Emergence of Classical Black Hole Thermodynamics using Monodromic Analysis

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Overview

- The present discussion will focus on deriving the classical variational entropy of a Black Hole using asymptotic exponents, e.g. monodromies:

$$X(x \to x_i) \sim (x - x_i)^{\beta_i} \iff X(x) = C(x - x_i)^{\beta_i}(1 + O(x - x_i))$$

- Why analyze Asymptotics?
  - Characterizing asymptotics is often **much** simpler than determining exact solutions. Further, asymptotic analysis can provide insight into singularity/boundary features.
    - In fact, it will be shown that the asymptotic exponent of our solution encodes the classical 2nd Law of Black Hole Thermodynamics:

$$T_i \delta S = \delta E_i - \Omega_i \delta J_i$$

    - In particular, the asymptotic exponent is directly proportional the variational entropy: $\delta S_i = 4\pi\alpha_i$

Local Field Equations

- Presently, we consider Flat and AdS-Kerr solutions within pure GR. These are uncharged, rotating, stationary Black Holes.
  - We choose the massless scalar Klein-Gordon field as our Thermodynamic Probe because it is:
    - a Lorentz-covariant, quantized scalar field with minimal spacetime coupling (it interacts through the scalar amplitude; i.e., the probe interacts via a single, Lorentz-covariant scalar parameter).
    - Massless: Does not perturb the metric (spacetime remains fixed as we probe, i.e., the system remains in Equilibrium).
    - Further, in these cases, its Euler-Lagrange equation is separable.
- Klein-Gordon Equation for mass-less scalar field:

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right)=0$$

Black Hole Spacetime Metrics

- **Flat:**

$$ds^2 = \frac{\Sigma}{\bar{\Delta}}dr^2 - \frac{\bar{\Delta}}{\Sigma}(dt - a\sin^2\theta d\phi)^2 + \Sigma\, d\theta^2 + \frac{\sin^2\theta}{\Sigma}\left((r^2+a^2)d\phi - a dt\right)^2$$

$$\bar{\Delta} = r^2 + a^2 - 2Mr = (r - r_-)(r - r_+) \text{ and } \Sigma = r^2 + a^2\cos^2\theta$$

  - Two trapped surfaces: r_+ is the Event Horizon, r_- is the Cauchy Horizon

- **AdS:**

$$ds^2 = \frac{\Sigma}{\Delta}dr^2 - \frac{\Delta}{\Sigma}(dt - \frac{a}{\Xi}\sin^2\theta d\phi)^2 + \frac{\Sigma}{\Delta_\theta}d\theta^2 + \frac{\Delta_\theta}{\Sigma}\sin^2\theta\left(\frac{(r^2+a^2)}{\Xi}d\phi - a dt\right)^2$$

$$\Delta_\theta = 1 - \frac{a^2}{l^2}\cos^2\theta, \qquad \Xi = 1 - \frac{a^2}{l^2}, \qquad \Sigma = r^2 + a^2\cos^2\theta,$$

$$\Delta = (r^2+a^2)(1 + \frac{r^2}{L^2}) - 2Mr = \frac{1}{L^2}\prod_{i=1}^{4}(r - r_i)$$

  - Similarly, r_1 is the Event Horizon, $r_{i=2,3,4}$ are the AdS-Cauchy Horizons[3].

Radial Ansatz

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right)=0$$

- It is straight-forward to expand our KG-field in temporal and azimuthal coupling constants:

$$\psi(t,r,\theta,\phi)=e^{-i\omega t+im\phi}R(r)S(\theta)$$

  - By regarding the polar function's dependence on ω as a (not necessarily small) perturbation, the discrete set of polar eigenvalues, K_l, have been found.[1]

- The Kerr Radial Ansatz[2]:

  - **Flat:**

$$[\partial_r^2+\frac{\partial_r\bar{\Delta}}{\bar{\Delta}}\partial_r+\frac{r_+-r_-}{\bar{\Delta}}(\frac{\tilde{\alpha}_+^2}{r-r_+}-\frac{\tilde{\alpha}_-^2}{r-r_-})+\frac{\Delta^*-K_l}{\bar{\Delta}}]R(r)=0$$

$$\bar{\Delta}=(r-r_+)(r-r_-)\,,\quad \Delta^*(r)=r^2+2M(2M+r)\,,\quad \alpha_\pm=\pm\frac{2Mr_\pm\omega-am}{r_+-r_-}$$

