem intuitive, there are some paradoxes that arise. One example is Russell’s paradox, named for Bertrand Russell.

Russell’s paradox: The set containing all sets that do not contain themselves does not exist. That is that \( x: x \) is a set and \( x \notin x \) does not exist. In the lesson plan, the class can discuss why this set does not exist.

Most other paradoxes of naïve set theory are another form of Russell’s paradox or deal with infinite sets. For more information about the other paradoxes of naïve set theory see http://en.wikipedia.org/wiki/Naive_set_theory#Paradoxes.
Lesson Plan (three to four 50 minute lessons)

There are no topics that are prerequisite for the understanding of this section.

An optional worksheet with added instructions and guided notes is attached (Attachment C.1). The worksheet can be modified to include other sets of interest with objects and figures that are not familiar to students that might have them exploring outside of class. Also make sure you read all instructions as a class, even when allowing the students to work individually or in small groups.

Give the students the following two columns. Fill in the information specific for your class and/or school in some of the cells to give specifications. You can also add or delete rows that will help demonstrate the difference between the two columns.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
<tr>
<td>list the members of the class</td>
<td>Mr./Ms. &lt;your name&gt;’s discrete mathematics class</td>
</tr>
<tr>
<td>Paul McCartney, John Lennon, George Harrison, Ringo Starr</td>
<td>The musical group that started in 1960 in Liverpool, England whose band name is The Beatles</td>
</tr>
<tr>
<td>2,1</td>
<td>The collection of real solutions to the equation $x^2 + 2 = 3x$</td>
</tr>
<tr>
<td>Barrack Obama, Michelle Obama</td>
<td>The couple of the President and First Lady of the United States from 2009-</td>
</tr>
<tr>
<td>Hearts, Spades, Diamonds, Clubs</td>
<td>The collection of suits in a standard 52-card deck</td>
</tr>
<tr>
<td>The pink with purple spots dragon in the back corner of the classroom</td>
<td>The collection of real solutions to the equation $\frac{21}{x} = 0$</td>
</tr>
<tr>
<td>list the administrators of the school by name</td>
<td>The administrators for &lt;school name&gt;</td>
</tr>
</tbody>
</table>

You will note that column A contains the elements of the sets described by column B.

Have students get into groups of three to four students. Let the groups discuss the differences between column A and column B. Walk about the room and listen to the students discussions.
Answer students question and ask questions of your own based on the discussion. Try not to answer questions from students about whether or not their conjecture about the differences in the columns is correct or if students are thinking on the right track. Answer questions that help students reason out the subtle differences between the wording for each cell in column B as opposed to column A.

After students have discussed the differences for a sufficient time, gather the class and have individuals and groups present conjectures. Lead the discussion to discuss how column A gives specifics and column B is an overall description each element in column A. In the end, you may need to use direct instruction to explain the differences between a set and the elements contained in the set. Also during this discussion it is important to show that sets are not well-defined and the definition we usually give for them is very circular.

Teach the class about the concept of a set and elements of set. Above are several sets that can be introduced and used as well as the set of natural numbers, integers, perfect squares, and many other sets (note that the set of natural numbers is represented by the symbol $\mathbb{N}$ and the integers by the symbol $\mathbb{Z}$).

Introduce the notation of sets using curly braces, lists, and descriptions and the notation of inclusion of an element into a set. Ask the students why we don’t continually write instead “$x$ is an element of the set $A$”.

Introduce the binary operations of two sets, union, intersection, and Cartesian product. Give the definition of these operations using set notation. Give the students two sets; e.g. $2, 3, 5, 7$ and George Washington, Abraham Lincoln. Use the two sets as a base for the students to explore the definitions given. Once again point out to the students how using set notation in the definition saves time and shows us that these binary operations yield sets.

Now is a very appropriate time to introduce Venn diagrams. Show how it is easy to draw diagrams for two or three sets and their relationship similar to that below.
Start with the Venn diagram representing two sets and then move to the diagram representing three sets. Label the diagram and note the difference between a region \((A \cap \overline{B})\) and a set \((A)\). Point out how it can illustrate the definitions of the operations we have introduced. Using the diagram point out the portion that represents the set difference between two of the sets, i.e. \(A \setminus B\).

Ask the students to define this area using English and then using set notation. Which one is more concise and which is more vague? Help the students to refine the definition as the discussion continues. Repeat this process for the complement of a set. Another resource that can be used, especially for homework is the National Library of Virtual Manipulatives; the address is below.

http://nlvm.usu.edu/en/nav/frames_asid_153_g_4_t_1.html?open=instructions&from=category_g_4_t_1.html

Create another diagram that illustrates the idea of a subset. Again have the students define the relationship using English and then using set notation and logic.

You can illustrate how notation is precise and time saving as opposed to English. Also at this time you could give the students a challenge to draw Venn diagrams that shows all possible intersections of 4 or 5 sets. Examples of such are given on Wikipedia key word Venn diagram.

Students at some point my want to denote the size of the set they are talking about. Introduce cardinality when appropriate for the class. Although infinite sets are interesting and the cardinality of infinite sets can raise more information on topics such as functions and bijections, limit the scope of the discussion to finite set. Discuss with the students what numbers are likely candidates for the cardinality of a set. If the empty set was not discussed when constructing the concept of a set, introduce the empty set as the only set with a cardinality of zero. You can use any of the sets presented in the table to give examples of sets with different sizes. Include in the discussion and presentation the notation used to denote cardinality.
Sets are also the foundation for logic implications. If $A$, then $B$ or notationally $A \Rightarrow B$, this means the objects or characteristics contained in the set $A$ are also contained in set $B$. In other words $A$ is a subset of $B$, $A \subseteq B$.

Introduce the idea of an implication using sets and subsets. Ask the students to draw a Venn diagram for crows and birds with black feathers. This can be a difficult concept especially when trying to show that a false implies a true, but the Venn diagrams and sets shows this more clearly.

Present to students the one axiom of naïve set theory. Present this as intuitive. Introduce Russell’s paradox. As a class discuss why this is a paradox. Choose an arbitrary set and determine if it belongs to $A = x : x$ is a set and $x \notin x$. After several sets that do or do not belong to the set, determine if $A$ itself belongs to the set.

If desired introduce the idea of axiomatic set theory, here.

Throughout the lesson find the uses of set theory and the usefulness of set notation and set operation. Find definitions, conjectures, and theorems that are easily described using sets and introduce them to the class.

Attached is a worksheet that can guide you in your class discussions and lessons.

Extensions:

- Axiomatic set theory
- Functions
- Counting

References

1. Consider the table below and answer the questions below.

<table>
<thead>
<tr>
<th>Column A</th>
<th>Column B</th>
</tr>
</thead>
<tbody>
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</tr>
<tr>
<td>list the administrators of the school by name</td>
<td>The administrators for &lt;school name&gt;</td>
</tr>
</tbody>
</table>

Compare column A with column B. Conjecture about what separates the items in column A from the items in column B.

What words in the description of the items helped you to determine your conjecture about the difference between the columns?

STOP! DO NOT MOVE ONTO THE NEXT SHEET UNTIL INSTRUCTED TO DO SO.
2. Write down the conjecture(s) that the class and individuals in the class presented on the differences between column A and column B.

3. Describe what a set is.


5. Try to develop an "if and only if" definition for a set.
6. Define happiness in your own words.

Is your definition a description? Would someone else define happiness differently? Explain.

7. We cannot define a set; we can merely describe it in vague terms, as a collection of objects. Notice how circular the description is. Are there any other mathematical objects that do not have a clear definition?

8. The objects in a set are often called the elements of the set or members of the set. List the elements of the set given below.
   a. The set of suits in a standard 52-card deck
   b. The set of prime numbers less than 10
   c. The set of real solutions to the equation $x^2 = -4$
9. Often we name a set so that we do not have to continue to describe it as with the sets described in 7.

Sets are denoted with a capital letter and can be described in words or as a list of elements enclosed in curly braces.

Example: \( S = \) suits in a standard 52-card deck or \( S = \spadesuit, \heartsuit, \diamondsuit, \clubsuit \).

Strictly using notation, please describe the set of prime numbers less than 10.

10. In the above question, you will note that 2 is an element of the set

\[ P = \{ x : x \text{ is prime and } x \leq 10 \} . \]

Using notation the inclusion of an element in a set is written \( 2 \in P \).

What numbers are not in the set \( P \)? How would you state this with reasonable notation?

11. List the elements that belong to the set of real solutions to the equation \( \frac{21}{x} = 0 \).

How many elements are there in the set of real solutions to the equation?

There exists a unique set that contains no elements; it is called the empty set. The empty set is denoted by \( \emptyset \).

Conversely there exists a universal set, the set \( U \); the set \( U \) contains all the elements in the context of the situation.
Set Operations and Relationships

12. Note the diagram below. It is called a Venn diagram.

Each circle and its interior represent a set and the sets elements. The sets in this diagram are labeled $A$ and $B$. The numbers represent the areas that will be discussed.

a. Describe the total shaded area of sections 1, 2, and 3 in words and using set notation.

This is known as the union of set $A$ and set $B$ and is denoted $A \cup B$.

b. Describe area 2 in words and using set notation.

This is known as the intersection of sets $A$ and $B$ and is denoted $A \cap B$. 
c. Describe area 1 in words and using set notation.

This is known as the set theoretic difference of $A$ and $B$ and is denoted by $A \setminus B$.

d. Describe the total area of 3 and 4 in words and using set notation.

This is known as the complement of $A$ and is denoted by $\overline{A}$ or $A^c$.

13. The union, intersection, and theoretic set difference of sets are binary operation on sets. Another binary operation is the Cartesian product, and it is defined as follows,

$$A \times B : = \{ x, y : x \in A \text{ and } y \in B \} .$$

Given $P = 2, 3, 5, 7$ and $Q = \text{Paul Erdös, Søren Kierkegaard}$.

