A Study of Electron Plasma Oscillations Using the NIMROD Code

McKay Murphy
Utah State University

Follow this and additional works at: https://digitalcommons.usu.edu/gradreports

Part of the Physics Commons

Recommended Citation
https://digitalcommons.usu.edu/gradreports/1690

This Report is brought to you for free and open access by the Graduate Studies at DigitalCommons@USU. It has been accepted for inclusion in All Graduate Plan B and other Reports by an authorized administrator of DigitalCommons@USU. For more information, please contact digitalcommons@usu.edu.
A Study of Electron Plasma Oscillations Using the NIMROD Code

A plan B Masters Report by

McKay Murphy

Physics Department
Utah State University
August 2022
A Study of Electron Plasma Oscillations Using the NIMROD Code

October 12, 2022

1 Introduction

A plasma, whether "hot" or "cold," magnetized or unmagnetized, electrostatic or electromagnetic, exhibits normal modes of oscillation. Critical to understanding the stability of a plasma is the study of these normal modes. Waves originate from the long-range electric interactions between charged particles. This work will consider a class of waves known as Langmuir waves. These waves occur when a group of electrons are displaced with respect to the ions, with the electric Coulomb force playing the role of the restoring force of the oscillation [1]. Since the mass of ions is much greater than that of electrons, we can approximate the ions as stationary.

This work will begin by developing a set of equations based on our assumptions. We will derive the characteristic plasma frequency equation while ignoring pressure effects, an assumption used in studying cold plasmas. Using numerical methods with NIMROD [2], we will show how solutions converge in the spatial dimension. We then expand upon the plasma frequency equation by including the pressure gradient in the equation of motion. This results in a dispersion relation for electron plasma oscillations known as Langmuir waves. The work will conclude with numerical solutions for this dispersion relation being compared to the expected theoretical dispersion relation.

2 Plasma Frequency and the Vlasov Equations

Anatoly Vlasov, an early pioneer in plasma physics, derived a set of equations used to characterize plasmas [3]. Plasmas at the time were described using the kinetic theory of gasses, where collisions are elastic and particles are distributed via the Boltzmann function. Experimental results disagreed with this approach and Vlasov theorized this stemmed from the long-range nature of Coulomb interactions. Thus, Vlasov’s approach sought to improve on the existing model by accounting for this long-range interaction. We follow his process briefly before deriving our desired plasma equations.

2.1 Derivation of the Dispersion Relation

We begin with a kinetic approach which uses independent velocity \(v\) and position \(r\) coordinates, as well as \(t\). The state of the plasma is thus governed by a distribution function \(f_a(r,v,t)\) where \(a\) denotes the species of the particles (electrons, ions, etc.). The distribution function is defined such that

\[
dN = f_a(r,v,t) \, d^3x \, d^3v
\]

is the probable number of particles in a certain phase space volume \(d^3x \, d^3v\). The time evolution of this distribution function is described using the Boltzmann equation:

\[
\frac{\partial f_a}{\partial t} + v \cdot \frac{\partial f_a}{\partial r} + \frac{q_a}{m_a} (E + v \times B) \cdot \frac{\partial f_a}{\partial v} = C(f_a)
\]

where \(C(f_a)\) is the collision operator. We have assumed here that the dominant force amongst particles is the Lorentz force. The collision operator accounts for the rate of change of the distribution function caused by particle collisions. It is important to note that solving for the distribution function \(f_a\) is
complicated for even a two species plasma. Vlasov recognized this and his solution consisted of taking velocity moments of the Boltzmann equation to transition from a microscopic to a macroscopic view of the plasma. Taking the zeroth moment of Eq. 2 and integrating over all velocity space yields the continuity equation

$$\frac{\partial n_a}{\partial t} + \nabla \cdot \mathbf{U}_a n_a = 0.$$  
(3)

