An Investigation of Active Learning on Students' Understanding of Infinite Series Convergence

Zachary Coverstone
Utah State University

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An Investigation of Active Learning on Students’ Understanding of Infinite Series Convergence

Zachary M. Coverstone
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A project submitted as partial fulfillment for the Master of Science in Mathematics degree at Utah State University.
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An early mentor of mine said to me as we parted ways that she hoped I would “become the teacher I wanted to be”. I believe that I am on that journey. As part of that journey, many people have been a part of my mathematics and teaching education, both those I have met and those I have not. Trying to list the contributions of each of the people mentioned would take a thesis, and perhaps more, of itself, so I intend to only list their names here as a way to acknowledge their contributions to helping me be the teacher I currently am today. I thank them all publicly, as well as all of the others who I have neglected to mention for any reason. I also especially thank those with asterisks next to their names for help in editing this document.

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Kay Hand
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Carla Haws
Lindsey Henderson
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Pamela Hill
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Sarah Johns
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Tim Jones
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Pam Kenyon
 Bowen Kerins
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Melissa Newberry
Samila Nickell
Zhaohu Nie*
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Renee Seegmiller
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Randy Stillman
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Stephanie Strait
Trisha Syverson
Dawn Teuscher
Sara Van Der Werf
Robert Wadley
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Zhi-Quang Wang
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Chapter 1

Introduction

1.1 Motivation: A Personal Narrative

As a first-semester graduate student, my first teaching assignment was to be a recitation leader for second-semester calculus lectures. I was nervous. I thought back to all of my previous experiences with infinite series, which was the source of the majority of my concern. To say I was nervous about this assignment is an understatement, but I took the challenge with faith I could succeed.

In engaging with the course content with a new perspective, I found that students were also struggling by and large with infinite series convergence, power series, and related concepts. The initial fears I had as a graduate teaching assistant appeared to be in my students. The student struggle was manifest again in the subsequent semester where I had the same recitation assignment.

I had seen success in other parts of the country implementing active learning methods, especially using a model developed for use with the Park City Mathematics Institute’s Teacher Leadership Program, which will be described in chapter two. Additionally, several researchers had investigated elementary understandings of infinite series convergence, but no formal curriculum that I could find had used these ideas in a complete way; some of these initial studies are summarized and referred to in section 1.7.

This study is motivated by my previous experience as a recitation leader and in examining current materials used in teaching infinite series convergence. Extant materials seemingly focused less on constructing concepts and discovering relationships (Cangelosi, 2002), and experience working individually with undergraduate calculus students taught me that students struggled with infinite series convergence and often confused the ideas of series and sequences. Also, since power series posed to be especially challenging for students, and since geometric series play a large role in understanding power series (i.e. Ratio Theorem), students need a strong understanding of geometric series convergence to be able to do mathematics with power series. These considerations, along with research-based considerations mentioned in the literature review, motivated the need to build a curriculum that accounted for appropriate conceptual foundations.
1.2 Project Goals

The goals for this project are:

1. To develop a curriculum that provides foundational understanding of infinite series convergence that includes appropriate concept-building using models; and,

2. To analyze student work to determine revisions for the curriculum in the future.

In chapter one, I discuss the mathematics of infinite series, including potential challenges students encounter when first learning about the content. Along with this mathematical description, included is a brief review of history and applications and relevant literature about what is known about student understanding with respect to infinite series. Also, active-learning pedagogies are briefly discussed, which is the format for the developed unit.

Chapter two discusses the study itself, including discussing relevant themes extracted from the student work.

Chapter three dives into a discussion specifically of student work with respect to the six themes mentioned in chapter two. These themes form a part of the foundational understanding for infinite series convergence. Each section includes a description of how the curricular materials were developed specifically to include each of the themes and an analysis of relevant student work from the the implementation of the curricular materials as described in chapter two.

Finally, in chapter four, I discuss results from the study and a discussion of potential future directions of the study.

1.3 Finite Sums

Summing series with finite terms is an experience many elementary school students have. Finding the average (arithmetic mean) temperature over a month in a city, calculating the total money spent on a trip to a grocery store, or finding the total amount of money to budget for a family vacation are common examples of processes requiring finding sums of finitely many terms.

Secondary students generally engage with series in a finite sense, specifically using arithmetic and geometric progression. These can be encountered at various points in the high school curriculum, but generally first appear in connection with linear and exponential functions for algebra or pre-calculus students. For a finite series, the series always converges, or has a finite, real-numbered value; it will never diverge.

1.4 Infinite Series

Infinite series generally are first encountered in second-semester calculus, usually applied to power series (i.e. Taylor and Maclaurin series). Infinite series differ from finite series in that usual properties, such as commutativity and associativity of addition, do not apply in
this context. Because the theorems and rules surrounding infinite series are different than
with finite series, students need to engage in a careful study of correct properties of infinite
series.

What follows illustrates some of the challenges of understanding infinite series convergence
from a novices’ perspective

**Infinite Sequences and Infinite Series**

There is often confusion between the sequence

$$
\left( \frac{1}{n^2} \right)_{n=1}^{\infty} = \left( \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \ddots \right)
$$

and the series

$$
\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots
$$

The first refers to a function $a_n : \mathbb{N} \rightarrow \mathbb{R}$ and the second refers to the limit of partial
sums of $a_n$. The partial sums themselves form a sequence which can converge.

In some cases, infinite sequences and series converge and at other times they will diverge.
One of the key questions of studying infinite series convergence is to determine when series
converge and when they diverge.

In the above cases, both the sequence and series converge. However, for series with more
complicated summands, determining convergence can seem daunting. With this realization,
it is necessary to consider what can be done pedagogically about this. The following three
examples illustrate some of the difficulties in trying to demonstrate what happens when
summing infinitely many terms.

**Infinitely Many Terms Converge**

Another counter-intuitive fact is that summing infinite terms yields a finite (convergent)
sum. For instance, the fact that $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, or even that it has an upper bound, is
not obvious, especially for those who are new to infinite series. The series represents a sum
of infinitely many terms; novice students may struggle with the process of adding infinitely
many things or even that said series converges at all. It may also be difficult for students to
recognize that the convergent value of the series is actually defined using the limit of partial
sums.

**Adding Infinitely Many Terms**

Summing series with infinite terms requires a different way of thinking. It might not
be obvious to the novice learner what it means to add infinitely many numbers. Careful
attention is needed to help novice students be able to add infinitely many terms. As an example, consider the following infinite series:

$$\sum_{n=0}^{\infty} 2 \cdot (-1)^n = 2 + (-2) + 2 + (-2) + \cdots$$

A novice student, say Eli, may think that the series converges to zero, since pairs of zero terms $2 + (-2)$ can be made infinitely, in the following way:

$$[2 + (-2)] + [2 + (-2)] + [2 + (-2)] + \cdots$$

By a similar argument and starting zero pairs with the second term, another student, say Nora, could argue that the sum of the series is 2:

$$2 + [(-2) + 2] + [(-2) + 2] + \cdots$$

Both of these processes yield different sums which leads to contradictory results. Incidentally, this series diverges, partially because of these contradictory results.

Thinking about the series in either of these ways violates fundamental theorems associated with infinite series. For both of these students’ reasoning, the associative property does not apply to infinite series in the same way as finite series, unless the series is absolutely convergent. This can be solved by analyzing the sequence of partial sums, which can be shown to diverge.

**Indeterminant Forms**

Consider the series

$$\sum_{n=2}^{\infty} \frac{2}{n^2 - 1}$$

Using partial fraction decomposition, it is possible to rewrite the series in the following way

$$\sum_{n=2}^{\infty} \left( \frac{1}{n-1} - \frac{1}{n+1} \right)$$

This series is telescoping when evaluated as a series using the associative property:

$$\left( 1 - \frac{1}{3} \right) + \left( \frac{1}{2} - \frac{1}{4} \right) + \left( \frac{1}{3} - \frac{1}{5} \right) + \cdots$$

In this particular case, the sum of the series is $\frac{3}{2}$. This is precisely because of zero pairs, but the zero pairs are spaced apart (i.e. $\frac{1}{3}$ and $-\frac{1}{3}$), and the only terms without pairs are 1

4
and \(\frac{1}{2}\). In addition, the terms in the series approach zero as \(n \to \infty\). However, if the series is split into two series:

\[
\sum_{n=2}^{\infty} \frac{1}{n-1} - \sum_{n=2}^{\infty} \frac{1}{n+1}
\]

A subtle rearrangement happened and terms in the first version were commuted. This violation of the commutative property causes the contradictory result. Indeed, the rearrangement is impossible because the series is not absolutely convergent. However, individually, these are divergent series to infinity. The subtraction of these two divergent series yields an indeterminant form: \(\infty - \infty\). The novice learner may see the difference as zero without a strong understanding of infinity and indeterminant forms. Thus, care must be taken to make clear that rearrangement of terms is not possible, except under specific conditions.

1.5 Series Convergence

In this and future sections, the word "theorem" is used in place of "test" to emphasize the "if...then" structure of the relationships between series.

The three previous examples illustrate potential difficulties for students in understanding what it means for an infinite series to converge. What follows is some of the understandings behind infinite series that students need to correctly construct concepts and discover relationships about.

Geometric series and "p-series" (i.e. infinite series with terms of the form \(\frac{1}{n^p}\), such as \(\sum_{n=1}^{\infty} \frac{1}{n^2}\)), have specific criteria to determine convergence as these patterns have predictable structures. In the former case, the specific sum of the series can be determined in a systematic way, namely that \(\sum_{n=0}^{\infty} ar^n = \frac{a}{1 - r}\).

When a series does not have one of these two exact structures, theorems relating a known series’ convergence or divergence are required to determine when the series converges or diverges. Determining a relationship between the convergence or divergence of unknown series and the convergence or divergence of a known series is required. For example, Auguste Cauchy, one of the pioneers of modern real analysis, often used comparisons to geometric series as a way to justify the Root and Ratio Theorems (Katz, 2009). A “p-series” can be used often as a comparison using the Limit Comparison Theorem, for “p-series” have a specific criteria for convergence that are often scalar multiples, in the long run, of other series.

The mathematical underpinnings of comparing a series to a geometric series, therefore, are grounded in comparison of series. This comparison is done by bounding series above or below by another series of known convergence or divergence to determine what the series with unknown convergence or divergence does. Mathematically, this requires a deep understanding of the greater than and less than operations. Particularly within study of infinite
series, understanding when a series is “greater than” or “less than” another series requires careful definition. Defining these notions of inequality often is done using “term-by-term” comparison, or considering corresponding terms of series and comparing their values in the usual way (i.e. \(a_1 < b_1, \ a_2 < b_2,\) and so on).

Many of the series theorems rely on the concept of comparison of infinite series. The Integral Theorem, for example, provides a way to use the area under a curve as an upper (or lower) bound for a series. This is appropriate since integrals are, in a way, series. The convergence or divergence of the integral determines whether the corresponding series converges or diverges. In this way, the Integral Theorem is directly an application of the Direct Comparison Theorem. Similar arguments can be made about the Ratio, Root, and Limit Comparison Theorems.

Students encountering infinite series for the first time, therefore, must make decisions, even if indirectly, about what series to use as a comparison. Various students of mine have made comments explaining that the hardest part of learning about infinite series convergence is to determine which series to use as a comparison. Making the decision of what to compare to requires having students gain intuition about what series converge or diverge and then use the intuition to make decisions about comparisons. A formulaic attempt to determine convergence or divergence in an algorithmic way often fails, as theorems unexpectedly yield inconclusive results.

1.6 Brief History of Infinite Series Convergence and Applications

Infinite series provide a logical foundation for the theory of calculus (real analysis), specifically in integration theory and for the development of power series approximations of functions. In order to adequately address the convergence of infinite series, it is essential to understand why the concept exists in the curriculum.

The history of calculus started in a non-rigorous way with Newton’s method of fluxions. However, later nineteenth-century mathematicians were unsatisfied with Newton’s imprecision. Weierstrass, Cauchy, and others felt a need to make convergence ideas precise. Thus, the sequence idea was born and the limit of a sequence became the standard of what it meant for a sequence to converge. The limit concept also provides a set-theoretic foundation for derivatives and integrals.

Infinite series provide a theoretical backbone to the foundations of calculus. They serve as foundational ideas to building Taylor and Fourier series. Series convergence arises in the context of approximating functions and in the theory of integration. Yet, in their own right, their study provides students an opportunity to engage with the foundations of calculus in a way that proves later useful in mathematical study, such as in determining a power series solutions to a differential equations.
1.7 Literature Review

In deciding how to design the unit, several scholars have previously written about their use of instructional techniques that promote active learning and others have written about infinite series in terms of models and how students understand infinite series convergence. A few of the projects and papers that have influenced the design of this unit are listed below.

Active Learning Techniques

MIT overhauled their introductory Electricity & Magnetism physics courses to include primarily a lab-based experience, with minimal lecturing (MIT, n.d.). Engaging in problem-based curriculum has shown to require adjustment on the part of students, but ends with positive results, even to the point of adopting problem-based lesson plan design for a presentation when given the choice of direct instruction (Pilgrim, 2014). Indeed, Pengelley (2020) advocates for collegiate mathematics instructors to implement active learning techniques in their classes. There is also a need for students to have ownership of mathematics and for instructors to speak minimally about relationships that are discoverable (Reinhart, 2000). Ideas surrounding mathematical play have also shown to be productive in promoting student engagement (Francis, 2019).

Infinite Series Convergence

Textbook authors often focus on procedural fluency, rather than on conceptual understanding (Biza et al., 2009). However, there are several models (Amal Sharif-Rasslan, 2016; Beata Randrianantoanina, 2004; Lindaman & Gay, 2012; Schielack, 1992) that can be used to assist student understanding of geometric series. One specific example Erickson (2020) used the context of a bouncing ball to explore exponential functions and convergence.

Student understanding of infinity is dependent on the type of infinity; students perceive “infinitely many”, “infinitely large”, and “infinitely close” as different (Manfreda Kolar & Hodnik Čadež, 2012).

Pre-service mathematics teachers in Turkey (Ergene & Özdemir, 2020) often showed an inconsistency in student understanding of the theory of the divergence of the harmonic series \[ \sum_{n=1}^{\infty} \frac{1}{n} \] and applying this theory. In a separate study, Turkish teachers also attempted to determine whether various series converge or diverge and researchers (Genc & Akinci, 2020) categorized common errors the teachers made. Additionally, understanding the meaning of a power series has shown to be challenging (Kung & Speer, 2013).
Chapter 2
Methodology

2.1 Research Questions

The two research questions addressed in this paper are:

1. How could a problem-based curriculum be developed to help students develop correct concept images surrounding infinite series convergence?

2. When applied, what does the designed curriculum surface about students' concept images for infinite series convergence?

2.2 Unit Design

The design for the problem sets of the unit is based on my personal experience of attending several seminars for secondary mathematics teachers (McLeod et al., [2021]). These seminars discussed the need to have students engage in problems to develop mathematical ideas. Typically these problems are computational or exploratory in nature which allows for engagement in inductive reasoning. After sufficient generalizations occur, then theorems can be reasoned about deductively in class discussions, through homework, or through future problem sets.

The problem sets for the three-week unit were split into three major sub-units, with each sub-unit intended to span approximately one week—the equivalent of four, 50-minute class meetings. The general aspects each of the sub-units developed is as follows:

1. An introduction to series and sequences as a concept using geometric series as a specific type of series and sequence to study;

2. An introduction to p-series and using geometric series and p-series as lower and upper bounds of series to show convergence or divergence; and,

3. An introduction to alternating series and summarizing the learning of the unit.
The intent of these problem sets is to use them during the time allotted in a “traditional” course in calculus discussing infinite series convergence, excluding content regarding power series. More details about the unit objectives and the focus of each day is located in Appendix A.

The design of the assessments for the unit was based on the principles of Understanding by Design (Wiggins & McTighe, 2005) using both performance and objective assessments. The objective assessments were designed to check procedural and conceptual understanding of content in each of the problem sets. The performance assessment was designed to allow for students to transfer their knowledge of content from the unit to a previously unencountered context. The assessment items are located in Appendix C.

### 2.3 Problem Set Design

The problem sets for this curriculum were designed similarly to a portion of the Park City Mathematics Institute’s (PCMI) Teacher Leadership Program (TLP), typically held in Park City, Utah. During the three-week program, in-service teachers engage with problems for two hours each day where direct instruction is not given, but rather students investigate problems that lead to mathematical ideas. A similar process was used for the infinite series convergence unit.

The problem sets were developed quasi-adaptively: the unit as a whole, with its goals and objectives, was developed in general before the unit started, with the goals, objectives and tasks being refined as the three-week unit unfolded and as student thinking emerged throughout the unit. The quasi-adaptiveness was necessary so students could receive relevant instruction that built on what was previously learned and what was discussed as a whole group. This pattern follows the PCMI TLP.

Because the course’s content was largely fixed by the institution, one control of objectives was to have a common set of homework problems used by other sections of the same course so that students could be assured that they were learning standard infinite series content and for quality control among sections. Students also completed common midterms and a common final with other non-experimental sections of the same course.

The prompts in the problem sets were generated based on student errors (Genc & Akinci, 2020) and the difference in understanding between theoretical and applied problems (Ergene & Ozdemir, 2020), as well as other considerations from the literature review. Wherever possible, students were introduced to ideas using a context that models infinite series with fidelity. Students were also given opportunities to engage with infinite series theoretically. The specific problem sets are located in Appendix B.