Write out $P \times Q$ by listing the elements.
Note that the binary operations on sets yields

14. Consider the Venn diagram below.

[T is called a subset of S which is denoted by $T \subseteq S$.]
Questions and Tasks

1. Construct a Venn diagram containing three sets, where each set is not a subset of the other two sets.

2. In the Venn diagram constructed above, label each region with a number. How many regions are there?

3. Label the sets in the Venn diagram above. Write out what each region represents using notation.
4. Construct a Venn diagram containing four sets, where each set is not a subset of the other three sets. Make sure your diagram contains all possible intersections.

5. In the Venn diagram constructed above, label each region with a number. How many regions are there?

6. If a Venn diagram was constructed with $n$ sets, how many regions would there be? First consider how many regions there are for $n = 0, 1, 2, 3,$ and $4$. Is there a pattern?

Can you generalize the pattern to $n$?

Explain your answer to the original question.
7. Construct a Venn diagram representing the following sets $P = \text{people}$, $F = \text{human females}$, $M = \text{human males}$, and $T = \text{mathematicians}$.

8. Using your diagram above and the fact that $x \in F$, identify the other sets of which $x$ is a member.

9. Please write a sentence that involves the relationship you presented in prompt 8.

Using the relationships illustrated in the Venn diagram above one could write: if $y \in T$, then $y \in P$; or in other words, if $y$ is a mathematician, then $y$ is a person. This relationship is an implication.
10. Let the statement $y \in T$ be represented by $p$ and the statement $y \in P$ be represented by $q$.

The notation to write the statement if $p$ then $q$ is given by $p \Rightarrow q$ ($y \in T \Rightarrow y \in P$).

Use words and the notation given above to write another implication using the Venn diagram constructed in prompt 7.

11. Use the given sets to write an implication.

   cities in Texas, Dallas, Houston, Austin, San Antonio

12. Fill out the table below to determine the “truth” about the statement $A \subseteq B$, given that $A$ and $B$ are sets.

   \[
   \begin{array}{|c|c|c|}
   \hline
   x \in A & x \in B & A \subseteq B \\
   \hline
   \text{True} & \text{True} & \text{True} \\
   \text{True} & \text{False} & \text{False} \\
   \text{False} & \text{True} & \text{True} \\
   \text{False} & \text{False} & \text{True} \\
   \hline
   \end{array}
   \]
13. Use the sets in prompt 11 to give an example that explains your reasoning for the entries in
the table.

14. Given that $x$ is an object, idea, or characteristic and $A$ is a set, derive one rule for the
relationship between $x$ and $A$.

15. Does your rule cover any element, $x$, and any set?

16. Up to this point, you have studied sets and their behavior by making observations. Now you
are deriving rules based on observations. You are a set theorist.
Sets like all mathematical structures follow a set of rules called axioms that are "self evident"
and whose truth is taken for granted. The an axiom of naïve set theory is as follows: for some
object $x$ and set $A$, either $x \in A$ or $x \notin A$, not both.
Is the rule you created in prompt #13 congruent to the axiom given above?
If no, how can you derive your rule from the axiom?
17. Call a set $S$ abnormal if it contains itself, that is $S \in S$, and call a set normal if it does not contain itself, that is $S \notin S$.

Determine if the sets below are normal or abnormal.

(a) $A = \text{one-legged ducks}$

(b) $A^c$

(c) $\emptyset$

(d) objects not in this room

18. Let $R = \text{normal sets}$. Is $R$ a normal set or an abnormal set?

Argue that $R$ is normal.

Argue that $R$ is abnormal.
Homework

1. Construct a Venn diagram containing five sets, where each set is not a subset of the other two sets. Make sure your diagram contains all possible intersections. Use prompt 6 from the "Questions and Tasks" section to help construct all the regions needed.

2. The converse of \( p \implies q \) is given as \( q \implies p \).

   Give an example where \( p \implies q \) is true and \( q \implies p \) is also true.

3. Lookup "Russell's paradox" on Wikipedia.

   Read the "Informal presentation".

   Did you do something similar for prompt 18 in the "Questions and Tasks" section? How is it the same how is it different?

4. From the "Russell's paradox" entry link to the Barber paradox. Write the paradox using sets and set notation.
APPENDIX D
INTRODUCTION TO GRAPH THEORY

Goals:

- Students construct the concept of a graph.
- Students construct the concept of a directed graph and a tournament.
- Students construct the concept of a transitive directed graph and cycles in a directed graph.
- Students construct the concept of an interval graph, specifically a unit interval graph.
- Students discover the relationship between a graph and its complement.
- Students learn by direct instruction the concept of graph coloring.
- Students discover the relationship between graph coloring and cycles of even and odd length.
- Students discover the greedy coloring algorithm.

Introduction:

A graph is an ordered pair $G = \langle V, E \rangle$ where $V$ is a set of vertices and $E$ is a set of edges which are two-element subsets of $V$, that is an unordered pair of vertices. Often time a graph is drawn or depicted as a set of points for the vertices and line segments for the edges. Consider Figures 1.

![Figure 1: Graph, G](image)

In Figure 1, the graph is named $G$ with $V = \{u, v, x, y, z\}$. The edges have been labeled $a, b, c, d, e, f,$ and $g$ for the purpose of discussion. The vertex $u$ is adjacent to the vertices $v$ and $x$, that is there is an edge between $u$ and $v$ and $u$ and $x$. The edge $c$ is incident with the vertices $v$ and $x$, that is $v$ and $x$ are the ends or end vertices of the edge $c$. Any edge can be written be a letter or by the pair of end vertices, that is $vx$ is the same as edge $c$. Also in the pair is unordered thus $vx$ and $xv$
represent the same edge in a graph. A characteristic of vertices that is often of interest is the vertex degree. The vertex degree is the number of edges incident with a vertex. The degree of vertex \( z \) is two, denoted \( \text{deg} \ z = 2 \), while the degree of vertex \( y \) is three, \( \text{deg} \ y = 3 \).

A **simple graph** (or more often just a graph) has no loops and no parallel edges. A loop is when a vertex is adjacent to itself or there is an edge from a vertex back to itself. Parallel edges occur when there is more than one edge between two vertices. A graph of particular note is the **complete graph** on \( n \) vertices. In the complete graph any two ordered pairs are adjacent. Again the complete graph is a simple graph.

Graphs are often used for modeling real world networks and connections. For example the vertices of some graph can represent a subset of airports in the U.S. An edge can represent whether there are flights between the two airports daily. Also, edges can be weighted based on the context for example distance between cities or average cost for ticket between the cities.

A directed graph or **digraph** is an order pair \( D \ V, A \) where \( V \) is a set of vertices and \( A \) is a set of arcs (directed edges) which are ordered pairs of vertices. Thus \( u, v \) is the arc from \( u \) to \( v \), or in the parlance of competition, \( u \) "beats" \( v \). Another way to express the relationship is by \( u \rightarrow v \). A directed graph gives more information than a graph, as the direction of an arc can explain a precedence or direction. A **tournament** is a directed graph where there is exactly one arc between any two vertices. It is the directed version of a complete graph.

The complement of a graph \( G \) is denoted \( G \). The compliment contains the same vertex set as \( G \), \( V \ G \), and two vertices are adjacent in \( G \) if and only if the two vertices are not adjacent in \( G \). Consider Figure 2 of the graph labeled \( H \) with its complement.

![Figure 2: The graph H and its complement](image-url)
An *interval graph* is a graph whose vertices represent intervals of the real number line. Adjacency of vertices exists if and only if their corresponding intervals intersect.

A *graph coloring* is a labeling of the vertices or edges. Specifically a vertex coloring of a graph is a labeling or “coloring” such that any two adjacent colors do not have the same color. A coloring that uses at most $k$ colors is called a $k$-coloring. The chromatic number of a graph $G$ is the smallest number of colors required to color $G$.

There are efficient algorithms to find colorings and the chromatic number for certain types of graphs. Graph coloring is computational a hard problem. A greedy coloring algorithm considers vertices in a specific order and gives each vertex the “smallest” color possible. Consider the colorings of the two graphs given in Figure 3 and the ordering of the colors. With a larger graph the ordering can have more of an impact than one color.

![Figure 3](image)

Note that finding the chromatic number of a graph is an optimization problem.

Further lessons will develop other topics in graph theory. There are many results that have been developed in the last 50 years and many open problems. The study is exciting in that students are not learning results that are over 150 years-old, but modern mathematics.

Lesson Plan Ideas:

(Four to Five 50 minute lessons)

There are no prerequisites for this lesson, but an understanding of set theory and basic binomial coefficients would be helpful.
These lessons are an introduction to graph theory. There are many other subtopics in graph theory that can be explored. As a class, determine what other parts of graphs you would like to study. This unit could be a subunit or as part of a relations unit. Also, there are several optional worksheets attached.

Special thanks to Angela Brock for collaborating on the activities and task sheets of this lesson.

Day 1: Introduction & Definitions

Group students into groups of about 5 students. Instruct students in each group to write down all the different connections they have with other members of their group. Examples might include playing on the same team or living in the same neighborhood. Exclude the connection of having this class together to add to the interest of the graph students will construct. As the teacher, you might want to group students who don’t normally sit by each other or talk to each other. Some students may have no connections with the other students in the group.

After students have recorded the connections have them draw the connections on a graph with 5 vertices. If a student has one or more connections with other students in the graph, draw a line between those students. Here is where students will visually see the connections between each other. An extension would be to weight the edges with number of connections.

From here, introduce terminology of graphs: vertices, edges, vertex degree. Since each group has a subgraph of the graph of the entire class, bring to light that each subgraph could be connected to the other subgraphs. Ask each group to draw their graph on the white board or butcher paper, so that the graph may be examined other days without losing information. Have the students make one or two connections (if possible) from each subgraph to the next. It might be easier to make only one connection between each subgraph to create the class graph, otherwise all the connections may get in the way of seeing what is happening. The graph constructed is also a subgraph, because it is unlikely that all edges are shown. Now that you have all students in the class on the graph, see if there is a path connecting all students together (Hamiltonian path).

Conclusions to this lesson might include a discussion about networking in the real world.
For homework, have the students find something in their lives that can be modeled with a graph. The vertices could be people, places, or items where relationships can be modeled such as activities done together, places connected by highways, or items that can be done simultaneously. After the graph is constructed have the students find any path through all of the vertices and determine each vertex’s degree.

