Enhance Variational Asymptotic Method for Unit Cell Homogenization (VAMUCH) for Real Engineering Structures and Materials

Zheng Ye
Utah State University

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ENHANCE VARIATIONAL ASYMPTOTIC METHOD FOR UNIT CELL HOMOGENIZATION (VAMUCH) FOR REAL ENGINEERING STRUCTURES AND MATERIALS

by

Zheng Ye

A dissertation submitted in partial fulfillment of the requirements for the degree of

DOCTOR OF PHILOSOPHY

in

Mechanical Engineering

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2013
Abstract

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Zheng Ye, Doctor of Philosophy
Utah State University, 2013

Major Professor: Dr. Wenbin Yu
Department: Mechanical and Aerospace Engineering

Modern technologies require the materials with combinations of properties that can not be met by conventional single phase materials. This requirement leads to the development of composite materials or other materials with engineered microstructures, such as polymer composites and nanotube.

Though the well-established finite element analysis (FEA) has the ability to analyze a small portion of such material, for the whole structure, the total degrees of freedom of a finite element model can easily exceed the bearable time in analysis or the capability of the best mainstream computers. To reduce the total degrees of freedom and save the computational efforts, an efficient way is to use a simpler and coarser mesh at the structure level with the micro level complexities captured by a homogenization method.

Throughout the dissertation, the homogenization is carried on by variational asymptotic method which has been developed recently as the Variational Asymptotic Method for Unit Cell Homogenization (VAMUCH). This methodology is also expandable to the structure analysis as long as a representative structural element (RSE) can be obtained from structure. In the present research, the following problems are handled: (1) Maximizing the flexibility of choosing a RSE; (2) Bounding the effective properties of a random RSE; (3)
Obtaining the equivalent plate stiffnesses for a corrugated plate from a RSE; (4) Extending the shell element of relative degree of freedom to analyze thin-walled RSE.

These problems covered some important topics in homogenization theory. Firstly, the rules need to be followed when choosing a unit cell from a structure that can be homogenized. Secondly, for a randomly packed structure, the efficient way to predict effective material properties is to predict their bounds. Then, the composite material homogenization and the structural homogenization can be unified from a mathematical point of view, thus the repeating structure can be always simplified by the homogenization method. Lastly, the efficiency of analyzing thin-walled structures has been enhanced by the new type of shell element. In this research, the first two topics have been solved numerically through the finite element method under the framework of VAMUCH. The third one has been solved both analytically and numerically, and in the last, a new type of element has been implemented in VAMUCH to adapt the characteristics of a thin-walled problem. Numerous examples have demonstrated VAMUCH application and accuracy as a general-purpose analysis tool.

(188 pages)
Public Abstract

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Utah State University, 2013

Major Professor: Dr. Wenbin Yu
Department: Mechanical and Aerospace Engineering

The applications of heterogeneous materials and other materials with engineered microstructures growth rapidly in all industries to achieve better performance. These materials and structures are defined such as composites and nanotube. With the increasing of computing power, though the well-established commercial finite element analysis (FEA) has the ability to analyze such material of a small portion. It is not feasible for the structure level, since the computing requirements of a finite element model can easily exceed the bearable time in analysis or the capability of the best mainstream computers. To reduce the computational efforts, an efficient way is to use a simpler and coarser mesh at the structure level with the micro level complexities captured by a homogenization method.

The homogenization method covers two parts. The first one is to calculate the equivalent material properties from heterogeneous materials or structures as an input for structural level analysis, and the second one is to use the behaviors from the structural lever to recover the local behavior in the heterogeneities.

The main point in the dissertation is to extend the application of homogenization method, which is based on variational asymptotic method developed recently as the Variational Asymptotic Method for Unit Cell Homogenization (VAMUCH), to some real en-
gineering structures and materials. A unit cell could be a small potion of heterogeneous material or a representative microstructure, which is further defined as a representative structural element (RSE). In the present research, the following problems are presented: (1) Maximizing the flexibility of choosing a RSE; (2) Bounding the effective properties of a random RSE; (3) Obtaining the equivalent plate stiffnesses for a corrugated plate from a RSE; (4) Extending the shell element of relative degree of freedom to analyze thin-walled RSE.

These problems covered some important topics in homogenization theory. Firstly, the rules need to be followed when choosing a unit cell from a structure that can be homogenized. Secondly, for a randomly packed structure, the efficient way to predict effective material properties is to predict their bounds. Then, the composite material homogenization and the structural homogenization can be unified from a mathematical point of view, thus the repeating structure can be always simplified by the homogenization method. Lastly, the efficiency of analyzing thin-walled structures has been enhanced by the new type of shell element. In this research, the first two topics have been solved numerically through the finite element method under the framework of VAMUCH. The third one has been solved both analytically and numerically, and in the last, a new type of element has been implemented in VAMUCH to adapt the characteristics of a thin-walled problem. Numerous examples have demonstrated VAMUCH application and accuracy as a general-purpose analysis tool for real engineering problems.
To my parents
Long Ye & Ling Xu
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Zheng Ye
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Chapter 1

Introduction

1.1 Motivation

The modern technologies require the materials with combinations of properties that cannot be met by conventional single phase materials. In aerospace applications, the structural materials are required to be strong, stiff, low thermal expansion, abrasion and impact resistant, yet low densities. This requirement leads to the development of composite materials and some structures with engineered microstructures. The judicious combination of materials or specific types of engineering structures provide the tailored properties for different components. As for composite materials, these include multiphase metal alloys, polymeric materials and ceramics; for engineering structures, these include corrugated plates, different forms of trusses, sandwich structures, etc. Such groups of materials or structures, which are strongly heterogeneous and highly complex, feature prominently in cutting edge industry and require an efficient way to carry out the structural analyses.

Although the scale of ingredient in composites could be very small, it is true the length scales are much larger than the characteristic length of the atomic spacings, so we still consider the heterogeneities behave under the framework of continuum mechanics. It is effective to ignore its discrete nature without introducing any significant errors for the behavior of normal engineering materials. Structural analysis is developed for continuous media with calculating the deformation and stress within a suitably constrained solid object under the action of applied loads. The three-dimensional (3D) formulation for linear static structural analysis includes a set of fifteen equations: six kinematic relations, three equations of motion, and six constitutive equations. By use of the constitutive equations the stresses can be expressed in terms of strains. The kinematic relations express the strains as functions of displacements. By substitution of kinematic and constitutive relations into the equations
of motion, the displacement components represent the only unknowns in the governing differential equations. Direct analytical solutions to the governing equations with boundary conditions are usually not available. The finite element analysis (FEA) is proposed as a numerical technique for finding approximate solutions for these partial differential equations with proper boundary conditions. Its abilities to handle complicated geometries and boundaries lay a foundation of FEA in industry to deal with practical problems.

In composite materials and some complex engineering structures, FEA could analyze the structure directly but it is not efficient. This means that one takes into consideration each component separately with a fine mesh would inevitably result in a finite element model with a huge number of degrees of freedom. In practice, capturing the behaviors in a large scale and discard the fluctuation due to inhomogeneities usually attracts more interest. Taking the composite snowboard as an example, without any simplification, 3D FEA is chosen to simulate this problem. The average waist is 250 millimeters and the average fiber diameter is 15.7 microns, which means we need at least 16000 elements along the waist direction to capture the material difference. However, if equivalent 3D material properties can be extracted from the properties of constituents in composite material and assigned to a much coarser fictitious homogenized structure mesh, say 50 elements along the waist direction, the problem is largely simplified and a lot of computational efforts can be saved. Furthermore, if we take the advantage of the fact that the thickness is much smaller than the waist and the edge, the equivalent two-dimensional (2D) plate stiffnesses can be applied to the snowboard and the analysis can be finished in a much more efficient way. This is the mission of homogenization: applying the equivalent properties to the structure analysis, saving computational time, and obtaining the global structural behavior with reasonable accuracy.

The first step to do homogenization is to identify a represent structural element (RSE) from the original structure. Choosing a RSE from a strictly periodic heterogeneous composite is a relative easy task and straightforward (Fig. 1.1). Although not all composites have periodic RSE, and in reality most composites are aperiodic. Nevertheless, when an
analyst decides to use micromechanics model to obtain effective properties, he or she al-
ready made an inherent assumption that some building block, i.e., the RSE exists. In such
a way, the fiber reinforced composite material can be decomposed as a bunch of building
blocks, and also the corrugated plate can be rebuilt by repeating one corrugation. Despite
the scale differences between the composite material and engineering structures. Math-
ematically speaking, the periodic parameters can be considered as fast oscillating variables
and the overall behavior of the structure can be considered as slow variables. The initial
partial differential equations with fast oscillating variables can be replaced by another par-
tial differential equations with constant variables that characterize the homogenized one.
To solve these constant variables (so-called effective material properties), we can formulate
the variational statements of the RSE through an asymptotic expansion of the energy func-
tional. For realistic problems, RSE are usually very complex and techniques such as finite
element method should be used to solve the validation statement for a numerical solution.
If the local fields within the RSE are of interest, a properly constructed model should accu-
rately recover those fields based on the global behavior. One thing needs to keep in mind is
that this description is correct only asymptotically, in the condition that the characteristic
wavelength of the field is large compared with the size of the periodic RSE and may not be
valid near the boundary of the composite sample.

Fig. 1.1: Regularly packed microstructure.
1.2 Review of Previous Work

1.2.1 Rigorous bounds

Before the availability of accurate homogenization results, rigorous bounds provide the estimate of effective properties given a limited amount of microstructural information. Rigorous upper bounds and lower bounds are useful because: (1) they give the constraint of effective properties; (2) they become progressively narrower when more microstructural information is provided; (3) one of the bounds usually can provide relatively accurate estimate of the effective properties; (4) they give the clue that how to choose different phases and the topology of microstructure in computer simulation or experiments. Nemat-Nasser and Hori [1] showed that for any general boundary conditions, the elastic strain energy and the complementary energy associated to lower bound and upper bound respectively.