### 2.4 Context of the Implementation Study and Data

Students participated in a three-week unit, studying infinite series convergence through interactive problem sets.
The student data collected from the unit includes specific homework assignments designed for the study, the problem set work from each class meeting (of which there were twelve), and assessment data. Three objective assessments were given in a take-home quiz format, and one performance assessment was given as a mini-project applying infinite series content to a novel context. The text of these assessments, problem sets, and homework specific to the study are attached in the appendices.

2.5 Implementation of Course Materials

The students in this study came from a second-semester calculus course at a large research university in the Rocky Mountain region. Out of the 44 students enrolled in the course, 27 students consented to be in the study. For the purposes of data analysis, five students’ work will be analyzed. The pseudonyms of these students are: Carlos, Charles, Sabrina, Thomas, and Victoria.

Students met four days a week for 50-minute class periods for three weeks, resulting in a total of 12 class meetings. For all but the last two sessions, students worked for 40 minutes (starting at the beginning of class) in groups of variable size based on a seating chart, etc. My role was that of observer, monitoring student thinking (Smith & Stein, 2018) and asking questions where appropriate. During the final 10-minute section of each class meeting, I chose students to discuss their thinking about problems and then used the student thinking as a transition to formalize ideas related to infinite series convergence.

One constraint I needed to account for is how to address the traditional Calculus II course content in the same time as a traditional, lecture-based course. I mitigated this partially through the advanced planning of objectives and having a general view of the three-week unit, so that an appropriate endpoint could be determined. Indeed, a common midterm exam across all sections of this course was used to help control for content coverage. Some common homework assignments across all sections were given as well.

2.6 Theme Selection

Student data were anonymized and cataloged by student and ordered chronologically. This was then analyzed to determine the progression of conceptual learning about infinite series convergence throughout the three-week unit. Thematic analysis (Braun & Clarke, 2006) was used to determine themes concerning how students develop what it means for an infinite series to converge or diverge. The purpose in using this approach is to see how student thinking is maintained, or changes, throughout the three-week unit. The themes used in this study are outlined in Table 2.1.
<table>
<thead>
<tr>
<th>Theme</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sequence Convergence</td>
<td>Demonstrates interaction with the general idea of what it means for a sequence to converge.</td>
</tr>
<tr>
<td>Series Convergence</td>
<td>Demonstrates interaction with the general idea of what it means for a series – as a sequence of partial sums – to converge.</td>
</tr>
<tr>
<td>Geometric Series</td>
<td>Demonstrates interaction with an infinite series with a common ratio.</td>
</tr>
<tr>
<td>Series Divergence</td>
<td>Demonstrates interaction with divergence of an infinite series.</td>
</tr>
<tr>
<td>Sequence/Series Distinction</td>
<td>Demonstrates interaction with the concepts of series convergence and sequence convergence together.</td>
</tr>
<tr>
<td>Series Comparison</td>
<td>Demonstration of interaction of two series using a less than, greater than, or equal to relationship.</td>
</tr>
</tbody>
</table>

table 2.1: A listing of themes used to categorize student work through the course of the three-week unit.

2.7 Limitations

Because students worked in teams, it at times difficult to see if the team had the misconception rather than the individual. This is one reason that each student submitted their own work for analysis. Additionally, while it seems that much of the work is the student’s own, it can be difficult to parse whether work was done by students individually, in teams, or if the work came from students during the discussion phase of each session.

One other consideration is that only five students’ work are analyzed in this paper. Analyzing the other available data would likely provide more examples of potential correct or incorrect understandings. This study does not attempt to be comprehensive, but rather descriptive of what was observed in these five students.
Chapter 3

Data Analysis

In this chapter, I will discuss the design of the prompts in the problem sets to help students construct concepts and discover relationships (Cangelosi, 2002). The focus will be on specific implementations and how they achieve the learning goals of the unit.

I will also present five students’-given pseudonyms Victoria, Carlos, Thomas, Sabrina, and Charles- work in this section, portraying evidence of correct and incorrect understandings of infinite series convergence throughout the three-week unit. Each of the six themes mentioned in Table 2.1 will be analyzed. A sampling of work will be provided, rather than a comprehensive analysis.

Throughout the data analysis section, there are two sections for each of the codes. The first of these describes a sampling of the specific prompts in the unit where each of the coded themes occurs. I describe the kinds of activities students engage in during the unit with respect to the theme described by session and prompt number. For example, the fifth prompt from the third day of the unit is referred to as Session 2-3, prompt 5. Note that the first two (i.e. “Session 2-“) refers to the fact that the unit is the second out of three in the entire semester-long course. Each synchronous class meeting is referred to as a session.

The second section analyzes student work from relevant sections of the unit and draw conclusions for teaching about infinite series content in the classroom. The figures in this chapter are labeled with the pseudonym and the prompt number from the unit for cross-referencing.
3.1 Sequence Convergence

Unit Design Considerations

On day one, block pattern problems (cf. Session 2-1, prompts 1 and 2) and the dividing sandwiches context (cf. Session 2-1, prompt 3) are meant to introduce students to the idea of a sequence in a concrete, intuitive way, as a function \( a_n : \mathbb{N} \to \mathbb{R} \). This is done by making tables where one row is represented by inputs to a sequence (in \( \mathbb{N} \)) and the outputs are represented in another row. Prompts 1 and 2 are both divergent sequences to infinity.

The fact that a sequence can converge is introduced in Session 2-1, prompt 4 and emphasized throughout the first four days of the unit. At first, the focus is on determining what “the sequences seemingly ‘approach’”. The word “approach” is used as a scaffold for a limit as \( n \to \infty \). Later, during Sessions 2-3 and 2-4, limits are used to formalize sequence convergence (cf. Session 2-4, prompts 1, 2, 6 and 7).

After some initial work using concrete models, including blocks and dividing sandwiches, the next tasks introduce students to sequences abstractly by inviting them to look at lists of numbers and determine what the “terms in the sequences seemingly approach” (cf. Figure 3.1). Prompts 1, 2, and 3 from Session 2-1 then mirror prompts 4E, 4F and 4C, respectively, acting as a self-check on the student’s understanding. This design feature is common in PCMI’s problem sets (McLeod, 2021).

Student Work Analysis

![Figure 3.1: Sabrina demonstrates proficiency with inferring the “long term” behavior of the sequences (Session 2-1, prompt 4).](image)

Sabrina’s work in Figure 3.1 illustrates a generally correct understanding of infinite sequence convergence. The estimates in prompts 4A-4D all show accurate limits of convergence for each of the sequences. In particular, Sabrina demonstrates reasoning about better estimates between terms in prompt 4A by using overbars to represent the distance between
terms. The differences, 0.5 and 0.25 are smaller as sequence progresses, suggesting an emerging understanding of Cauchy sequences due to the difference between terms decreasing towards zero.

Two particular examples of student thinking with regards to sequence convergence seem particularly important to note. Early in the unit, Charles’ work demonstrates emergent reasoning on what it means for a sequence to converge and diverge.

Figure 3.2: Charles demonstrating that divergent sequences behave differently than convergent sequences (Session 2-1, prompt 4).

Figure 3.3: Charles’ work when asked to define sequence convergence/divergence after using the instructor’s definition as a template (Session 2-3, prompt 1).

In Figure 3.2, prompts B, C, and D all demonstrate Charles’ correct concept image for what it means for a sequence to converge in that the limit of the terms in the sequence approaches the listed value in the long run. However, prompts E, F, and G demonstrate next-term thinking, rather than considering divergence as approaching infinity or some undetermined limit. Interestingly, on day three, Charles defined divergence in terms of an “irregular [number]” (See Figure 3.3), but listed appropriate ways of divergence: approaching infinity and undefined limits. The way I defined sequence convergence at the beginning of day three was not in terms of an “irregular [number]”. The use of the word “irregular” suggests that the end behavior of the sequence is erratic or unpredictable which is arguably accurate. Calling the end behavior a number, however, is generally incorrect.

In the first quiz (Quiz 5), Thomas wrote about behavior at a point when talking about sequence convergence (see Figure 3.4). Thomas wrote that “if measuring values of $n$ from 0 to $\infty$, then it diverges when $n = 0$ to DNE because $\frac{1}{0}$ isn’t a real number”. The next
sentence suggests that Thomas is examining “long term” behavior (as $n \to \infty$). The fact that Thomas suggests that $n = 0$ is important. This is because it shows the thinking that the entire domain $\mathbb{R}^+ \cup \{0\}$ must be completely examined for continuity in order for a limit to exist, thus showing that a sequence converges. Perhaps this misunderstanding is due to the conflating of the convergence and continuity concepts.

The student work analyzed thus suggests that care needs to be taken to emphasize the end behavior of the sequence and to carefully define what it means for a sequence to converge, as opposed to diverge. Attention needs to be paid particularly to end behavior, rather than individual points and next terms.
3.2 Geometric Series

Unit Design Considerations

Students demonstrated understanding about criteria for convergence or divergence of geometric series. This was primarily using a context surrounding dividing sandwiches. Initially, students were invited to divide a sandwich into pieces between two friends and record the portion each friend received (cf. Session 2-1, prompt 3). On day four of the unit, students were asked to generalize criteria for when geometric series converge (cf. Session 2-4, prompt 3). Homework 2-4 then invited students to develop a general relationship for calculating the sum of a geometric series.

Student Work Analysis

Figure 3.5: Thomas describing when the sandwich division can happen, versus when it cannot, using percentages (Session 2-4, prompt 3).

Figure 3.6: Thomas attempted to formalize the criteria for determining convergent and divergent geometric series (Session 2-4, prompt 8).

When generalizing the sandwich sharing problem on day four, Thomas used percentage reasoning to make sense of when sandwich sharing make sense. Specifically, Thomas reasoned that “...taking less than 100% of the sandwiches makes sense, when > 100% it doesn’t because you can’t create more sandwiches” (See Figure 3.5). This falls in line correctly with criteria for a convergent geometric series.

This idea is made more precise in Figure 3.6, where Thomas uses ratio thinking to describe when a geometric series converges. Specifically, Thomas recognizes that the common ratio 1 is the separation between convergent geometric series ratios and divergent ones for positive
common ratios. Thomas illustrates this in Figure 3.7 by creating a geometric series with the common ratio’s magnitude greater than one. However, Thomas does not make clear what happens to geometric series with common ratios that are negative in general. In Figure 3.8, specifically in row G, Thomas demonstrates understanding, even if tentatively, that a geometric series with a negative common ratio converges (see column 4).

\[
\text{Figure 3.7: Thomas uses “scalar multiple” reasoning to describe when a geometric series converges (Quiz 5, prompt E).}
\]

Carlos, when talking about geometric series, describes the convergence criteria similarly to Thomas in that only positive common ratios are mentioned in Figure 3.9 and all negative common ratios converge. Both Thomas and Carlos’ work suggest the need to be clear with students about negative common ratios and emphasizing contexts where negative common ratios make sense.

In contrast to Thomas and Carlos, Charles makes clear (in 3.10) that the magnitude of the common ratio, independent of sign, is what causes a geometric series to converge. The use of the correct notation (absolute value bars) indicates that it is the magnitude that matters.
A geometric series \( \sum_{n=0}^{\infty} ar^n \) converges when \( r < 1 \) and diverges when \( r \geq 1 \). The geometric series \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \), where \( r \) is called the common ratio.

Figure 3.9: Carlos demonstrates understanding of a common ratio of one separating converging geometric series from diverging geometric series (Session 2-8, prompt 2).

- True
- False

E. A geometric series is always convergent.

If \( |r| < 1 \) then series converges to zero
If \( |r| \geq 1 \) then series diverges (infinity)

Figure 3.10: Charles emphasizes that the magnitude, rather than only the positive side, of the common ratio being less than one causes convergence and that any common ratio outside of this range diverges (Quiz 5, prompt E).

At the end of the unit, students completed a performance assessment about fractals. The intent of this performance assessment was to have students transfer their knowledge about infinite series convergence to an unfamiliar context. When engaging with this task, Carlos (in Figure 3.11) uses visual reasoning to show that \( \sum_{n=1}^{\infty} \frac{1}{2n+1} \) converges to \( \frac{1}{2} \). The use of nested squares is a way to show that the series converges to half of a unit square.

These pieces of student work suggest that there is a need to attend not only to positive common ratios, but negative common ratios in determining geometric series convergence. Emphasizing the magnitude of the common ratio is essential, specifically that whether positive or negative, the common ratio of a geometric series needs to be less than one in order to show convergence of said series.
Figure 3.11: Carlos visually depicts what it means for a geometric series converge (Performance Assessment, task 1).
3.3 Series Convergence

Unit Design Considerations

In order to develop the convergence concept, at least three representations need to be connected:

- The sequence of partial sums;
- The limit of partial sums; and,
- The sum of infinitely many terms.

As an example, the sequence

\[ 1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \ldots \]

converges, as does the limit

\[ \lim_{n \to \infty} \frac{2^{n+1} - 1}{2^n} = \lim_{n \to \infty} \left( 2 - \frac{1}{2^n} \right) = 2 \]

and the series

\[ 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \]

All three of these are representations of the same series, \( \sum_{n=1}^{\infty} \frac{1}{2^n} \); each of these representations converge to 2. In general is it not required to show what the series converges to, but rather that the series converges.

The design of the unit lends itself to promoting these three concept images together for what it means for an infinite series to converge. The first three sessions are designed such that explicit connections are made between these three representations (see, for example, Session 2-3, prompts 2, 3, and 4). Additionally, the use of the equal sign to mean convergence begins in Session 2-4, but continues through subsequent sessions.

Other elements of series convergence are woven into later parts of the unit. “P-series” and alternating series are introduced and corresponding theorems (i.e. The “P-series” Theorem and The Alternating Series Theorem) are developed in the context of a water tank, as seen in and adapted from Ergene and Özdemir (2020).

Student Work Analysis

Charles created an example of a series (see Figure 3.12) students did not discuss in class. However, Charles’s series is geometric in nature, in that the common ratio between terms is \( \frac{1}{10} \) and the initial term is \( \frac{3}{10} \). This novel application suggests that Charles recognized an infinite decimal representation as a finite value.
In Figure 3.13, Sabrina responded to a prompt from the first quiz as to whether $\sum_{n=0}^{\infty} \frac{1}{3^n} < \frac{3}{2}$. The design of the prompt is to see whether students recognize what the equal sign means in that the limit of the sequence of partial sums converges to $\frac{3}{2}$. In other words, the prompt assesses whether the limit of partial sums is equal to $\frac{3}{2}$, or if the sequence of partial sums is always less than $\frac{3}{2}$. Sabrina expresses that this geometric series will be equal to $\frac{3}{2}$, but then says “I am not quite sure how an infinite series can become a finite value.” This statement suggests that what it means for a series to converge is unclear, even though Sabrina knows the series converges to $\frac{3}{2}$.

Interestingly, Sabrina uses the context of “halvies” sandwiches to reason about the same prompt as Charles in Figure 3.12. In Figure 3.14, Sabrina correctly notes that the series $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$ and uses the sandwich context to help demonstrate understanding of how the series converges.

The difference between Sabrina’s two instances of student thinking is striking, especially since they are essentially assessing the same content: determining what it means for a series to converge. Sabrina’s work suggests that there is nuance to using the equal sign to mean convergence in the classroom. Paying particular attention to the equal sign and defining it in terms of convergence is essential.

Later, in week three of the unit, Charles (see Figure 3.15) reasons about a tank filling task adapted to model an alternating series, in this case the alternating harmonic series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$.

Charles creates a visual representation of the first four states, and then plots the number of gallons in the tank over the number of days. There are two dotted black lines on the graph: one at the vertical axis value of 1 and the other at a vertical axis value presumably between 0 and 1. This work suggests that Charles understands the sequence of partial sum is bounded above by 1 and bounded below by 0. In addition, the exponential decay of the sinusoid curve in the graph suggests that the partial sums will converge eventually to some number.
The ultimate limit of partial sums is not labeled, and is unimportant for the purposes of this unit; what is important is that the series converges. Charles demonstrates that the series converges using the sequence of partial sums, although not formally.

In contrast to Charles’ work, Thomas’ work in Figure 3.16 shows a comparison between the different representations of series as the value a sequence partial sums approaches, the limit of the partial sum written as a formula, and using addition notation. Interestingly, prompt G is analyzed to be neither convergent or divergent, which has implications for previous sections of the work, but of more interest is the recognition in Figure 3.17 that all the representations are “all variations of the same series”. The point of interest here is that an alternating series is seen as converging differently when compared to other kinds of convergence.

All of these examples of student work suggests that helping students develop infinite series convergence requires careful attention to the equal sign and that the use of the models in class has been effective in developing concept images for these students.
Figure 3.14: Sabrina using the “halvsies” (sandwich division) context to reason about why \( \sum_{n=1}^{\infty} \frac{1}{2^n} = 1 \) (Quiz 5, prompt A).
Figure 3.15: Charles’ reasoning about alternating sequences and series (Session 2-9, prompt 1).
Figure 3.16: Thomas' work comparing different representations of series (Session 2-3, prompts 2-6).