Day 2: Preferences, Directed Graphs and Tournaments

Start this lesson with a bell ringer activity where students answer 5 -10 questions about preferences. Do you prefer this to that? How about such and such to this? Etc. To add interest, don’t ask for preferences between all the listed objects. Students will infer these relationships later. An example is given on the task sheet attached (Attachment D.1).

As a class answer a different set of questions about preferences. (A vote could decide which thing is preferred.) Once preferences have been decided, draw the directed graph that represents these preferences. Here, it is natural to introduce the concept of one vertex ‘beating’ another and describe the digraph as directed. Take note of any cycles that occur in the graph. Discuss: What does this mean? (No real preference can be determined.) If no cycles exist, we can call the digraph transitive because there IS a real preference ordering. Now discuss what can be inferred about edges that do not exist. What relationships does the digraph imply? What relationships does the digraph tell you nothing about?

Now that students know about directed graphs, have them draw their own for the preferences they stated at the beginning of class. They should also be able to determine whether their digraph is a transitive or not. A series of questions about their graph and the inferences that could be made, should accompany the digraph making. It would be interesting to ask students what the graph infers about their preferences and if that inference is in fact true. *This could bring up logic.

For homework have the students play a round robin tournament of a simple game like “Paper, Rock, Scissors” or Tic-Tac-Toe with three other friends (in the class or not). Have the student create a directed graph of the tournament. Ask the students to analyze the graph by finding any cycles and determining a “winner” to the pairwise game. Also ask if the students can identify one
vertex/player such that for every other vertex/player the first player beats it or beats a vertex that beats it. Is there more than one such vertex?

Day 3: Indifference with Interval and Compliment graphs

Group the class into groups of four or five students. Give each student a topic such as pizza toppings, salt on food, sports, and music. In each topic there will be categories such as for pizza toppings, no toppings, 1, 2, 3, etc. Each student will be given a number line on a piece of paper numbered from 1 to 10. The students individually are to put the categories on the number line from least liked being near 1 to favorite being near 10. If the students don’t feel that any category deserves a 10, they don’t have to put one there. Then as a group the students will come up with a length of indifference. The indifference length will be long enough to reach two items on the number line that the students have no preference between; it can be one unit, 2 units, ½ units, or whatever they choose. Then the students will be instructed to make intervals, centered at where they put the category and with lengths that are the predetermined indifference length they came up with. While the students are performing the activity monitor the progress of the groups and select students whose ideas will be presented to help construct graphs.

After most of the groups have finished, select 2 or 3 students from different groups to reproduce their number lines on the board. Then with the help of the students, create a graph of indifference. Explain that the graph’s vertices represent the categories and the graph’s edges will be constructed whenever the intervals between two categories overlap. Individually, the students will then complete their own indifference graphs.

Bringing the class back together as a large group, probe the students on how we could make this into a preference graph such as the day before. Remind them that the preference graph was directed. Give the students 2 or 3 minutes to write down a response to the prompt and then after monitoring, have a student start off a discussion. Interject and steer the discussion towards creating a compliment of the graph and a rule on how we would direct the edges. After students have established a rule for converting preference graphs and interval graphs, question students about any cycles that might persist in the graph or if the graph is always transitive.

Students have now been introduced to creating and using interval graphs and making a compliment graph that can be directed. This can also be tied to logic in another unit or a later lesson. Question students if one category is indifferent to two others, does that mean that those
two others are indifferent? If you are given the directed compliment graph, could you create an interval representation for the vertices?

For homework, have the students find interval representations for the following graphs. Let students know that the intervals they create do not have to be a unit length, but can vary in length.

Day 4: Coloring of Graphs and Algorithms

As an introduction give students an assignment to color cycles of different (but given) lengths with as few colors as possible. The rule for coloring is that vertices that connect have to be different colors. Lead students to discover that any odd cycle will need 3 colors, while any even cycle only needs 2 colors. This activity can be done in groups (of 3 or less) or individually.

Help students to extend the idea of coloring to graphs that are not just cycles. As a large group, draw on the board a six cycle, add edges to the cycle, and ask the students “How does adding the edge change the coloring?” Present to students other graphs (such as the Peterson graph) for the students to consider coloring. Also explain to students what a complete graph is and ask what its coloring must be. These ideas can lead to a discussion on algorithms that allow us to color graphs.

After the discussion, put students in groups of three; have them come up with an algorithm to color graphs. The students can work on this for homework and the next day in class. Then have the students present their algorithms to the class. Give the students a graph and use each algorithm to color the graph to see which one gives a smaller coloring. Then discuss with the students how coloring a graph is a hard problem (NP-Hard, in fact) and that even for computers it takes a long time. As a wrap up to the mini-unit, discuss with students about looking into other graphs, graph problems, and other problems. The idea is to have the students become interested in other fields in math, computer science, etc.
*This unit could be extended one more day to talk about coloring intervals, so that only intervals of different colors that overlap are adjacent. Doing so would extend the concepts of coloring, intervals, and even partitioning the graph into independent sets of vertices.

Other extensions:

- Tournaments
- Hamiltonian Paths and Cycles
- Strong Graphs
- Trees
- Mathematical modeling with graphs

Assessment Ideas:

Give the students the following information on chemicals and transportation. Then ask the students to organize the information in the table into a graph. Also ask the students to find the complement of the graph (no directed edges). With each graph ask what the vertices and edges represent. Then have the students find the minimum number of sand barges required to freight the chemicals with the specifications where one car of each type of chemical will be transported.

A manufacture of chemicals ships its products by railroad tank cars. To reduce the danger that might occur through accidental spills of chemicals, the company specifies that the train must be up in segments in such a way that the following hold.

(a) No two chemicals in the cars of each segment react dangerously.

(b) Two open gondola carloads of sand must precede each segment to separate dangerously reactive chemicals in case of a derailment or other emergency.

(c) Two open gondolas of sand must separate the last segment of chemical cars from the caboose.

Below is the table of information on the 10 chemicals to be transported and a list of the chemicals being shipped.
1. Toluene  
2. Acetone  
3. Phosphoric Acid  
4. Sulfuric Acid  
5. Potassium cyanide  
6. Sodium hydroxide  
7. Dinitrogen tetroxide  
8. Nitrogen  
9. Chlorine  
10. Potassium dichromate

Reaction Table: S (relatively safe when mixed); U (unsafe when mixed)

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Preferences and Directed Graphs

Tasksheet

Circle your preference for each pairing (which one do you like better?)

- Soccer
- Baseball

- Volleyball
- Football

- Tennis
- Football

- Basketball
- Tennis

- Tennis
- Soccer

- Basketball
- Baseball

Using this information, draw a directed graph representing your preferences.
Now, find a partner in the class about whom you know little. Copy their preference graph here:

Notice that not all sports have a connection drawn between them. Using on the information in the graph, what connections can you infer? If you can't make the inference, explain why.

Does this person prefer Soccer or Volleyball? How can you tell?

Does this person prefer Volleyball or Baseball? How can you tell?

Does this person prefer Volleyball or Basketball? How can you tell?

Does this person prefer Volleyball or Tennis? How can you tell?
Does this person prefer Football or Baseball? How can you tell?

Does this person prefer Football or Soccer? How can you tell?

Does this person prefer Football or Basketball? How can you tell?

Does this person prefer Basketball or Tennis? How can you tell?

Does this person prefer Soccer or Basketball? How can you tell?

Add in any edges that you can infer.
Does the individual whose graph you have drawn, have a favorite sport? How do you know?

Now, talk to the person who’s preference graph you are drawing.

Are those connections that you made correct? If not, what does this tell you about human preferences?
Attachment D.2

Coloring Assignment:

The rule for coloring these cycles is that vertices that connect have to be different colors. Color in each dot using this rule. Try to use as FEW colors as possible.

3 vertices Number of colors: 4 vertices Number of colors:

6 vertices Number of colors: 5 vertices Number of colors:
Before you color the next two, predict how many colors you think it will take to color.

7 vertices number of colors: 8 vertices number of colors

Was your prediction correct?

Now, without coloring, how many colors are needed to color a cycle with 108 vertices? 555 vertices? 1082 vertices?

How did you make these predictions?

What did you notice about the cycles that you did color that lead to you this conclusion?
APPENDIX E

SEQUENCES AND FUNCTIONS

Goals:

- Students construct the concept of a sequence.
- Students construct the concept of the terms of a sequence.
- Students discover the relationship between sequences and functions from a subset of the natural numbers.
- Students use mathematical notation to communicate information about sequences, subsequences, and terms of a sequence.
- Students appreciate the use of sequence notation and the use of sequences.
- Students are introduced to recurrence relations and their relationship to sequences.

Introduction:

A sequence is an ordering of elements such as numbers, geometric figures, objects, etc. Commonly, a sequence is an ordering of an infinite number of elements and a string is an ordering of a finite number of elements. In this lesson we will refer to finite and infinite orderings as sequences; the instructor may wish to distinguish in further lessons between a string and sequence.

Since sequences are orderings of elements, one can discuss the first, second, third, or $n^{th}$ element. In contrast a set has no ordering thus one can only state that an element belongs to a set and not its position in the set. This has many advantages for discussing sequences and discovering relationships.

A sequence is denoted like a function with a name, usually by a letter, and can also be denoted by its elements. For example, $a_{n \geq 1}$ is the name of an infinite sequence, specifically $a_{n \geq 1}$ consider to be the sequence with terms being the positive odd numbers with the natural ordering of greater than; therefore $a_n = 2n - 1$ where $n \geq 1$. If the sequence is finite the notation would be $s_{n=0}^{11}$ where the indices range from 0 to 11 that is there is a 0$^{th}$ term and the sequence ends at the 11$^{th}$ term. Naming a sequence and having a closed form formula for the terms are different. Often times, as shall be seen in the lesson on generating functions, the goal is to create a sequence in order to count something and then identify the closed form formula.