  - **AdS:**

$$\left[\partial_r^2+\frac{\Delta_l'}{\Delta_l}\partial_r+\sum_i\frac{\alpha_i^2\Delta_l'(r_i)}{\Delta_l(r-r_i)}+\frac{\Delta_l^*-K_l}{\Delta_l}\right]R(r)=0$$

$$\Delta_l=\frac{1}{L^2}\prod_{i=1}^{4}(r-r_i)\,,\quad \Delta_l^*=-l^2\Xi w^2+\frac{a^2m^2}{l^2}\,,\quad \alpha_i=\frac{(r_i^2+a^2)}{\Delta_r'(r_i)}(w-\Omega_i m)$$

Thermodynamics

- The KG separation constants ω,m represent the time and azimuthal-rotation quasinormal modes.
    - As such, they are the generators of the KG-field energy and angular momentum operators: $\omega = \delta E$, $m = \delta J$.

- Noting the 2nd Law of BH Thermodynamics:

$$T_i \delta S_i = \delta E_i - \Omega_i \delta J_i$$

    - *T_i is defined as the Hawking temperature at the i^{th} horizon*
    - And, rewriting the α_i in terms of physical parameters:

$$4\pi\alpha_i = \frac{\omega - \Omega_i m}{T_i} = \frac{\delta E_i - \Omega_i \delta J_i}{T_i} \Rightarrow \delta E_i = T_i(4\pi\alpha_i) + \Omega_i \delta J_i$$

- α_i is therefore directly proportional to the classical variation of the Black Hole Entropy!

$$\boxed{\delta S_i = 4\pi\alpha_i}$$

Radial Ansatz

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\psi\right)=0$$

- It is straight-forward to expand our KG-field in temporal and azimuthal coupling constants:

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- The Kerr Radial Ansatz[2]:
  - **Flat:**

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$$\bar{\Delta}=(r-r_+)(r-r_-)\,,\quad \Delta^*(r)=r^2+2M(2M+r)\,,\quad \alpha_\pm=\pm\frac{2Mr_\pm\omega-am}{r_+-r_-}$$

  - **AdS:**

$$\left[\partial_r^2+\frac{\Delta_l'}{\Delta_l}\partial_r+\sum_i\frac{\alpha_i^2\Delta_l'(r_i)}{\Delta_l(r-r_i)}+\frac{\Delta_l^*-K_l}{\Delta_l}\right]R(r)=0$$

$$\Delta_l=\frac{1}{L^2}\prod_{i=1}^{4}(r-r_i),\ \Delta_l^*=-l^2\Xi w^2+\frac{a^2m^2}{l^2},\quad \alpha_i=\frac{(r_i^2+a^2)}{\Delta_r'(r_i)}(w-\Omega_i m)$$

Fuch's Relation

- For any second order differential equation, the solutions local to any regular point z_0 are[1]:

$$w(z) = (z - z_0)^{\sigma}\big(1 + O(z)\big)$$

  - Then the **Monodromy**, σ, solves the indicial equation for a second order ODE:

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0 \quad \Longrightarrow \quad \sigma(\sigma-1) + f_0\sigma + g_0 = 0$$

$$f_0 := \lim_{z\to z_0} f(z)(z - z_0)$$

$$g_0 := \lim_{z\to z_0} g(z)(z - z_0)^2$$

    - If every pole is regular, Fuch's relation applies:

$$\sum_{b,i}^{n,r} \sigma_{bi} + \sum_{b}^{n} \sigma_{b\infty} = \frac{n(n-1)(r-1)}{2}$$

      - Here, b indexes the multiplicity of the pole: for an nth order ODE, there are n linearly independent solutions (here, b=1,2); i indexes each of the r singularities.