Voigt and Reuss bounds

The most elementary bounds are Voigt upper [2] and Reuss lower [3] bounds. They are the simplest bounds which can be considered as first-order bounds because they only require the information of one-point correlation functions which represent volume fraction of constituents. They are assuming uniform strain and uniform stress respectively in the heterogeneous material. This approach takes only the influence of volume fraction. In terms of isotropic bulk ($K$), shear ($G$) moduli, volume fractions ($f$) of a two phase composite ($a$ and $b$), these bounds can be expressed as

\[
\begin{align*}
\bar{K}_{Upper} &= \langle K \rangle = f_a K_a + f_b K_b, \\
\bar{G}_{Upper} &= \langle G \rangle = f_a G_a + f_b G_b, \\
\bar{K}_{Lower} &= \left\langle \frac{1}{K} \right\rangle = \frac{f_a}{K_a} + \frac{f_b}{K_b}, \\
\bar{G}_{Lower} &= \left\langle \frac{1}{G} \right\rangle = \frac{f_a}{G_a} + \frac{f_b}{G_b}.
\end{align*}
\] (1.1)

Here noted that $f_a + f_b = 1$. 

Hashin-Shtrikman bounds

Hashin [4] and Shtrikman [5] obtained better bounds based on a solution that can be given by the gradient of a scalar function on the polarization field with variational principles. The variational bounding method is based on the minimum energy principle. They are the best estimates of overall elastic behavior that can be obtained if no geometrical or statistical details are provided about the arrangement within heterogeneity. With the same symbols used in Voigt and Reuss bounds, we further assume that $K_b > K_a$ and $G_b > G_a$

\[
\bar{K}_{Upper} = K_b + \frac{f_a}{K_a - K_b} + \frac{3f_b}{3K_b + 4G_b},
\]

\[
\bar{G}_{Upper} = G_b + \frac{f_a}{G_a - G_b} + \frac{6(k_b + 2G_b)f_b}{5G_b(3K_b + 4G_b)},
\]

\[
\bar{K}_{Lower} = K_a + \frac{f_b}{K_b - K_a} + \frac{3f_a}{3K_a + 4G_a},
\]

\[
\bar{G}_{Lower} = G_a + \frac{f_b}{G_b - G_a} + \frac{6(k_a + 2G_a)f_a}{5G_a(3K_a + 4G_a)}.
\]

Third-order and higher-order bounds

Physical properties of heterogeneous materials highly depend on distribution and orientation of the constituents, thus Silhutzer [6] and Milton [7–9] have derived improved bounds with additional microstructural geometric information. The Silhutzer bounds, which is known as third-order bounds, depend on an integral involving the three-point correlation function and the Milton bounds include an integral involving the four-point correlation function and it is known as fourth-order bounds. However, higher-order bounds such as third-order and fourth-order bounds diverge with increasing contrast of constituent properties, even though they are definitely an improvement over the lower-order bounds.

Torquato [10–12] proposed a theoretical formalism for systematically representing the $n$-point correlation functions which an infinite set of $S_1, S_2, ..., S_n$ is used to capture the characteristics of microstructure in order to find the rigorous bounds. He also discussed the numerical and experimental difficulties in obtaining higher-order point correlation functions. The lowest order is the one-point correlation function, which represents the phase
volume fraction that is commonly used in classical homogenization methods to capture the material’s heterogeneity. To take into account the material’s heterogeneous morphology, it is necessary to incorporate higher order probability functions. Berryman [13–15] derived an efficient way to find three-point correlation functions in various situations in order to calculate third-order bounds and some recent discovery can be found in Ref. [16]. The Torquato third-order bounds for the random distributed impenetrable spheres [12] can be expressed as,

\[
\bar{K}_{\text{Upper}} = \langle K \rangle - \frac{3f_a f_b (K_a - K_b)^2}{3 \langle K \rangle + 4 \langle G \rangle \zeta},
\]

\[
\bar{G}_{\text{Upper}} = \langle G \rangle - \frac{f_a f_b (G_a - G_b)^2}{\langle G \rangle + \theta},
\]

\[
\bar{K}_{\text{Lower}} = \langle K \rangle - \frac{3f_a f_b (K_a - K_b)^2}{3 \langle K \rangle + 4 \langle \frac{1}{G} \rangle \zeta},
\]

\[
\bar{G}_{\text{Lower}} = \langle G \rangle - \frac{f_a f_b (G_a - G_b)^2}{\langle \frac{1}{G} \rangle + \tau},
\]

where,

\[
\theta = \frac{2 \langle K \rangle \zeta \langle G \rangle^2 + \langle K \rangle^2 \langle G \rangle \eta}{\langle K + 2G \rangle},
\]

\[
\tau = \frac{1}{2 \langle \frac{1}{K} \rangle \zeta + \langle \frac{1}{G} \rangle \eta},
\]

with the angle bracket defined as,

\[
\langle M \rangle = M_a f_a + M_b f_b,
\]

\[
\langle \tilde{M} \rangle = M_a f_b + M_b f_a,
\]

\[
\langle M \rangle \zeta = M_a \zeta_a + M_b \zeta_b,
\]

\[
\langle M \rangle \eta = M_a \eta_a + M_b \eta_b,
\]

\[
(1.3)
\]

\[
(1.4)
\]

\[
(1.5)
\]
and the parameters $\zeta_b$ and $\eta_b$ depend on:

\[
\begin{align*}
\zeta_b &= 0.21068 f_b - 0.04693 f_b^2 + 0.00247 f_b^3, \\
\eta_b &= 0.48274 f_b, \\
\zeta_a &= 1 - \zeta_b, \quad \eta_a = 1 - \eta_b.
\end{align*}
\] (1.6)

From these equations, we can see that the third order bounds depend on four extra geometric parameters $\zeta_a, \zeta_b$ and $\eta_a, \eta_b$ which derived from three-point correlation functions. The details of how to use a triangle with two sides and formed angle to determine the correlation function can be referred to [12].

**Dual variational principles**

The application of dual variational principles is to obtain the inequality of the strain energy form, thus the upper bound and lower bound can be obtained accordingly. The main point of a dual variational principle is to construct a function with two variables $\Phi(u,v)$ as a connection between the minimization and the maximization problems. Consider the minimax problem

\[
\tilde{I} = \min_{u \in M} \max_{v \in N} \Phi(u,v).
\] (1.7)

where $\tilde{I}$ is the solution of the variational problem and $u, v$ are two variables. Assume the order of maximum and minimum values in Eq. (1.7) can be changed,

\[
\tilde{I} = \max_{v \in N} \min_{u \in M} \Phi(u,v).
\] (1.8)

Denote

\[
\begin{align*}
I(u) &= \max_{v \in N} \Phi(u,v), \\
J(v) &= \min_{u \in M} \Phi(u,v).
\end{align*}
\] (1.9)
Then the minimization and maximization problems,

\[
\hat{I} = \min_{u \in M} I(u),
\]
\[
\tilde{I} = \max_{v \in N} J(v),
\]

are defined as a dual variational problem. Thus, we get an estimate

\[
J(\tilde{v}) \leq \max_{v \in N} J(v) = \hat{I} = \min_{u \in M} I(u) \leq I(\tilde{u}),
\]

where \(\tilde{v}\) and \(\tilde{u}\) are arbitrary values in the domain \(N\) and \(M\) respectively. So the rigorous bounds can be expressed as

\[
J(\tilde{v}) \leq \hat{I} \leq I(\tilde{u}).
\]

This approach will be proposed in this thesis to solve the randomly distributed heterogenous problem. As in the energy expression, the domain \(N\) and \(M\) will be chosen from strain and stress fields respectively.

The inherent characteristics of bounds is that the distance between them increases with the divergence of properties of the heterogeneities. In the extreme case with a rigid phase leads to the increasing to infinity of upper bound. This invokes the requirements of more accurate homogenization results.

### 1.2.2 Homogenization for randomly distributed field

In reality, heterogeneities are not regularly packed in the RSE and its response is highly influenced by both the material behavior and geometrical arrangement of distinct phases, thus the randomness of microstructure has been an active research area recently. Computational models of random heterogenous materials rely on simulating the random microstructures such as Voronoi tessellation [17] which is a large group in random simulation and used to mimic polycrystals, cells, organs, etc. A majority of the studies consider particles of regular geometric shape and size, such as cylinders or spheres that are randomly distributed in a matrix phase [18]. Torquato and Stell studied impenetrable and penetrable
spheres with \( n \)-point correlation functions [10]. There are two major targets to access the homogenized properties. One is to utilize rigorous bound theory, as introduced in previous section, to obtain the fork of equivalent material properties, and the other is to predict the probability distribution of each individual values of equivalent properties. Theoretically, the complex real microstructure can be replaced by a substantially simpler RSEs, which statistically represents the real microstructure. With these RSEs, one can do homogenization analysis independently and yield the expectation of the effective properties. This method is straight forward but it is computationally expensive to get convergent results.

For demonstrative purposes, an RSE of 2D microstructure with randomly distributed inclusions is under investigation using variational asymptotic method for unit cell homogenization (VAMUCH) and compared with literature [19]. The microstructure is simplified as containing random distributed circular inclusions in an Aluminum matrix with following assumptions: (1) Only one type of micro inclusion is considered in the matrix material of microstructure, so this is a two-phase composite; (2) The shape of the inclusion is simplified as a simple circle in 2D SRVE, represented using the center position as descriptor factors. Other attributes such as orientation and irregular inclusions are ignored; (3) The inclusion material is SiC, and matrix material is Al2618-T4, which are simplified as isotropic materials. No variation in the properties are considered. Periodic boundary conditions are applied to the RSE by enforcing the fluctuation functions of two opposite boundaries remain identical during simulation. The center of each inclusion is generated by the combination of Gaussian random field [20] and revised Random Sequential Absorption Algorithm [21]. The material properties of inclusion and matrix are listed in Table 1.1, according to Kari’s paper [19].