Also, unlike a set, elements of a sequence do not have to be a distinct. For example, the sequence 0,1,0,1,1,0,1,1,1,0... contains terms of only 0 and 1 while as a set 0,1,0,1,1,0,1,1,1,0... would be more succinctly given as 0,1.
Clearly, a sequence is a function. Recall that a function is a set of ordered pairs, \( x, y \), such that there is only one output value for each input value, or more commonly there is only one \( y \) for each \( x \). There is only one first term in a sequence, only one second term, and more generally only one \( k \)th term where \( k \) is a natural number. Thus a sequence is a function from some subset of the natural numbers to any nonempty set. Also, this means that a sequence does not necessarily have a pattern in the terms. For example the sequence \( Jemma, Carlos, Li, Karen \) can be either the children Ms. Buckner has given birth to in the order of age or it could be the current lists of the presidents of the Mountain Biking club at Wabasha High School in order of membership. Though patterns may be unclear, often times the terms in practical purposes have some relationship to each other.

A recurrence relation is a formula to give a desired term of a sequence using previous terms and initial conditions. The most famous is the recurrence relation that gives the Fibonacci numbers, \( F_n = F_{n-1} + F_{n-2} \) for \( n \geq 2 \) and with initial conditions \( F_0 = F_1 = 1 \).

Lesson Plans:

(Three or four 50 minute lessons)

Prerequisite: Set Theory and simple counting techniques.

Begin by grouping students into groups of five or six students. Give each group of students a group of numbers given as a cluster (see figure 1). Ask the students to only use seven of the numbers and to arrange the numbers in some order.

```
2   6   8
0   -2  5
12  14
-1  11
```

Figure 1

Use the puzzle method of groups to split the first groups into new groups where no two students are in the same group as they were previously. Give each new group one of the previous group orderings. Have the students complete the following task.

- Label the ordering of the first seven numbers.
- Determine an eighth, ninth, and tenth number that should follow this ordering.
- Determine if there is a pattern and provide a formula if a pattern exists.
- Answer the following questions.
  - Is the ordering of numbers, including the numbers your group added, a set?
  - Is Gauss, Euclid, Erdős, Euler ordered?
    - If so, what is the ordering?
    - If not, give the set an ordering.
Explain to the class that the orderings they created are called a sequence. As an entire class define a sequence based on the experiment. Ask the class to discuss if a sequence can be infinite or must be finite. With the discussion, have the class give examples and write them on the board. Include some of your own sequences in the discussion including sequence with nonnumeric terms, for example regular polygons or the line of authority in the school. Use the sequences on the board to lead the class to discuss how it would be helpful to have names for each sequence and how to discuss each term.

Explain again that mathematician use notation to more efficiently and easily discuss certain aspects of a mathematical object such as a sequence. Use three sequences from the discussion and label them as $a_n$, $b_n$, and $c_n$. These are the names of sequences. Ask each student to write what they suspect $a_1$, $b_1$, $c_3$, $a_4$, and $c_k$ represent with respect to the sequences. Then discuss as a class that the subscript gives the term of the sequence named. Give the example of creating a sequence from the set Gauss, Euclid, Erdös, Euler and call it $m_1$. Ask the students to give the notation that represents Euclid, i.e. $m_1$, or $m_2$.

Discuss with the students that in an individual sequence there is only one first term, second term, etc. Then introduce that we can represent the sequence as a set of ordered pairs, that is the sequence labeled by $S$ can be written as $S = (0, s_0), (1, s_1), (2, s_2), (3, s_3), ... n, s_n, ...$. Discuss with the students that a set of ordered pairs represents a relation. Continue the discussion by finding the domain and range of the relation. Then use the fact that there is only one first term, second term, etc to discover that a sequence is a function from a subset of the natural numbers to the terms of the sequence.

Introduce the idea of a recurrence relation. An easy example is an arithmetic sequence or a geometric sequence. Arithmetic sequence is of the form $a_n = a_{n-1} + d$, where $d$ is the difference between any two consecutive terms; a geometric sequence is of the form $a_n = r \cdot a_{n-1}$, where $r$ is the ratio of two consecutive terms. These sequences are simple examples but both can be rewritten in a closed form formula given the difference or ratio and the initial term of the sequence $a_0$. Work out as a class a closed form formula for each given only in terms of the difference (ratio) and initial term.

Attached is a creative-thinking lesson plan that can be used to enhance the understanding of sequences (Attachment E.1).

Extensions:

- In depth recurrence relations
- Generating Functions
- Counting problems

Mini-Experiments:
To assess students' knowledge, consider the following prompts. Choose from the prompts and modify for your class or create your own.

1. Create a sequence that is neither increasing nor decreasing from term to term. List the first seven terms of the sequence and then construct a function from the natural numbers to the terms of the sequence. Explain how you constructed the sequence.

2. Explain in a sentence what $a_{16}$ means. Construct a formula for the sequence $a_n$. Find $a_{16}$.

3. The following recurrence relation comes from a counting problem, $g_n = g_{n-2} + n$ where $n \geq 1$. $g_1 = 1$ and $g_2 = 3$. Find $g_7$ and $g_8$. Is $g_n$ a natural number, integer, or real number? Explain your reasoning.

4. Create a recurrence relation that is not arithmetic or geometric. Give the initial conditions with the equation. Write the first seven terms of the sequence and determine if the terms of the sequence are always increasing, decreasing, or neither.
Attachment E.1

Creative-Thinking Lesson Plan

Objective: The student invents patterns for novel sequences of numbers.

This lesson will be implemented after students have constructed the concept of a sequence.

The instructor will direct students to create formulas that define sequences of numbers. Then some of the students will present their sequences to the class. The set of the class will then be partitioned into subsets containing three or four students. The groups will work together to answer the following questions about two or three of the sequences that were presented to the class:

- How is <student’s name> sequence like a roller coaster?
- You are a term in the sequence mentioned above; what are your feelings about the terms on either side of you? Are you friends, enemies, etc.? Why is that the status of your relationship?
- Would someone call this sequence “discrete-continuous”? Explain your reasoning.
- Write a question that you want your classmates to answer about the sequence of <students’ name>.
- As individuals create one more sequence of numbers, with or without its formula.

After the students answer the questions in their groups, the class will reconvene as a large group and discuss the responses. The instructor will choose one or two more sequences that the class will look at. The instructor will then present the following questions for the students to answer on their own at home and the class will discuss them the following class period.

- Imagine that you are the sequence. Describe your movement from one of your terms to another; focus on what is happening as your sequence goes to infinity.
- How is the sequence like a bowl of spaghetti?
- Explain how sequences would be different if the terms of sequence were $a_j$ and $j \in \text{irrationals}$. 
APPENDIX F

GENERATING FUNCTIONS, SEQUENCES, AND COUNTING

Goals:

- Students construct the concept of a generating function
- Students discover the relationship between a generating function and a sequence.
- Students derive the power series expansions for common generating functions.
- Students apply the relationship between a generating function and a sequence to solve counting problems.

Introduction:

A generating function is a function whose coefficient on \( x^n \) is the \( n \)th term of a sequence of interest. We denote this by \( \left[ x^n \right] f(x) = a_n \) where \( f(x) \) is the generating function and \( a_n \) is the sequence of interest.

For example, if \( s_n = 0 \) is the sequence of possible ways the standard six-sided dice can have a face that displays \( n \), then the sequence is just 0, 1, 1, 1, 1, 1. Thus the generating function is \( f(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 \). If a new six-sided dice had faces of 1, 2, 2, 2, 4, 4 instead then the sequence would be 0, 1, 3, 0, 2, 0, 0 and the generating function would be \( f(x) = x + 3x^2 + 0x^3 + 2x^4 + 0x^5 + 0x^6 \) or \( f(x) = x + 3x^2 + 2x^4 \). If you roll the two dice together and add their face values the sequence would be 0, 0, 1, 4, 4, 6, 6, 5, 2, 2 and thus has a generating function of \( g(x) = x^2 + 4x^3 + 4x^4 + 6x^5 + 6x^6 + 6x^7 + 5x^8 + 2x^9 + 2x^{10} \). Notice that \( g(x) = x + x^2 + x^3 + x^4 + x^5 + x^6 \cdot x + 3x^2 + 2x^4 \).

Another example is a geometric sequence \( a_n = r \cdot a_{n-1} \) for \( n \geq 1 \) and \( a_0 \) given. In a previous lesson we noted that this can be rewritten in a closed form formula as \( a_n = a_0 r^n \). If \( g(x) \) is the generating function for the geometric sequence, then \( g(x) = a_0 + a_0 rx + a_0 r^2 x^2 + ... + a_0 r^n x^n + ... \); obviously, this generating function is a power series.

From calculus, it is shown that \( f(x) = \frac{1}{1-x} \) can be expand as a power series as follows,

\[
\frac{1}{1-x} = 1 + x + x^2 + x^3 + ... = \sum_{n=0}^{\infty} x^n.
\]

Thus the sequence where each term is 1, has a generating function of \( f(x) = \frac{1}{1-x} \). This can also be shown algebraically using the additive inverse
property and factoring by grouping. Using only this generating function, the following can be developed:

\[
\frac{1}{1-ax} = \sum_{n=0}^{\infty} a^n x^n, \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n, \quad \text{and} \quad \frac{1}{1-x} = \sum_{n=0}^{\infty} n+1 x^n.
\]

Other algebraic manipulations are possible; these are just a few examples. Returning to the example above of the geometric sequence and its generating function, \( g(x) \) can easily be found explicitly as

\[
g(x) = \frac{a_0}{1-rx}.
\]

Another powerful facet of generating functions is in forming binomial coefficients. By the binomial theorem,

\[
1 + x^n = \sum_{k=0}^{n} \binom{n}{k} x^k.
\]

Isn't this a generating function for the sequence of ways to choose a \( k \)-set from an \( n \)-set?

Lesson Plan:
(Two or three 50 minute lessons)

Prerequisite: Sequences

This lesson can be a unit once the students and instructor discover the power of a generating function.

Expect to spend between three to five days on this lesson. Attached is an optional task sheet from which you can use examples, definitions, etc. (Attachment F.1).