<table>
<thead>
<tr>
<th></th>
<th>Problem 2</th>
<th>Problem 3</th>
<th>Problem 4</th>
<th>Problem 6B (Common Ratio)</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>[ \frac{63}{64} \Rightarrow 1 ]</td>
<td>1</td>
<td>1</td>
<td>[ \frac{1}{2} ]</td>
</tr>
<tr>
<td>B</td>
<td>Can 2</td>
<td>2</td>
<td>2</td>
<td>[ \frac{1}{2} ]</td>
</tr>
<tr>
<td>C</td>
<td>Can 4</td>
<td>4</td>
<td>4</td>
<td>[ \frac{1}{2} ]</td>
</tr>
<tr>
<td>D</td>
<td>Can ( \frac{1}{9} )</td>
<td>( \frac{1}{9} )</td>
<td>( \frac{1}{9} )</td>
<td>[ \frac{1}{2} ]</td>
</tr>
<tr>
<td>E</td>
<td>Can ( \infty )</td>
<td>( \infty )</td>
<td>( \infty )</td>
<td>[ \frac{3}{2} ]</td>
</tr>
<tr>
<td>F</td>
<td>Doesn't diverge or converge</td>
<td>DNE</td>
<td>DNE</td>
<td>[ \frac{1}{2} ]</td>
</tr>
<tr>
<td>G</td>
<td>Neither</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>[ \frac{1}{3} ]</td>
</tr>
<tr>
<td>H</td>
<td>1</td>
<td>( \frac{2}{3} )</td>
<td>( \frac{2}{3} )</td>
<td>[ \frac{1}{4} ]</td>
</tr>
</tbody>
</table>

Figure 3.17: Thomas' work summarizing the table in Figure 3.16 (Session 2-3, prompt 5).

5. What connections do you notice between questions 2, 3 and 4? Use the table on page 5 to note any connections.

They are all variations of the same series.
3.4 Sequence/Series Divergence

Unit Design Considerations

Sequence and series divergence is generally defined as the opposite of convergence. However, there are different ways for a sequence or series to diverge. For example, it is possible for a series to diverge because there is an inconsistent limit (i.e. does not exist), or it is possible for a series to diverge because there is a limit that consistently is unbounded (i.e. approaches infinity). Often, however, the definition of divergence is given as not convergence, which does not help students realize the differing kinds of divergence that are possible.

As such, various kinds of sequences and series in the first four sessions are presented to help students experience these different kinds of divergence. Intentionally having to grapple with these different kinds of divergence helps students develop a complete concept image of what it means for a sequence or series to diverge.

Student Work Analysis

Early in the unit, Sabrina reasoned about infinite series using three different representations: sequences of partial sums (See Figure 3.18), limits of the terms in the sequence of partial sums (See Figure 3.19), and using the terms of the sum of a series (See Figure 3.20).

![Figure 3.18](image)

Figure 3.18: Sabrina’s reasoning that the sequence of partial sums diverges in two senses: to infinity and that the limit does not exist (Session 2-3, prompt 2).

Sequences E and F both diverge, but in different ways. When the sequence diverges toward infinity (i.e. Sequence E), Sabrina shows consistency among the three representations of the series. However, in sequence F, Sabrina’s responses in Figures 3.18 and 3.19 show divergence, but in Figure 3.20 demonstrates convergence. In the sequence and limit of partial sums representations of the series \( \sum_{n=0}^{\infty} (-1)^n \) it is clear that the eventual value of the sequence and limit does not exist. The infinite addition representation yields a convergence to zero, which mirrors the reasoning in the introduction section: Sabrina likely saw that an equal number of zero pairs are represented, and therefore concluded the series converges to zero. No explicit use of the associative property is present, so this is only an inference.

Victoria’s work in Figure 3.21 seems to confirm what Sabrina sees. In series D, Victoria correctly notices that the series diverges to infinity. In contrast, Victoria labels series E
Figure 3.19: Sabrina’s reasoning that the limit of terms in the sequence of partial sums approaches infinity and does not exist (Session 2-3, prompt 3).

![Image showing calculations and limits]

Figure 3.20: Sabrina’s reasoning that the infinite series approaches infinity and zero, respectively (Session 2-3, prompt 4).

“undefined”. Victoria likely notices, as Sabrina does, that the kind of divergence, namely a limit of partial sums that does not exist, is different from a divergence to infinity.

Both Victoria and Sabrina’s work suggest that attention to kinds of divergence is required, specifically in the case of when the sequence of partial sums of a series does not exist. Without a clear understanding of the different kinds of divergence, students may believe that different ways to diverge should be given a different name, as in Victoria’s Figure 3.21 or think that the two different kinds of divergence are unrelated.
Figure 3.21: Victoria using “undefined” as a category distinct from convergence or divergence (Session 2-4, prompt 5).

C. $\sum_{n=1}^{\infty} \left( \frac{1}{2} \cdot \left( \frac{1}{2} \right)^n \right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \ldots \text{ converge to } \frac{1}{2}$

D. $\sum_{n=1}^{\infty} (1) = 1 + 1 + 1 + 1 + 1 + \ldots \text{ diverge to } \infty$

E. $\sum_{n=1}^{\infty} ((-1)^{n+1} \cdot n) = 1 + (-2) + 3 + (-4) + 5 + (-6) + \ldots \text{ undefined}$
3.5 Sequence/Series Distinction

Unit Design Considerations

In order to distinguish between sequences and series, sequences are first introduced with series being a specific kind of sequence, namely a sequence of partial sums. Session 2-1, prompt 3 uses the sandwich division ("halvsies") context to introduce both a sequence and a series as specific kinds of sequences. However, as the unit moves to Session 2-4, there is a gradual development of the concept that a series is a sequence of partial sums and the three representations of a series mentioned in Chapter 3.3 are all essential to constructing the appropriate concept image.

Student Work Analysis

Sabrina’s work from an early problem set (see Figure 3.22) demonstrates understanding of series convergence (in spite of a typographical error in prompt 5A). Sabrina demonstrates in her work here that “sums not limits” are important. This likely refers to the fact that the limit of terms (approaching zero) is not what is important, but rather what the sum of the terms approaches, an important distinction between sequences and series.

Figure 3.22: Sabrina recognizes that the sum of the series, not the limit of the terms in the series, is what causes convergence of a series (see Session 2-4, prompt 5)). Note that there is a typographical error in prompt 5A that has been fixed in the Appendices.

In Figure 3.23, Thomas defines a sequence as a “function”. While a sequence is a function, Thomas’ response is unclear as to what the domain of this function is, whether it is \( \mathbb{N} \) or not. There is a hint to what Thomas understands about the domain from the second sentence, suggesting that sequences start at “\( x = 1 \)”, but the domain of the function is still unclear. Additionally, the series being defined as the “sum of the sequence after \( x = 1 \)” is somewhat limiting, as infinite series can be evaluated starting at different lower limits than 1. This suggests that there is a need to ensure examples are given that do not always have a lower limit starting at 1 and that it is clear that the sum is over \( \mathbb{N} \).

Charles demonstrated an unclear concept image of series and sequence concepts in two different pieces of work. In Figure 3.24, Charles uses limits to think about both the sequence
Figure 3.23: Thomas’ work describing the difference between a sequence and series (Session 2-7, introduction).

Figure 3.24: Charles’ work demonstrating a conflation between series and sequences (Quiz 5, prompt D).

\[
\left(\frac{1}{2n}\right) \text{ and the series } \sum_{n=1}^{\infty} \frac{1}{2n}, \text{ but the use of the limit notation in both situations is used incorrectly. For instance, } \lim_{n \to \infty} \frac{1}{2n} = 0, \text{ but Charles uses this notation to think about the terms in the sequence. However, no final limit is calculated. The series also uses a limit notation, but this is inappropriate as the definition of the symbol } \sum_{n=1}^{\infty} = \lim_{k \to \infty} \sum_{n=1}^{k} \text{ already requires a limit, so the extra limit symbol is unnecessary. It seems that the limit was not really used in this work.}

Additionally, Charles’ work evaluates the sum of the harmonic-like series as 2, but it is unclear how this is determined. It could be that Charles saw the sequence of terms in \( \left(\frac{1}{2n}\right) \) had a difference of two in the denominator, confusing this for a geometric series with a common ratio of \( \frac{1}{2} \), which also has a sum of 2. Indeed, of the first four terms listed before the ellipsis, three of these terms are in the geometric series with common ratio \( \frac{1}{2} \). The difficulty in using this reasoning to explain Charles’ work is that the first term is when \( n = 1 \), and the sum of the series would be 1.
In spite of this, using the sum of 2, Charles concludes that this sum is not zero, which means that the series is divergent and thus that the sequence is divergent. This conflation of using the sum of a series as a way to determine whether a sequence converges is a misuse of the Divergence Theorem for Series: namely that if \( \lim_{n \to \infty} a_n \neq 0 \), then \( \sum_{n=1}^\infty a_n \) diverges.

<table>
<thead>
<tr>
<th>( n \geq 0 )</th>
<th>( a_n = \frac{3^{n+1}}{2^{n+1}} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( n \geq 0 )</td>
<td>( \sum_{n=0}^{\infty} \left( \frac{2^n}{2^{n+1}} \right) )</td>
</tr>
</tbody>
</table>

Figure 3.25: Charles’ work on the performance assessment.

In completing the performance assessment at the end of the unit (see Figure 3.25), Charles uses series to determine the area of a fractal shaded black in the long run. When creating the model \( \frac{2^n - 1}{2^{n+1}} \), Charles was representing the value shaded in the nth stage; in other words, \( \frac{2^n - 1}{2^{n+1}} = \sum_{n=0}^\infty a_n \). However, Charles says that the “sequence converges to \( \frac{1}{2} \)”, when in reality adding the phrase “of partial sums” would be more appropriate, as evidenced by the “\(+\frac{1}{8}\)”.

![What fraction is shaded black as n→∞?](image)
“+ $\frac{1}{16}$”, “+ $\frac{1}{32}$” in the upper-left corner. Additionally, Charles’ conclusion that the series diverges is representing a double summation, rather than the summation desired to calculate area, although the Divergence Theorem for Series is correctly used in this case. Thus, the meaning of Charles’ series in the context of the fractal is unclear, but Charles’ demonstrates a correct application of the Divergence Theorem for Series.
3.6 Series Comparison

Unit Design Considerations

Comparison of series is introduced in the second week of the unit. Throughout this section of the unit, the focus for comparison is on positive series, one of the criteria for using the Direct Comparison and Integral Theorems for Series. Series with negative terms are not addressed in much detail until week three, specifically in the context of alternating series. Additionally, coming into this unit, students had a working understanding of integration, which provided a way to bound above and below various series using improper integrals.

At the beginning of Session 2-5, students are introduced to the meaning of a series being “greater than” or “less than” another series by finding geometric series whose terms are greater than or less than \( \sum_{n=0}^{\infty} \frac{1}{5^n} \). Using what students know about geometric series from the first week, students can create series with common ratios that are both above and below \( \sum_{n=0}^{\infty} \frac{1}{5^n} \). Students recognize that not all positive series greater than \( \sum_{n=0}^{\infty} \frac{1}{5^n} \) converge, but all positive series less than \( \sum_{n=0}^{\infty} \frac{1}{5^n} \) converge. In so doing, students begin to determine criteria for convergence or divergence using upper and lower bounds, respectively.

Throughout the remainder of the second week, students also use integrals to help determine whether series converge, ultimately developing the Integral Theorem for Series (traditionally known as “The Integral Test”). The use of the Integral Theorem for Series permits analysis of “p-series”, as the integral of a function of the form \( \frac{1}{x^p} \) is possible to evaluate using elementary integral rules, and therefore developing conditions for a “p-series” integral’s convergence (\( p > 1 \)) and divergence (\( p \leq 1 \)) is possible.

Student Work Analysis

In Figure 3.26, Sabrina notes that the integral \( \int_{1}^{\infty} \frac{1}{x}dx \) diverges (prompt 2) and that the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) is an overestimate of the integral’s area under the curve. Sabrina uses the word “generalizing” to mean that the \( n \)-th rectangle representing each term in the harmonic series, with length 1 and height \( \frac{1}{n} \), is a coarse version of the integral (i.e. \( \Delta x = 1 \) in a Riemann sum). The fact that the left-hand sum (in this case, the harmonic series) is always greater than the integral, and that the integral diverges, suggests that the series will diverge. To use this reasoning is an emergent form of the Direct Comparison Theorem. In this way, the Direct Comparison Theorem acts as the core conceptual underpinning of the Integral Theorem for Series.

In the first day of week two, Thomas (in Figure 3.27) describes series that are greater
C. What does problem 2 and problem 3B have to do with the convergence or divergence of the series \( \sum_{n=1}^{\infty} \frac{1}{n} \)?

They both will diverge to infinity at a painfully slow rate but the will diverge. The summation will be an overestimate because it is generalizing all values from 1-1.9 as 1.

Figure 3.26: Sabrina uses overestimates to determine divergence of the harmonic series (Session 2-6, prompt 3C).

than or less than \( \sum_{n=0}^{\infty} \frac{1}{5^n} \). In doing this, Thomas notes that each of the geometric series greater than \( \sum_{n=0}^{\infty} \frac{1}{5^n} \) converge, but then uses the value of the series as the comparison point, rather than a term-by-term method. Reasoning using the convergent value of a series as a comparison point contains a fundamental error in applying the properties of geometric series. To illustrate this error, consider the following two geometric series:

\[
\sum_{n=0}^{\infty} \frac{1}{2^n} = \frac{1}{1 - \frac{1}{2}} = 2
\]

\[
\frac{3}{2} \sum_{n=0}^{\infty} \frac{1}{4^n} = \frac{3 \cdot \frac{1}{4}}{1 - \frac{1}{4}} = \frac{3}{2} \cdot \frac{4}{3} = 2
\]

These two series sum to the same value, but term-by-term, for all \( n > 0 \):

\[
\frac{3}{2} \sum_{n=1}^{\infty} \frac{1}{4^n} < \sum_{n=1}^{\infty} \frac{1}{2^n}
\]

Ignoring the \( n = 0 \) term of each series is fine, as it is the eventual trend of terms that matters. Thus, there is a need to be clear that comparing the convergent sum is not the same as comparing the eventual trend of terms in both series using term-by-term comparison.

Another instance of Thomas performing comparison to a series is in Figure 3.28. Technically, Thomas is talking about the sequence \( d_n = \frac{1}{(n+1)!} = \left( \frac{1}{2^0 \cdot 6^1 \cdot 24^2 \cdot \cdots} \right) \) and arguing that the terms converge to a number less than one, which is true but not a requisite criteria.
Figure 3.27: Thomas describes series that are greater than or less than $\sum_{n=0}^{\infty} \frac{1}{5^n}$ (Session 2-5, prompts 1B and 1C).

for convergence. However, considering that the context is to evaluate whether $\sum_{n=0}^{\infty} \frac{1}{(n+1)!}$ converges or not, it’s unclear what Thomas means by $d_n$ being “less than 1”.

There are two possibilities for why Thomas may think this. Realistically, the first is that the terms themselves are all less than one which means that the sequence converges. Indeed, all of the terms in the sequence ($d_n$) are less than (or equal to) 1. By itself, this is not enough to show convergence. However, taking an optimistic view, it may be that Thomas is attempting to use geometric series reasoning, in that the magnitude of common ratios of successive terms being less than 1 will cause the geometric series to converge. In this case, the use of a factorial means that there is no common ratio between terms. However, each consecutive pair of terms in the series has a common ratio less than 1, implying that Thomas may be attempting use of the Ratio Theorem for Series. The optimism for this comes from the fact that so much work has been done with geometric series to this point.

Note that the series $\sum_{n=0}^{\infty} \frac{1}{(n+1)!}$ is the first series containing factorials in the unit, and no procedure to this point has been shown. Attempting to generalize ratio thinking for geometric
series may make sense here, as this is the way the unit was constructed: to use geometric series as comparison wherever possible. In this case, using ratio thinking of successive terms is an appropriate thing to do. Ratio thinking is related to the Direct Comparison Theorem and describes Cauchy’s original reasoning as to why the Ratio Theorem works as it does.

Figure 3.28: Thomas reasoning about the series \( \sum_{n=0}^{\infty} d_n \), where \( d_n = \frac{1}{(n+1)!} \) (Session 2-5, prompt 5G).

3C. Will the series you created in prompt 3A and your work in prompt 3B help determine the convergence of \( \sum_{n=1}^{\infty} \frac{3}{n+100} \)? Why or why not?

Yes it tells us that \( \sum_{n=1}^{\infty} \frac{3}{n+100} \) diverges. Using The Integral Test we can say that if \( \sum_{n=1}^{\infty} f(n) \) diverges, then the convergence of the series \( \sum_{n=1}^{\infty} a_n \) is the same. So if \( a_n = b_n \), then the convergence of the series \( \sum_{n=1}^{\infty} b_n \) is the same. We know that \( \sum\frac{3}{n+100} \) diverges, but \( \sum\frac{3}{n} \) is not greater than \( \sum\frac{3}{n+100} \), so we can make \( \sum\frac{3}{n+100} \) greater than \( \sum\frac{3}{n} \) because the constant (k) rule says that the convergence of any sum is the same with any constant k. So since \( \sum\frac{3}{n} \) diverges, so does \( \sum\frac{3}{n+100} \).

Figure 3.29: Thomas reasoning about why \( \sum_{n=1}^{\infty} \frac{3}{n} \) shows divergence (Quiz 6, prompt 3C).

