Linearized Conformal Gravity

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Introduction

Weyl gravity [1,2] is a theory of gravity based on the conformally invariant action

$$S_{Weyl} = \int C^{\mu\nu\alpha\beta} C_{\mu\nu\alpha\beta} \sqrt{-g}\, d^4x$$

quadratic in the Weyl tensor, with metric variation leading to the fourth-order Bach equation,

$$W^{\mu\nu} = D_\alpha D_\beta C^{\mu\alpha\nu\beta} - \frac{1}{2} C^{\mu\alpha\nu\beta} R_{\alpha\beta} = \kappa T^{\mu\nu}$$

Vacuum solutions to the Einstein equation (Figure 1) are also solutions to the Bach equation. However, counterexamples to the converse are known. While specific solutions give some insights, we take a different approach to the relations between general relativity and conformal gravity, linearization.

Methods

The linearization of general relativity around flat space, $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$, is well-established [3].

$$G_{\alpha\beta} = \frac{1}{2}(\bar{h}_{\beta\nu} - \bar{h}^{\alpha}_{\beta,\alpha\nu} - \bar{h}^{\alpha}_{\nu,\alpha\beta} + \eta_{\beta\nu}\, \bar{h}^{\alpha\beta}_{,\alpha\beta}$$

The infinitesimal coordinate freedom of the metric yields the much simpler equation

$$\Box \bar{h}_{\alpha\beta} = \sigma T_{\alpha\beta}$$

where $\bar{h}_{\alpha\beta}$ is the trace-reversed metric perturbation. Using Green's functions, we can determine solutions to the linearized Einstein equation with source (Figure 3). On the other hand, linearization of the Bach equation for conformal gravity is less understood. By linearizing both general relativity and Weyl gravity, we can compare large classes of solutions, both with and without sources.

Results

In addition to coordinate covariance, Weyl gravity allows conformal transformations. By changing the metric by a conformal factor near unity, performing an infinitesimal coordinate transformation, and constructing a transverse, traceless gauge choice for arbitrary Weyl metric perturbations, the linearized expression for the Bach tensor becomes

$$\frac{1}{2} \Box\Box h_{\alpha\beta} = \kappa T^{\mu\nu}$$

By imposing certain initial conditions and boundary conditions, we find general solutions to the linearized Bach equation using Green's functions (Figure 3).

$$h^E_{\alpha\beta}(x,t) = \int_V G^{\mu\nu}_{\alpha\beta}(x,t;x',t') T_{\mu,\nu} d^4x'$$
$$-\frac{1}{4\pi} \int_S \frac{\partial G^{\mu\nu}_{\alpha\beta}(x,t;x',t')}{\partial n'} h_{\mu\nu}(x',t') d^3x'$$

$$h^W_{\alpha\beta} = \kappa \int_V d^4x'' G^{\rho\sigma}_{\alpha\beta}(x,t;x'',t'') \cdot$$
$$\int_V G^{\mu\nu}_{\beta\sigma}(x'',t'';x',t') T_{\mu\nu}(x',t') d^4x''$$

Figure 3. Solutions to the linearized Einstein equation (top) and the linearized Bach equation (bottom) using a Green function G.

$$G_{\alpha\beta} = \sigma T_{\alpha\beta}$$

Figure 1. The Einstein equation, where $G_{\alpha\beta}$, $T_{\alpha\beta}$, and σ are the Einstein tensor, the stress energy tensor, and the curvature, respectively.

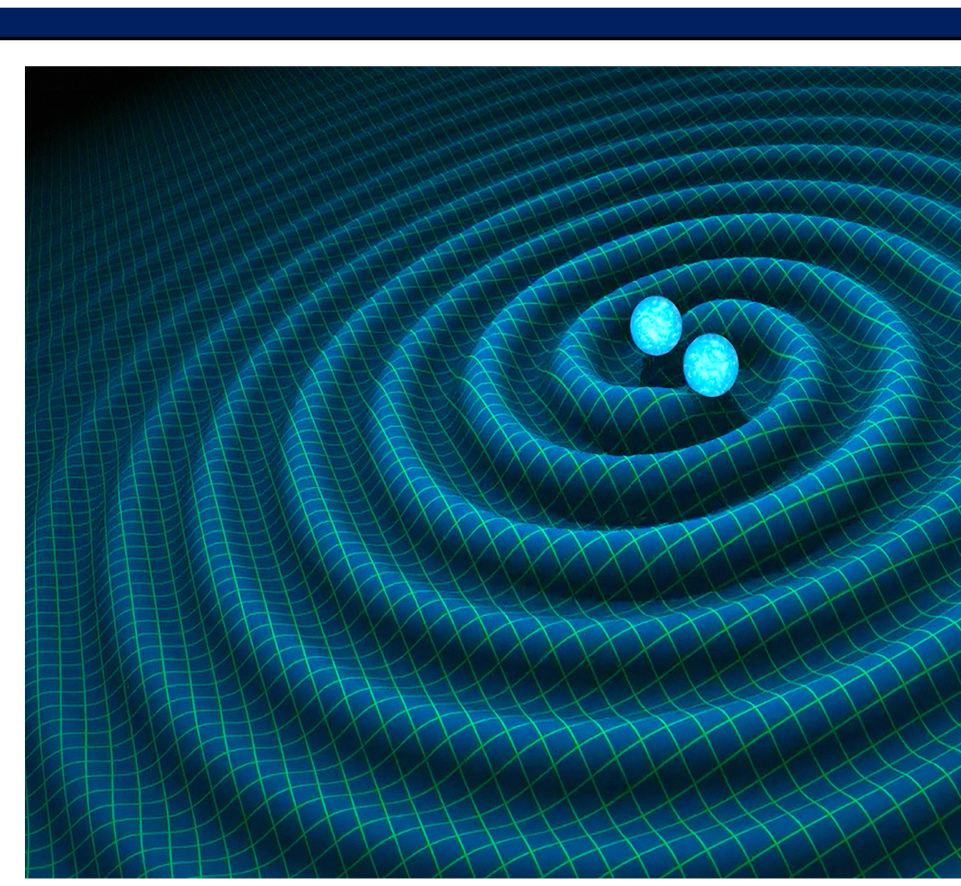

Figure 2. Linearization of general relativity around flat space provides a description of phenomenon such as gravitational waves. We see above an artist's depiction of a gravitation wave.

Conclusions

Outside of specific examples, the differing boundary conditions on the solutions in Figure 3 make it difficult to characterize the relationship between solutions to the linearized Einstein equation and the linearized Bach equation. However, by restricting to only the source term and suitably restricting the class of perturbations we consider, we have gained some interesting insights which are the subject of current investigations.

References

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Acknowledgments
Noah gives his deepest thanks to Dr. James Wheeler from Utah State University's Department of Physics for his help and mentorship throughout his work on this project.