On the first day present the topics of binomial coefficients (i.e. \( \binom{n}{k} \)), sequences, and polynomials to the class. Review the concept of a sequence and the notation used. In regards to polynomials, make sure the class understands the vocabulary of coefficient, factor, and term, as well as the algebra of polynomial multiplication and factoring.

After the review, give the students a teaser problem to motivate the students. For this problem ask the students how many ways they can make $2 in change using only pennies, nickels, dimes, and quarters. Obviously there are many ways, using eight quarters, 20 dimes, 40 nickels, or 200 pennies. But what about using ten pennies, eight nickels, five dimes, and four quarters? There are many combinations, but how many? Obviously counting them all would take more time than you or your students have in a math class, but you can do it for something as low as 15¢ or 20¢. Explain to the students that though we could derive a sequence, it would also take too long, eventually after discovering generating functions, the outcome just requires computational software.

To begin constructing the concept of a generating function, begin with the familiar binomial coefficients and a set of three objects (two seems too small and four too many). Label the objects
o_1, o_2, o_3. The class should easily be able to find the sequence \( a_n \) \( n \geq 0 \) where \( a_n \) is the number of subsets of size \( n \) from the set \( o_1, o_2, o_3 \), the sequence is 1,3,3,1,0,0,... This generating function can be created by looking at the product of \( 1 + o_1 \) \( 1 + o_2 \) \( 1 + o_3 \). Notice that each factor can be written instead as \( o_1^0 + o_1^1 \) that is object \( o_j \) is not in the set, \( o_1^0 \), or is in the set being created, \( o_1^1 \). When the product is expanded, the polynomial is \( 1 + o_1 + o_2 + o_3 + o_1 o_2 + o_1 o_3 + o_2 o_3 + o_1 o_2 o_3 \). Ask the class what each term represent; obviously, it gives the possible subsets created by the three objects. Ask what happens if each object \( o_j \) is replaced by \( x \) and then one adds like terms. Ask the students to look at the coefficient on \( x^2 \), have the class discuss what that coefficient represents. You can do this with all of the coefficients showing the students that the coefficients are the number of subsets of different sizes. The students have created their first generating function.

At this point you can look at the binomial theorem, if you feel that the class is ready for that discussion. Use the proof given in the attached task sheet or use the example above to show what is going on with those binomial coefficients and why they are called that.

Give the students the definition of a generating function. In the task sheet are examples of going from sequence to generating function and from generating function to sequence. Have the students work through one example of each type as a class, and then have the students work alone or as groups to work the other examples. Walk around the room and answer questions, be aware of the constant and what it represents. Ask the students after performing the exercise specifically about the constant and what it means. Why does it matter?

Here is where you can introduce the idea of using generating functions to count things. You can start with the care package or fruit basket example to show the idea of counting. If the polynomials in the task sheet are too difficult for your students to compute come up with simpler examples or less items. Note, the goal is not to see if the students can multiply polynomials, but to find the relationship between polynomials and sequences. Use one of the examples given or come up with one on your own for the students to work through as homework. When students bring in their resulting generating functions, use a computational software package to expand the polynomials and solve the problem.

Another great example is the dice. This requires more deductive reasoning not just creating polynomials. Use the two six-sided dice example to show that it is possible to find other “weird” or nonstandard dice that give the same distribution. Make sure you review the process of finding the weird dice and why the process works. For a reference look at Joseph Gallian’s article “Weird Dice” in Math Horizons February 1995 issue.

After the dice example, you can move onto the infinite case. Be sure to include summation notation as part of your discussion. Introduce students to the idea of what a summation tells the
reader especially the indices. If students struggle with factoring by grouping, use direct
instruction to give that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$. From here work the exercises to come up with the general
generating functions that can be derived by replacing $x$ with another term. After students derive
the generating functions have the class identify the sequence. As a class work through the
process of finding generating functions for geometric sequences with given ratios and $a_0$ term.

Now the students are ready to finish the problem introduced about the change. Only the setup is
required. After students have worked the example, but before expanding the function using a
computer, go to the Online Encyclopedia of Integer Sequences, http://www.research.att.com/~njas/sequences/index.html. Use the first 10 or so terms of the
sequence to search for the sequence on making change, i.e. 1,1,1,2,2,2,2,4,4,4,... Have the
students look at the page, identify the generating function given at the site, look at the list
provided, and even listen to it. Then show that the list doesn't give $2$, and a computational
package will be required.

Extensions:

- In depth recurrence relations
- More counting problems
- Introduction to exponential generating functions

Assessment:

Use any of the homework prompts to assess the students understanding of generating functions
and sequences. Or create your own prompts to solve generating function problems in both the
finite and infinite cases.

Another assessment is to give the students the quote attached to the task sheet. Ask the students
to use their own words in a paragraph to explain what Herbert Wilf meant. Require students to
give an example of a generating function and sequence to illustrate their point.
Give points on clarity and content as well as points for the example presented and its accuracy.
Generating Functions

*A generating function is a clothesline on which we hang up a sequence of numbers for display.*

— Herbert Wilf, *Generatingfunctionology*

Guided Notes and Worksheet

Name:

Review
Please answer each prompt on review material.

1. Define a sequence.

2. What does the notation $\binom{n}{k}$ represent?

3. Give a closed form formula for a geometric sequence with ratio $r$ and initial sequence value $a_0$.

4. Recall: a coefficient is the numeric part of a term in an algebraic expression.
5. Find the coefficient for each of the monomials
   a. $9x^2$
   b. $\frac{x}{13\sqrt{y}}$
   c. $\frac{4}{3z^3}$
   d. 0

6. Recall: a polynomial is an algebraic structure where the power on each term is a nonnegative integer. A polynomial in one variable of degree $n$ often is represent as $P(x) = a_0 + a_1x + ... + a_{n-1}x^{n-1} + a_nx^n$ where $a_0, a_1, ..., a_{n-1}, a_n$ are the coefficients of the polynomial $P$.

Write a paragraph (three or more sentences) on how a polynomial's structure might be connected to a sequence, especially consider the notation used for both a polynomial's coefficient and a sequence’s terms.

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Problem 1 and Motivation

1. How many ways can you make $2 in change using pennies, nickels, dimes, and quarters? What about $1? Or 50¢? Or 15¢?
2. List the ways to make 15¢ out of pennies, nickels, and dimes.

3. One goal of this lesson is to answer the original question on how many ways to make $2 in change.

Discovery and Exploration

1. Consider creating a set from three objects labeled \(o_1, o_2, \text{ and } o_3\) respectively. Each object is either in the set to be created or is not. Thus create a sequence such that \(a_k\) gives how many \(k\)-sets can be created from the objects.

2. Recall that in counting problems "or" is often denoted using addition and "and" using multiplication. Let \(o_1^0\) denote the absence of \(o_1\) and \(o_1^1\) that \(o_1\) is included once. A representation of the absence of the first object OR including the object in the set is \(o_1^0 + o_1^1\). Simplify the expression using your knowledge about exponents of 0 and 1.

3. Interpret into English the following mathematical notation into English:
   
   \[1 + o_1, 1 + o_2, 1 + o_3\]
4. Using algebra expand the product, \( 1 + o_1 \quad 1 + o_2 \quad 1 + o_3 \), into a polynomial, remember to add like terms.

5. Write a paragraph explaining what you think the terms of the sequence in prompt 1 represent with connection to building subsets from the set \( o_1, o_2, o_3 \).

6. If we did not want to distinguish between elements, we could call each one simply \( o \) or \( x \). By combining like terms, what would the polynomial be by expanding \( 1 + o \quad 1 + o \quad 1 + o \)?
7. Write a paragraph explaining what you think the terms of the sequence in prompt 1 represent with connection to building subsets from the set $o_1, o_2, o_3$.

8. Below is a proof for the binomial theorem using combinatorics, read the argument below and consider what you have seen above in the examples while reading the proof.

Claim: $1 + x^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Proof:

Let $S$ be an $n$-set. To create a $k$-set from an $n$-set by definition is $\binom{n}{k}$.

Consider expanding by polynomial multiplication $1 + x^n$. Imagine doing the distribution all at once, thus from each of the factors in $1 + x^n$, all of which are $1 + x$, one element will contribute to the term. To construct the $x^k$ term, select the $k$ factors that will contribute an $x$ and the $n-k$ terms that will contribute only 1. How many ways can the $k$ factors be selected that contribute an $x$? It is the same as creating a $k$-set from an $n$-set; therefore there are $\binom{n}{k}$ ways to select the $k$ factors that contribute. This is true for $k = 1, 2, \ldots, n$

Thus when expanded, $1 + x^n = \binom{n}{0} + \binom{n}{1} x + \binom{n}{2} x^2 + \ldots + \binom{n}{n-1} x^{n-1} + \binom{n}{n} x^n$ and in summation notation it is written as $1 + x^n = \sum_{k=0}^{n} \binom{n}{k} x^k$. 
9. Explain in a short paragraph the differences between building the subsets of $a_1, a_2, a_3$ and the proof above to explain the same sequence.

10. Use the above definition to create a generating function for the sequence given where each sequence begins with $a_0$.

a. $0, 1, 2, 3, 4, 5$

b. $1, 2, 4, 8, 16, 32, 64$

c. $1, 0, 1, 0, 1, 0$
11. Use the definition of a generating function to construct a sequence given a generating function.

a. \( f \ x = x + x^2 + 2x^3 + 3x^4 + 5x^5 \)

b. \( g \ x = x^1 + 2x^3 + 3x^5 + 4x^7 \)

c. \( h \ x = 1 - x + x^2 - x^3 + x^4 - x^5 + x^6 \)

Problem 2

1. You and your friends serve in the Key Club. You are making care packages for children in the local hospital. The care packages are made of yo-yo’s, coloring books, fruit snack packs, and animal cracker containers. The hospital administration has strict rules on the number of each item that can be in the care package. The packages can have at most three yo-yo’s; one or two coloring books; three, five, or six fruit snack packs; and an even number and no more than four animal cracker containers.

How many ways can you create a care package with eight items in it if you stick to the hospital’s rules?