Later, Thomas shows command of the Direct Comparison Theorem for Series by using two facts: The Direct Comparison Theorem and The Limit Comparison Theorem. Initially Thomas recognizes that \( \sum_{n=1}^{\infty} \frac{3}{n+100} \leq \sum_{n=1}^{\infty} \frac{3}{n} \), which is not helpful since \( \sum_{n=1}^{\infty} \frac{3}{n} \) has known divergence. However, scaling \( \sum_{n=1}^{\infty} \frac{3}{n+100} \) so that this series is greater than \( \sum_{n=1}^{\infty} \frac{3}{n} \) would permit the comparison. To do this, Thomas multiplies \( \sum_{n=1}^{\infty} \frac{3}{n+100} \) by the scalar 101. In
doing so, Thomas implicitly uses the Limit Comparison Theorem without formally evaluating a limit, but using the essential reasoning behind the Limit Comparison Theorem.

Thomas’ and Sabrina’s work suggests that there is room for students to inductively reason about correct mathematics, but there is a need to formalize student thinking using the tools of calculus, such as using limits for the Limit Comparison Theorem and the Ratio Theorem.
Chapter 4

Discussion

Developing a problem-based curriculum intended to help students surface the appropriate concept images proved to be a challenge, especially in helping students discover relationships that are traditionally taught directly. The process of creating the problems in this curriculum required understanding how students potentially would think about infinite series and sequences and creating low thresholds to entry that students could approach with minimal knowledge. The use of the sandwich division model gave students a computational and visual method for thinking about positive-term geometric sequences and series. This context provided students a way to check their reasoning intuitively, rather than merely abstractly. Ergene and Özdemir (2020) also address the harmonic series using the model of filling a water tank, providing students a contextualized way to think about the harmonic series diverging, and, more broadly, criteria for when a p-series converges and diverges. The implementation of both of these models, among others, provided students a lower threshold to entry.

The extensive use of calculational reasoning in this unit also aided in student learning and provided useful insight into the participants’ thinking throughout the unit. Students were able to predict the end behavior of sequences, and therefore series. In performing the predictions, several students revealed much about what they knew and did not know about about infinite series. In Sabrina’s case, the difference between the sequences of partial sums and the series was not immediately clear. However, Sabrina also demonstrated that a geometric series converged using the sandwiches model. Charles demonstrated novel understanding of geometric series convergence using rational numbers, but in the performance assessment demonstrated the challenge of differentiating between sequences and series, as with Sabrina. Thus, care must be taken to formalize the connections between sequences and series, while recognizing that they are different.

To succeed in this unit, students need specific prerequisite knowledge. The intent was for this to be minimal. The prerequisites include an understanding of evaluating integrals, including basic integral evaluating techniques, like trigonometric substitution, and an understanding of how to evaluate a limit analytically. These techniques were used only when necessary, but were essential for the unit’s success, suggesting that a minimum of first-semester calculus is necessary, and that some knowledge of integration is needed before this unit. The use of computational methods in the early class sessions of the unit allowed students to leverage their current understanding of addition for finite series.
The use of models strategically also provided students geometric and real-world intuition, giving students a way to check the reasonableness of their responses. This is one of the defining features of the PCMI problem sets: the ability to check your work without having to use an answer key (McLeod, 2021). There were some moments where the built in answer checking did not go as planned, for example with Sabrina’s reasoning about the various representations of infinite series convergence. However, this is where the use of the classroom discussion phase is helpful, specifically where the instructor can use student work from class to promote classroom discussion (Smith & Stein, 2018).

Various examples of student work show that students are able to reason about infinite series convergence without direct instruction. In this way, a theorem-proof-example approach is unnecessary. However, as several students’ work suggests, students need assistance with formalities, which is one place the instructor can use whole-class time (Reinhart, 2000).

The problem-based curriculum gave insights into how students understood infinite series convergence. In particular, attention needs to be paid between the various representations of infinite series: as partial sums and using addition of infinitely many terms. Students also successfully reasoned about geometric series with positive terms, but struggled with negative term series; putting special emphasis on negative series in the future would be helpful.

The student work analyzed suggests that even though explicit attention is paid to defining features of infinite series convergence, students do not have consistent concept images of the various facets of infinite series convergence.

Later studies could potentially investigate what students understand about power series, as the unit only developed content providing a foundation in infinite series convergence. One of the goals of the course is to discuss power series as it relates to Taylor series; thus, an expansion to include content about power series would be relevant and helpful. It would also be worth investigating the role of technology (i.e. applets) in helping student develop concept images about infinite series convergence. Mathematically, it would be interesting to look into students’ use of reasoning similar to Cauchy sequences (much like Sabrina’s emergent reasoning) as they examine sequence and series convergence.
Bibliography


Francis, K. (2019). Play and mathematics. *Canadian Association for Curriculum Studies, 17*(1), 75–89. [https://doi.org/10.11575/PRISM/37414](https://doi.org/10.11575/PRISM/37414)


Appendix A

Unit Plan

Unit 2-1: Infinite Series

Unit Plan Outline

This unit is designed to be completed in twelve sessions over three weeks. It is assumed that each of the sessions for each of the three weeks are fifty minutes long for at least four distinct days each week and that at least twenty-four hours separates each of these sessions.

Unit Goal

The purpose of this unit is to help students recognize when an infinite series converges or diverges, and to justify using an appropriate test or theorem when this occurs.

Unit Objectives: Throughout Unit

IV-1. Students will, when presented with an infinite series, choose an appropriate test to determine its convergence or divergence. [Application]

IV-2. Students will, when presented with an infinite series, articulate their reasons for why a series converges or diverges using appropriate tests, stating the conditions for the test they use. [Comprehension and Communication]

IV-3. Students will, when using a test or theorem, state and verify the conditions for using it. [Simple knowledge/Algorithmic skill]

IV-4. Students will classify series as one or more of the following: geometric, alternating, increasing, decreasing, convergent, divergent, absolutely convergent, conditionally convergent, telescoping. [Comprehension and Communication]

IV-5. Students will use sigma notation appropriately to represent infinite series in their mathematical work. [Comprehension and Communication]
Unit Objectives: End of Unit

V-1. Students will investigate infinite series’ convergence or diverges in fractals. [Application]
**Unit Plan: Week 1**

**This Week’s Objectives**

I-1. Students will define a (real) sequence as an ordered list of numbers (i.e. \( f : \mathbb{N} \rightarrow \mathbb{R} \)) and a series as the sum of the terms in the corresponding sequence. [Construct a concept]

I-2. Students will conclude that a sequence \((a_n)\) converges when the associated limit \( \lim_{n \to \infty} a_n = L \), where \( L \in \mathbb{R} \). The sequence will diverge otherwise. [Construct a concept]

I-3. Students will recognize that an infinite series can sum to a finite value, and that this value is the limit of the sequence of partial sums. [Discover a relationship]

I-4. Students will develop conditions for when a series diverges. [Discover a relationship]

I-5. Students will develop the fact that a geometric series converges to \( \frac{a}{1-x} \) for a common ratio \(|x| < 1\). [Discover a relationship]

**Session 1: To \( \infty \ldots \) but not beyond!**

Objectives I-1, I-2, I-3 will be addressed.

Students will begin to develop an understanding of what it means for a sequence to converge and a sequence to diverge. Students will be introduced to a series, albeit informally, as a sequence of partial sums. Students will begin to connect that determining whether a series or sequence converges requires evaluating limits.

**Session 2: Ad \( \infty \)-um**

Objectives I-1, I-2, I-3 will be addressed.

Students will continue to investigate infinite sequences to determine when sequences converge or diverge. Students will be able to determine when sequences are increasing, decreasing, or are of neither group. Students will more fully investigate series, specifically geometric series, to determine conditions on their convergence and divergence.

---

1 Implicit in this definition is that these are infinite sequences and series.
**Session 3: Running to ∞**

Objectives I-3, I-5, I-6 will be addressed.

Students will connect a sequence of converging partial sums to a convergent series. Students will determine conditions on which a geometric series converges and diverges. Students will review L’Hopital’s rule and indeterminant forms.

**Session 4: Summing ∞**

Objectives I-4, I-5, I-6 will be addressed.

Students will be introduced to series notation and formalize that a geometric series converges for \( \frac{a}{1-x} \) where \(|x| < 1\). Students will also develop conditions for when a series diverges ("the divergence test") and develop properties of infinite series, including summing two infinite series and multiplying an infinite series by a scalar.
Unit Plan: Week 2

This Week’s Objectives

II-4. Students will perform term-by-term comparison between series to determine when $\sum a_n \leq \sum b_n$. [Algorithmic skill]

II-5. Students will compare series to one another and determine upper and lower bounds for infinite series. [Discover a relationship]

II-3. Students will use integrals (areas between curves and the x-axis) to bound areas of corresponding series. [Discover a relationship]

II-1. Students will deduce that the harmonic series is the “dividing point” among p-series to “separate” p-series that converge or diverge; in other words, a p-series $\sum \frac{1}{n^p}$ converges when $p > 1$ and diverges when $p \leq 1$. [Discover a relationship]

II-2. Students will recognize that a series whose terms converge to zero is a necessary, but not sufficient condition, for convergence. [Discover a relationship]

Session 5: Can You Bound It?

Objectives II-1, II-2 will be addressed.

Students will determine series that are upper bounds or lower bounds for other series. They will also be able to determine that a series that acts as a lower bound for another series cannot show convergence, and likewise a series that acts as an upper bound cannot show divergence.

The key assumption that all terms are positive (i.e. the ordered pairs representing series terms are in the first quadrant of the Cartesian plane) needs to be explicitly pointed out in this session.
Session 6: Staying in Quadrant I

Students will begin to investigate how series and integrals are related, noticing that at times integrals can acts as lower or upper bounds to series, depending on context. For example, students need to realize that a convergent, lower-bound series cannot be used to show divergence, and vice-versa. Additionally, the p-series theorem/test acts as a way to determine convergence for power series functions in the denominator, as opposed to exponential functions, which use the geometric series test.  

Students will formalize the integral theorem/test and will develop the p-series theorem/test as a corollary to the integral version.

Session 7: Which is Greater?

Students will build off of the foundation of integrals and determine series that are upper or lower bounds that have known convergence for other series. Students will also formalize the direct comparison test.

Session 8: Does it Compare?

Students will investigate and formalize the limit comparison test. Students will connect this test to the direct comparison test.

---

2This relates to the unassessed standard III-B.
Unit Plan: Week 3

This Week’s Objectives

III-1. Students will recognize that an alternating series has terms where every other term changes sign. [Construct a concept]

III-2. Students will recognize that a series that is absolutely convergent when the sum of the absolute value of all its terms converges. This happens when the terms \(|a_n|\) are decreasing and \(\lim_{n \to \infty} a_n = 0\). [Discover a relationship]

III-3. Students will recognize that a series that absolutely converges also converges. [Discover a relationship]

III-4. Students will differentiate between absolutely convergent and conditionally convergent series. [Construct a concept]

III-5. Students will connect the use of the ratio and root tests to determining whether a series “acts like” a geometric series. [Discover a relationship]

Session 9: Entering the Fourth Quadrant

Students will begin this week investigating what happens when a series has negative terms and develop conditions for the alternating series tests.

Session 10: Uncommon Ratios

Students will continue investigating alternating series and will begin to investigate the ratio test.

Session 11: \(\cdots, \infty - 1, \cdots\)

Students will review infinite series by creating their own series that converge or diverge in order to show understanding to include in Session 12’s problems. Students will also continue to investigate the ratio test, along with the root test.

Session 12: The End of \(\infty\)...?

Students will continue to review infinite series by investigating problems created by their peers in session 11. Students will also formalize the ratio and root tests.

---

\(3\)This standard relates to how, historically, Cauchy compared a series behavior to a geometric series to show series convergence (Katz, 2009, p. 773-774).
Assessment

Objective Assessment

Students will participate in three quizzes to determine their understanding of determining whether a series converges or diverges, with the different theorems/tests assessed at appropriate times.

Performance Assessment

Students will apply their knowledge of whether a series converges by investigating a variation on a Sierpinski carpet whether the sequence of areas in a variation will converge or diverge and why, using series reasoning.

Ideas for future semesters: the Wallis sieve[^4], the Van Koch snowflake, or other fractals (Cantor set, etc.) and their area/perimeter relationships, determining whether the associated series converge or diverge.

Course Journal

Students will reflect daily on the learnings in the course and submit these periodically for review. The purpose of the journal entries is to see how student understandings about their work over each of the sessions changes.

**Unassessed Standards**

The following standards will be unassessed, but mentioned or discussed, in class.

II-A. Students will estimate errors on partial sums using improper integrals. [Application]

III-A. Students will estimate errors on series using the next term. [Application]

III-B. Students will compare and rank the relative “strength” of functions and determine when one function family overpowers another, including: logarithmic, power, exponential, and factorial functions. [Discover a relationship]
Appendix B

Problem Sets

The problem sets presented in this section are presented without major modification. Typographical errors and other edits were made during the course of teaching and at the end of teaching, but no major edits were made. The norms at the beginning of the problem sets (cf. Session 2-1) are based on norms from the PCMI workshops; for example, at (“Day 1: Flip The Script,” 2016).
Before You Begin

Welcome to class! Read the following boxed statements with your team, out loud so all can hear and discuss. Some of the norms listed here are adapted from “Day 1: Flip The Script” (2016). They are used under a CC-BY-NC-SA-4.0 license, whose text is available from https://creativecommons.org/licenses/by-nc-sa/4.0/.

During this unit, we will be discussing infinite series convergence, but in a problem-based, collaborative way. The purpose of these problems is to help you engage with mathematical concepts. The way it will be done may be different from previous math classes you’ve taken, so if there is some trepidation about doing mathematics, that is OK right now.

Each class session will be divided into (approximately) a 40-minute block and a 10-minute block. The first 40 minutes are meant to be spent engaging with the problems assigned. The final 10 minutes will be class discussion on various ideas encountered in the day’s problems, with students presenting their work as asked by your course instructor.

Your course instructor will largely not be giving direct instructions, but will be monitoring your learning throughout the course session, and may ask questions or give clarifications as needed. However, your course instructor will not be “teaching” in the traditional sense: behind a lectern, or having you take notes. Direct hints on “how to do the problem” will generally not be given. The purpose of these problems if for you to engage with the mathematical ideas of this unit in an authentic way, as mathematicians do with problems they don’t know how to solve.

Before starting any problems, it is important to set some norms for discussion:

**You are a mathematician.** Mathematicians are able to recognize patterns and reason through problems they have never encountered before. This is an essential skill for anyone taking a math class to develop.

**Give others time to think.** Sometimes it might be helpful for everyone in your team to work alone for a few minutes before discussing. Sometimes it might be helpful to talk through something together. Determine among yourselves which is best based on where you are at any given time. Feel free to break away from your group to explore something if you feel you need or want to.

**People want to be heard and valued.** If a colleague has an idea, please take the time to listen to them. Who knows? It might help you along a productive path to solving a problem. Listen carefully to each person’s idea as you work today and every day.

**Don’t worry about finishing all the problems.** There are more problems in this problem set than can be completed in 40 minutes. The purpose of these problem sets is to think deeply about the mathematical ideas contained in the problems. Occasionally, some problems will be assigned as homework.

**If you think you know the math, don’t try to “teach”**. Telling information to someone before they’re ready may be counter-productive. Rather, let the problem do the talking and help your col-
league to see what you see in the context of the problem you’re working on. Help them see what you see, rather than telling them your answer.

**Sometimes technology might be helpful.** Desmos and GeoGebra are two, free online graphing tools that might help with your work.

---

[a] Unless, of course, the problem has an error in it.
[b] As defined in a prior paragraph.

0. Before you begin work on any problems, complete the following:

A. As a team, are there any other norms you feel are important so you can work together well? Discuss these now.

B. Determine a team leader who manages the discussion as a team. This will help you stay organized.

C. Find a dedicated space (i.e. a spiral notebook, a legal pad, a binder) to write all your thoughts down on the problems, as these will be collected for participation credit each day via Canvas.

---

**Don’t begin any problems until you have completed problem 0.**
Start Here

1. Clive is building with some square blocks. He makes the following patterns on the floor:

   Stage #1
   Stage #2
   Stage #3
   Stage #4
   Stage #5

A. Fill in the following table

<table>
<thead>
<tr>
<th>Stage #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>12345</th>
</tr>
</thead>
<tbody>
<tr>
<td>Term</td>
<td>$b_1$</td>
<td>$b_2$</td>
<td>$b_3$</td>
<td>$b_4$</td>
<td></td>
<td></td>
<td></td>
<td>$b_{12345}$</td>
<td></td>
</tr>
<tr>
<td>Blocks in this stage</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

B. Let’s call this sequence $(b_n)$. What does the equation $b_2 = 3$ mean in the context of the problem?

C. What does the $n$th block pattern look like? How many blocks are in the $n$th stage?

2. Clive decides to put all of his squares together after each step and see what happens to the number of blocks.

A. Fill in the following table

<table>
<thead>
<tr>
<th>Stage #</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>10</th>
<th>12345</th>
</tr>
</thead>
<tbody>
<tr>
<td>Blocks in this stage</td>
<td>1</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

| Total blocks from stage 1 to the current stage $t_n$ | 1 | 4 | 25 |   |   |   |   |     |       |

B. Let’s call this sequence $(t_n)$. What does the equation $t_5 = 25$ mean in the context of the problem?

C. What does the $n$th block pattern look like? How many blocks are in the $n$th stage?
3. Cecily is playing a game of “halvsies” with her brother Jeffrey. They get a whole sandwich and decide to take turns taking off half of what they are given and putting it on their plate for eating after they’re done splitting it up, then giving the remainder to the other sibling. Cecily starts the halving with a complete sandwich. Fill in the totals below.¹

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>C</th>
<th>J</th>
<th>C</th>
<th>J</th>
<th>10</th>
<th>100</th>
<th>n</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>10</td>
</tr>
<tr>
<td>How much of the sandwich the sibling got</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{8}$</td>
<td>$\frac{1}{4}$</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much total has been eaten of the whole sandwich</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{3}{4}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

4. The following are some sequences. What do the terms in the sequences seemingly approach? Make your estimate within 0.001.