<table>
<thead>
<tr>
<th></th>
<th>Al2618-T4</th>
<th>SiC</th>
</tr>
</thead>
<tbody>
<tr>
<td>( E ) (GPa)</td>
<td>70</td>
<td>450</td>
</tr>
<tr>
<td>( \nu )</td>
<td>0.3</td>
<td>0.17</td>
</tr>
</tbody>
</table>
It is noted that the simulation carried out giving volume fraction of inclusion 10%, the same as what is needed by Kari in [19]. The number of inclusions vary from 10, 20, 30, ···, 90 in RSE. After the analysis of random structure of each category, VAMUCH has the ability to give all the equivalent material properties. Without losing generality, the transverse Young’s modulus $E_{22}$ of each random microstructure is plotted in Fig. 1.2. To determine how many simulations of random configurations are needed, the criteria of convergence of mean and standard deviation of effective properties is used. With the increasing number of sampling, the mean value becomes convergent in Fig. 1.3. The convergence criteria is considered that the variance of mean value is less than 5%, which is the point when the random configuration generation stops. The transverse Young’s modulus of Kari’s paper [19] is 81.66 GPa, which agrees fairly well with the convergent result in Fig. 1.3. By taking into account the relation of microstructure parameters and effective material properties, different volume fractions are also studied. Compared with the random position factor, volume fraction is the most critical microstructure parameters to describe random structures. A demonstration analysis shows that when volume fraction increase to 13.5%, expectation of $E_{22}$ becomes to 86.1 GPa.

Another way is to find a descriptor such as the probability density function to capture the characteristics of the random heterogeneous materials. Then, the distribution of effective properties can be obtained. Sakata et al. [22] analyzed the major influence of microscopic uncertainty such as Young’s modulus and Poisson’s ratio of fiber and matrix, volume fraction
and size of fiber. Based on the first order perturbation method, different influence on overall material from different microscopic uncertainties is calculated and compared. Li et al. [23] estimated the equivalent mechanics properties of a composite material based on statistical multi-scale analysis. Shoukry et al. [24, 25] studied random distributed unit cell containing both spherical and ellipsoid particles using ANSYS. Pan et al. [26] investigated the effect of the interaction between two over-crossing fibers on the overall elastic properties of composite and local stresses by using Abaqus.

The RSE has to include a very large number of trial fields (possible microstates) to cover enough diversity of heterogeneities, so that it becomes statistically homogeneous and ergodic. The main difficulty of modeling heterogeneous materials with random microstructures is associated with stochastic representative volume elements [27]. A synthetically creating microstructure can be obtained experimentally or numerically. The algorithm for generating random fields has been studied in the literature, like Random Sequential Absorption Algorithm [21], Monte Carlo simulation [28] and image reconstruction technique [29], etc. Because it is hard to determine how large is enough to satisfy ergodic assumption, a large number of trial fields (see a sample of sketch in Fig. 1.4) are usually used to obtain an accurate description of the random heterogeneous material.
The assumption of microstructure periodicity of the randomly distributed material, on the contrary, may appear to be rather artificial and inappropriate for real materials. The numerical studies of Terada et al. [18] revealed that the periodicity conditions are well suited for the analysis of materials with disordered microstructure. Thus we stick to the periodic boundary conditions for randomly distributed unit cells. For a bunch of random cells, $n$-point [10] correlation functions cannot describe the change of positional variances. Furthermore, the assumption of the absence of any long-range order [30] in $n$-point correlation functions always violates the periodic assumption in analysis. Hence, a new microstructure descriptor is required in calculating the bounds for the random cell structure and will be studied in a later section.

### 1.2.3 Other approaches

A tremendous amount of literature exists on obtaining effective properties either theoretically or experimentally. Classical analytical estimates for the macroscopic overall response of composites often make use of the fundamental result of Eshelby [31] concerning the fluctuation strain field in an inclusion of ellipsoidal shape. These approaches are in many circumstances restricted, especially with respect to the geometry of the representative
micro-structure and its constitutive response that is often assumed to be linearly elastic. Halpin and Tsai \[32\] obtained the effective properties by using invariant concepts along with quasi-isotropic laminate theory. The dilute \[33\], self-consistent \[34\], differential \[35\] and Mori-Tanaka methods \[36\] are the main micromechanics approaches that have received the most attention. They are all based on a two-phase model of heterogeneous materials containing an inclusion phase and matrix phase. The generalized self-consistent method is a more sophisticated micromechanics approach \[37,38\]. Different from the aforementioned micromechanics methods based on the two-phase model, the generalized self-consistent method is based on the three-phase model. The calculation is narrowed to a unidirectional fiber reinforced composite with Mori-Tanaka theory \[36,39\] combined with Eshelby tensor in case of continuous long unidirectional fiber.

On the computational side, several numerical methods have been developed which discretize the local fields on the microstructure of a composite. The governing equations can be solved numerically to find the effective properties of a heterogeneous material. The stresses and strains are averaged over the RSE domain \( \Omega \), thus the effective properties \( \bar{C}_{ijkl} \) can be extracted from the relation

\[
\langle \sigma_{ij} \rangle = \bar{C}_{ijkl} \langle \epsilon_{kl} \rangle ,
\]

where the average stresses and strains can be calculated as,

\[
\langle \sigma_{ij} \rangle = \frac{1}{\Omega} \int_{\Omega} \sigma_{ij} d\Omega ,
\]

\[
\langle \epsilon_{ij} \rangle = \frac{1}{\Omega} \int_{\Omega} \epsilon_{ij} d\Omega .
\]

Adams \[40\] applied finite difference approach to regular arrays of circular fibers which modeled for transverse normal moduli and longitudinal shear moduli. Due to the improved discretization of different geometries, the finite difference approach replaced by finite element approach, at beginning, most finite element simulations are using square or hexagonal arrays of fibers in a 2D problem. With the increasing of computational ability, complex
2D and 3D problems have been investigated. The correct boundary conditions need to be well taken care of to obtain the correct answer. A brief review of finite element approaches can be found here in Ref. [41]. A Fourier transform based numerical approach is used by Moulinec and Suquet [42] to solve the unit cell problem. The solution is obtained by alternating between the real and Fourier spaces and has been applied to 2D problems. The method of cells (MOC) developed by Aboudi [43–45] has been used to model the micromechanical behavior of different types of composites. The advantage of MOC is considered that it can get the full set of material properties in one calculation instead of solving multiple problems with different boundary conditions. The basic assumption of MOC is that the displacement vector within each subcell varies linearly with local coordinates. The continuity condition of displacements and traction is imposed at the interfaces between subcells as well as between the repeating cells on average sense. Later on, the MOC has been extended as Generalized Method of Cells (GMC) [46] by using a single subcell to represent multiple subcells, which gained more computational efficiency than finite element analyses. The High fidelity GMC (HFGMC) [47] is developed to improve the accuracy in GMC where the coupling between normal and shear stresses are being neglected. The HFGMC used quadratic displacement field to account for coupling effect, hence, the HFGMC enables simulation of stress and strain with greater accuracy. Mathematical homogenization theory (MHT) which is developed since the 1970’s provides a rigorous mathematical framework for analysis of heterogeneous materials. It solves a series equilibrium equation within the separated same order [48] and the formalization of its mathematical foundations can be found in Benssousan et al. [49]. A second-order mathematical homogenization theory, or so called a gradient-enhanced computational procedure, that extends the classical MHT is proposed by Kouznetsova et al. [50]. The current macroscopic configuration is obtained using a Taylor series expansion based on the first-order relation up to second-order, so that it uses both the macroscopic deformation tensor and its gradient (second-order term) to prescribe kinematic boundary conditions in a unit cell. As a result, this theory allows the finite deformation on the microstructural level. However, in elastic problems, the additional calculation for
gradient of deformation tensor is not necessary and cost additional computational efforts.

A unique approach to modeling of structures and materials is based on the variational asymptotic method to avoid apriori assumptions, which are commonly invoked in other approaches. Exploiting the smallness of microstructures, the VAMUCH [51–54] is developed for prediction of the effective properties of heterogeneous materials. The companion code VAMUCH can be used to efficiently yet accurately predict effective properties of heterogeneous materials with 1D, 2D, or 3D unit cells, and recover the micro fields. Taking advantage of the smallness of some dimensions of structures, the original 3D analysis can be mathematically split into a local analysis over the small dimensions for constitutive modeling and a global analysis over the large dimensions for global responses. A rigorous assessment of the accuracy of the latest homogenization method can be found in Ref. [55].