2. Find the following sequences and the corresponding generating functions to help find the number of ways to create the care packages.

Remember: some items don’t have to be in the care packages.

a. Let \( y_n \) be the number of ways to have \( n \) yo-yo’s in the care packages.

b. Let \( c_n \) be the number of ways to have \( n \) coloring books in the care packages.
c. Let $f_n$ be the number of ways to have $n$ fruit snack packs in the care packages.

d. Let $a_n$ be the number of ways to have $n$ animal cracker containers in the care packages.

3. For each of the sequences determine the generating function that corresponds to it.
   a. $y_n$
   b. $c_n$
   c. $f_n$
   d. $a_n$

4. By multiplying the generating function for the sequence $y_n$ and the sequence $f_n$, the result is the polynomial $x^3 + x^4 + 2x^5 + 3x^6 + 2x^7 + 2x^8 + x^9$. This polynomial is the generating function for the sequence that gives number of care packages that can be created with $n$ objects using only yo-yo's and fruit snack packages under the rules of hospital.

What does the coefficient on the $x^6$ term mean in context of the care package problem?
5. By multiplying all four generating functions for the sequences \( y_n \), \( c_n \), \( f_n \), and \( a_n \), the result is the polynomial
\[
x^4 + 2x^5 + 4x^6 + 7x^7 + 9x^8 + 11x^9 + 11x^{10} + 8x^{11} + 5x^{12} + 3x^{13} + x^{14} + x^{15}.
\]

a. For what sequence is the above polynomial a generating function?

b. Answer the original question, how many ways are there to create a care package that has eight items and follows the hospital’s rules?

6. For homework, use the care package example as a template to work the following problem:

Sienna and Hector want to create fruit baskets to impress their clients. Each fruit basket can have an even number of plums, but no more than six; an odd number of mangos but no more than three; less than five kiwi; and one, two, or three strawberries.

a. By looking only at the restrictions for each type of fruit, what is the smallest number of fruit that a fruit basket can have?
   What is the largest number of fruit a basket can have?
   Explain your reasoning.
b. Create a sequence and a corresponding generating function for each type of fruit, where the \( n^{th} \) term in the sequence is the number of ways \( n \) pieces of the specific type of fruit can be put in the basket.

   i. Plums

   ii. Mangos

   iii. Kiwis

   iv. Strawberries

c. Find an expression for the generating function corresponding to the sequence \( f_n \) where \( f_n \) is number of ways to construct an \( n \)-piece fruit basket with given restrictions.

d. Be prepared to present the expression for the generating function in class to continue the problem.
Problem 3

1. Consider playing a game that requires two dice such as Monopoly™, Clue™, or Settlers of Catan™. Your token moves the total sum of the face values or the sum of the face values dictates game play. If the dice are fair or unbiased, then the distribution is as follows.

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a. Using the table, create a sequence for the number of ways to roll the two dice to have a sum of \( n \). That is, \( a_n \) is the number of ways two dice can be rolled and the sum of their faces is 3.

b. Give the sequence AND corresponding generating function, call it \( f(x) \), for the numbers of ways to have a face value of \( n \) when only one dice is rolled.
c. Find \( f(x^2) \) by polynomial multiplication, remember to add like terms.

d. For what sequence is \( f(x^2) \) the generating function? Explain why this is.

2. Consider if your friend introduced new dice to the game. She claims the sum of the faces appears the same number of times, but that each dice has different faces than a standard die. For example one die has faces of 1, 2, 2, 3, 3, 4 and the other has faces of 1, 3, 4, 5, 6, 8. Does this change the game? How so?
   a. Show dice distributions using a table like the one above.
b. What is the generating function corresponding to the first dice?

c. What is the generating function corresponding to the second dice?

d. Find the product of the two generating functions.

e. Though the distribution is the same, what is the chance of getting doubles? Would this affect certain games?

3. Consider instead the possibility of a game that requires a three-sided dice. (Does a 3-sided dice exist?) A standard three-sided dice has the faces of 1, 2, 3. Consider the distribution of rolling two three-sided dice and summing the face values.

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a. Are there other possible face values for two nonstandard three-sided dice that give the same distribution? By applying the method of generating functions and basic knowledge of factoring, investigate the above question. First what is the sequence for the distribution?

b. What is the corresponding generating function to the sequence? Call the generating function $p(x)$.

c. Let $f_1, f_2, f_3$ be the face values of one three-sided dice and $g_1, g_2, g_3$ be the face values for the other die. Thus if the dice are going to give the same distribution,

$$x^{f_1} + x^{f_2} + x^{f_3} + x^{g_1} + x^{g_2} + x^{g_3}$$

must be equal to the generating function $p(x)$.

Write an equation setting the two polynomials equal.

d. This is hard to solve with only one equation and six unknown values. Thus we will use factoring to solve.

Check the factoring of the original generating function given below.

$$x^2 + 2x^3 + 3x^4 + 2x^5 + x^6 = x^2 + 2x + 3x^2 + 2x^3 + x^4$$

$$= x^2 + 1 + x + x^2$$
e. Therefore by algebra \( x^q + x^r + x^s = x^q 1 + x + x^2 \) where \( 0 \leq q \leq 2 \) and \( 0 \leq r \leq 2 \).
Write an argument for the above equation and conditions on \( q \) and \( r \).

f. The equation above is true for all \( x \). Evaluate each side of the equation when \( x = 1 \).

```
\begin{align*}
\text{Left Side:} & \quad x^q + x^r + x^s = 2 + 1 + 1 = 4 \\
\text{Right Side:} & \quad 1 + 1 + 1 = 3 \\
\end{align*}
```

\( \text{The equation is not true for } x = 1. \)

\( g. \) Why must \( r = 1 \)? Or why is \( r \) not 0 or 2?

h. Evaluate each side of the equation in e. when \( x = 0 \).

\( i. \) Why is \( q \) not 0?

j. If \( q = 2 \), what is the smallest value for a sum that is possible?
k. Thus $q = 1$ and $r = 1$. Therefore $x^6 + x^7 + x^8 = x^2 + x^3 - x + 1 + x + x^2 = x + x^2 + x^3$.

According to this generating function what are the face values of the first die?

l. How many distinct pairs of three-sided dice are there that give the distribution above?

4. For homework work through the following prompts on four-sided dice and be prepared to discuss with the class the results.
A standard four-sided dice has the faces of 1, 2, 3, 4. Consider the distribution of rolling two four-sided dice and summing the face values. The distribution is given below

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Are there other possible face values for two nonstandard four-sided dice that give the same distribution?

a. What is the sequence for the distribution?

b. What is the corresponding generating function to the sequence? Call the generating function $p(x)$. 
c. Let $f_1, f_2, f_3, f_4$ be the face values of one four-sided dice and $g_1, g_2, g_3, g_4$ be the face values for the other die. Thus, if the dice are going to give the same distribution,
$$x^{f_1} + x^{f_2} + x^{f_3} + x^{f_4} = x^{g_1} + x^{g_2} + x^{g_3} + x^{g_4}$$
must be equal to the generating function $p(x)$.

Write an equation setting the two polynomials equal.

d. This is hard to solve with only equation and eight unknown values. Thus we will use factoring to solve.

Check the factoring of the original generating function given below.
$$x^2 + 2x^3 + 3x^4 + 4x^5 + 3x^6 + 2x^7 + x^8 = x^2 (1 + 2x + 3x^2 + 4x^3 + 3x^4 + 2x^5 + x^6)$$
$$= x^2 (1 + x^2) (1 + x^2)$$

e. Therefore by algebra $x^{f_1} + x^{f_2} + x^{f_3} + x^{f_4} = x^q 1 + x^r 1 + x^s$ where $0 \leq q \leq 2, 0 \leq r \leq 2,$ and $0 \leq s \leq 2$.

Write an argument for the above equation and conditions on $q, r,$ and $s$. 
f. The equation above is true for all $x$. Evaluate each side of the equation when $x = 1$.

g. Why must $r + s = 2$?

h. Evaluate each side of the equation in e. when $x = 0$.

i. Why is $q$ not 0?

j. If $q = 2$, what is the smallest value for a sum that is possible?

k. Thus $q = 1$ and $r + s = 2$.
   
   If $r = 0$ then $s =$ _______.
   
   If $r = 1$ then $s =$ _______.
   
   If $r = 2$ then $s =$ _______.
1. Up to this point it has been shown that
\[ x^{f_i} + x^{f_i} + x^{f_i} + x^{f_i} + x^{g_i} + x^{g_i} + x^{f_i} + x^{g_i} = p \quad \text{and} \quad p \quad \text{is factored; thus} \]
\[ x^{f_i} + x^{f_i} + x^{f_i} + x^{f_i} + x^{g_i} + x^{g_i} + x^{f_i} + x^{g_i} = x^2 + x^2 + 1 + x^2 + 2. \]

Explain to yourself why
\[ x^{g_i} + x^{g_i} + x^{g_i} + x^{g_i} = x^{2-q} + 1 + x^{2-r} + 1 + x^{2-s}. \]

m. If \( r = 1 \), what are the values of \( f_1, f_2, f_3, f_4 \)? Note that \( f_i = f_j \) is possible.

What are the face values of \( g_1, g_2, g_3, g_4 \)?

n. If \( r = 0 \), what are the values of \( f_1, f_2, f_3, f_4 \)? Note that \( f_i = f_j \) is possible.

What are the face values of \( g_1, g_2, g_3, g_4 \)?
If $r = 2$, what are the values of $f_1, f_2, f_3, f_4$? Note that $f_i = f_j$ is possible. What are the face values of $g_1, g_2, g_3, g_4$?

Besides the standard dice, how many other distinct dice pairs gives the distribution of sum of values desired? Use a table(s) to show the distribution of the sum of face values.

Generating Functions for Infinite Sequences

1. Often a sequence is not finite and therefore its generating function has an infinite number of terms.
   A geometric sequence is not finite. Given $a_0 = 1$ and $r = 3$, write the first five terms of the sequence and give the generating function.
   Use "..." to show that the sequence and the generating function are not finite.
2. The generating function for these infinite sequences is not necessarily a polynomial, often it is a power series and is represented by summation notation.