The first moment (weighted by $\mathbf{U}$ of Eq. 2 yields the momentum equation

$$m_a n_a \left( \frac{\partial \mathbf{U}_a}{\partial t} + \mathbf{U}_a \cdot \nabla \mathbf{U}_a \right) = q_a n_a (\mathbf{E} + \mathbf{U}_a \times \mathbf{B}) - \nabla P_a - \nabla \cdot \mathbf{\Pi}_a + \mathbf{R}_a.$$  
(4)

Here, $\mathbf{U}_a$ is the flow velocity defined as

$$n_a \mathbf{U}_a = \int f_a \mathbf{v} \, dv.$$  
(5)

The first moment gives rise to multiple new terms including the pressure scalar $P_a$, the stress tensor $\mathbf{\Pi}_a$, and the friction force density $\mathbf{R}_a$.

Like Vlasov, we are considering a quasi-neutral ($\sum_a q_a n_{a0} = 0$) plasma where collisions are negligible [3]. Furthermore, we will consider electrostatic oscillations along $\mathbf{B}$ such that $\mathbf{U}_a \times \mathbf{B} = 0$. Thus, the terms involving $\mathbf{B}$ and $\mathbf{R}_a$ are discarded resulting in

$$m_a n_a \left( \frac{\partial \mathbf{U}_a}{\partial t} + \mathbf{U}_a \cdot \nabla \mathbf{U}_a \right) = q_a n_a \mathbf{E} - \nabla P_a - \nabla \cdot \mathbf{\Pi}_a.$$  
(6)

Although the stress tensor can have a damping effect on Langmuir oscillations, it will be left out in this study. Assuming isotropic pressure means $\nabla \cdot \mathbf{\Pi}_a = 0$. Neglecting collisions allows us to treat the plasma as an adiabatic gas; energy is only transferred via work. We can thus use the adiabatic equation of state:

$$P = P_0 \left( \frac{n_a}{n_{a0}} \right)^{\gamma_a}.$$  
(7)

The factor $\gamma$ is used for an ideal adiabatic gas and is equal to $\gamma = \frac{n+2}{n}$ where $n$ represents the number of degrees of freedom. In addition to Eqs. 3, 6, and 7, Gauss’s law with the substitution $\rho = \sum_a n_a q_a$ is also used:

$$\nabla \cdot \mathbf{E} = \frac{1}{\epsilon_0} \sum_a n_a q_a.$$  
(8)

We assume that the resulting wave will have a small amplitude. This allows us to linearize as follows:

$$P = P_0 + P_1$$  
(9)

$$n_a = n_{a0} + n_{a1}$$  
(10)

$$\mathbf{U}_a = \mathbf{U}_{a1}$$  
(11)

$$\mathbf{E} = \mathbf{E}_1$$  
(12)

$$\mathbf{B} = \mathbf{B}_1.$$  
(13)

Here, $\mathbf{E}_1$ accounts for the oscillations in the electric field. We make these substitutions into Eqs. 3, 6, 7, and 8 and assume oscillations along the constant, uniform magnetic field $\mathbf{B}_0$ to obtain

$$\frac{\partial n_{a1}}{\partial t} + n_{a0} \nabla \cdot \mathbf{U}_{a1} = 0,$$  
(14)

$$m_a n_{a0} \left( \frac{\partial \mathbf{U}_{a1}}{\partial t} \right) = q_a n_{a0} \mathbf{E}_1 - \nabla P_{a1},$$  
(15)
\[ P_{a1} = T_{a0} \gamma_a n_{a1}, \quad (16) \]

and

\[ \nabla \cdot E_1 = \frac{1}{\epsilon_0} \sum_a n_{a1} q_a. \quad (17) \]

Equation 16 introduces temperature by means of the ideal gas law, \( P_a V = n_a k_B T_a \) where \( n_a = \frac{N_a}{V} \).