1.2.4 Homogenization of corrugated plates

A shell or plate is a thin 3D body bounded by two, relatively close, curved or flat surfaces where the thickness is relatively small compared to the dimensions of the surfaces. Simplifications can be obtained by exploiting the small thickness $h$ compared with the in-plane size. The in-plane stresses become dominant and a two-dimensional theory can be used to approximate the original three-dimensional problem. However, in engineering applications, thin flat plates lack of ability to resist shear stress, and absorb compression and bending load, which can be remedied by engineering corrugation into the structures. The expanded applications of corrugated shapes include fiberboards [56–62], folded roofs, container walls, sandwich plate cores [63, 64], bridge decks [65], ship panels [66], flexible wings [67–69], and so forth. In addition, corrugated plates have potential applications for their thermal stress-alleviating properties as well as for their shear-carrying properties, in sound reduction, and in a sandwich-covered wing structure where a relatively large crushing strength and a large shear buckling strength are needed [70]. Compared with flat plates, corrugated plates have all the characteristics of plates with an additional one — repeated curvature changing in plane, normally along one direction, and the behavior of corrugated plates is primarily governed by this curvature. Owing to the curvature of the middle surface,
two typical features of a corrugated plate are the large discrepancy between the bending and extension stiffness of the plate along the wavy direction and its perpendicular direction. The bending stiffness along the direction perpendicular to the wavy direction is usually two or three orders of magnitude greater than that along the wavy direction, meanwhile the extension stiffness perpendicular to the wavy direction is also two or three orders of magnitude greater than that along the wavy direction. This dramatical difference in bending and extension stiffness is due to the fact that the bending moments, which is along the perpendicular to wavy direction, are mostly balanced by the membrane stresses distributed along the plate thickness and the extension displacements along the wavy direction are largely caused by the bending of the wavy structure instead of in-plane stretching. In general, the bending of a corrugated plate cannot be separated from its stretching in certain directions, which makes its analyses much more complex than that of a flat plate. Furthermore, for a composite corrugated plate, more unique features can be discovered and more widely applications could be found. A sandwich structure with a corrugated core is frequently suggested to increase out-of-plane stiffness. However, at this stage to obtain the analytical solution of effective properties, our main focus is restricted to isotropic corrugated plates.

Typically, the shape of corrugated plate can be sinusoidal, cycloidal, parabolic, circular, semi-elliptical, and trapezoidal depending on the cross-sections shown in Fig. 1.5. The direct-FEA involving these corrugated structures will entail the requirement of a large finite element model and it is not efficient particularly for design purpose. An effective flat plate model, as illustrated in Fig. 1.6, with much less number of elements in the global analysis can dramatically reduce the total degrees of freedom in the finite element model of corrugated plates by replacing the original corrugated plate with a flat plate with corresponding equivalent plate stiffness, which is obtained from the corrugated cross-section. Then, the overall behavior of corrugated plates, such as extension, deflection, vibration, buckling, and several others can be predicted using FEA or closed-form solutions.

Despite many contributions to the theory of corrugated plates over the last few decades, the establishment of efficient homogenized modeling techniques for simulating the global as
Fig. 1.5: Unit cross-sections of different types of corrugated plates.

Fig. 1.6: Homogenization of corrugated plate.
well as the local responses of corrugated plates still remains a challenging task, specifically in view of their expanding applications. Kirchhoff-Love and Reissner-Mindlin plate theory are widely accepted and used in engineering. For corrugated plates, the existing analytical homogenized models can be fitted in this two categories. Generally speaking, a thin corrugated plate is suitable to be modeled using the Kirchhoff-Love plate theory and for a relatively thick plate, one needs to consider transverse shear effects and model it using the Reissner-Mindlin plate theory. In the present work, we focus on thin corrugated plates.

**Effective bending stiffnesses**

In additional to the Kirchhoff-Love assumptions commonly used for thin plates/shells such as transverse normal remains normal to mid-surface during deformation, and transverse stresses (both normal and shear stresses) are negligible comparing to in-plane stresses in constitution relation, we need to make additional assumptions regarding corrugated plates to model it with an equivalent flat plate. It is assumed that the deflection of the corrugated plate is defined by the homogenized surface of the flat plate (Fig. 1.6), and this deflection is assumed small compared with the thickness so that the curvature of the deflected surface is also very small compared with unity. Another assumption is that the length of the corrugated period is much smaller than the wave length of the deflection of the homogenized plane. Based on these assumptions, the original corrugated plate can be modeled using an equivalent flat plate with a set of plate stiffness which can be obtained through analytical approaches. Because of wavy structure, the corrugated plates feature direction dependent properties. The equivalent material properties differ in two mutually perpendicular directions, thus, it is also commonly called orthotropic plate.

Analytical approaches provide a set of closed-form formulas to compute equivalent stiffness of corrugated plates to be used in a Kirchhoff-Love plate theory, which can be
expressed in the following general form

\[
\begin{bmatrix}
  N_{xx} \\
  N_{xy} \\
  N_{yy} \\
  M_{xx} \\
  M_{xy} \\
  M_{yy}
\end{bmatrix} =
\begin{bmatrix}
  A_{11} & A_{12} & A_{13} & B_{11} & B_{12} & B_{13} \\
  A_{12} & A_{22} & A_{23} & B_{12} & B_{22} & B_{23} \\
  A_{13} & A_{23} & A_{33} & B_{13} & B_{23} & B_{33} \\
  B_{11} & B_{12} & B_{13} & D_{11} & D_{12} & D_{13} \\
  B_{12} & B_{22} & B_{23} & D_{12} & D_{22} & D_{23} \\
  B_{13} & B_{23} & B_{33} & D_{13} & D_{23} & D_{33}
\end{bmatrix}
\begin{bmatrix}
  \varepsilon_{xx} \\
  2\varepsilon_{xy} \\
  \varepsilon_{yy} \\
  \kappa_{xx} \\
  2\kappa_{xy} \\
  \kappa_{yy}
\end{bmatrix} =
\begin{bmatrix}
  A & B \\
  B & D
\end{bmatrix}
\begin{bmatrix}
  \varepsilon_{xx} \\
  2\varepsilon_{xy} \\
  \varepsilon_{yy} \\
  \kappa_{xx} \\
  2\kappa_{xy} \\
  \kappa_{yy}
\end{bmatrix}.
\]

(1.15)

The purpose of homogenization is to find solutions to \( A, B, D \). Here notations, typical for presentations of the theory of laminated plates, can be found in textbook by Reddy [71].

The force resultants \( (N_{xx}, N_{xy}, N_{yy}) \) and moment resultants \( (M_{xx}, M_{xy}, M_{yy}) \) are related to the strains \( (\varepsilon_{xx}, \varepsilon_{xy}, \varepsilon_{yy}) \) and curvatures \( (\kappa_{xx}, \kappa_{xy}, \kappa_{yy}) \) through extension stiffness \( A \), bending stiffness \( D \), and the coupling \( B \) between bending and extension. It is admitted the removal of \( A_{12}, A_{23}, D_{12}, \) and \( D_{23} \) when the direction of corrugation coincides with one of in-plane coordinates. For orthotropic plate, torsional rigidities are different between the \( x \)- and the \( y \)- directions and the constitutive relation for \( D_{22} \) represents the average effect. \( D_{13} \) represents the poisson effect of the bending moments on the torsion curvature. Furthermore, if the plate is symmetric with the origin of the thickness coordinate, then \( B = 0 \) and the bending behavior is decoupled from extension which is governed by the following fourth-order partial differential equation.

\[
D_{11} \frac{\partial^4 w}{\partial x^4} + 2(D_{13} + 2D_{22}) \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_{33} \frac{\partial^4 w}{\partial y^4} = p,
\]

(1.16)

where \( x, y \) are the Cartesian coordinates used to describe the reference plane, \( w \) is the transverse displacement, and \( p \) is the pressure load on the reference plane. For a homogeneous flat plate made of isotropic materials, Eq. (1.16) will be simplified to be

\[
D \left( \frac{\partial^4 w}{\partial x^4} + 2 \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} \right) = p,
\]

(1.17)
where $D$ is the bending stiffness calculated as $D = \frac{Eh^3}{12(1-\nu^2)}$ with $E$ denoting the Young’s modulus, $\nu$ the Poisson’s ratio, and $h$ the plate thickness.

The equivalent bending stiffness $D_{ij}$ shown in Eq. (1.16) are first studied in literature because they directly relate to the deflections of corrugated plates in most applications. In an early stage, free body diagram analysis to obtain equivalent bending stiffness is adopted by several researchers [72–76]. Three load cases, pure bending along $x$ direction, pure bending along $y$ direction and pure torsion, are needed to obtain the bending stiffness in Eq. (1.16) where $D_{13} + 2D_{22}$ is lumped as total torsional rigidity $H$ in pure torsion behavior. Then, the deflection of the plate can be solved by closed-form solution procedures or the FEA. Before presenting the analytical solutions from literature, the geometric parameters of corrugated plate are introduced first.

Take a sinusoidal corrugated plate as an example (Fig. 1.7). The geometric characteristics are described in coordinates $(x, y, z)$ where $x$ lies on the wavy direction, $y$ is pointing inside of $x-z$ plane. The width of corrugated plate along $y$ axis is considered as infinite. $h$ is the constant thickness measured perpendicular to its sinusoidal middle surface, $T$ is the rise of the corrugations measured to middle surface, $S$ is arc length and $\varepsilon$ is the projected length of the corrugation. This sinusoidal shape can be expressed as $z = T\sin(2x\pi/\varepsilon)$ with periods $\varepsilon$.

The earliest estimation of equivalent bending stiffness, is found in 1923, where Huber [72] postulated an analytical method to determine the coefficients in the orthotropic plate partial differential equation in Eq. (1.16) to model the bending behavior of corrugated plates. Seydel [73] and Huber treated the corrugated plates as a uniform thickness orthotropic homogeneous plate, which possesses orthorhombic symmetry composed of one plane of symmetry mid-way as homogenized surface and two orthogonal planes of symmetry normal to the homogenized surface. The non-homogeneity is neglected as the characteristic dimensions of the corrugations are small in comparison to the planar dimensions $\varepsilon$ of the whole plate. A pure bending linear theory was also assumed considering there is no local buckling and no in-plane forces exist.
Fig. 1.7: Unit cell of a sinusoidal corrugated plate (with a sinusoidal corrugation).