For example, \( \sum_{n=0}^{\infty} a_n x^n \) is a power series representation of a generating function of the sequence \( a_n \). That is, \( \sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \ldots \). Notice how the coefficient on the \( x^n \) term is \( a_n \).

3. The most common power series and the one from which others are derived is

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \ldots
\]

What is the corresponding sequence?

4. To derive the power series above only requires knowledge of additive inverses and factoring by grouping.

Examine the argument below at each line of the equation justify the algebra with your own words.

\[
1 = 1 - x + x - x^2 + x^3 - \ldots \\
= 1 - x + x - x^2 + x^3 + \ldots \\
= 1 - x + x + x^2 - x^3 + x^4 - \ldots \\
= 1 - x + x^2 + x^3 + \ldots \\
= 1 - x \sum_{n=0}^{\infty} x^n \\
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n
\]
5. Changing variables doesn't change the value; thus \( \frac{1}{1-z} = \sum_{n=0}^\infty z^n \).

Below are variable substitutions that give generating functions. Examine the generating functions, expand the power series to the fourth term, \( x^4 \), and give the corresponding sequence.

a. Let \( z = ax \), then
\[
\frac{1}{1-ax} = \sum_{n=0}^\infty a^n x^n.
\]

b. Let \( z = x^k \), then
\[
\frac{1}{1-x} = \sum_{n=0}^\infty x^n = \sum_{n=0}^\infty x^{kn}.
\]

c. Consider
\[
\frac{a}{1-x} = a \sum_{n=0}^\infty x^n = \sum_{n=0}^\infty a x^n.
\]

d. Let \( z = -x \), then
\[
\frac{1}{1-(-x)} = \sum_{n=0}^\infty (-x)^n = \sum_{n=0}^\infty (-1)^n x^n.
\]
6. Using the generating functions developed above, find the generating function for a geometric sequence with \( a_0 = \frac{3}{2} \) and \( r = 4 \).

7. For homework find the generating functions for the following sequences. Note \( k = 1, 2, 3, 4, \ldots \)
   a. Let \( p_k \) be the number of ways to use only pennies to make \( k \) cents.
   b. Let \( n_k \) be the number of ways to use only nickels to make \( k \) cents.
   c. Let \( d_k \) be the number of ways to use only dimes to make \( k \) cents.
   d. Let \( q_k \) be the number of ways to use only quarters to make \( k \) cents.
e. Let $c_k$ be the number of ways to use pennies, nickels, dimes, and quarters to make $k$ cents.

f. As a class, use a computer with computational software and the generating function for $c_k$ to answer Problem 1.
APPENDIX G

VOTING THEORY AND TACTICAL VOTING

This lesson can begin a unit on social decision making and game theory.

Goals:

- Students discover the preference ordering relationship as an asymmetric and transitive relation and communicate preference relations using proper notation.
- Students review and demonstrate knowledge of counting techniques.
- Students apply voting strategies to situations.
- Student appreciate and model the axioms of a social welfare function.
- Students appreciate discrete modeling as a method of solving real world problems.

Introduction:

Discrete mathematics is often the tool for modeling social problems. One such social phenomenon is that of using the democratic process to make a decision. It is reasonable that we would want to have an algorithm that would allow us to find a ranking or preference that satisfies the preferences of each member in the decision making process.

Given a set \( I \) of individuals, labeled 1, 2, ..., \( k \), that have to make a decision between the objects in set \( A \). Each individual \( 1 \leq i \leq k \) has a preference ordering \( P_i \), individuals \( i \)'s preference profile, that ranks the objects in the set \( A \). \( aP_i b \) means that the \( i \)th individual prefers \( a \) to \( b \) or ranks \( a \) over \( b \). The preference profile of any individual allows for ties. If \( a \) and \( b \) are tied by individual \( i \), then we write \( aT_i b \). An example of \( P_i \) on the set \( A = \{ r, s, t, u \} \) is as follows, where a hyphen between two objects indicates a tie:

\[
\begin{array}{c}
P_i \\
s \\
t - u \\
r \\
\end{array}
\]

Notice that every ranking \( P \) is transitive (if \( aPb \) and \( bPc \) then \( aPc \) ) and asymmetric (if \( aPb \) then not \( bPa \) ).
Given a group, the rankings \( P_1, P_2, \ldots, P_k \) of the members of the group define the group's profile. \( \hat{P} \ A \) is defined to be the set of all possible individual profiles on the elements of set \( A \) and \( \hat{P}_k \ A = \hat{P} \times \hat{P} \times \ldots \times \hat{P} \), \( k \) times; thus \( \hat{P}_k \ A \) is the set of all preference profiles for a group with \( k \) individuals.

The social welfare function is the goal of any group decision making. Given a group with profile \( P_1, P_2, \ldots, P_k \), the goal is to determine a winner or more generally a reasonable ranking based on \( P_1, P_2, \ldots, P_k \). In other words, the object is to find a reasonable function, \( F \) such that \( F : \hat{P}_k \ A \rightarrow \hat{P} \ A \). We should note that finding a ranking is more difficult than finding a winner. Below we will consider several voting strategies and as you teach you can consider whether these are appropriate for your class and the activities you will carry out.

Voting systems, as given by the Wikipedia Article of the same name:

- **Plurality**: a single voting system where each voter gives one vote to produce a single winner who receives the most votes or "relative majority".
- **Runoff**: a single voting system where voting takes place in multiple rounds of plurality voting to ensure that the winner receives a majority of votes, with each round eliminating a set number of candidates/options. Often the runoff process occurs until a single candidate or option receives a majority of the votes.
- **Borda count**: a ranked voting method in which the candidates receive points based on the ranking given by the voter. Usually when there are \( n \) choices, the top choice receives \( n \) points, next preferred choice receives \( n-1 \), and so on to the last choice receiving 1 point.
- **Primary**: two round runoff process, the two candidates receiving the most votes move on to the general election and a winner is never chosen in the primary
- **Range voting**: a ranked system where the voters give numeric ratings to each option, and the option with the highest total score wins. Thus a voter can choose how much they prefer one candidate to another.
- **Approval voting**: a ranked system where voters may vote for as many candidates as they like. Approval voting is range voting where the only ratings are 0 and 1.
- **Condorcet pairwise methods**: ranked voting methods where we compare every option pairwise with every other option, one at a time, and an option that defeats every other option is the winner. An option defeats another option if a majority of voters rank it higher on their ballot than the other option.

The above voting strategies are attempts by mathematicians/voting theorists such as Borda, Condorcet, and Laplace at finding a reasonable social welfare function.
Arrow’s impossibility theorem uses four axioms, each of which seems reasonable for a social welfare function as given by Wikipedia and Fred Roberts:

- **Positive Association of Social and Individual Values**: if any individual reorders her preference to rank option $a$ higher, then in the group rating $a$ will be ranked higher or will remain unchanged.
- **Independence of Irrelevant Alternatives**: the social welfare function provides the same ranking of preferences among a subset of options as it would for a complete set of options. Changes in individuals’ rankings of irrelevant alternatives (ones outside the subset) should have no impact on the societal ranking of the relevant subset.
- **Citizens’ Sovereignty**: whenever each individual in the group ranks $a$ over $b$, the social welfare function also ranks $a$ over $b$.
- **Nondictatorship**: social welfare function accounts for multiple voter preferences, it does not duplicate the preference order of a single individual.

Arrow’s impossibility theorem states that if $|A| \geq 3$ and if there are two or more individuals making the decision, there is no societal welfare function that satisfies all four axioms.

A social welfare function is not always achievable, nor is it always desirable. Thus at times we might abandon sincere voting to achieve a desirable outcome. Below are tactical voting strategies to be considered that do not involve sincere voting.

**Tactical Voting Strategies:**

The following strategies are summarized from the Wikipedia article of the same name.

- **Compromising** (sometimes "useful vote") is a type of tactical voting in which a voter insincerely ranks an alternative higher in the hope of getting it elected.
- **Burying** is a type of tactical voting in which a voter insincerely ranks an alternative lower in the hopes of defeating it.
- **Push-over** is a type of tactical voting in which a voter ranks a perceived weak alternative higher, but not in the hopes of getting it elected. This primarily occurs in runoff voting when a voter already believes that her favored candidate will make it to the next round – the voter then ranks an unpreferred, but easily beatable, candidate higher so that her preferred candidate can win later.
- **Bullet voting** is a type of tactical voting used in elections where voters select more than one representative from a pool of candidates. In this instance a voter can cast as many votes as there are representatives to be elected. The winners are the candidates that receive the highest vote totals. By not casting the maximum number of votes, a voter helps his or her preferred candidate by not supplying votes to potential rivals.
Lesson Plan Ideas:

Prerequisite: Sets and simple counting techniques

(Three to four 50 minute lessons)

Optional task sheets and information are given in attachments to the lesson (G.1-G.5)

Group the students into a minimum of six groups (optimal group size is 3 to 4 students). Give the students a set of candidates in an upcoming election at the state or city level or some other set of objects to vote on that pertains to the students’ lives. Let the size of the set of options be at least three and at most five. Also give each group a distinct ordering of the candidates in terms of preference. Explain that the goal of each group is to get their first or second choice elected and not their last.

Give the groups one particular voting system, single or ranked and any sub category of either of these, if you are not sure which method to choose we suggest a two round runoff system. Have the students take 15 or so minutes to discuss a strategy for getting their candidate to win. As the groups discuss, give each group a manila folder that has the preference ordering of at least two other groups. Let the students use the new knowledge to come up with a detailed strategy in writing. (See Attachment G.1).

After the groups have finished strategizing, allow the students to mingle and discuss the election prior to voting. Inform the students that anything goes except personal attacks of other students and any behavior not part of the school’s standards of conduct. Have the students return to their groups and write down their perceptions of other groups strategies. Then allow the students to vote as individuals, or skip the voting depending on time and other factors.

For homework, have the students investigate the presidential election of 1912 and the history behind it. Also have them read Arrow’s Impossibility Theorem, just for an introduction. Both can be found in easily accessible articles on Wikipedia. Either print these out for students or have the students research them at home.