The adiabatic equation of state allows us to write:

\[ m_a n_a \left( \frac{\partial U_{a1}}{\partial t} \right) = q_a n_a E_1 - \nabla T_{a0} \gamma_a n_{a1}. \quad (18) \]

We now take the Fourier transforms of our remaining equations. This is affected by writing perturbed quantities as \( \exp(i \mathbf{k} \cdot \mathbf{r} - \omega t) \). The resulting system is

\[ -i \omega n_{a1} + i n_{a0} \mathbf{k} \cdot \mathbf{U}_{a1} = 0 \quad (19) \]

\[ -i \omega m_a n_{a0} \mathbf{U}_{a1} = q_a n_{a0} E_1 - i k T_{a0} \gamma_a n_{a1} \quad (20) \]

\[ i \mathbf{k} \cdot \mathbf{E}_1 = \frac{1}{\epsilon_0} \sum_a n_{a1} q_a. \quad (21) \]

Thus \( n_{a1}, \mathbf{U}_{a1}, \) and \( E_1 \) now represent the Fourier transform quantities of themselves. Reducing these three transformed equations further, we take \( \mathbf{k} \cdot \mathbf{U}_{a1} \) of Eq. 20:

\[ i n_{a0} \mathbf{k} \cdot \mathbf{U}_{a1} = -\frac{q_a n_{a0}}{m_a} \mathbf{k} \cdot \mathbf{E}_1 + \frac{i k^2 T_{a0} \gamma_a n_{a1}}{m_a} \quad (22) \]

which allows substitution into Eq. 19,

\[ -i \omega^2 n_{a1} m_a - q_a n_{a0} \mathbf{k} \cdot \mathbf{E}_1 + i k^2 T_{a0} \gamma_a n_{a1} = 0. \quad (23) \]

Our final two equations can be combined by solving Eq. 23 for \( n_{a1} \), thus yielding

\[ \mathbf{k} \cdot \mathbf{E}_1 = \left[ \sum_a \frac{\omega^2}{\omega^2 - k^2 T_{a0} \gamma_a} \right] \mathbf{k} \cdot \mathbf{E}_1. \quad (24) \]

We are now left with the resulting general dispersion relation for collisionless, electrostatic plasma oscillations along a uniform magnetic field [4]. Here, \( \omega_{pe} \) is defined as \( \omega_{pe} = \sqrt{\frac{n_e e^2}{m_e \epsilon_0}} \) which is the electron plasma frequency. In the limit of stationary ions \( (m_i \rightarrow \infty) \), and a cold plasma \( T = 0 \), the electrons oscillate back and forth harmonically at \( \omega_{pe} \).

### 2.1.1 Derivation of Langmuir Waves

Ignoring any ion motion allows us to simplify Eq. 24 further. By setting \( \alpha = e, \gamma = 3 \) for a 1-dimensional case and seeking only non-trivial solutions, we arrive the following equation [1]:

\[ \omega^2 = \omega_{pe}^2 + 3 k^2 v_{e,th}^2 \quad (25) \]

where the electron thermal speed squared is defined as \( v_{e,th}^2 = \frac{k_B T_e}{m_e} \) and the electron frequency as \( \omega_{pe}^2 = \frac{n_e e^2}{\epsilon_0 m_e} \). This result is our desired dispersion relation. We can see from this equation that in addition to the plasma frequency, we have a quadratic dependence on the wavenumber, \( k \), which arises from the pressure gradient term in the electron momentum equation.
Note that the plot approaches the asymptote \( \frac{\omega}{k} = \sqrt{\frac{3k_B T_e 0}{m_e}} \), or the group velocity. Also of note is the case when \( k = 0 \), or the “cold electron” case. If the wavenumber is zero, or if the thermal velocity is zero (such as for an electron plasma with temperature \( T_e 0 = 0 \)), we are left with simply \( \omega = \omega_{pe} \). Both hot and cold plasmas are of interest to us and we will use numerical methods to verify our results.