A. $(15, 15.5, 15.6, 15.75, 15.8, 15.83, \cdots)$

B. $(1, 0.5, 0.25, 0.125, 0.0625, \cdots)$

C. $(\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \cdots)$

D. $(8, 8.0, 8.00, 8.000, 8.0000, \cdots)$

E. $(1, 3, 5, 7, 9, 11, \cdots)$

F. $(1, 4, 9, 16, 25, \cdots)$

G. $(-2, 2, -2, 2, -2, 2, \cdots)$

H. $\left(\frac{\sqrt{2}}{2}, 1, \frac{\sqrt{2}}{2}, 0, -\frac{\sqrt{2}}{2}, -1, -\frac{\sqrt{2}}{2}, 0, \frac{\sqrt{2}}{2}, \cdots\right)$

5. What does the $\cdots$ mean in the sequences above?

6. Evaluate the following limits. Technology may be helpful if you’re stuck.

A. $\lim_{n \to \infty} 16 - \frac{1}{n}$

B. $\lim_{n \to \infty} \left(\frac{1}{2}\right)^n$

C. $\lim_{n \to \infty} \frac{2^n - 1}{2^n}$

D. $\lim_{n \to \infty} 8$

E. $\lim_{n \to \infty} 2n - 1$

F. $\lim_{n \to \infty} n^2$

G. $\lim_{n \to \infty} (-1)^n \cdot 2$

H. $\lim_{n \to \infty} \sin\left(\frac{n\pi}{4}\right)$

¹If you want to use a piece of paper as a way to model playing “halvsies”, you may get some.
Important Problems

7. Revisit the functions of the limits in problem 6. What do the graphs of these sequences look like? How does this connect to the value of the limits?

8. In problem 3, how much of the whole sandwich did Cecily get? How much did Jeffrey get? Does this surprise you?

9. Revisit Cecily’s block situation again (cf. problem 1). How does $b_{n+1} = b_n + 2$ connect to her sequence? Asked another way: What is the meaning of $b_n$? What is the meaning of $b_{n+1}$? Why is there a “+2” in this equation?

10. Revisit Cecily’s total block situation again (cf. problem 2). How does $t_{n+1} = t_n + (2n - 1)$ connect to her new sequence? Break down the meaning of each of the parts of this equation and connect those parts to the context, as in question 9.

11. What patterns, if any, do you notice in this sequence: $a_n = \frac{2n-1}{2n-2}$. Let $a_1 = 10$ and $a_2 = 5$.

Advanced Problems

12. Here’s a sequence: $(a_n) = (\frac{1}{n^2})$.

   A. For what value of $N$ will the sequence be within 0.012345 of what the limit approaches for all terms after $N$? (Is your $N$ the lowest $N$ possible?)

   B. Same question as part A, but now see if you can be within 0.0000001 of what the limit approaches.

   C. If you’re given some real number $\epsilon > 0$, what value of $N$ should you choose so that all terms following the $N$th term are within $\epsilon$ of the actual limit?

13. Here’s another sequence: $(a_n) = (\frac{1}{n^2-n-20})$. Answer the same questions as problem 12. Go!

References

**Before You Begin**

Review briefly what happened last time in case someone was absent.

Check to make sure you have uploaded your work from Session 2-1 to Canvas. If you’ve done that, then go to problem 1!

**Start Here**

1. Cecily decides to play a game of “thirdsies” with her brother Jeffrey. This is played similarly to “halvesies” (cf. Session 2-1, problem 3), except that each sibling takes a third of the remaining portion of the sandwich on each turn. In other words, they get a sandwich and take turns taking off a third of what they are given and eating it, then giving the remainder to the other sibling. Cecily starts the dividing with a whole sandwich. Fill in the totals below:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>C</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1</td>
<td>2 3 4 5 6 10 100 n</td>
</tr>
<tr>
<td>How much of the sandwich the sibling got</td>
<td>1/3</td>
<td></td>
</tr>
<tr>
<td>How much total has been eaten of the whole sandwich</td>
<td>1/3 19/27</td>
<td></td>
</tr>
</tbody>
</table>

2. Cecily and Jeffrey now play “two-thirdsies” together. Cecily starts the dividing with two whole sandwiches. Fill in the totals below:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>C</th>
<th>J</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1</td>
<td>2 3 4 5 6 10 100 n</td>
</tr>
<tr>
<td>How much of the two sandwiches the sibling got</td>
<td>4/3</td>
<td></td>
</tr>
<tr>
<td>How much total has been eaten of the two sandwiches</td>
<td>4/3</td>
<td></td>
</tr>
</tbody>
</table>
3. Cecily and Jeffrey now play “three-halfsies” together. Cecily starts the dividing with a whole sandwich. Fill in the totals below:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>C</th>
<th>J</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
<td>10</td>
<td>100</td>
</tr>
<tr>
<td>How much of the sandwich the sibling got</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much total has been eaten of the whole sandwich</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Isn’t something strange about this round of the game? If something isn’t making sense, why isn’t it making sense?

4. What makes problem 3 different from problems 1 and 2? Based on this reflection, create a variation on problem 3 that would also be equally awkward.
5. The following are some sequences. What do the terms in the sequences seemingly “approach”? Make your estimate within 0.00001 if possible. Calculators are encouraged.

A. \((\frac{2}{3}, \frac{4}{9}, \frac{8}{27}, \frac{16}{81}, \frac{32}{243}, \ldots)\)

B. \((-\frac{1}{4}, \frac{1}{16}, -\frac{1}{64}, \frac{1}{256}, \ldots)\)

C. \((\sqrt{10}, \sqrt{\frac{20}{3}}, \sqrt{\frac{30}{5}}, \sqrt{\frac{40}{7}}, \sqrt{\frac{50}{9}}, \ldots)\)

D. \((\frac{4}{3}, \frac{16}{9}, \frac{52}{27}, \frac{160}{81}, \ldots)\)

E. \((3, 4, 3, 4, 3, 4, 3, 4, \ldots)\)

F. \((100, 99, 98, 97, 96, \ldots)\)

G. \((-2, 4, -8, 16, -32, 64, \ldots)\)

H. \((\frac{3}{2}, \frac{9}{4}, \frac{27}{8}, \frac{81}{16}, \ldots)\)

6. A sequence is considered decreasing when \(a_n \geq a_{n+1}\). A sequence is considered increasing when \(a_n \leq a_{n+1}\). Both of these definitions require that the inequalities hold for all \(n\). Determine if the sequences in problem 5 are increasing, decreasing, or neither.

\(^1\)For example, \(a_2 \geq a_3\). Another way to interpret this is that each successive term is less than or equal to the one before it.
7. Evaluate the following limits. A graphing calculator might be helpful here to check your reasoning. (Using $x$ instead of $n$ might be helpful when graphing.)

A. $\lim_{n \to \infty} \left(\frac{2}{3}\right)^n$

B. $\lim_{n \to \infty} \left(-\frac{1}{4}\right)^n$

C. $\sqrt{\lim_{n \to \infty} \frac{10n}{2n - 1}}$

D. $\lim_{n \to \infty} \frac{2(3^n - 1)}{3^n}$

E. $\lim_{n \to \infty} \left(\frac{7}{2} + (-1)^n \frac{1}{2}\right)$

F. $\lim_{n \to \infty} (100 - n)$

G. $\lim_{n \to \infty} (-2)^n$

H. $\lim_{n \to \infty} \left(\frac{3}{2}\right)^n$

8. A sequence $(a_n)$ **converges** if $\lim_{n \to \infty} a_n = A$, where $A$ is a real number. Otherwise, the sequence **diverges**. Determine if the sequences in question 5 are **convergent** or **divergent**.
Important Problems

9. Cecily and Jeffrey are both playing “halvesies” again, but with some slightly different rules. The amount that each sibling is allocated on their turn is half of what the other sibling’s turn. On Cecily’s turns, she will take away what is allocated to her away from Jeffrey, instead of from the original sandwich; Jeffrey always takes from the original sandwich. Fill out the following table:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>J</th>
<th>C</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>How much of the sandwich the sibling was allocated</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich belongs to Jeffrey</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich belongs to Cecily</td>
<td>0</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich is unallocated</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

What patterns do you notice after filling in this table? Why do these patterns occur?

10. Consider the following for, separately, any \( r \in (0, 1) \) and any \( r \in [1, \infty) \):

   A. What do the terms in the sequence \( (r, r^2, r^3, r^4, r^5, \ldots) \) seemingly approach?

   B. Evaluate \( \lim_{n \to \infty} r^n \)

   C. Are there any connections between prompts 10A and 10B?

Advanced Problems

11. What is the sum of the following? How can you prove this is true?

   \[
   1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots
   \]

12. What is the sum of the following? How can you prove this is true?

   \[
   1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \cdots
   \]
Start Here

1. Fill in the blanks. (cf. Problem Set 2-2, problem 8, if needed)

<table>
<thead>
<tr>
<th>A sequence converges when the terms in the sequence</th>
<th>A sequence diverges when the terms in the sequence</th>
</tr>
</thead>
<tbody>
<tr>
<td>________________</td>
<td>_____________________</td>
</tr>
</tbody>
</table>

For questions 2-4 and 6, use the attached table to write notes.

2. What are the next few terms in these sequences? Do they converge? If so, to what?
   A. \((\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots)\)
   B. \((1, \frac{3}{2}, \frac{7}{4}, \frac{15}{8}, \frac{31}{16}, \frac{63}{32}, \ldots)\)
   C. \((2, 3, \frac{7}{2}, \frac{15}{4}, \frac{31}{8}, \frac{63}{16}, \frac{127}{32}, \ldots)\)
   D. \((\frac{1}{8}, \frac{3}{16}, \frac{7}{32}, \frac{15}{64}, \frac{31}{128}, \frac{63}{256}, \frac{127}{512}, \ldots)\)
   E. \((1, \frac{5}{2}, \frac{19}{4}, \frac{65}{8}, \frac{211}{16}, \ldots)\)
   F. \((1,0,1,0,1,0,1,\ldots)\)
   G. \((1, \frac{1}{2}, \frac{3}{4}, \frac{5}{8}, \frac{11}{16}, \frac{21}{32}, \ldots)\)
   H. \((\frac{1}{2}, \frac{5}{8}, \frac{21}{32}, \frac{85}{64}, \frac{340}{128}, \ldots)\)

3. Evaluate the following limits. A graphing calculator might be helpful to check your reasoning. (Using \(x\) instead of \(n\) might be helpful when graphing.)
   A. \(\lim_{n \to \infty} \frac{2^n - 1}{2^n} = \lim_{n \to \infty} \left(1 - \frac{1}{2^n}\right)\)
   B. \(\lim_{n \to \infty} \frac{2^{n+1} - 1}{2^n}\)
   C. \(\lim_{n \to \infty} \frac{2^{n+2} - 1}{2^n}\)
   D. \(\lim_{n \to \infty} \frac{2^n - 1}{8 \cdot 2^{n-1}}\)
   E. \(\lim_{n \to \infty} \frac{3^n - 2^n}{2^n}\)
   F. \(\lim_{n \to \infty} \frac{1 + (-1)^n}{2}\)
   G. \(\lim_{n \to \infty} \frac{2^{n+1} - (-1)^{n+1}}{3 \cdot 2^n}\)
4. What do you notice about these sums? Do they converge? If so, to what? Note that these sums are called **infinite series**. Using a calculator to make **partial sums**—sums that stop at a specific term—might be helpful.

A. \( \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \cdots \)

B. \( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots \)

C. \( 2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \)

D. \( \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \frac{1}{64} + \frac{1}{128} + \cdots \)

E. \( 1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots \)

F. \( 1 + (-1) + 1 + (-1) + 1 + (-1) + \cdots \)

G. \( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} + \cdots \)

H. \( \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \frac{1}{512} + \cdots \)

5. What connections do you notice between questions 2, 3 and 4? Use the table on page 5 to note any connections.

6. Each series in problem 4 is called a **geometric series**. A geometric series is a series where each term is a scalar multiple of the previous term. For example, the scalar multiples (often called the **common ratio**) of problem 4A is \( \frac{1}{2} \), since \( 1 \cdot \frac{1}{2} = \frac{1}{2}, \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4} \), etc. Notice that not all of these series converge however.

A. Determine the common ratios for each of the series in problem 4. Fill in the common ratios in the attached table.

B. Is there a pattern between the common ratios of the series and when the series converge? If there is, why is this the case? (How does this connect to the sandwich problems from problem set 2-2?)

C. When a series diverges, what happens to the limit of the individual terms? (Hint: Look at problems 3E and 3F)
7. Make your own geometric series (different from the ones above) that you know converges. Why does it converge?

8. Make your own geometric series (different from the ones above) that you know diverges. Why does it diverge?

**Important Problems**

9. Joseph and Juliette are walking their dog Apollo on a three mile long trail. Joseph and Juliette start on opposite ends of the trail and Apollo start running from Joseph to Juliette. Once Apollo reaches Juliette, he (quickly) turns around and runs to Joseph. Once Apollo reaches Joseph, he again turns around and runs to Juliette. This process continues until Joseph and Juliette meet.¹

   A. How far does Apollo run?

   B. How does this problem connect to an infinite series? Make a series model for this problem to demonstrate the connection.

¹Adapted from *Calculus* (Garner, 2003).
10. Joseph and Juliette go on another walk along the same one mile trail, but Juliette decides to go twice as fast as Joseph under the same conditions as problem 9. How far does Apollo run? How do you know?

11. Consider this sequence: \((1, \frac{2+\sqrt{2}}{2}, \frac{3+\sqrt{2}}{2}, \frac{6+3\sqrt{2}}{4}, \frac{7+3\sqrt{2}}{4}, \frac{14+7\sqrt{2}}{8}, \ldots)\)
   
   A. Does this sequence converge? If it does, what does it converge to? If not, why not?
   
   B. What infinite series is this a representation for?

12. Evaluate the following limits, simplifying the expressions if needed. Technology might be helpful.
   
   A. \(\lim_{n \to \infty} \frac{n^3}{n^2}\)
   
   B. \(\lim_{n \to \infty} \frac{n^2}{n^3}\)
   
   C. \(\lim_{n \to \infty} \frac{2n^5 - 5n}{3n^5}\)
   
   D. \(\lim_{n \to \infty} \frac{3n^5}{2n^5 - 5n}\)

11. Note that each pair of limits in problem 7, if evaluated directly, represent a limit that approaches \(\frac{\infty}{\infty}\) which is called an indeterminant form. However, each of the limits in problem 7 have a finite value. It is therefore ambiguous what the true limit is if the result is \(\frac{\infty}{\infty}\) without further simplification. **L’Hospital’s Rule** states that, if the limit of a function approaches \(\frac{\infty}{\infty}\) or \(0\) \(\infty\), then the derivative of the rational function’s numerator and denominator can be evaluated in order to determine the limit’s value. Use L’Hospital’s rule to evaluate each of the limits in problem 7.

**Advanced Problems**

12. Building off of problem 9, Joseph and Juliette go on another walk along a one mile trail, but Juliette and Joseph’s travel speeds are in ratio to \(m : n\), meaning that Juliette goes \(m\) mph as Joseph travels \(n\) mph. How far does Apollo run? How do you know? (Are there any assumptions about \(m : n\)?)

**References**

Before You Begin

You’re invited to use a calculator on these problems to facilitate efficient computation.

Start Here

1. Cecily and Jeffrey play a game of ”Three-Fourthsies” with three sandwiches, played in a similar way to ”Halvsies” (cf. Problem Set 2-1, Problem 3).
   A. What is the meaning of the sequence \((t_n) = (\frac{9}{4}, \frac{9}{16}, \frac{9}{64}, \frac{9}{256}, \ldots)\) in the context of the problem?
   B. What is the meaning of the sequence \((s_n) = (\frac{9}{4}, \frac{45}{16}, \frac{189}{64}, \frac{765}{256}, \ldots)\) in the context of the problem?
   C. What is \(\lim_{n \to \infty} t_n\)?
   D. What is \(\lim_{n \to \infty} s_n\)?
   E. What is the meaning of the series \(\frac{9}{4} + \frac{9}{16} + \frac{9}{64} + \frac{9}{256} + \cdots\) in the context of the problem?
   F. What is the sum of the series \(\frac{9}{4} + \frac{9}{16} + \frac{9}{64} + \frac{9}{256} + \cdots\)?