Applying constant curvatures $\kappa_{xx}$, $\kappa_{xy}$ and $\kappa_{yy}$ in three load cases respectively, the equivalent bending stiffness in corrugated plates can be expressed by relating corrugated geometry and material properties. Huber suggested orthotropic plate models for corrugated plates by using the following coefficients

$$D_{11} = \left(\frac{S}{h}\right)^2 \frac{Eh^3}{12(1-\nu^2)},$$
$$D_{33} = EI_y,$$
$$H = D_{13} + 2D_{22} = \left(\frac{S}{h}\right)^2 \frac{Eh^3}{12(1+\nu)},$$

where $I_y$ is the moment of inertia along $y$ axis. The same equations for sinusoidal corrugations were later presented by Lekhnitskii [74], Szilard [77], and Troitsky [76]. Then the problem becomes to obtain the values of $S$ and $I_y$ accurately. A sinusoidal corrugated shape, shown in Fig. 1.7, was also studied in a bunch of textbooks from Timoshenko et al. [78], Troitsky [76], Mcfarland [79], Szilard [75], Heins [80], and Jawad [81]. Based on the same formula from Huber and Seydel in Eq. (1.18), $S$ and $I_y$ in these textbooks were
calculated in Eq. (1.19) approximately

\[ S = \varepsilon (1 + \frac{\pi^2 T^2}{\varepsilon^2}), \]
\[ I_y = \frac{T^2 h}{2} (1 - \frac{0.81}{1 + 2.5(\frac{T}{\varepsilon})^2}). \]

(1.19)

Although in Timoshenko’s book, it is said that these equations were quoted from Seydel [73], the values of \( I_y \) from Eq. (1.19) in different configurations gave about 20 percent difference compared with Seydel’s results in Seydel’s original paper in Refs. [73,82]. Lau [83] and Lee [84] pointed out that Eq. (1.19) was not a good approximation and had some limitations. They improved the formula for calculating the arc length and moment of inertia. Mathematically, the exact arc length \( S \) and moment of inertia per unit length \( I_y \) can be calculated using the following expressions

\[ S = 2 \int_0^{\varepsilon/2} \sqrt{1 + \left( \frac{dz}{dx} \right)^2} \, dx, \]
\[ I_y = \frac{2}{\varepsilon} \int_0^{\varepsilon/2} z^2 \sqrt{1 + \left( \frac{dz}{dx} \right)^2} \, h \, dx. \]

(1.20)

The comparison of \( S \) and \( I_y \) between Eq. (1.19) and Eq. (1.20) are normalized and plotted in Fig. 1.8 which shows that they become significantly different with the increasing of corrugation \( 2T/\varepsilon \). For a shallow corrugated plate, where \( \frac{2T\pi}{\varepsilon} < 1 \) or \( \frac{2T}{\varepsilon} < 0.31 \), the arc length \( S \) and moment of area \( I_y \) can take

\[ S = \varepsilon (1 + \frac{\pi^2 T^2}{\varepsilon^2} - \frac{3}{16} \frac{\pi^2 T^2}{\varepsilon^2} + \cdots) \approx \varepsilon (1 + \frac{\pi^2 T^2}{\varepsilon^2}), \]
\[ I_y = \frac{T^2 h}{2} (1 + \frac{1}{2} \frac{\pi^2 T^2}{\varepsilon^2} - \frac{1}{16} \frac{\pi^2 T^2}{\varepsilon^2} + \cdots) \approx \frac{T^2 h}{2} \left(1 + \frac{1}{2} \frac{\pi^2 T^2}{\varepsilon^2}\right). \]

(1.21)
Table 1.2: Moment of inertia densities.

<table>
<thead>
<tr>
<th>Corrugated Shape</th>
<th>( I_y ) per unit length</th>
<th>Annotations</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cycloidal-type [88]</td>
<td>( \frac{8\varepsilon}{15}[1 + \frac{32}{7}(\frac{T}{\varepsilon})^2]T^2h )</td>
<td>Fig. 1.5</td>
</tr>
<tr>
<td>Parabolic-type [88]</td>
<td>( \frac{8}{155}[6 + \sqrt{1 + 64(\frac{T}{\varepsilon})^2}]T^2h )</td>
<td>Fig. 1.5</td>
</tr>
<tr>
<td>Circular-type [88]</td>
<td>( \frac{5\alpha}{48\sin\alpha}h^3 + \frac{2\alpha + 2\alpha\cos\alpha - \tan\alpha\tan\alpha}{4\sin\alpha(1-\cos\alpha)}T^2h )</td>
<td>Fig. 1.5</td>
</tr>
<tr>
<td>Semi-elliptical-type [89]</td>
<td>( \frac{\pi}{16}\left[(T + \frac{h}{2})^3(\varepsilon + h) - (T - \frac{h}{2})^3(\varepsilon - h)\right] )</td>
<td>Fig. 1.5</td>
</tr>
</tbody>
</table>
| Trapezoidal-type [90]       | \( \frac{h^3 + 2ha}{12T} + T^2bw + \frac{1}{3}\tan^2\theta((\varepsilon/2)^3 - (bw + \frac{T}{\tan\theta})^3) \)\)
\( - (2T + bw \tan \theta )\tan\theta((\varepsilon/2)^2 - (bw + \frac{T}{\tan\theta})^2) \)
\( + (2T + bw \tan \theta )^2((\varepsilon/2)^2 - (bw + \frac{h}{\tan\theta})^2) \)) | Fig. 1.9    |

Fig. 1.9: Unit cell of a trapezoidal corrugated plate.

Other than that, the sinusoidal corrugated plates have to be carefully studied before using approximate arc length \( S \) and moment of inertia density \( I_y \). Based on the values in Eq. (1.21), Lee [84] did the experiment and proved that the rigidities predicted by Huber’s model were pretty good compared with frequencies of corresponding shapes.

Besides sinusoidal shape, a set of related formulas for quickly determining the cross-sectional properties can also be found in Ref. [85]. Ranger [86] presented a method for calculating the moment of inertia for the corrugated medium which relies upon summing the moments of inertia of a set of rectangular areas making up the medium’s cross-sectional area. The approximate moments of inertia for some types of corrugation were calculated and summarized in Luo’s paper Ref. [87], which are listed in the Table 1.2.

Comparison of moments of inertia between the approximate values and exact values...
(Eq. (1.20)) are given in Fig. 1.10. The engineers need to be aware of the restriction when using the approximations. When $2T/\varepsilon < 0.6$, the approximate approach holds enough accuracy compared with exact solution for sinusoid, semi-elliptical and trapezoidal corrugated plate. For cycloidal and circular type, the $2T/\varepsilon$ is a constant $\pi$. Only when $T < 0.2$ the approximate values are close to the exact solution. For parabolic-type, when $2T/\varepsilon < 0.2$ the approximate values are close to the exact solution. Finally, to avoid loss of accuracy, it is recommended to use Eq. (1.20) and piecewise functions to calculate length $S$ and moment of inertia $I_y$, for their use in evaluating effective stiffness of corrugated plates using Eq. (1.18).
Briassoulis [91] also reported bending stiffnesses calculated as which are also close to his finite element results

\[
D_{11} = \left( \frac{S}{L} \right) \frac{Eh^3}{12(1-\nu^2)},
\]

\[
D_{33} = \frac{Eh^3}{12(1-\nu^2)} + \frac{Et^3}{2},
\]

\[
D_{13} = \nu D_{11},
\]

\[
D_{22} = \frac{Eh^3}{24(1+\nu)},
\]

(1.22)

Compared with Eq. (1.18), Briassoulis gave the specific values to \( D_{13} \) and \( D_{22} \) instead of lumping them as \( H \). \( D_{13} \) comes from Poisson’s effect from \( D_{11} \), and \( D_{22} \) represents torsional stiffness which is exactly the same formula as in isotropic plate case. This is due to his assumption that the corrugation does not affect the torsional behavior of the corrugated plate. For \( D_{33} \), the first part indicates the original flexural stiffness of the flat plate and the second part is an additional nonuniform moment due to the membrane forces \( N_{yy} \) developed in corrugation. The details of the derivation can be found in [91].

Norman et al. [92] gave a very simple assumption that the corrugations only affect the bending stiffness in the direction of the corrugations. It is augmented by the moment arm of the material about the center plane, giving an additional stiffness \( \alpha D \), where \( \alpha \) is a function of the shape of the corrugations. Evaluating the strain energy density of corrugated plates, \( \alpha \) can be expressed by initial corrugation curvature.

\[
D = \frac{Eh^3}{12(1-\nu^2)},
\]

\[
D_{11} = D,
\]

\[
D_{33} = D + \alpha D,
\]

\[
D_{13} = \nu D,
\]

\[
D_{22} = \frac{1}{2}(1-\nu)D,
\]

(1.23)

where \( \alpha D = \frac{1}{2} \int_0^L E z(x)^2 h dx \).

Instead of direct analysis of corrugated plate problem, the beam idea was introduced by Abbes and Guo [93] to obtain torsional stiffnesses, of which can be separated and calculated as combination of two beam-like torsional problems. The torsional curvature was separated
into two orthogonal beam torsional rates and torsional stiffnesses of the beams in both
directions were added together to describe the corrugated torsional behavior

\[ D_{22} = \frac{1}{4} \left( \frac{GJ_x}{b} + \frac{GJ_y}{a} \right), \]  

(1.24)

where \( GJ_x \) is the total beam torsional stiffness along \( x \), \( GJ_y \) is that of along \( y \), and \( a, b \) is
the plate length along \( x \) and \( y \) respectively. Because the poisson’s effect of the plate in two
directions can not be captured using a beam, and this method is not recommended.