The next day, or when you continue the topic, partition the class into sets of size 4 or 5 where the sets are not equivalent to those formed the previous day and then give each set two different voting systems and the definitions from the following set:

{plurality, run-off voting, primary, random ballot, Borda count, range voting, approval voting} .

Have the students discuss tactical voting strategies for the type of voting they have been given. Have each group write their ideas on a piece of paper and be prepared to present the strategy.

Union the class and discuss each of the strategies students came up with to have their candidate win. Then introduce the tactical voting strategies making comparisons to the groups’ strategies.
To this point, the ideas that have been discussed are presented as a tool for motivation. Discuss with the students how it is easy to see that not everyone’s preferences can be satisfied, but that should be the goal of any decision making. Thus we want to find a way to make a choice that best satisfies the needs and choices of the group making the choice, a social welfare function.

At this point we can revisit (or visit for the first time) preferences. Discuss the notation that $P_i$ is the preference order for the $i$th individual on a set of candidates or choices. You can also include that there could be no preference between any two choices in an individual’s preference order. This is also an opportunity to have the students review counting. Ask how many ways an individual can make a preference order from a set of $n$ choices. You can start off with the simple case of no ties and then progress to the case where ties are allowed. Include in your discussion the notation $aP_i b$ that is individual $i$ prefers choice $a$ to $b$. Ask students what properties hold for a preference relation, meaning is the preference relation symmetric, reflexive, and transitive. (Attachment G.3)

This is a good place to delve into what a social welfare function is and if a social welfare function can it be reasonable. First, introduce the notation for the profile of a group of $k$ individuals. Also introduce the notation for all possible profiles for a group of size $k$. This will be a transition from the counting done above.

Depending on time you can delve into a discussion of the social welfare function or for the students’ homework have them come up with different algorithms and/or voting strategies that would give a reasonable social welfare function, this can further the discussion for the next day.

Resuming discussion on social welfare functions. Have the class as a whole discuss when the Borda count causes problems that counteract intuition. Come up with concrete examples using the set $A = a, b, c, d, e$. If you and the class are struggling with examples think about when there are four individuals voting on more than three options; three of the voters rank $a$ first and $b$ second and the other one ranks $a$ last and $b$ first. Intuition says that $a$ should be the winner, but by the Borda count, $b$ is the winner.

Obviously this strategy has flaws, thus let’s look at a method that might fix this.

Introduce Condorcet methods as an alternative to a Borda count. With the class, look for a paradox using this method of finding a social welfare function.

Let’s consider the election of 1912. There were three candidates: Taft (Republican, $R$), Roosevelt (Progressive, $P$), and Wilson (Democrat, $D$). The following is the percentage of population that ranked the candidates and their preference orders:

<table>
<thead>
<tr>
<th></th>
<th>$D$</th>
<th>$P$</th>
<th>$R$</th>
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</thead>
<tbody>
<tr>
<td>ranked</td>
<td>45%</td>
<td>30%</td>
<td>25%</td>
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</tbody>
</table>

Let’s look at a concrete example using the set $A = a, b, c, d, e$. If you and the class are struggling with examples think about when there are four individuals voting on more than three options; three of the voters rank $a$ first and $b$ second and the other one ranks $a$ last and $b$ first. Intuition says that $a$ should be the winner, but by the Borda count, $b$ is the winner.

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</thead>
<tbody>
<tr>
<td>ranked</td>
<td>45%</td>
<td>30%</td>
<td>25%</td>
</tr>
</tbody>
</table>

R

45% ranked $P$ 30% ranked $R$ 25% ranked $P$
Using our methods, let’s consider the outcomes of plurality, Borda count, and Condorcet pairwise methods. With the Condorcet pairwise method, construct a tournament where each candidate represents a vertex and arc goes from candidate $a$ to candidate $b$ if the majority ranks $a$ over $b$. Let’s analyze the tournament, that is given below.

For homework, have the students analyze the following example using the plurality method, a Borda count, the Condorcet pairwise method, and a Runoff with elimination by plurality, where after each vote only one option is removed. Also, have the students answer the following questions

A company decided to boost employ morale, they are going to go on a retreat. Four individuals proposed plans. Armand suggested they go to an all inclusive resort for three days. Bergitta optioned for a camping trip with rock climbing and rafting ventures each day. Clyde hinted that some would prefer to go skiing for several days. Demosthenes suggested that they should take the money and give everyone a raise instead. 27 employs voted on the plan and the following are the preference profiles.

$$
\begin{array}{c}
10 & 7 & 6 & 3 & 1 \\
A & C & D & B & C \\
B & B & C & D & D \\
C & D & B & C & B \\
\end{array}
$$

Questions to Consider:

- If you were had the preference of $B$, which method of voting would you choose?
- Since you are a student of discrete mathematics, how could you change your voting preference in a runoff election so that either $C$ would win?
- Since you are a student of discrete mathematics, how could you change your voting preference in a Borda count so that $C$ would win?
- Since you are a student of discrete mathematics, how could you change your voting preference in a Condorcet pairwise method so that $B$ would win?
• Given that the members of the office chose to use a plurality voting system, write an argument against the system with your knowledge of everyone's preferences.

The next day, or whenever is convenient. Review the homework and what the different voting strategies produced. Do the voting systems only give a winner or can they produce a ranking? Have the students discuss why no outcome is given by the same method. Is there any method that is "right"? Introduce the idea of a social welfare function, a function that gives a ranked preference from any set of voter profiles. Give an introduction to Arrow's impossibility theorem that is axiom based. As a class or as an assessment for the students, come up with examples of the axioms using simple examples \( k = 2, |A| = 3 \). Analyze plurality, Borda count, or Condorcet pairwise and how it follows the axioms.

Related topics: Interval graphs, tournaments, game theory

Extensions:

This lesson can easily extend for several days by

✓ Proving Arrow's impossibility theorem.
✓ Having the student's come up with examples of why plurality, Borda count, and Condorcet methods are not reasonable social welfare functions.
✓ Discussing distance between rankings, the axioms and theorems associated with that.
✓ Discussing Gibbard–Satterthwaite theorem, closely related to Arrow's theorem but it also discuss tactical voting strategies.

Suggested Problems for the Classroom and Homework:

From Fred S. Robert's *Discrete Mathematical Models: with applications to social, biological, and environmental problems* 7.1 Exercises 1-5, 7, 8, 13, 14. Any of these modified to the classroom discussion would be beneficial for understanding.

Assessment:

Below is a possible assessment for this lesson. Modification for your classroom experience is expected.

Take the axioms for a reasonable social welfare function and come up with an example for each axiom with a group of 3 individuals making voting on four objects.

Analyze the plurality, Borda count, and Condorcet pairwise to determine if they follow the axioms.
More information on voting:


Wikipedia. Key word search: *Arrow’s impossibility theorem, voting systems, and/or tactical voting*
Voting Theory
Day 1

Names of Group Members

Candidates in the election:

Voting system to be used:

This group’s preferences:

Group _______’s preferences:

Group _______’s preferences:

Strategy to get desired outcome (use the back of this page)
Attachment G.2

Voting Theory
Day 2

Names of Group Members


Voting systems to be used:


Ideas for tactical voting using ________________ system.


Ideas for tactical voting using ________________ system.


Other tactics discussed as a class (write on the back)
Voting Theory
Preferences and Social Welfare Function

Name:
Preference:
  • Notation:
  • Relation Properties:
  • Ties
  • Examples:
    • How many ways can an individual have preference ordering on a set with \( n \) elements?
      • with no ties
    • with ties
Voting Theory
Preference and Social Welfare Function

Social Welfare Function:

- Helpful notation for the definition:
  - Preference profile of a group of size $k$:
  - Set of all possible profiles of a group of size $k$:

- Definition:

- Analyzing Voting Systems
  - Borda Count
  - Condorcet Pairwise
Voting Theory
Preference and Social Welfare Function

- Historical Perspective—Election of 1912
  - The Candidates

- Three profiles and the percent of the population who had the profile

- Outcomes with specific voting systems
  - Plurality

- Borda Count

- Condorcet Pairwise (construct a tournament)
Voting Theory
Preference and Social Welfare Function

- Axioms for a reasonable social welfare function
  - Positive Association of Social and Individual Values: if any individual reorders her preference to rank option \( a \) higher, then in the group rating \( a \) will be ranked higher or will remain unchanged.
  
  - Independence of Irrelevant Alternatives: the social welfare function provides the same ranking of preferences among a subset of options as it would for a complete set of options. Changes in individuals' rankings of irrelevant alternatives (ones outside the subset) should have no impact on the societal ranking of the relevant subset.
  
  - Citizens' Sovereignty: whenever each individual in the group ranks \( a \) over \( b \), the social welfare function also ranks \( a \) over \( b \).
  
  - Nondictatorship: social welfare function accounts for multiple voter preferences, it does not duplicate the preference order of a single individual.

The final assessment for this lesson:

Take the axioms for a reasonable social welfare function and come up with an example for each axiom with a group of 3 individuals making voting on four objects.

Choose one of the following voting methods: plurality, Borda count, or Condorcet pairwise. Use the method and create examples for each of the axioms. If no example exists explain your reasoning.
## Survey on Discrete Mathematics Lesson

**Set Theory**

Date: January 28, 2010

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<td>Activities</td>
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Based on your experience with the lesson, answer the following questions.

Would you take a discrete math course if it were offered next year at your school?  
Yes  No

Do you think discrete math is more applicable than algebra in your everyday life?  
Yes  No

What did you like the most about the lesson?

What did you like the least about the lesson?

Name one thing you learned about sets that you did not know before this lesson.
**APPENDIX I**

Table of Student Responses to the First Eight Prompts on the Survey

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<th>Clear Instruction</th>
<th>Level of Difficulty</th>
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APPENDIX J

Below are frequency tables of students’ responses to the first six prompts of the survey. One table is given for each aspect of the lesson students scored.

### Engaging

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### Informative

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Note that the "6" in the activities chart represents a no response from a student.