*Note: KG-AdS is everywhere regular, KG-Kerr is **not** regular at spatial ∞.*

$$\frac{d^2w}{dz^2} + f(z)\frac{dw}{dz} + g(z)w = 0$$

Indicial Equation

$$\sigma(\sigma - 1) + f_i\sigma + g_i = 0$$

Fuch's Relation

- Returning to our Ansatze:
  - ***AdS:*** $\Delta_l = \frac{1}{L^2}\prod_{i=1}^{4}(r - r_i)$

$$\left[\partial_r^2 + \frac{\Delta_l'}{\Delta_l}\partial_r + \sum_i \frac{\alpha_i^2 \Delta_l'(r_i)}{\Delta_l(r - r_i)} + \frac{\Delta_l^* - K_l}{\Delta_l}\right]R(r) = 0$$

$$\Rightarrow \quad f_i = 1 \qquad g_i = \alpha_i$$

  - ***Kerr:*** $\Delta = (r - r_+)(r - r_-)$

$$\left[\partial_r^2 + \frac{\Delta'}{\Delta}\partial_r + \frac{(r_+ - r_-)}{\Delta^2}\Big(\alpha_+^2(r - r_-) - \alpha_-^2(r - r_+)\Big) + \frac{1}{\Delta}\Big(\Delta^*\omega^2 - K_L\Big)\right]R(r) = 0$$

$$\Rightarrow f_\pm = 1 \quad g_\pm = \alpha_\pm$$

    - *Each finite horizon is regular:*
      - ***AdS:*** $\sigma_{bi} = i\alpha_i(-1)^{b-1}$
      - ***Kerr:*** $\sigma_{b\pm} = i\alpha_\pm(-1)^{b-1}$

$$\Rightarrow R_{AdS}(r) = (r - r_i)^{\pm i\alpha_i}\big(1 + O(r - r_i)\big)$$

$$\Rightarrow R(r) = (r - r_\pm)^{\pm i\alpha_\pm}\big(1 + O(r - r_i)\big)$$

Fuch's Relation

$$\sum_{b,i}^{n,r} \sigma_{bi} + \sum_{b}^{n} \sigma_{b\infty} = \frac{n(n-1)(r-1)}{2}$$

Indicial Equation

$$\sigma(\sigma - 1) + f_i \sigma + g_i = 0$$

Monodromies

- The Monodromies at infinity can be found by the Frobenius Method, giving:

❖ **AdS:**

$$\tilde{R}(r) = \begin{cases} 1 + O(\frac{1}{r}) \\ r^3(1 + O(\frac{1}{r})) \end{cases} \qquad \sigma_{b\infty} = 0, 3$$

◦ Then, because n=2 and there are exactly r=4 regular singularites, Fuch's Relation applies:

$$0 = \sum_{i=1}^{4}\sum_{b=1}^{2} \sigma_{bi} = i \sum_{i=1}^{4}\sum_{b=1}^{2} (-1)^b \alpha_i$$

❖ We note the sign pairings of $\{\sigma_{ib}\}$ yield this trivially.

❖ **Kerr:**

$$R(r) = e^{\pm i\omega r} r^{\sigma_{\infty\pm}} \left(1 + O(r^{-1})\right) \qquad \sigma_{\infty\pm} = \pm 2iM\omega$$

– Although infinity is **not** regular, if we consider the exponent of r to be the monodromy parameter, then:

$$\sum_{i=1}^{4}\sum_{b=1}^{2} \sigma_{bi} + \sum_{b=1}^{2} \sigma_{b\infty} = 0$$

These pair correlations beg for renormalization!

Fuch's Relation

$$\sum_{b,i}^{n,r}\sigma_{bi}+\sum_{b}^{n}\sigma_{b\infty}=\frac{n(n-1)(r-1)}{2}$$

Monodromy Frame

- Making the following analytic coordinate change[4]:

$$R(r)=\prod_i (r-r_i)^{-i\alpha_i}\tilde{R}(r) \Rightarrow \tilde{R}_i(r)=\begin{cases}1+O(r-r_i)\\ (r-r_i)^{2i\alpha_i}\left(1+O(r-r_i)\right)\end{cases}$$

  - *The Monodromies transform, accordingly, as:*

    - **AdS:**

$$\tilde{\sigma}_{ib}=0,2i\sigma_i$$
$$\tilde{\sigma}_{b\infty}=0,3$$
$$\Rightarrow \boxed{2i\sum_{i=1}^{4}\alpha_i=\frac{i}{2\pi}\sum_{i=1}^{4}\delta S_i=0}$$

    - **Kerr:**

$$\tilde{\sigma}_{ib}=0,2i\sigma_i$$
$$\tilde{\sigma}_{b\infty}=0,4iM\omega$$

By direct calculation:

$$\sum_{b,\pm}\tilde{\sigma}_{b\pm}=2i(\alpha_+-\alpha_-)=4iM\omega=\tilde{\sigma}_{\infty+}$$