### 2.2 Nondimensionalization of the Linearized Plasma Equations

We now seek methods of verifying our results numerically. This work will use the NIMROD (Non-Ideal Magnetohydrodynamics with Rotation, Open Discussion) code \[2\], which utilizes a finite-element approach to solving the plasma fluid equations. To verify Eq. 25 with NIMROD, we will first simplify it further through nondimensionalization. Nondimensionalization is a popular method of simplification that helps identify characteristic lengths and timescales. In our case, we select a dimensionless unit to use for our two dependent variables; \( \omega \) and \( k \). An ideal choice is to use units of inverse Debye length for \( k \) and plasma frequency for \( \omega \). The Debye length is defined as

\[
\lambda_{De}^2 = \frac{\epsilon_0 k_B T_e 0}{n_e q_e^2}.
\]

(26)

We can now make a substitution of nondimensionalized units into our original Vlasov equations and follow a similar process to that of obtaining Eq. 25. The resulting nondimensionalized equations will be of similar form to this, but with the solution in terms of our selected variables.

We begin our nondimensionalization with our definition of nondimensional units for \( t \) and \( x \):

\[
\bar{t} = \frac{\omega_{pe} t}{\lambda_{de}},
\]

\[
\bar{x} = \frac{x}{\lambda_{de}} = \frac{x}{L}.
\]

(27)

where \( \bar{x}, \bar{t} \) denote the normalized variables. We make a similar definition for \( U_{a1} \) as well, where we use the definition of thermal velocity \( v_{a,th} = \sqrt{\frac{k_B T_a}{m_a}} \).

\[
U_{a1} = \frac{U_{a1}}{v_{a,th}}.
\]

(28)
We make these substitutions into Eq. 29, resulting in
\[ \omega_{pe} n_0 \frac{\partial \mathbf{U}_1}{\partial t} + \frac{n_0 v_{th}}{L} \nabla \cdot \mathbf{U}_1 = 0. \] (29)

The remaining linearized plasma equations follow the same method and we are left with a set of linearized, nondimensionalized plasma equations:
\[ m_a n_0 v_{th} \omega_{pe} \left( \frac{\partial \mathbf{U}_1}{\partial t} \right) = -\frac{q_a^2 n_{a0}}{4\pi \epsilon_0 L \lambda} \mathbf{E} - \frac{1}{L} \nabla P_{a1} \] (30)
\[ P_{a1} = n_0 T_{a0} \gamma_a \pi_{a1} \] (31)
\[ \frac{\omega_{pe} q_e}{4\pi \epsilon_0 L^3} \nabla \cdot \mathbf{E}_1 = \frac{n_0}{\epsilon_0} \sum_a \pi_{a1} q_a. \] (32)

Several of these terms can be reduced for simplicity. We will also take their Fourier transform, now with \( \nabla = i \mathbf{k} \) and \( \frac{\partial}{\partial t} = -i \omega \), such that
\[ -\omega \omega_{pe} n_1 + \frac{v_{th}}{L} \mathbf{k} \cdot \mathbf{U}_1 = 0. \] (33)
\[ m_a n_0 v_{th} \omega_{pe} \mathbf{U} = \frac{i q_a^2}{4\pi \epsilon_0 L \lambda} \mathbf{E} + \frac{i}{L} n_{a0} \mathbf{k} P_{a1} \] (34)
\[ P_{a1} = n_0 T_{a0} \gamma_a \pi_{a1} \] (35)
\[ \frac{1}{4\pi L^3} \nabla \cdot \mathbf{E}_1 = n_0 \pi_1. \] (36)

We can eliminate \( P_{a1} \) by combining Eqs. 35 and 34 to obtain
\[ -im_a v_{th} \omega_{pe} \mathbf{U} = \frac{q_a^2}{4\pi \epsilon_0 L} \mathbf{E} + \frac{T_{a0} \gamma_a}{L} \mathbf{k} \pi_{a1}. \] (37)