**Effective extension stiffnesses**

To fully describe the behavior of corrugated plate, the effective extension stiffnesses \( A \)
were also needed. As long as \( B = 0 \), the membrane behavior can be decoupled from the
bending behavior in Eq. (1.15) and described in the following governing partial differential
equations

\[
A_{11} \frac{\partial^2 u^0}{\partial x^2} + A_{22} \frac{\partial^2 u^0}{\partial y^2} + (A_{13} + A_{22}) \frac{\partial^2 v^0}{\partial x \partial y} - q_x = 0, \\
(A_{13} + A_{22}) \frac{\partial^2 u^0}{\partial y^2} + A_{22} \frac{\partial^2 v^0}{\partial x^2} + A_{33} \frac{\partial^2 v^0}{\partial y^2} - q_y = 0,
\]

(1.25)

where \( u^0 \) and \( v^0 \) are the displacements along \( x \) and \( y \) directions, and \( q_x \) and \( q_y \) are external
force projections along \( x \) and \( y \) directions.

Sinusoidal corrugated plate in Fig. 1.7 will be used again as an example for illustration.
In an early stage, \( A_{11} \) was thought to be so small to be neglected. Shimansky [94] reported
that for thin-walled, large degree of corrugation corrugated plates, the extension stiffness
\( A_{11} \) along \( x \)-axis is usually small in comparison to the extension stiffness \( A_{33} \) which is par-
allel to the corrugation (\( y \)-axis) because the most deformation is due to bending induced
by the corrugated geometry. However, for a thick wall or small degree of corrugation, the
extension stiffness \( A_{11} \) cannot be neglected because the deformation will mostly due to in-
plane extension. He also gave the relationship between extension stiffnesses \( A_{11} \) and \( A_{33} \)
depending on corrugation ratio and thickness ratio. The ratio \( (A_{11}/A_{33}) \) changes dramati-
cally from 1 to 0.0005. Free body diagram analysis again applied to several load cases and
the analytical expressions of extension stiffnesses [95–98] are

\[ A_{11} = E_x h = \frac{E}{1(1-\nu^2)} \left( \frac{h}{T} \right)^2 h, \]
\[ A_{33} = E_y h = \frac{S}{\varepsilon} Eh, \]
\[ A_{13} = \nu A_{11}, \]
\[ A_{22} = \frac{\varepsilon}{S} \frac{E \rho}{2(1+\nu)} h, \]

where \( \rho \) means the reduction factor of shear stiffness and \( \rho = 1 \) for ideal case without fasteners and connections. Briassoulis [91] simulated with shell finite elements by applying states of constant nominal strain to a representative element of corrugated plate and calculating the stresses. The extension stiffnesses were compared with the analytical solution [95–98] which showed some discrepancy and he proposed that the stiffnesses in Eq. (1.26) along \( x \) direction neglected the axial force \( q_x \) term in strain energy and bending moment respect to \( x \) induced by \( q_x \). In order to solve the problem, he made the constant slope assumption = \( \tan(\arccos(\varepsilon/S)) \) in half of the corrugation \( \varepsilon \), then derived expressions for extension stiffness of \( A_{11} \). For shear stiffness \( A_{22} \), he kept the same form as an isotropic flat plate which is different in Eq. (1.26).

\[ A_{11} = E_x h = \frac{E h}{[1+(T/h)^2(1-\nu^2)][(S/\varepsilon)^2-(S/2\pi\varepsilon)\sin(2\pi S/\varepsilon)]^{1/2}} , \]
\[ A_{33} = E_y h = \frac{S}{\varepsilon} Eh, \]
\[ A_{13} = \nu A_{11}, \]
\[ A_{22} = \frac{E}{2(1+\nu)} h. \]

Liew et al. [90, 99] used these formula from Briassouli [91], Eq. (1.27) and Eq. (1.22), to predict the buckling load in trapezoidal and sinusoidal corrugated plates. Most results showed good agreement with the FEA using ANSYS. The nonlinear results were also shown in Liew’s later paper [100] with the same equivalent properties. The free vibration analysis of corrugated plates was studied in his more recent paper Ref. [101].
Thin shell approach

Thin shell approach uses shell theory to model the corrugated plate and it is a step forward compared with the traditional analytical approach. Because the shell theory provides a closer modeling of the real corrugated structure, we can directly see the interaction between the bending and extension energy involved in a shell model, which reflects and explains the behaviors of corrugated plate more accurately. Another advantage is that tedious multiple free body diagram analyses are replaced by one energy homogenization which makes the process simple and straightforward. In Eq. (1.15), assumption of no coupling between extension and bending \( B = 0 \) is not true in general, and shell theory do not need this ad hoc assumption. The value of coupling stiffnesses \( B \) can be obtained simultaneously along with extension stiffnesses \( A \) and bending stiffnesses \( D \).

Moreover, the thin shell approach establishes the global and local connections which can be used to recover local behavior of the corrugated plate, such as strains or stresses. Libove proposed that a corrugated plate need to be viewed as a thin shell and a trapezoidal corrugated plate can be viewed as an assemblage of flat plate elements. He also developed total potential energy expression when corrugated plates are viewed as such. Later on, two of his students Perel and Hussain followed up his ideas and they worked on trapezoidal corrugated plates and curvilinear corrugated plates respectively. Giordano \[102\] compared the critical buckling loads from Perel \[103\], Wittrick and Williams \[104\] which showed the buckling load predicted by Perel is higher. This was commented later by Perel and Libove that the higher buckling loads are due to the assumption of zero transverse strains. After that, thin shell approach is adopted by Wu \[105\] and Hsiuo \[106\] to treat corrugated plates.

Kinematic assumptions had been made in relating shell displacements \( u_i \) to the homogenized plane displacements \( v_i \). The shell strain energy can be expressed by shell displacements \( u_i \), and the coefficients of the energy can be averaged over a period of corrugation by minimizing the fluctuations between \( \psi_i \). The calculus of variations was also used to obtain the governing differential equations regarding proper boundary conditions. Lee \[84\] proposed this idea to find the relation between the shell stresses and homogenized plane
displacement function of sinusoidal and triangular shape corrugated section, but the result
was dominated by the unexpected second-order derivative term of $u_z$ which made his ideal
unsuccessful. Andrianov et al. [107] applied asymptotic homogenization method in corru-
gated plates problem because the stress-strain state can be represented as sums of slow and
rapid variables. The projection of initial equations from shell to the homogenized plane was
used to build the homogenized equations. The homogenized bending moment compared
with oscillated moment was given as an example in their paper.

1.3 Present Work and Outline

As reviewed in the above section, the theory of homogenization is motivated by the
engineering applications and provides straight useful solutions to many different engineering
problems. In this dissertation, our efforts will be put on maximizing the freedom of choosing
RSE, using the VAM to homogenize randomly distributed heterogeneous material, and
homogenize a corrugated plates both analytically and numerically. Here is a brief description
of following chapters.

- Chapter 2 discusses the theoretical foundations of the present work.

- Chapter 3 presents the idea to maximize the freedom to choose a RSE and how to
generate periodic boundary condition in the RSE which lacks of paired nodes along
the edges/surfaces.

- Chapter 4 presents homogenization of the random cell structure using VAM which
gives the upper bound and lower bound.

- Chapter 5 presents the application of VAM to solve corrugated plate problem both
analytically and numerically.

- Chapter 6 presents the construction of SERDF in solving thin-walled RSE problems.

- Chapter 7 summarizes the conclusions and offers recommendations for future work.
Chapter 2

Theoretical Foundation

In a word, a homogenization problem can be described as using proper method to solve proper domain with proper boundary conditions. These three factors, proper method as variational asymptotic method; proper domain as the concept of RSE; proper boundary conditions as periodic boundary conditions, compose the fundamental of current research presented in this research.

2.1 Variational-Asymptotic Method (VAM)

A review of VAM is presented here in order to provide an insight to the theory used in present work.

In the static case, Hamilton’s variational principle degenerates into the principle of minimum total potential energy (PMTPE). The admissible variations (functions of the space variables only) are those that satisfy continuity conditions and geometric boundary conditions. The advantage of variational method is that only geometric boundary conditions need to be satisfied and the disadvantage of this method is its restriction to conservative system (path independent). A static conservative system is in equilibrium if its potential energy is stationary. The principle of minimum potential energy states: “Of all displacements satisfying the given boundary conditions, those which satisfy the equilibrium equations make the potential energy a minimum.” The equilibrium equations can be derived from the variation of the total potential energy following the normal procedure of calculus of variations. They are referred to as the Euler-Lagrange equations according to the calculus of variations.

An asymptotic series is a good approximation to original series when a certain parameter is small. When this small parameter vanishes, the series exactly reproduces the first term
of original series. Because the mathematical models used in physics often lead to problems with which the solutions are hard to obtain. This is even worse when some small parameters are present. The motivation of asymptotic method is to simplify the solution process and provide a way to obtain an approximate solution of the initial model. After that, the effective properties can be used with other numerical technics, such as FEA to obtain the solutions of the global system. In most cases, the first asymptotic approximation can give a good balance between the efforts and necessary accuracy, while higher approximations are required when the dominant parameters changed by situations. The asymptotic analysis requires one to learn how to introduce small parameters into a system. Here are some well known small terms in physical problems:

- In beam theory, $h/l$, with $h$ as the characteristic size of the cross-section, $l$ the characteristic wavelength of the deformation along the beam reference line.

- In shell theory, $h/l$, with $h$ as the thickness, $l$ the characteristic wavelength of the deformation of the reference surface.

- In micromechanical analysis, $h/l$, with $h$ as the characteristic size of the UC and $l$ as the characteristic wavelength of the deformation of the macroscopic material.

Besides geometric parameters, there also exist possibilities that more parameters could be great or small depending on the analyzers’ experience and understanding of the problem. Such as for sandwich plate, the ratio of elastic moduli of the core and the skin could be a small parameter [108], which will change the asymptotic analysis of the system. However, with the introductions of more small parameters, one needs to keep in mind the limitations of these assumptions and the proper application domain. On the other hand, if the small parameter is not naturally exist, it still possible to choose certain objects such that the solution appear to close to one of them [109].