Kerr is a smooth AdS-limit, and this looks Fuchs-like, so…

Cauchy Horizons

- The flat-limit is variationally smooth, and in it the sum of the complex AdS-monodromies auto-normalize to the variational Kerr entropy at infinity: $\lim_{L\to\infty}(\alpha_3+\alpha_4)=-4\pi\delta S_\infty$

  - Further, the complex horizons and corresponding monodromies, diverge conjugately to complex infinity:

$$\lim_{L\to\infty} r_{n=3,4}=-r_0-(-1)^n i\lim_{L\to\infty} L$$

$$\lim_{L\to\infty} \alpha_{n=3,4}=-M\omega-(-1)^n \lim_{L\to\infty}\frac{iL\omega}{2}$$

    - Together, these divergences create two poles of opposite direction and sign in the same neighborhood on the boundary: the entropy on the boundary is realized as a contour deformation about this point.

Monodromy Frame Relation

- Intriguingly, we've found an additional constraint:
  - **AdS:** $$\delta \sum_{i=1}^{4} S_i = 0$$
    - The sum of the Horizon Entropies is a function **independent** of the Black Hole parameters (mass and angular momentum).
    - Further, in the flat limit the monodromy **sums** approach their counterparts:

$$\lim_{L\to\infty}(\alpha_1+\alpha_2)=\alpha_+ + \alpha_- \ , \quad \lim_{L\to\infty}(\alpha_3+\alpha_4)=\frac{i}{2}\tilde{\sigma}_{\infty+}$$

and we conclude: $$0=\lim_{L\to\infty}\delta\sum_{i=1}^{4}S_i=\delta(S_+ + S_-)+2i\pi\tilde{\sigma}_{\infty+}$$

  - **Kerr:** $$0=\delta(S_+ + S_-)+2i\pi\tilde{\sigma}_{\infty+}$$
    - The result is derived directly from the Kerr geometry; indeed

$$\delta S_\infty = -2i\pi\tilde{\sigma}_{\infty+} = 8\pi M\omega = \delta(4\pi M^2)$$

$$\Longrightarrow \delta(S_+ + S_-) = \delta S_\infty = \delta(4\pi M^2)$$

    - This holds even though Kerr is **not** completely regular (spatial infinity is an irregular singularity in d=4 asymptotically flat spacetime).

Monodromy Conservation

$$\sum_i \alpha_i = 0$$

- We can arrive at the same constraint on the summed entropy using physical parameters alone:
  - With the Hawking temperature, entropy, and angular velocity of each event horizon are given, respectively, by

$$T_i = \frac{\Delta'(r_i)}{4\pi(r_i^2 + a^2)}, \qquad S_i = \frac{\pi(r_i^2 + a^2)}{\Xi}, \qquad \Omega_i = \frac{a(1 + r_i^2/L^2)}{r_i^2 + a^2}.$$

and, indeed, the **summed** Bekenstein-Hawking entropy of the AdS-Kerr BH is independent of the physical parameters:

$$\sum_{i=1}^{4} S_i = -2\pi L^2, \Rightarrow \delta \sum_{i=1}^{4} S_i = 0 \Rightarrow \sum_i \alpha_i = 0$$

  - In AdS, the **sum** of the coefficients of each quasinormal mode in the monodromies independently cancel, and thus the **sum** of the monodromies is uniformly zero:

$$4\pi\alpha_i = \frac{\omega - \Omega_i m}{T_i} = \frac{\delta E_i - \Omega_i \delta J_i}{T_i}$$

$$\sum_{i=1}^{4} \frac{\Omega_i}{T_i} = 0, \qquad \sum_{i=1}^{4} \frac{1}{T_i} = 0,$$

$$\sum_i \alpha_i = 0$$

Further Results

- Results have been found in a wide array of cases[4]:

$$4\pi\alpha_i = \frac{\delta E - \Omega_i \delta J}{T_i}$$

$$\sum_i \alpha_i = 0$$

$$\delta \sum_{i=1}^{4} S_i = 0$$

Black Hole	$\sum_i \delta S_i = \delta S_\infty$	$\sum_i T_i^{-1} = 0$	$\sum_i \Omega_i^a / T_i = 0$	$\sum_i \Phi_i^a / T_i = 0$
Schwarzschild	✓	✗	-	-
Kerr [13]	✓	✗	✓	-
Reissner-Nordstrom (RN)	✓	✗	-	✗
Kerr-Newman (KN) [12]	✓	✗	✓	✗
Schwarzschild $d > 4$ [14]	✓	✓	-	-
Myers-Perry [6]	✓	✓	✓	-
BTZ [7]	✓	✓	✓	-
Schw-(A)dS	✓	✓	-	-
Kerr-(A)dS [8]	✓	✓	✓	-
Schw-(A)dS$_d$	✓	✓	-	-
Kerr-(A)dS$_d$ [9, 15]	✓	✓	✓	-
RN$_d$[11]	✓	✓	-	✓
RN-(A)dS	✓	✓	-	✓
RN-(A)dS$_d$	✓	✓	-	✓
KN-(A)dS [11]	✓	✓	✓	✓
5d gauged SUGRA [16]	✓	✓	✓	✓
6d gauged SUGRA [17]	✓	✓	✓	✓

  - These results were found by calculating the relevant parameters directly from the metric; the above results are thus **completely** independent of the probe-field.

$$\sum_i \alpha_i = 0$$

Appendix A:
Further Analytic KG Results

- **Extremal AdS** ($r_+ \to r_-$):

$$\alpha_i = \frac{(r_i^2+a^2)}{\Delta_r'(r_i)}(w - \Omega_i m)$$

Event Horizon Asymptotic:

$$\tilde{R}_1(r) = 1 + O(r - r_0)$$
$$\tilde{R}_2(r) = e^{\frac{i\epsilon \sum_l (-1)^l \tilde{\alpha}_l}{r - r_0}} (r - r_0)^{2i \sum_l \tilde{\alpha}_l} (1 + O(r - r_0))$$

- **BTZ:**

$$\alpha_{\pm i} = \pm \frac{L^2 r_i(\omega - \frac{Jm}{2r_i^2})}{2(r_i^2 - r_j^2)}$$

Event Horizon Asymptotic:

$$\tilde{R}^{(1)}_{r \to r_i} = 1 + O(r - r_i)$$
$$\tilde{R}^{(2)}_{r \to r_i} = (r - r_i)^{2i\alpha_i}(1 + O(r - r_i))$$

- **Extremal BTZ:**

Event Horizon Asymptotic:

$$\tilde{R}_1(r) = (r - r_0)^{i \sum_l \alpha_l} \Big(1 + O(r)\Big)$$
$$\tilde{R}_2(r) = e^{\frac{2i\epsilon \sum_l (-1)^l \alpha_l}{r - r_0}} (r - r_0)^{i \sum_l \alpha_l - 1} \Big(1 + O(r)\Big)$$

Appendix B:
Monodromy Frame Radial Ansatze

- **AdS:**

$$\Big[\partial_r^2 + \Big(\sum_{i=1}^{4} \frac{1-2i\,\tilde{\alpha}_i}{r-r_i}\Big)\partial_r - \sum_{i}^{4}\sum_{j\neq i}^{4} \frac{\tilde{\alpha}_i(i+\tilde{\alpha}_j)}{(r-r_i)(r-r_j)} + \sum_{i=1}^{4} \frac{\tilde{\alpha}_i^2}{(r-r_i)^2}\left(\frac{L^2\Delta'(r_i)}{\prod_{j\neq i}(r-r_j)} - 1\right) + \frac{\Delta^*}{\Delta}\Big]\tilde{R}(r) = 0$$

- **Kerr:**

$$\Big[\partial_r^2 + \Big(\sum_{i=1}^{2} \frac{1-2i\tilde{\alpha}_i}{r-r_i}\Big)\partial_r - \frac{2\tilde{\alpha}_+\tilde{\alpha}_- + i(\tilde{\alpha}_+ + \tilde{\alpha}_-)}{\bar{\Delta}} + \sum_{i=1,j\neq i}^{2} \frac{\tilde{\alpha}_i^2}{(r-r_i)^2}\Big(\frac{r_i-r_j}{r-r_j} - 1\Big) + \frac{\bar{\Delta}^*}{\bar{\Delta}}\Big]\bar{R}(r) = 0$$

Thank You

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