2. Now, let’s play ”Four-Thirdsies” with three sandwiches.
   A. This version of the game is impossible to play. Why?
   B. What would the meaning of the sequence \((t_n) = (4, \frac{16}{3}, \frac{64}{9}, \frac{256}{27}, \ldots)\) be in the context of the problem?
   C. What would the meaning of the sequence \((s_n) = (4, \frac{28}{3}, \frac{148}{9}, \frac{700}{27}, \ldots)\) be in the context of the problem?
   D. What is \(\lim_{n \to \infty} t_n\)?
   E. What is \(\lim_{n \to \infty} s_n\)?
   F. What is the meaning of the series \(4 + \frac{16}{3} + \frac{64}{9} + \frac{256}{27} + \cdots\) in the context of the problem?
   G. What is the sum of the series \(4 + \frac{16}{3} + \frac{64}{9} + \frac{256}{27} + \cdots\)?

3. Under what conditions does playing ”Divide the Sandwich” (i.e. ”Halvsies”, ”Three-Fourthsies”) make sense? When does it not make sense? Why?
4. Instead of having to write out all the terms in a series, often the $\sum$ symbol is used to compress information. Fill in the blanks where requested.

Ex. $1 + 2 + 3 + 4 + 5 = \sum_{n=1}^{5} n$

A. $1 + 4 + \underline{+ 16} + \underline{+ 36} = \sum_{m=1}^{6} m^2$

B. $27 + 30 + \underline{+ 39} + \underline{+ 42} = \sum_{g=3}^{14} 3g$

C. $4 + 1 - 2 - 5 = \sum_{a=1}^{4} \underline{a}$

D. $\frac{3}{8} + \underline{+ \frac{3}{35}} + \underline{+ \frac{3}{48}} + \cdots = \sum_{v=3}^{\infty} \frac{3}{v^2 - 1}$

E. $1 + \underline{+ \frac{1}{4}} + \underline{+ \frac{1}{8}} + \underline{+ \cdots} = \sum_{i=0}^{\infty} \frac{1}{2^i}$

5. Examine the following series. Do they converge or diverge? Why?

A. $\sum_{n=1}^{\infty} \left(3 \cdot \frac{3}{4} \left(\frac{1}{4}\right)^{n-1}\right) = \frac{9}{4} + \frac{9}{16} + \frac{9}{64} + \frac{9}{256} + \cdots$

B. $\sum_{n=1}^{\infty} \left(3 \cdot \left(\frac{4}{3}\right)^n\right) = 4 + \frac{16}{3} + \frac{64}{9} + \frac{256}{27} + \cdots$

C. $\sum_{n=1}^{\infty} \left(\frac{1}{2} \cdot \left(\frac{1}{2}\right)^n\right) = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$

D. $\sum_{n=1}^{\infty} (1) = 1 + 1 + 1 + 1 + 1 + \cdots$

E. $\sum_{n=1}^{\infty} ((-1)^{n+1} \cdot n) = 1 + (-2) + 3 + (-4) + 5 + (-6) + \cdots$
6. Evaluate the following limits:

A. \[ \lim_{n \to \infty} \left( 3 \cdot \frac{3}{4} \left( \frac{1}{4} \right)^n \right) \]

B. \[ \lim_{n \to \infty} \left( 3 \cdot \left( \frac{4}{3} \right)^n \right) \]

C. \[ \lim_{n \to \infty} \left( \frac{1}{2} \left( \frac{1}{2} \right)^n \right) \]

D. \[ \lim_{n \to \infty} 1 \]

E. \[ \lim_{n \to \infty} (-1)^{n+1} \cdot n \]

7. Consider how the following blanks could be filled in to make a mathematically correct statement, based on the results of problems 5 and 6:

A series \( \sum a_n \) diverges when the corresponding sequence of terms, \((a_n)\), has the following property:
\[ \lim_{n \to \infty} a_n \]

8. Go back to Problem Set 2-3, Problems 7-8 and complete them if you were unable to earlier. Then, fill in the following blanks with a conjecture:

A geometric series (cf. Problem Set 2-3, Problem 6) converges when
\[ \text{___________________________} \] and diverges when
\[ \text{___________________________} \]

Important Problems

9. Determine whether the following equations are true or false.

A. \( \left( 2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots \right) = 2 \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \)

B. \( \left( \frac{3}{5} + \frac{3}{10} + \frac{3}{20} + \frac{3}{40} + \cdots \right) = \frac{3}{5} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \)

C. \( \left( -1 - \frac{1}{2} - \frac{1}{4} - \frac{1}{8} + \cdots \right) = -1 \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \)

D. \( \left( k + \frac{k}{2} + \frac{k}{4} + \frac{k}{8} + \cdots \right) = k \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \right) \), for any \( k \in \mathbb{R} \).

E. \( \sum_{n=1}^{\infty} \left( k \cdot \frac{1}{2^{n-1}} \right) = k \sum_{n=1}^{\infty} \left( \frac{1}{2^{n-1}} \right) \), for any \( k \in \mathbb{R} \).
10. A **telescoping series** is a series what has adjacent terms add to zero, leaving only the first and the last terms. Consider the following telescoping series:

\[
\sum_{n=1}^{\infty} \left( \frac{1}{n(n+1)} \right) = \sum_{n=1}^{\infty} \left( \frac{1}{n} - \frac{1}{n+1} \right)
\]

A. Why are is the left hand side of the equation equal to the right hand side? (Hint: Use partial fraction decomposition.)

B. What is the series’ sum (either the left hand side or the right hand side)? Why?

**Advanced Problems**

12. What is the sum of the following? How can you prove this is true?

\[
1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \cdots
\]

13. What is the sum of the following? How can you prove this is true?

\[
1 + \frac{1}{16} + \frac{1}{81} + \frac{1}{256} + \frac{1}{625} + \cdots
\]
Complete the following homework before Session 2-5 and submit to Canvas.

1. Consider the following sums:
   \[ S_3 = 1 + \frac{1}{3} + \frac{1}{9} \]
   \[ \frac{1}{3}S_3 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \]
   A. Demonstrate why \((1 - \frac{1}{3})S_3 = 1 - \frac{1}{27} = \frac{26}{27}\).
   B. Since \(\frac{2}{3}S_3 = \frac{26}{27}\), solve for \(S_3\). (Does \(S_3\) have the same value as \(1 + \frac{1}{3} + \frac{1}{9}\)?)
   \[ S_4 = 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} \]
   \[ \frac{1}{3}S_4 = \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{81} \]
   C. What is \((1 - \frac{1}{3})S_4\)?
   D. What is \(S_4\)?
   E. What is \(S_5\)? (Hint: What numbers aren’t subtracted out?)
   F. What is \(S_n\)?
   G. What is \(\lim_{n \to \infty} S_n\)?

2. Consider the following more general argument for a geometric series with a common ratio \(|r| < 1\).
   \[ S_3 = 1 + r + r^2 \]
   \[ rS_3 = r + r^2 + r^3 \]
   A. Demonstrate why \((1 - r)S_3 = 1 - r^3\).
   B. What is \(S_3\)?
   \[ S_4 = 1 + r + r^2 + r^3 \]
   \[ rS_4 = r + r^2 + r^3 + r^4 \]
   C. What is \((1 - r)S_4\)?
   D. What is \(S_4\)?
   E. What is \(S_5\)?
   F. What is \(S_n\)?
   G. What is \(\lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1 - r^n}{1 - r}\)? (Hint: If a number \(r \in (-1,1)\), i.e. \(-1 < r < 1\), is raised to a large power, what happens?)
3. What are the sums of the following geometric series? (Hint: Look for a common ratio between terms and use the result from 2G.)

A. \[
1 + \frac{1}{4} + \frac{1}{16} + \frac{1}{64} + \cdots = \sum_{n=0}^{\infty} \left( \frac{1}{4} \right)^n
\]

B. \[
1 - \frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \cdots = \sum_{n=0}^{\infty} \left( -\frac{2}{3} \right)^n
\]

C. \[
\frac{3}{4} + \frac{3}{8} + \frac{3}{16} + \frac{3}{32} + \frac{3}{64} + \cdots = \frac{3}{4} \left( 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots \right) = \frac{3}{4} \sum_{n=0}^{\infty} \left( \frac{1}{2} \right)^n = \sum_{n=0}^{\infty} \frac{3}{4} \left( \frac{1}{2} \right)^n
\]

D. \[
\frac{7}{9} - \frac{21}{36} + \frac{63}{144} - \frac{189}{576} + \cdots = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{7}{9} \left( \frac{3}{4} \right)^n
\]

E. Writing out the first few terms of this series might be helpful; also, rewriting might be helpful:
\[
\sum_{n=0}^{\infty} \frac{4 \cdot 2^n}{3^{n+1}}
\]

F. Pay attention to the starting value of \( n \):
\[
\sum_{n=2}^{\infty} \frac{5^n}{7^{n-1}}
\]
Solutions to Problem 3

A. \[ \frac{1}{1 - \frac{1}{4}} = \frac{4}{3} \]

B. \[ \frac{1}{1 - \left( -\frac{2}{3} \right)} = \frac{3}{5} \]

C. \[ \frac{3}{4} \cdot \frac{1}{1 - \frac{1}{2}} = \frac{3}{4} \cdot \frac{3}{1 - \frac{1}{2}} = \frac{3}{2} \]

D. \[ \frac{\frac{7}{9}}{1 - \left( -\frac{3}{4} \right)} = \frac{4}{9} \]

E. 4

F. \[ \frac{25}{2} \]
Before Your Begin

Note that \( n! \) is read “\( n \) factorial”. An example of the meaning of this operation is seven factorial: 
\[ 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1, \] 
\[ \text{or } 7! = 7 \cdot 6! \]. In a similar way, \( n! = n \cdot (n - 1)! \), \( n \geq 1 \).

Recall also the criteria for determining whether a geometric series converges [Write this here; cf. Problem Set 2-4, Problem 8]:

Recall also that the sum of a geometric series is determined by \( \frac{a}{1 - r} \), for a common ratio \(|r| < 1\) and an initial term \( a \). [cf. Homework 2-4, Problem 2G, 3C-3F].

When two series are compared, often this is done by comparing corresponding terms. Example:

\[
\frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \sum_{n=2}^{\infty} a_n \\
5 + 7 + 9 + \cdots = \sum_{n=2}^{\infty} b_n
\]

Note that \( a_2 = \frac{1}{4} \leq 5 = b_2 \). Also, \( a_3 = \frac{1}{9} \leq 7 = b_3 \). Indeed, for all \( n \geq 2 \),

\[
1 \leq 2n^3 + n^2 \\
1 \leq n^2 (2n + 1) \\
\implies \frac{1}{n^2} \leq 2n + 1 \\
\implies a_n \leq b_n
\]

This means that \( \sum_{n=2}^{\infty} a_n \leq \sum_{n=2}^{\infty} b_n \).
Start Here

1. The geometric series \( \sum_{n=0}^{\infty} \frac{1}{5^n} = 1 + \frac{1}{5} + \frac{1}{25} + \cdots \) converges to \( \frac{1}{1 - \frac{1}{5}} = \frac{5}{4} \).

   A. Discuss why the geometric series converges to \( \frac{1}{1 - \frac{1}{5}} \) [Hint: The sum of a geometric series with initial term 1 is determined by the common ratio using the following expression: \( \frac{1}{1 - r} \); remember homework 1-4].

   B. Find two geometric series whose common ratios are positive that are, when corresponding terms are compared, less than (or equal to) \( \sum_{n=0}^{\infty} \frac{1}{5^n} \).

   C. Find two geometric series whose common ratios are positive that are, when corresponding terms are compared, greater than (or equal to) \( \sum_{n=0}^{\infty} \frac{1}{5^n} \).

D. Graph \( f(x) = \frac{1}{5^x} \) and graph the corresponding functions for the geometric series in problem 1B. What do you notice?

E. Do both of your series in 1B converge? Why or why not?

F. Will every series possible for 1B converge? Why or why not?

G. Graph \( f(x) = \frac{1}{5^x} \) and graph the corresponding functions for the geometric series in problem 1C. What do you notice?

H. Do both of your series in 1C converge? Why or why not?

I. Will every series possible for 1C converge? Why or why not?
2. Consider the reasoning of Chloe and Sam in describing whether \( \sum_{n=0}^{\infty} \frac{1}{5^n} \) converges or diverges.

**Chloe:** I’m thinking that \( \sum_{n=0}^{\infty} \frac{1}{5^n} \) converges.

**Sam:** Really? I think it diverges.

**Chloe:** Really? Why is that?

**Sam:** Considering that \( 1 + 1 + 1 + 1 + 1 + \cdots \) diverges, and \( \frac{1}{5^n} \leq 1 \) for all non-negative values of \( n \), \( 1 + \frac{1}{5} + \frac{1}{25} + \cdots \) is always less than something that diverges. What I mean is the first term in the “\( \frac{1}{5} \) series” is always greater than or equal to the first term in the ”one series”: \( 1 \leq 1, \frac{1}{5} \leq 1, \frac{1}{25} \leq 1 \), and so on. This is true for any positive integer, \( n \).

**Chloe:** I did a comparison too, but I used a series made from the powers of \( \frac{4}{5} \): \( 1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \cdots \) converges, and \( \frac{1}{5^n} \leq \frac{4^n}{5^n} \) (since \( 1 \leq 4^n \)) for all non-negative values of \( n \), \( 1 + \frac{1}{5} + \frac{1}{25} + \cdots \) converges. It also helps that the terms of the ”one-fifth” series are separated by a common ratio of \( \frac{1}{5} \) between each term, which is a convergent geometric series.

Whose reasoning is correct? Why is this?

3. Consider the reasoning of Chloe and Sam in describing whether \( \sum_{n=0}^{\infty} 5^n \) converges or diverges.

**Chloe:** I’m thinking that \( \sum_{n=0}^{\infty} 5^n \) converges.

**Sam:** Really? I think it diverges.

**Chloe:** Really? Why is that?

**Sam:** Considering that \( 1 + 1 + 1 + 1 + 1 + \cdots \) diverges, and \( 1 \leq 5^n \) for all non-negative values of \( n \), \( 1 + 5 + 25 + \cdots \) is always greater than or equal to something that diverges. What I mean is the first term in the ”5 series” is always greater than or equal to the first term in the ”one series”: \( 1 \leq 1, 1 \leq 5, 1 \leq 25 \), and so on. This is true for any \( n \). It also helps that a geometric series with a common ratio of 5 diverges, since 5 \( \geq 1 \).

**Chloe:** I did something similar, but I used powers of \( \frac{4}{5} \): \( 1 + \frac{4}{5} + \frac{16}{25} + \frac{64}{125} + \cdots \) converges, and \( \frac{4^n}{5^n} \leq 5^n \) for all non-negative values of \( n \), \( 1 + 5 + 25 + \cdots \) converges.

Whose reasoning is correct? Why is this?
4. Fill in the blanks with a conjecture:

All series (with positive terms) that have corresponding terms less than a convergent series are __________________________. All series (with positive terms) that have corresponding terms greater than a divergent series are __________________________.

5. Consider the following series and fill in the blanks that follow based on the reasoning of Chloe and Sam:

\[
1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots = \sum_{n=0}^{\infty} a_n, \text{ with } a_n = \frac{1}{2^n}
\]

\[
1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \cdots = \sum_{n=0}^{\infty} b_n, \text{ with } b_n = \frac{1}{3^n}
\]

\[
1 + 1 + 1 + 1 + 1 + 1 + \cdots = \sum_{n=0}^{\infty} c_n, \text{ with } c_n = 1
\]

\[
1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \cdots = \sum_{n=0}^{\infty} d_n, \text{ with } d_n = \frac{1}{(n+1)!}
\]

A. Circle the correct inequality: Every corresponding term in \(a_n\) is \((\geq \text{ or } \leq)\) \(b_n\) for every \(n \geq 0\).

B. Circle the correct inequality: Every corresponding term in \(a_n\) is \((\geq \text{ or } \leq)\) \(c_n\) for every \(n \geq 0\).

C. Circle the correct inequality: Every corresponding term in \(a_n\) is \((\geq \text{ or } \leq)\) \(d_n\) for every \(n \geq 0\).

D. Does \(\sum_{n=0}^{\infty} a_n\) converge or diverge? Why?

E. Does \(\sum_{n=0}^{\infty} b_n\) converge or diverge? Why?

F. Does \(\sum_{n=0}^{\infty} c_n\) converge or diverge? Why?

G. Based on your answer to 5C and 5D, what do you conclude about \(\sum_{n=0}^{\infty} d_n\): Does it converge or diverge? Why?

H. The same argument for task 5G (except using 5B and 5D) also works when considering whether \(\sum_{n=0}^{\infty} b_n\) converges or diverges. Why is this so?

6. Consider the series in the problem 5.

\[1\text{Note that } a_5 = \frac{1}{32} \text{ and } b_4 = \frac{1}{64}.\]
A. Find two geometric sequences \((s_n)\) where \(b_n < s_n < c_n\), for all \(n \geq 0\). (For the sake of this problem, don’t use \((a_n)\).)

B. Does the geometric series \(\sum_{n=0}^{\infty} s_n\) converge or diverge? Why is this true?

7. The series \(1 + \frac{3}{2} + \frac{9}{4} + \frac{27}{8} + \frac{81}{16} + \cdots = \sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n\) diverges.

