An Exploration Into Polynomial Functional Graph Mapping Problems

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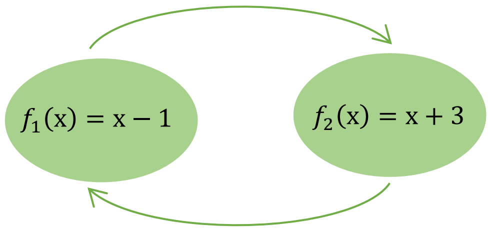
Problem Description

Given a directed graph in which vertices are polynomials, does there exist a walk which maps a given real x to a given real y? Where the mapping of a walk is equivalent to the mapping of the composition of functions in the walk.

Example

Does there exists a walk which maps 5 to 10 in the following graph G_1?



Graph G_1

Yes, the walk $\{f_2, f_1, f_1\}$ maps 5 to 10.

$$f_2\left(f_1\left(f_2(5)\right)\right) = 10$$

Approach

As any directed graph with a closed walk will have an infinite number of walks, the approach used translates repetition between walks into equations. If there are a finite number of equations and the equations are solvable for the repetition count, then it is possible to determine if a walk which maps x to y exists in the graph.

Converting Graphs To Equations

As the mapping of walks is determined by the composition of the functions comprising the walk, the approach is to create equations from repetitions. The author uses a special walk called a **cycle** along with a new operator called the **cycle operator** to create the equations.

Cycles

A **cycle** is a special walk described by its ability to be composited with itself any integer number of times and result in a walk that is still in the graph. A cycle *c* is considered a **base cycle** if there is no other cycle in the graph which, when composited with itself, creates *c*.

Example – Base Cycles of G_1

The following shows the base cycle walks and compositions of the base cycles of G_1.

$\{f_1, f_2\}$ -> $c_1 = f_2(f_1)$ = x + 2
$\{f_2, f_1\}$ -> $c_2 = f_1(f_2)$ = x + 2

As counter examples, $\{f_1, f_2, f_1, f_2\}$ is not a base cycle because it is equivalent to $c_1(c_1)$. And $\{f_1\}$ is not a cycle because the walk $\{f_1, f_1\}$ does not exist in G_1.

Cycle Operator

Much like how multiplication is repeated addition, the cycle operator is repeated composition. The cycle operator takes the from:

Number $\circlearrowleft$ Function

Using this operator is equivalent to taking the composition of the Function with itself the Number of times. The Number must be a positive integer or zero. If Number is zero, then the resulting function is f(x)=x. The result of the cycle operator is a function.

Example - Cycle Operator

Using the cycle c_1 from the base cycles of G_1.

$0 \circlearrowleft c_1$ = x
$1 \circlearrowleft c_1 = c_1$=x+2
$2 \circlearrowleft c_1 = c_1(c_1)$=x+4

Creating Equation Set From Graph

This pseudo code generates the equation set E for a graph in which a max of one cycle needs to be accounted for in any walk.

Let E be the empty set
For each possible starting function f_s:
  For each possible ending function f_e of walks starting with f_s (labeled as $w_{s,e}$):
    For each function f_t in walk $w_{s,e}$:
      For each walk base cycle c_t starting at f_t:
        add y= $w_{t,e}(\, c_t(w_{s,t}(x)))$ to set E
      If there is no base cycle c_e starting at f_e :
        add y= $w_{s,e}(x)$ to set E

Equation Set Of Example Set G_1

{
y = $c_2(f_1(x))$ = (n $\circlearrowleft c_2$)(x-1) = x-1+2n
y = $c_1(f_2(f_1(x)))$ = (n $\circlearrowleft c_1$)(x+2) = x+2(n+1)
y = $c_1(f_2(x))$ = (n $\circlearrowleft$ c_1)(x+3) = x+3+2n
y = $c_2(f_1(f_2(x)))$ = (n $\circlearrowleft c_2$)(x+2) = x+2(n+1)
}

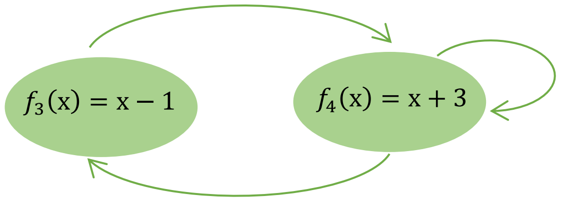
Solving The Equation Set Mapping

All equations in the set will have at least one cycle operator. As such, it is necessary to solve for the number of repetitions for the cycle. And, as you can not apply a part of a cycle, the number of repetitions must be a positive integer. The only way the author was able to find to do so, was to find a closed form for the recursion. Finding a closed form is specific per function, and as such, is not covered here. For example, the cycles which are 3rd degree polynomials have the name quadradic maps, and not all of those can be expressed in a closed form.

Case Of Infinite Base Cycles

The algorithm shows that there are at least as many equations as the max of the number of functions and base cycles in the graph. For certain graphs, there are an infinite amount of base cycles, which will result in an infinite amount of equations to solve. This occurs when base cycles are created from a combination of other base cycles. In general, problems containing graphs with an infinite number of base cycles remain open.

Example: Infinite Base Cycles



Graph G_2

This graph has infinite base cycles. For example $\{f_3, f_4\}$, $\{f_3, f_4, f_4\}$, and $\{f_3, f_4, f_4, f_3, f_4\}$ are base cycles.

Special Case: Linear Diophantine Equations

There is a special case when cycles are communitive. **Communitive cycles** are sets of cycles which have the following properties for some cycles c_j and c_i:

$c_j(c_i) = c_i(c_j)$

In the case of communitive cycles, if all cycles in the base cycles set can be described as a combination of finite subset of the base cycle set, then it is possible to create a finite set of equations describing all mappings of all the walks in the graph. This is the case when all functions in the graph take the form $f_i(x) = x + c_i$ where c_i is some real number. In such cases the walks will result in linear Diophantine equations.