Here, the fundamental symbols and definitions of asymptotical analysis will be introduced: $O$, $o$, and $\sim$. Functions $f(x)$ and $g(x)$ are continuous functions defined on some
domain and possessing limits as \( x \to x_0 \) in the domain. We can define the following shorthand notation for the relative properties of these functions in the limit \( x \to x_0 \).

- \( f(x) = O(g(x)) \) as \( x \to x_0 \) if \( |f(x)| \leq K|g(x)| \) in the neighborhood of \( x_0 \) with \( K \) denoting a constant. We say that \( f(x) \) is asymptotically bounded by \( g(x) \) in magnitude as \( x \to x_0 \) or \( f(x) \) is of the order of \( g(x) \).

- \( f(x) = o(g(x)) \) as \( x \to x_0 \) if \( |f(x)| \leq \epsilon|g(x)| \) in the neighborhood of \( x_0 \) for all positive value \( \epsilon \). We say that \( f(x) \) is asymptotically smaller than \( g(x) \).

- \( f(x) \sim g(x) \) as \( x \to x_0 \) if \( f(x) = g(x) + o(g(x)) \) in the neighborhood of \( x_0 \). We say that \( f(x) \) is asymptotically equal to \( g(x) \).

To correctly recognize small terms, we not only need to know the asymptotic order of the functions, but also often need to know the asymptotic order of their derivatives.

Consider a function \( f(x) \) defined for \( x \in [a, b] \) and sufficiently smooth in this domain. We denote the amplitude of change of \( f(x) \) on \([a, b]\) as the maximum difference of the function evaluated at any two points in its domain, \( i.e. \)

\[
\tilde{f} = \max_{x_1, x_2 \in [a, b]} |f(x_1) - f(x_2)|.
\] (2.1)

Then for a sufficiently small number \( l \), the following inequality holds

\[
\left| \frac{df}{dx} \right| \leq \frac{\tilde{f}}{l}.
\] (2.2)

The largest constant \( l \) satisfying the above inequality is termed the characteristic length of function \( f(x) \) in its own definition domain. If we need to estimate higher derivatives, then the corresponding terms are included in the definition of \( l \), and the characteristic length is the largest constant satisfying the following inequalities

\[
\left| \frac{df}{dx} \right| \leq \frac{\tilde{f}}{l}, \quad \left| \frac{d^2f}{dx^2} \right| \leq \frac{\tilde{f}}{l^2}, \quad \cdots, \quad \left| \frac{d^k f}{dx^k} \right| \leq \frac{\tilde{f}}{l^k}.
\] (2.3)
where $k$ is the highest derivative we want to estimate the asymptotic order. This definition of characteristic length can be easily generalized to functions of multiple variables.

The VAM is a synthesis of variational method and asymptotic analysis proposed by Prof. Berdichevsky decades ago [110]. It allows one to consider the minimization problems for functions of a finite number of variables and the problems for differential equations possessing the variational structure. The first advantage of this method is simplicity. It simplifies the solving process from putting asymptotic solving process in a system of differential equations with different orders to solving one variational problem asymptotically. The second advantage of this method is that it is variational which is easy to be implemented numerically by FEA, thus the applicable area is not constrained to simple problem with analytical solutions. The VAM is based on neglecting small energy terms. However, when neglecting small terms, it is important to understand how the small terms affect the next approximations, how to handle the loss of uniqueness or the existence of the solutions and how the accuracy to be evaluated. A recent book [111,112] covers all aspects of this theory, and more details may be found there for interested readers. Although, some asymptotic methods give the same result to VAM, but they are difficult to apply, having long series equilibrium equation divided by different orders, and more problematic when dealing with corresponding terms of the series.

**An example using VAM**

Here we use an example to show the basic idea and procedure of the VAM. This example has been used in [111] for illustration of VAM. Let a functional $f(u, \varepsilon)$ depending on a small parameter $\varepsilon$ be given at some set $M$ of elements $u$. Using the variational-asymptotic method to investigate the stationary points of the function of one variable $u$

$$f(u, \varepsilon) = u^2 + u^3 + 2\varepsilon u + \varepsilon u^2 + \varepsilon^2 u,$$

with $\varepsilon$ as a small parameter.
The stationary points of the function $f(u, \varepsilon)$ can be exactly solved as

$$\tilde{u} = \frac{1}{3} \left( -1 - \varepsilon \pm \sqrt{1 - 4\varepsilon - 2\varepsilon^2} \right). \quad (2.5)$$

This can be expanded asymptotically in terms of $\varepsilon$ as

$$\tilde{u} = \begin{cases} 
- \frac{2}{3} + \frac{\varepsilon}{3} + \varepsilon^2 + o(\varepsilon^2) \\
0 - \varepsilon - \varepsilon^2 + o(\varepsilon^2) 
\end{cases} \quad (2.6)$$

Then, we use VAM to approximate the solution and compare with it.

- **Zeroth-order approximation:** $f_0(u) = f(u, 0) = u^2 + u^3$. The two stationary points are $\tilde{u}_0 = 0$ and $\tilde{u}_0 = -\frac{2}{3}$ and they matches the first terms in stationary points in Eq.(2.6).

- **First-order approximation:** Let us express $u = \tilde{u}_0 + u'$, where $u' \to 0$ for $\varepsilon \to 0$. Substitute this expression into original function and keep the leading terms containing $u'$. In the neighborhood of $-\frac{2}{3}$, from $u = -\frac{2}{3} + u'$ we obtain the following function

$$f(-\frac{2}{3} + u', \varepsilon) = -u'^2 + \frac{2u'\varepsilon}{3} + u'^3 + u'^2\varepsilon + u'^2\varepsilon^2 + \frac{4}{27} - \frac{8\varepsilon}{9}. \quad (2.7)$$

The double underlined terms are additive constants that will not affect the stationary points and can be simply dropped. The underlined terms are much smaller than those non-underlined terms. To be specific,

$$|u'^3| \ll |u'^2|, \quad |u'^2\varepsilon| \ll |u'^2|, \quad |u'^2\varepsilon^2| \ll \left| \frac{2u'\varepsilon}{3} \right|, \quad (2.8)$$

in view of the fact that both $u'$ and $\varepsilon$ are small. Keeping the leading terms with respect to $u'$ in the function $f(-\frac{2}{3} + u', \varepsilon)$, we arrived at the following function

$$f_1(u', \varepsilon) = -u'^2 + \frac{2u'\varepsilon}{3}. \quad (2.9)$$
It is stationary when $u' = \frac{1}{3} \varepsilon$. Note that the asymptotic order of $u'$ is not assumed a priori, but is determined as the stationary point of the function $f_1(u', \varepsilon)$. Hence, we have obtained the first-order approximation of the stationary point in the neighborhood of $-\frac{2}{3}$ as

$$\tilde{u}_0 + \tilde{u}_1 = -\frac{2}{3} + \frac{1}{3} \varepsilon + o(\varepsilon).$$

(2.10)

The first-order approximation in the neighborhood of 0, which is the other solution of $\tilde{u}_0$, can be obtained analogously. Setting $u = 0 + u'$, we obtain the following function

$$f(u', \varepsilon) = u'^2 + 2 \varepsilon u' + u'^3 + \varepsilon \varepsilon + 2 \varepsilon u'.$$

(2.11)

The underlined terms are much smaller than those non-underlined terms. That is

$$|u'^3| \ll |u'^2|, \quad |u'^2 \varepsilon| \ll |u'^2|, \quad |u' \varepsilon^2| \ll |2u' \varepsilon|,$$

(2.12)

in view of the fact that both $u'$ and $\varepsilon$ are small. Keeping the leading terms with respect to $u'$ in the function $f(u', \varepsilon)$, we arrive at the following function

$$f_1(u', \varepsilon) = u'^2 + 2 u' \varepsilon.$$

(2.13)

It is stationary when $u' = -\varepsilon$. Hence, we have obtained the first-order approximation of the stationary point in the neighborhood of 0 such that

$$\tilde{u}_0 + \tilde{u}_1 = 0 - \varepsilon + o(\varepsilon).$$

(2.14)

Till now, we have reproduced that the first two terms of the asymptotic expansion of the exact solution. We can continue this process to find higher-order approximations.

• **Second-order approximation**: Similarly, let us express $u = \tilde{u}_0 + \tilde{u}_1 + u''$, where $u'' \to 0$ for $\varepsilon \to 0$. Substitute this expression into original function and keeping the leading terms containing $u''$. In the neighborhood of $-\frac{2}{3}$, from $u = -\frac{2}{3} + \frac{\varepsilon}{3} + u''$ we
obtain the leading term
\[
f_2(u'', \varepsilon) = 2\varepsilon^2 u'' - (u'')^2.
\] (2.15)

It is stationary when \( u'' = \varepsilon^2 \), such that
\[
\ddot{u}_0 + \ddot{u}_1 + \ddot{u}_2 = -\frac{2}{3} + \frac{1}{3}\varepsilon + \varepsilon^2 + o(\varepsilon^2).
\] (2.16)

In the neighborhood of 0, we set \( u = -\varepsilon + u'' \). Keeping the leading terms, we get the functional
\[
f_2(u'', \varepsilon) = u''^2 + 2u''\varepsilon^2.
\] (2.17)

The stationary point of \( f_2(u'', \varepsilon) \) is \( u'' = -\varepsilon^2 \), such that
\[
\ddot{u}_0 + \ddot{u}_1 + \ddot{u}_2 = 0 - \varepsilon - \varepsilon^2 + o(\varepsilon^2).
\] (2.18)

To this point, it shows we reconstruct the exact solution the same as Eq.(2.6) up to the order of \( o(\varepsilon^2) \).

This example demonstrates that the main difficulty in the asymptotic analysis is to recognize the leading terms and the negligible terms. Usually, this is the most important and most difficult part of the asymptotic analysis. To determine which terms are negligible, we need to consider the following two conditions.