We take the dot product of \( \mathbf{k} \) and Eq. 37, allowing for substitution into Eq. 33:
\[ \mathbf{k} \cdot \mathbf{U} = \frac{1}{im_a v_{th} \omega_{pe}} \left( \frac{q_a^2}{4\pi \epsilon_0 L} \mathbf{k} \cdot \mathbf{E} + \frac{T_{a0} \gamma_a^2}{L} \mathbf{k} \pi_{a1} \right) \] (38)

and
\[ \omega_{pe} n_0 \pi_1 = \frac{n_0}{L m_a \omega_{pe}} \left( \frac{q_a^2}{4\pi \epsilon_0 L} \mathbf{k} \cdot \mathbf{E} + \frac{T_{a0} \gamma_a^2}{L} \mathbf{k} \pi_{a1} \right) \] (39)

This can be solved for \( n_0 \pi_1 \) and substituted into our dedimensionalized Gauss’s Law equation:
\[ \frac{1}{4\pi L^3} \mathbf{k} \cdot \mathbf{E}_1 = \left( \omega_{pe} - \frac{1}{L m_a \omega_{pe}} \frac{T_{a0} \gamma_a}{L} \right)^{-1} \frac{n_0}{L m_a \omega_{pe}} \frac{q_a^2}{4\pi \epsilon_0 L} \mathbf{k} \cdot \mathbf{E} \] (40)

Similar to our initial dispersion relation derivation, we desire only non-trivial solutions to Eq. 40. Luckily, the solution becomes incredibly simplified when we incorporate our definitions for \( \omega_{pe}, \lambda_{De}, \) and \( v_{th} \). The result is:
\[ \omega^2 = \frac{\omega_{pe}^2}{\omega_{pe}^2} = 1 + c \gamma k^2 \] (41)
where \( c \) is a constant that depends on the definition of the thermal speed. Our dimensionalized oscillation equation is complete and provides solutions in terms of our previously picked units, namely, the plasma frequency \( \omega_{pe} \) and \( \lambda_{De} \). We are now ready to implement this into NIMROD for numerical analysis, which will be covered in the next section. Note it is Eqs. 29 - 32 that NIMROD is solving to verify both the plasma frequency and the dispersion relation for Langmuir oscillations.
3 NIMROD Plasma Simulations of Electron Oscillations

This section will demonstrate NIMROD’s ability in simulating plasma oscillations. We focus first on a specific type of oscillation in which the plasma is ‘cold,’ or where \( T_e = 0 \). This specific solution results in the plasma frequency being equal to the electron frequency (see Eq. 25 with \( T = 0 \)).

We will begin with a test of spatial convergence of NIMROD’s plasma frequency solutions. This is then followed with a demonstration of how NIMROD evolves the plasma frequency by stepping electron density, \( n_e \), and flow velocity, \( U \), forward in time. We conclude the section by comparing the dispersion relation of Eq. 41 to that of NIMROD’s numerical solutions.

3.1 spatial Convergence of Plasma Frequency Calculations

Our previous derivation of the plasma oscillation equation, and hence the plasma frequency equation, will be used in NIMROD as an important benchmark in this work. We will show that varying the spatial grid in NIMROD results in converging solutions for the plasma frequency. This is done through two parameters; the number of finite elements, \( m_y \), and the polynomial order used between grid points, \( \text{poly
degree} \). These quantities were varied in the direction of the initial perturbations in the plasma, which is listed as \( V_z \) but is physically our \( y \)-direction. Each run of NIMROD varied these parameters and the results are shown below.