A. Find a convergent series that is a lower bound for \(\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n\).

B. Find a divergent series that is a lower bound for \(\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n\).

C. Find a series that is an upper bound for \(\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n\).

D. Can a convergent series show that a series diverges? Why or why not?

8. Based on your results from the homework for problem set 2-4, fill in the blank.

The sum of a geometric series, when it converges, is ____________________.

Important Problems

9. Does \(\sum_{n=0}^{\infty} \frac{2^n}{3^n + 10}\) converge or diverge? Why?

10. Which of these series is, when comparing corresponding terms, is “largest”? Which is “smallest”?

\[\sum_{n=2}^{\infty} \frac{1}{\sqrt{x - 1}} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{x}} \quad \sum_{n=2}^{\infty} \frac{1}{\sqrt{x + 1}}\]

Advanced Problems

11. Go back to problem sets 2-1 through 2-4 and work on the advanced problems from before.
Start Here

You will want some graph paper for today’s problems! Make sure you have some nearby!

1. Becki is filling an initially empty tank of water. The tank is filled in the following pattern: on the first day a gallon of water is poured in; on the second day, a half gallon of water is poured in; on the third day, a third gallon of water is poured in; this pattern continues ad infinitum (Ergene & Özdemir, 2020).

   A. Can Becki fill a 4-gallon water tank?

   B. Can Becki fill a 9-gallon water tank?

   C. What is the largest water tank Becki can fill? Why?

2. What is the area under the curve for \( f(x) = \frac{1}{x} \) on \([1, \infty)\), as illustrated below?
3. Consider the following series:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9} + \cdots = \sum_{n=1}^{\infty} a_n, \text{ with } a_n = \frac{1}{n} \]

A. Below is a graph of the terms in \((a_n)\). Describe the meaning of the width and height of each rectangle.

B. How does the graph of \(f(x) = \frac{1}{x}\) compare to \(\sum_{n=1}^{\infty} \frac{1}{n}\) in the graph below?

C. What does problem 2 and problem 3B have to do with the convergence or divergence of the series \(\sum_{n=1}^{\infty} \frac{1}{n}\)?
4. Consider the following series:

\[ 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \frac{1}{125} + \frac{1}{216} + \frac{1}{343} + \frac{1}{512} + \frac{1}{729} + \cdots = \sum_{n=1}^{\infty} c_n, \text{ with } c_n = \frac{1}{n^3} \]

A. Why are the terms of \((c_n)\) not geometric?

B. Below is a graph of the terms in \((c_n)\). How was this graph made?

C. How does the graph of \(\frac{1}{x^3}\) compare to \(\sum_{n=1}^{\infty} \frac{1}{n^3}\) in the graph below?

D. Visually, what does \(\int_{1}^{\infty} \frac{1}{x^3} \, dx\) represent in the above diagram?

E. How does \(\sum_{n=1}^{5} \frac{1}{n^3}\) compare to \(\int_{1}^{5} \frac{1}{x^3} \, dx\)?

F. Does \((c_n)\) converge or diverge? Why?
5. Make a graph/drawing of the following integrals and then evaluate the integrals. Then, make a graph/drawing of the corresponding series (using rectangles). Do the integrals converge or diverge? What does that tell you about the corresponding series converging or diverging? (Does the picture confirm the fact that each integral converges or diverges?)

A. \( \int_1^\infty \frac{1}{x^2} \, dx \)  \( \sum_{n=1}^{\infty} \frac{1}{n^2} \)

B. \( \int_1^\infty \frac{1}{\sqrt{x}} \, dx \)  \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \)

C. \( \int_1^\infty \frac{1}{x^3} \, dx \)  \( \sum_{n=1}^{\infty} \frac{1}{n^3} \)

D. \( \int_1^\infty \frac{1}{x} \, dx \)  \( \sum_{n=1}^{\infty} \frac{1}{n^{-1}} \)
Important Problems

6. Evaluate the following integral:

\[ \int_1^\infty \frac{1}{x^p} \, dx \]

7. In problems 3-6, you may notice that the conditions for convergence of the integral depends on the value of the exponent \( p \) of \( x \). Find a value of \( p \) in (not used so far in problem set 2-6) that will cause \( \int_1^\infty \frac{1}{x^p} \, dx \) to converge. Then, find a value of \( p \) that causes the integral to diverge. Why does this happen?

8. What is the \( \lim_{x \to \infty} \frac{1}{x^p} \) for various values of \( p \)? Do you notice any connections with regards to question 6? \(^1\)

9. When is it advantageous to use a “left-hand sum” or a “right-hand sum” of a series when using an integral?

10. If \( \int_1^\infty f(x) \, dx \) converges, what can be said about \( \int_4^\infty f(x) \, dx \)? Why is this true?

Advanced Problems

11. What is the total area shaded below, assuming the nested pattern continues ad infinitum? You may assume that:

\(^1\)Using technology like Desmos or GeoGebra might be helpful!
• the triangles are equilateral and are inscribed in circles, with the largest circle being of diameter 1 unit; and,

• circles are tangent to the respective sides of equilateral triangles.

References

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1. Make a graph/drawing of the following integrals and then evaluate the integrals. Then, make a graph/drawing of the corresponding series (using rectangles) on the same graph as the function. Do the integrals converge or diverge? What does that tell you about the corresponding series converging or diverging? (Does the picture confirm the fact that each integral converges or diverges?)

A. \[ \int_1^\infty \frac{1}{x^2} \, dx \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n^2} \]

B. \[ \int_1^\infty \frac{1}{\sqrt{x}} \, dx \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{\sqrt{n}} \]

C. \[ \int_1^\infty \frac{1}{x^3} \, dx \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n^2} \]

D. \[ \int_1^\infty \frac{1}{x^{-1}} \, dx \quad \text{and} \quad \sum_{n=1}^\infty \frac{1}{n^{-1}} \]
Important Problems

2. Evaluate the following integral:

\[ \int_1^{\infty} \frac{1}{x^p} \, dx \]

3. In problems 1-2, you may notice that the conditions for convergence of the integral depends on the value of the exponent \( p \) of \( x \). Find a value of \( p \) in (not used in question 1) that will cause \( \int_1^{\infty} \frac{1}{x^p} \, dx \) to converge. Then, find a value of \( p \) that causes the integral to diverge. Why does this happen?
Partial Solutions

1A. Both the integral and series converge.
1B. Both the integral and series diverge.
1C. Both the integral and series converge.
1D. Both the integral and series diverge.
Before Your Begin

Review the results of Homework 2-6 with your team and fill in the following blanks:

A series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when _______________________________ and diverges when _______________________________. These are called p-series.

With your team, discuss the difference between a sequence and a series. Write your thoughts here about the difference.

Also read the following statement with your team:

So far, there has been minimal concern for what the series converge to; only that the series converge, except for geometric series. In many cases, it is challenging to determine the numerical value a series converges to (cf. Advanced Problems in Problem Sets 2-1 through 2-4). However, determining whether a series converges or diverges is possible by comparing a series with known convergence or divergence to a series with an unknown status.
Start Here

1. Jason is filling an initially empty tank of water. The tank is filled in the following pattern: on the first day a gallon of water is poured in; on the second day, a quarter gallon of water is poured in; on the third day, a ninth of a gallon of water is poured in; this pattern continues ad infinitum (Ergene & Özdemir, 2020).

   A. Can Jason fill a 4-gallon water tank?

   B. Can Jason fill a 9-gallon water tank?

   C. What is the largest water tank (using tanks of only a whole numbers of gallons) Jason can fill? Why?

   D. Why are the solutions different from Becki’s version of this situation? (cf. Problem Set 2-6, Problem 1)
2. Compare the following integral and series.

A. Does \( \int_1^{\infty} \frac{1}{x^2} \, dx \) converge or diverge?

B. What does this tell you about the series \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converging/diverging?

C. It’s incorrect to say that \( \sum_{n=1}^{\infty} \frac{1}{n^2} = 1 \). Why?

3. Are these statements true or false? Why? [Hint: Is there a picture you could draw to convince yourself?]

A. \( \sum_{n=1}^{\infty} a_n \) converges, then \( k + \sum_{n=1}^{\infty} a_n \) converges, where \( k \) is any real number.

B. \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{10} b_n + \sum_{n=1}^{\infty} a_n \) converges, where \( (b_n) \) is a sequence of real numbers.

C. If \( \int_1^{\infty} f(x) \, dx \) converges, then \( \sum_{n=5}^{\infty} f(n) \) converges.
D. If $\int_{5}^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} f(n)$ converges.

4. Consider the following series:

$$1 + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \frac{1}{4\sqrt{4}} + \frac{1}{5\sqrt{5}} + \cdots$$

A. Predict whether this series converges or diverges. (Spend no more than 30 seconds on this.)

B. What is $\int_{1}^{\infty} \frac{1}{x^{\frac{3}{2}}}dx$?

C. What does the result from 3B tell you about the convergence or divergence of the series $\sum_{n=1}^{\infty} \frac{1}{n^{\frac{3}{2}}} = \sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$?
5. Compare the following integral and series.

A. Does \( \int_{2}^{\infty} \frac{1}{x \ln(x)} \) converge or diverge?

B. What does this tell you about the series \( \sum_{n=2}^{\infty} \frac{1}{n \ln n} \) = converging/diverging?

6. Does the following series converge or diverge? Why? [Hint: Does the numerator have a function that is always an upper or lower bound to it?]

\[
\sum_{n=1}^{\infty} \frac{|\sin(n)|}{n^2}
\]

7. Do the following series converge or diverge? Why?

A. \( \sum_{n=1}^{\infty} \frac{1}{n^3 + n} \)

B. \( \sum_{n=1}^{\infty} \frac{n}{n^3 + n} \)

C. \( \sum_{n=1}^{\infty} \frac{n^2}{n^3 + n} \)
Important Problems

8. Determine whether these statements are true or false:

A. If a series \( \sum a_n \) converges, then \( \frac{1}{5} \sum a_n \) also converges.

B. If a series \( \sum a_n \) converges, then \( 5 \sum a_n \) also converges.

C. If a series \( \sum a_n \) diverges, then \( k \sum a_n \) also converges for any real number \( k \).

9. Rank these series from least to greatest using term-by-term comparison.

\[
\sum_{n=2}^{\infty} \frac{5^n}{7^n} \quad \sum_{n=2}^{\infty} \frac{1}{e^n} \quad \sum_{n=2}^{\infty} \frac{1}{n^5} \quad \sum_{n=2}^{\infty} \frac{1}{\ln(n)} \quad \sum_{n=2}^{\infty} \frac{1}{n!}
\]

Advanced Problems

10. Prove the integral theorem (“test”) (for series): If \( \int_1^{\infty} f(n) \, dn \) converges, then \( \sum_{n=1}^{\infty} a_n \) does the same, where \( f(n) = a_n \) for positive integers \( n \). The same is true for divergence.

References


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Start Here

1. True or False: If \( \lim_{n \to \infty} a_n = 0 \), the associated series \( \sum a_n \) converges. Why?

2. Below are some general statements about series. Fill in the blanks to make the statement true.

- **The Divergence Theorem**: If the terms of a series \( \sum a_n \) have the property \( \lim_{n \to \infty} a_n \neq 0 \), the series ____________________.

- A geometric series \( \sum_{n=0}^{\infty} ar^n \) converges when ______________________ and diverges when ______________________. The geometric series \( \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r} \), where \( r \) is called the ______________________.

- A p-series \( \sum_{n=1}^{\infty} \frac{1}{n^p} \) converges when ______________________ and diverges when ______________________.

- Let \( k \) be a real number (constant). If a series \( \sum a_n \) converges, then \( k \sum a_n \) ______________________. Additionally, if a series \( \sum a_n \) diverges, then \( k \sum a_n \) ______________________.

- **The Integral Theorem I**: A series \( \sum a_n \) converges when a series \( \sum b_n \) (or integral \( \int b_n \, dn \)) ______________________ and \( \sum a_n \leq \sum b_n \) (or \( \sum a_n \leq \int b_n \, dn \)).

- **The Integral Theorem II**: A series \( \sum a_n \) diverges when a series \( \sum b_n \) (or integral \( \int b_n \, dn \)) ______________________ and \( \sum a_n \geq \sum b_n \) (or \( \sum a_n \geq \int b_n \, dn \)).
3. Consider the following two series: \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) and \( \sum_{n=1}^{\infty} \frac{2}{n^2} \).

A. Fill in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>50</th>
<th>500</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \frac{1}{n^2} )</td>
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<td></td>
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<tr>
<td>( \frac{2}{n^2} )</td>
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<tr>
<td>Ratio of terms</td>
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</tr>
</tbody>
</table>

B. What is \( \lim_{n \to \infty} \frac{\frac{2}{n}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{2n^2}{n^2} \)?

C. Does \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converge or diverge? Why?

D. Based on the results of 3B and 3C, does \( \sum_{n=1}^{\infty} \frac{2}{n^2} \) converge or diverge? Why?

E. What is \( \lim_{n \to \infty} \frac{\frac{1}{n}}{\frac{2}{n^2}} \)?

F. Does \( \sum_{n=1}^{\infty} \frac{2}{n^2} \) converge or diverge? Why?

G. Based on the results of 3E and 3F, does \( \sum_{n=1}^{\infty} \frac{1}{n^2} \) converge or diverge? Why?
4. Consider the following two series: \( \sum_{n=1}^{\infty} \frac{1}{n+4} \) and \( \sum_{n=1}^{\infty} \frac{1}{n} \).

A. Fill in the following table:

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
<tbody>
<tr>
<td>( \frac{1}{n+4} )</td>
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<tr>
<td>( \frac{1}{n} )</td>
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</tbody>
</table>

Ratio of terms

B. What is \( \lim_{n \to \infty} \frac{1}{n+4} \) ?

C. Does \( \sum_{n=1}^{\infty} \frac{1}{n} \) converge or diverge? Why?

D. Based on the results of 4B and 4C, does \( \sum_{n=1}^{\infty} \frac{1}{n+4} \) converge or diverge? Why?
5. Consider the series:

\[
\sum_{n=1}^{\infty} \frac{6n^2}{2n^3 - 5}
\]

Melissa and Richard demonstrate two different ways they think the series diverges.

**Melissa**

I know that if the integral on the interval \([2, \infty)\) converges or diverges, then so will the series. The function \(f(x) = \frac{6n^2}{2n^3 - 5}\) is positive for all \(n \geq 2\), continuous, and is always decreasing, so calculating this integral is OK.

\[
\int_{2}^{\infty} \frac{6x^2}{2x^3 - 5} \, dx = \lim_{a \to \infty} \int_{2}^{a} \frac{1}{u} \, du = \ln \left| \frac{2a^3 - 5}{11} \right| \to \infty
\]

Because the integral diverges, the associated series diverges.

A. Why are the conditions positive, continuous and decreasing necessary to calculate the integral?

B. Why is the conclusion that the series diverges?
Richard

With the exception of the first term of the series, all of the terms in \( \sum_{n=1}^{\infty} \frac{6n^2}{2n^3 - 5} \) are positive.

Looking at the numerator and denominator of terms in the series, the terms in the series seems to grow similarly to \( \sum_{n=1}^{\infty} \frac{1}{n} \). I know that \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges, since this is a p-series with \( p = 1 \), and p-series with \( p \leq 1 \) all diverge. What’s interesting is that the terms in the series \( \sum_{n=1}^{\infty} \frac{1}{n} \) seem to be only off by a multiple of \( \sum_{n=1}^{\infty} \frac{6n^2}{2n^3 - 5} \), as shown in the table below.

<table>
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<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>50</th>
</tr>
</thead>
<tbody>
<tr>
<td>6n^2</td>
<td>6</td>
<td>24</td>
<td>54</td>
<td>96</td>
<td>150</td>
<td>15000</td>
</tr>
<tr>
<td>2n^3 - 5</td>
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<td>49</td>
<td>123</td>
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<td>249995</td>
<td></td>
</tr>
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<td>1</td>
<td>1</td>
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</tr>
<tr>
<td>Ratio</td>
<td>-2</td>
<td>48</td>
<td>162</td>
<td>384</td>
<td>150</td>
<td>150000</td>
</tr>
</tbody>
</table>

It seems that the ratio of terms in each of the series approaches 3. To confirm this, I will calculate the ratio of the terms \( \lim_{n \to \infty} \frac{6n^2}{2n^3 - 5} = \lim_{n \to \infty} \frac{6n^3}{2 - \frac{5}{n^3}} = 3 \). Therefore, since \( \sum_{n=1}^{\infty} \frac{1}{n} \) diverges and \( \sum_{n=1}^{\infty} \frac{6n^2}{2n^3 - 5} \) is essentially, eventually, a scalar multiple of \( \sum_{n=1}^{\infty} \frac{1}{n} \), \( \sum_{n=1}^{\infty} \frac{6n^2}{2n^3 - 5} \) diverges.

C. Richard makes the claim that “the terms in the series seems to grow similarly to \( \sum_{n=1}^{\infty} \frac{1}{n} \)”. How does he know that?

D. Justify why \( \lim_{n \to \infty} \frac{6n^3}{2n^3 - 5} = \lim_{n \to \infty} \frac{6}{2 - \frac{5}{n^3}} \)
Important Problems

6. Often making the decision which of the theorems to use to determine series convergence or divergence is the most difficult. Based on the ideas developed thus far (summarized on page 1), determine which of these series converge or diverge. Explain why each series converges or diverges.

A. \[ \sum_{n=1}^{\infty} \frac{n}{n^3 + n - 1} \]

B. \[ \sum_{n=1}^{\infty} \frac{4^n}{5^n + 6} \]

C. \[ \sum_{n=1}^{\infty} \frac{8n}{n^2 + 7} \]

D. \[ \sum_{n=1}^{\infty} \frac{8n^2}{n^2 + 7} \]

E. \[ \sum_{n=1}^{\infty} \frac{3n}{e^n} \]

7. In determining whether \[ \sum_{n=1}^{\infty} \frac{1}{n^2 + 1} \] converges, Caleb decides to use the series \[ \sum_{n=1}^{\infty} \frac{1}{n^4} \], which he knows converges, and use a comparison of limits to determine convergence. When using the limit comparison method that Richard explained, Caleb finds the limit of terms approaches \( \infty \). What does this mean?

8. In determining whether \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n} + 7} \] converges, Elizabeth decides to use the series \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \], which she knows diverges, and use a comparison of limits to determine divergence. When using the limit comparison method that Richard explained, Elizabeth finds the limit of terms approaches \( \infty \). What does this mean?

Advanced Problems

Have you solved the previous advanced problems? Give those a try!
Before You Begin

You will want some graph paper for today’s problems! Make sure you have some nearby!

Start Here

1. Bailey is filling an initially empty, 2-gallon tank of water. The tank is filled in the following pattern: on the first day a gallon of water is poured in; on the third day, a third gallon of water is poured in; on the fifth day, a fifth gallon of water is poured in; this pattern continues ad infinitum. However, due to a water-sharing agreement, Kylie is able to take from Bailey’s water tank in the following way: on the second day, Kylie takes a half gallon of water out; on the fourth day, a fourth gallon of water; on the sixth day, a sixth gallon of water; this pattern continues ad infinitum. (Ergene & Özdemir, 2020).

Will Bailey be able to fill the 2-gallon water tank? Why or why not?
2. Consider the following series:

\[ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{1}{n} \]

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \]

A. What does \((-1)^{n+1}\) mean?

B. Calculate partial sums of the alternating harmonic series.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial sums of (\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n})</td>
<td>(1)</td>
<td>(\frac{5}{6})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

C. Is there a partial sum that acts as an upper bound for the sum of the series? If so, what is it? If not, why not?

D. Is there a partial sum that acts as a lower bound for the sum of the series? If so, what is it? If not, why not?

E. Make a plot of partial sums \(S_n\) on a vertical axis and the partial sum number \(n\) on the horizontal axis, with each ordered pair being \((n, S_n)\). One of your points should be \((3, \frac{5}{6})\). What do you notice? Why?

F. What is \(\lim_{n \to \infty} \frac{1}{n}\)? Does this tell you anything about either series’ convergence?
3. Consider the following series:

\[
\sum_{n=1}^{\infty} \frac{1}{n^2} \quad \text{and} \quad \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}
\]

B. Calculate partial sums of the \textbf{alternating series}.

<table>
<thead>
<tr>
<th>(n)</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>Partial sums of (\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2})</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

C. Is there a partial sum that acts as an upper bound for the sum of the series? If so, what is it? If not, why not?

D. Is there a partial sum that acts as a lower bound for the sum of the series? If so, what is it? If not, why not?

E. Make a plot of partial sums \((S_n)\) on a vertical axis and the partial sum number \((n)\) on the horizontal axis, with each ordered pair being \((n, S_n)\). One of your points should be \((3, \frac{5}{6})\). What do you notice? Why?

F. What is \(\lim_{n \to \infty} \frac{1}{n^2}\)? Does this tell you anything about either series’ convergence?
4. Investigate the following pairs of series. Does each series converge or diverge? How do you know?

A. \( \sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1} \) and \( \sum_{n=2}^{\infty} \frac{(-1)^n n^2}{n^3 + 1} \)

B. \( \sum_{n=1}^{\infty} n \) and \( \sum_{n=1}^{\infty} (-1)^n n \)

C. \( \sum_{n=0}^{\infty} \frac{e^n}{2^{n+1}} \) and \( \sum_{n=0}^{\infty} (-1)^n \frac{e^n}{2^{n+1}} \)
Important Problems

5. Consider the convergent alternating harmonic series introduced in problem 2. After the partial sum of 10 terms, how far off from the true, convergent sum do you imagine the series to be? Why is this true?

6. Cecily and Jeffrey are both playing “halfsies” again, but with some slightly different rules. The amount that each sibling is allocated on their turn is half of what the other sibling’s turn. On Cecily’s turns, she will take away what is allocated to her away from Jeffrey, instead of from the original sandwich; Jeffrey always takes from the original sandwich. Fill out the following table:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>J</th>
<th>C</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>10</th>
<th>100</th>
<th>∞</th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1/2</td>
<td>1/4</td>
<td>1</td>
<td>3</td>
<td>8</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>How much of the sandwich the sibling was allocated</td>
<td>1</td>
<td>1/2</td>
<td>4</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much of the sandwich belongs to Jeffrey</td>
<td>1/2</td>
<td>1/4</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much of the sandwich belongs to Cecily</td>
<td>0</td>
<td>1/4</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much of the sandwich is unallocated</td>
<td>1</td>
<td>1/2</td>
<td>3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

What patterns do you notice after filling in this table? Why do these patterns occur?
Advanced Problems

7. What is the exact value of \( \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \)? Prove that your result is correct.

Copyright

Problem 1 on this problem set is adapted under a Creative Commons License: CC-BY-NC-4.0, available from https://creativecommons.org/licenses/by-nc/4.0/. The original problem can be seen in Ergene and Özdemir (2020).

References

Start Here

1. What makes an alternating series alternating? Give an example of an alternating and non-alternating series.

2. Investigate the following pairs of series. Does each series converge or diverge? How do you know?

A. \[ \sum_{n=2}^{\infty} \frac{n^2}{n^3 + 1} \quad \text{and} \quad \sum_{n=2}^{\infty} \frac{(-1)^n n^2}{n^3 + 1} \]
B. \[ \sum_{n=1}^{\infty} n \text{ and } \sum_{n=1}^{\infty} (-1)^n n \]

C. \[ \sum_{n=0}^{\infty} \frac{e^n}{2^n+1} \text{ and } \sum_{n=0}^{\infty} (-1)^n \frac{e^n}{2^n+1} \]
Important Problems

3. Consider the convergent alternating harmonic series introduced in Problem Set 2-9, Problems 1 and 2. After the partial sum of 10 terms, how far off from the true, convergent sum do you imagine the series to be? Why is this true?

4. Cecily and Jeffrey are both playing “halvesies” again, but with some slightly different rules. The amount that each sibling is allocated on their turn is half of what the other sibling’s turn. On Cecily’s turns, she will take away what is allocated to her away from Jeffrey, instead of from the original sandwich; Jeffrey always takes from the original sandwich. Fill out the following table:

<table>
<thead>
<tr>
<th>Whose turn it is</th>
<th>J</th>
<th>C</th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Turn #</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich the sibling was allocated</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>How much of the sandwich belongs to Jeffrey</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
<td>1/2</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich belongs to Cecily</td>
<td>0</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
<td>1/4</td>
</tr>
<tr>
<td>How much of the sandwich is unallocated</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
<td>1/2</td>
</tr>
</tbody>
</table>

What patterns do you notice after filling in this table? Why do these patterns occur?

---

1Modeling this situation by playing the game with piece of paper may be helpful!
Advanced Problems

5. What is the total area shaded below, assuming the nested pattern continues ad infinitum? You may assume that:

- the triangles are equilateral and are inscribed in circles, with the largest circle being of diameter 1 unit; and,
- circles are tangent to the respective sides of equilateral triangles.
Start Here

You are team #

0A. If you’re in a group of larger than three today, partition your group into smaller groups of no more than three. (For example, a group of five could be split into a group of two and a group of three.)

0B. List the members of your team.

1. With your team, I invite you to create your own problem regarding infinite series convergence for inclusion in tomorrow’s problems. When you design your problem, it should be designed so that the majority of students in the class are able to solve it. Do not try to create a “hard” problem in the sense of trying to have a low percentage of students be able to solve it. Spend about 10-15 minutes maximum developing your problem and its solution. Once both the problem and its solution are complete, submit them to Canvas under the submission link for today. I will review the problems, edit them as needed, and then Session 2-12 will be designed so we all solve each other’s problems. Happy creating! (Use the back of this sheet if needed)
Appendix C

Assessments
Determine whether these statements are true or false. Justify your responses using mathematical reasoning, illustrating the reasoning with specific examples where applicable.

A. An infinite series can sum to a finite value.  
True False

B. The series associated with \( a_n = \frac{n}{n+1} \), with \( n \geq 1 \), is \((\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \cdots)\).  
True False

C. \( \sum_{n=1}^{\infty} 3 \) converges.  
True False
<p>| | | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>D.</strong> The sequence ( \frac{1}{2^n} ) converges.</td>
<td><strong>False</strong></td>
<td></td>
</tr>
<tr>
<td><strong>E.</strong> A geometric series is always convergent.</td>
<td><strong>False</strong></td>
<td></td>
</tr>
<tr>
<td><strong>F.</strong> ( \frac{3}{2} &gt; \sum_{n=0}^{\infty} \frac{1}{3^n} ).</td>
<td><strong>True</strong></td>
<td></td>
</tr>
<tr>
<td><strong>G.</strong> If ( \lim_{n \to \infty} a_n \neq 0 ), then the associated series ( \sum_{n=1}^{\infty} a_n ) diverges.</td>
<td><strong>False</strong></td>
<td></td>
</tr>
</tbody>
</table>
Objectives Distribution

<table>
<thead>
<tr>
<th>I-1</th>
<th>1B</th>
</tr>
</thead>
<tbody>
<tr>
<td>I-2</td>
<td>1D</td>
</tr>
<tr>
<td>I-3</td>
<td>1A, 1F</td>
</tr>
<tr>
<td>I-4</td>
<td>1C, 1G</td>
</tr>
<tr>
<td>I-5</td>
<td>1E</td>
</tr>
<tr>
<td>IV-4</td>
<td>*</td>
</tr>
<tr>
<td>IV-5</td>
<td>**</td>
</tr>
</tbody>
</table>

* The vocabulary words listed are used throughout the assessment, so no specific questions of vocabulary are necessary.

** Correct and incorrect instances of sigma notation will be noted throughout the assessment. There is no one specific question that assesses this standard, as it is a fluency standard.
1. Determine whether these statements are true or false and circle your response. Justify each of your responses using mathematical reasoning, illustrating the reasoning with specific examples or proof as appropriate.

True  False  A. \( 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots \) converges.

True  False  B. The series \( \sum_{n=1}^{\infty} \frac{1}{2n} \) converges.

True  False  C. For a sequence \((a_n)\), if \( \lim_{n \to \infty} a_n = 0 \), then \( \sum_{n=1}^{\infty} a_n \) converges.
2A. Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{x}}$ converge or diverge? Describe why your answer is correct.

2B. Which of the following figures justifies your response in prompt 2A? Describe mathematically how the picture you chose connects to your response in prompt 2A.
3A. Give an example of a series that is always greater than, term-by-term, \( \sum_{n=1}^{\infty} \frac{3}{n + 100} \). Demonstrate why your series is greater than \( \sum_{n=1}^{\infty} \frac{3}{n + 100} \).

3B. Does the series you created in prompt 3A converge or diverge. Use one of the series theorems to describe why your answer makes sense.

3C. Will the series you created in prompt 3A and your work in prompt 3B help determine the convergence of \( \sum_{n=1}^{\infty} \frac{3}{n + 100} \)? Why or why not?
Objectives Distribution

<table>
<thead>
<tr>
<th>II-1</th>
<th>I, B, 2A</th>
</tr>
</thead>
<tbody>
<tr>
<td>II-2</td>
<td>1C</td>
</tr>
<tr>
<td>II-3</td>
<td>2B</td>
</tr>
<tr>
<td>II-4</td>
<td>1D, 3A</td>
</tr>
<tr>
<td>II-5</td>
<td>1A, 3C</td>
</tr>
<tr>
<td>IV-1</td>
<td>3B</td>
</tr>
<tr>
<td>IV-4</td>
<td>*</td>
</tr>
<tr>
<td>IV-5</td>
<td>**</td>
</tr>
</tbody>
</table>

* The vocabulary words listed are used throughout the assessment, so no specific questions of vocabulary are necessary.

** Correct and incorrect instances of sigma notation will be noted throughout the assessment. There is no one specific question that assesses this standard, as it is a fluency standard.
1. Determine whether these statements are true or false. Justify your responses using mathematical reasoning, illustrating the reasoning with specific examples where applicable.

A. \[ \sin \left( \frac{\pi}{2} \right) + \sin \left( \frac{3\pi}{2} \right) + \sin \left( \frac{5\pi}{2} \right) + \cdots \] is an alternating series.

B. If \[ \sum_{n=1}^{\infty} |b_n| \] converges, then \[ \sum_{n=1}^{\infty} b_n \] converges.

C. If \[ \sum_{n=1}^{\infty} |b_n| \] converges, then \[ \lim_{n \to \infty} b_n = 0. \]
D. An absolutely convergent series is also conditionally convergent.

2. List the hypotheses of three of the following theorems (tests) to determine whether \( \sum_{n=1}^{\infty} a_n \) converges.

- P-series theorem
- Direct comparison theorem
- Limit comparison theorem
- Alternating series theorem
- Integral theorem

Choice for Theorem 1: ____________________________
Hypotheses:

Choice for Theorem 2: ____________________________
Hypotheses:

Choice for Theorem 3: ____________________________
Hypotheses:
3. Determine whether the following series conditionally converges, absolutely converges, or diverges. Describe which theorems you used to make this determination and how you applied them (including the checking of hypotheses).

\[
\sum_{n=2}^{\infty} \frac{(-1)^n n}{\sqrt{n^4 - 1}}
\]
## Objectives Distribution

<table>
<thead>
<tr>
<th>III-1</th>
<th>1A</th>
</tr>
</thead>
<tbody>
<tr>
<td>III-2</td>
<td>1D</td>
</tr>
<tr>
<td>III-3</td>
<td>1B, 1C</td>
</tr>
<tr>
<td>III-4</td>
<td>1D</td>
</tr>
<tr>
<td>III-5</td>
<td>[Not Assessed]</td>
</tr>
</tbody>
</table>

| IV-1 | 3 |
| IV-2 | 2 |
| IV-3 | 3 |
| IV-4 | * |
| IV-5 | ** |

* The vocabulary words listed are used throughout the assessment, so no specific questions of vocabulary are necessary.

** Correct and incorrect instances of sigma notation will be noted throughout the assessment. There is no one specific question that assesses this standard, as it is a fluency standard.
The purpose of this performance assessment is to help you apply content learned about infinite sequences and series to a problem you have not seen before. When working on these prompts, you are:

- Welcome to engage in solving these prompts with peers from the course;
- Required to submit your own solution;
- Encouraged to use the course materials provided (i.e. consider this an “open note” assessment).

When writing up your solution, you are encouraged to submit your solution using typewriting software (i.e. Microsoft Word, LaTeX). However, this is not required. Your solution needs to be submitted using fewer than five (5) pages. PDF files are encouraged.

The following prompts will be scored as $\frac{1}{4}$ of your overall second midterm score (5% of the overall grade), with the second midterm exam counting $\frac{3}{4}$ (15% of the overall grade).
Performance Assessment Task

The following pattern is an example of an infinite pattern called a fractal.

Task 1

What fraction of the original white square is shaded black if this pattern continues to stage infinity? Describe mathematically how you came to your conclusion.
Task 2

Consider what happens to the fractal in Task 1 if the second stage changes:

Stage 0 Stage 1
Stage 2 Stage 3

What fraction of the original white square is shaded black if this pattern continues to stage infinity? Describe mathematically how you came to your conclusion.
Appendix D

File Sorting Computer Code

Once data had been downloaded from Canvas, student work was sorted by student rather than by assignment and put into chronological order by file. The code here performed this process. The computer code included here is written in Ruby.

```ruby
# Sorting and Coding Files
# by Zachary Coverstone
require 'fileutils'
def find_code_name(file_name, code_name_hash)
  file_name_name = file_name.split('_')[0]
  return code_name_hash[file_name_name]
end
Dir.chdir('files')
# Load the code names
code_name_file = open('../code_names.csv', 'r')
code_names = {} for i in code_name_file
  my_array = (i.split('n')[0]).to_s.split(',')
  code_names[my_array[1] + my_array[0].downcase.delete('')] = my_array[2]
  FileUtils.mkdir '../sorted_files/' + my_array[2]
end
# Get all the files
all_the_files = Dir.glob('**') for i in all_the_files
  # Only read the directories
  if Dir.exists?(i)
    Dir.chdir(i)
    file_list = Dir.glob('**').sort
    for j in file_list
      # Read the files in the directory
      Dir.chdir(i)
      file_name = find_code_name(j, code_names)
      # If code name does not exist, then put the file in 'other' folder
      if file_name == nil
        # Do nothing
      else
        max_extension = j.split('.').length - 1
        extension = '.' + j.split('.').max_extension_length
        new_file_name = '../sorted_files/' + file_name + '/' + i + 'n'
        counter = Dir.glob(new_file_name + '*').length + 1
        new_file_name = new_file_name + counter + extension
        FileUtils.cp j, new_file_name
      end
    end
    Dir.chdir('..')
  else
    end
end
```