![Figure 2: spatial convergence of \( \omega_{pe} \) calculations. The variable \( h \) is defined as the spacing between points and the error term is the absolute value between the experimental and theoretical plasma frequencies.](image)

As to be expected for any polynomial degree, the solution is more accurate at a higher number of grid cells, \( m_y \). The rate of convergence is directly related to the slope of each individual line, proof that an increased polynomial degree further increases the rate of convergence. A final point of interest in the comparison in convergence rate between polynomial degree changes and spatial cell size. Polynomial degree convergence appears to happen exponentially, as can be seen in the distance between points of equal \( \ln(h) \). Conversely, convergence from increasing spatial grid appears linear. This is supported further through hp-FEM theory, in which B. A. Szabó and A. K. Mehta show that their numerical results converge ”rapidly in finite element approximations in which the finite element mesh is fixed and the order of polynomial displacement interpolations is increased” [5].

Whereas the above simulations showed how accurate NIMROD’s solutions are, it’s of equal import to demonstrate how NIMROD runs on a single plasma frequency case. As previously stated, the
spatial grid was varied in the direction of our initial perturbations. Depicted below is the flow velocity perturbation, $V_z$, along this axis at time $t = 0$.

![Image of Re Vz Along Slice](image)

Figure 3: Initial electron velocity $V_z$ perturbations at $t = 0$. This run used $dt = 1e-13$, $my = 64$, first order polynomial degree, and $ny = 4$ which yields two full Sine waves in the periodic $y$ direction.

As seen from our previous plasma frequency derivation, the electron density, $n_e$, directly controls the plasma frequency. NIMROD evolves both $V_z$ and $n_e$ forward in time. Depicted below is the resulting change in $V_z$ and $n_e$ as time progresses.

![Image of Re Vz vs. t](image)

Figure 4: Electron flow velocity time evolution using the same setup as the previous test. As we expect, the electron flow velocity oscillates at the frequency $w_{pe}$. Here $V_z(t)$ was taken at $iy/my = 0.25$ in Fig. 3.
Figure 5: Electron density evolution from the same simulation as Fig. 4.

The theoretical plasma frequency for the parameters in this simulation is $5.64 \times 10^{11}$ rad/s, consistent with our above results in Figures 4 and 5. A final note of significance, the timestep $dt$ was converged properly for both our spatial convergence and single case simulations. The resulting period from our plasma frequency is $\frac{2\pi}{\omega_{pe}} = 1.11 \times 10^{-11}$ s is much longer than our timestep, $dt = 1 \times 10^{-13}$ s, thus justifying our assertion of time convergence.

3.2 Plasma Oscillation calculations

Recall from Eq. 25 that the squared oscillation frequency of a warm plasma with only the electrons moving depends quadratically upon the wavenumber $k$. Additionally, we previously dedimensionalized this equation for easier implementation in NIMROD. To simulate points on the dispersion relation in NIMROD, we gradually increase the initial electron wavenumber using $\frac{\pi}{\lambda}$ as the number of wavelengths that fit the direction of propagation. We pair the results of NIMROD with our expected theoretical values in the below figure.

Figure 6: Comparison of theoretical data (blue dotted line) to NIMROD’s solutions (red points). The $R^2$ value listed is a measure of how well the numerical data matches the theoretical dispersion curve.

The numerical data is accurate by mere observation, with a $R^2$ value almost nearly equal to 1 to confirm this. NIMROD was successful in providing precise simulations that show the plasma
frequency dependence on wavenumber when the pressure gradient is included in the electron flow evolution equation.

4 Conclusion and Future Work

Recall that this plasma model only used a few higher-order terms. This model can thus be improved upon by including the stress tensor $\Pi_a$. Further improvements include allowing for collisional effects through the Coulomb collision operator and allowing the ions to move. The latter would require evolving NIMROD’s plasma flow evolution equation and a separate temperature equation for the ions. Solutions and approximations to those remaining terms currently give modern plasma physicists much to work on.

References


I would like to extend thanks to the Howard W. Blood family and the USU Physics department for making this summer of research possible. A huge thanks to Dr. Held for mentoring and encouraging me through the process. And lastly a thanks to my wife, Alma, for the endless support throughout my higher education.