- For two terms \( A(u, \varepsilon) \) and \( B(u, \varepsilon) \) which are summed in the functional \( I(u, \varepsilon) \), if
  \[
  \lim_{\varepsilon \to 0} \max_{u \in M} \frac{|B(u, \varepsilon)|}{|A(u, \varepsilon)|} = 0,
  \] (2.19)
  then \( B(u, \varepsilon) \) is negligible in comparison to \( A(u, \varepsilon) \) for all stationary points. Such terms are called globally secondary.

- Let \( \ddot{u} \to 0 \) for \( \varepsilon \to 0 \), and for any sequence \( \{u_n\} \) converging to \( u = 0 \). If
  \[
  \lim_{n \to \infty} \lim_{\varepsilon \to 0} \frac{|B(u, \varepsilon)|}{|A(u, \varepsilon)|} = 0,
  \] (2.20)
then \( B(u, \varepsilon) \) is negligible in comparison to \( A(u, \varepsilon) \) for the stationary point \( \tilde{u}_\varepsilon \). Such terms are called locally secondary.

In the illustrative example, the term \( \varepsilon u^2 \) is globally secondary with respect to \( u^2 \), the term \( \varepsilon^2 u \) is globally secondary with respect to \( 2\varepsilon u \) while \( u^3 \) is locally secondary with respect to \( u^2 \) in the neighborhood of the point \( u = 0 \).

### 2.2 Representative Structural Element (RSE)

In the literature, the concepts of a representative volume element (RVE) which contains size, shape, spatial distribution and properties of the microstructural constituents and their respective interfaces is widely used in micromechanics. The RVE is a micro-mechanical model to obtain the response of the corresponding homogenized macroscopic continuum, thus, the accuracy of the homogenized properties heavily depend on the proper choice of RVE. Hill [113] proposed that a RVE is structurally entirely typical of the whole mixture on average and contains a sufficient number of inclusions for the apparent overall moduli to be effectively independent of ‘macroscopically uniform’ surface values of traction and displacement. Hashin and Shtrikman [114] described their reference cube as an element taken from the whole body, which is large in comparison to the crystals yet small in comparison to the whole body. The mean value of strain or stress, is the same for the whole body and for the reference cube. Drugan and Willis [115] used the statistical nature of the real composite microstructures to argue that the smallest RVE needs to be sufficiently large to be statistically representative of the composite, which implies that the RVE needs to include a very large number of micro-heterogeneities (such as grains, inclusions, voids, cracks, fibers, etc.). Although many types of RVE are used in the literature including concentric cylinders [31], cubes [114], square arrays [116], square and hexagonal arrays [117], and rectangular models [118, 119]. RVEs with straight edges/surfaces are the popular choices, particularly for modeling realistic materials in a numerical fashion.

In the present work, the concept of RVE is expand to a broader concept, namely representative structural element (RSE), in which the scale is beyond microstructure, such
as representative for a structure (beam, plate, shell, foam, etc.). The application of homogenization theory should not be restricted to micromechanics when carrying asymptotic analysis, because the large and small parameters in asymptotic analysis are relative. For instance, let us consider the “displacement” from the earth to the moon, it is safe to neglect the difference from the highest mountain or from the lowest trench under the sea. Though this difference is large, when putting into the the distance between the moon and the earth, this fluctuation is relatively small. The key point in homogenization is separation of scales. Thus, we use the concept of RSE which covers all kinds of heterogeneous materials and the engineering structure with representative features. More specifically, the heterogeneous materials can be metal alloys, metal foams, composites, perforated materials with the holes, and etc. The representative engineering structure includes sandwich structure, corrugated plate, cranes (Fig. 2.1 (perforated structure with a very small amount of materials along the bars)), towers (Fig. 2.2 (perforated structures with big hole and very small amount of materials along the walls)) and etc. The RSE represents the basic building block of a structure.

![Fig. 2.1: Crane RSE.](image1)

![Fig. 2.2: Tower RSE.](image2)

The validity of the RSE assumption could be verified by a theoretical analysis, where some clear periodicity is identified or statistical homogeneity and ergodicity exists. Although these RSE definitions are based on valid arguments, we would like to define RSE as any block of material or structure the analyst wants to use for homogenization, giving the freedom to
the analyst in choosing a RSE of his or her convenience. Of course, we rely on the analyst’s judgement to ensure the representativeness of RSE so that the fundamental postulate of homogenization theory will not be violated. The term unit cell (UC) is also used extensively in the literature and sometimes it is used interchangeably with the concept RVE. Here and throughout the dissertation we define UC as the smallest RSE, which means an RSE can contain several UCs but still present the same effective material properties. For a strict periodic media or structure, such as a fiber-reinforced composite arranged in a periodic manner, one can always chooses a single fiber of regular shape with matrix surrounding it as a UC. However, in reality, the randomness in geometry always exists and the selected region might not necessarily be periodic in its geometry. Sab [120] claimed that any homogenization result holds for periodic media, holds also for statistically periodic ergodic and statistically homogeneous ergodic random media. Thus, as long as the heterogeneity satisfy the ergodic hypothesis that the moving average and the ensemble average are constants and equal, this moving window can be considered as a RSE and homogenization can be applied.

2.3 Hill-Mandel Condition and Boundary Conditions

The Hill-Mandel condition can be considered as a fundamental theorem of homogenization theory. It belongs to the law of conservation of energy which defines that the strain energy in heterogeneous RSE and homogenized RSE should be equal. This condition provides a link between an experimentalist’s and a theoretician’s viewpoint on homogenization as well as a way of spatial and statistical averaging of the RSE. This principle can be expressed through stress and strain tensors $\sigma_{ij}$ and $\epsilon_{ij}$ as

$$\frac{1}{2} \langle \sigma_{ij} \epsilon_{ij} \rangle = \frac{1}{2} \langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle.$$  \hspace{1cm} (2.21)

Here and throughout the dissertation, summation convention of Latin letter is applied. The angle bracket $\langle \cdot \rangle$ denotes the average value which is calculated as its integral over a specified region divided by the volume of the region. Indeed, the real stress and strain can
be expanded by their averages and fluctuations

\[ \sigma_{ij} = \langle \sigma_{ij} \rangle + \sigma'_{ij}, \quad \epsilon_{ij} = \langle \epsilon_{ij} \rangle + \epsilon'_{ij}. \]  

(2.22)

Note \( \langle \sigma'_{ij} \rangle = \langle \epsilon'_{ij} \rangle = 0 \), then

\[ \frac{1}{2} \langle \sigma_{ij} \epsilon_{ij} \rangle = \frac{1}{2} \langle \sigma_{ij} \rangle \langle \epsilon_{ij} \rangle + \frac{1}{2} \langle \sigma'_{ij} \epsilon'_{ij} \rangle. \]  

(2.23)

To satisfy Hill-Mandel Condition in Eq. (2.21), it requires \( \langle \sigma'_{ij} \epsilon'_{ij} \rangle = 0 \) which leads to

\[
0 = \langle \sigma'_{ij} \epsilon'_{ij} \rangle \\
= \frac{1}{\Omega} \int_{\Omega} \sigma'_{ij} \epsilon'_{ij} dV \\
= \frac{1}{\Omega} \int_{\Omega} (\sigma_{ij} - \langle \sigma_{ij} \rangle ) (u_i - \langle u_i \rangle )_{,j} dV \\
= \frac{1}{\Omega} \int_{\partial \Omega} (t_i - \langle \sigma_{ij} \rangle n_j)(u_i - \langle \epsilon_{ij} \rangle x_j) dS.
\]  

(2.24)

Here \( u_i \) denotes the displacement, strain is \( \epsilon_{ij} = \frac{1}{2} (u_{i,j} + u_{j,i}) \), and divergence theorem is also applied to change integration domain. The commonly used boundary condition can be extracted from Eq. (2.24)

1. \( t_i = \langle \sigma_{ij} \rangle n_j \). Static uniform boundary conditions, also called Neumann boundary conditions where uniform tractions are prescribed on the edges of the RSE.

2. \( u_i = \langle \epsilon_{ij} \rangle x_j \). Kinematic uniform boundary conditions, also called Dirichlet or essential boundary conditions where uniform displacements are applied to the boundary.

3. \( (t_i - \langle \sigma_{ij} \rangle n_j)(u_i - \langle \epsilon_{ij} \rangle x_j) = 0 \). Mixed boundary conditions, such as displacements are prescribed in one part of the boundary and forces are prescribed on the remaining part boundary.

4. Periodic boundary conditions, also a Dirichlet type boundary condition, can satisfy the \( (u_i^+ - u_i^-) = \langle \epsilon_{ij} \rangle (x_j^+ - x_j^-) \) with \( (u_i^+) = (u_i^-) \) as periodic fluctuation on the
boundaries $\partial \Omega^+$ and $\partial \Omega^-$ where $\partial \Omega = \partial \Omega^+ \cup \partial \Omega^-$. As a consequence, FE-mesh on one side of a RSE, must be completely mirrored by the mesh on the other side.

The boundary conditions have to be carefully chosen to obtain accurate effective material properties in the homogenization process. Mixed boundary conditions 3 are mostly used in experimental settings. Boundary conditions 1, 2 and 4 have been used to solve homogenization problem analytically or numerically. The static uniform boundary underestimates the stiffness while the kinematic uniform boundary overestimates the actual ones. Hill [121] estimated the difference between these two predictions as the order of $(\delta/\varepsilon)^3$, where $\delta$ is the characteristic length of heterogeneity and $\varepsilon$ the size of RSE. From this relation, we can conclude that there are size effects related with these two boundary conditions and only if the size of RSE is large enough compared with the size of heterogeneities, two predictions are close to each other. But, in most cases, the “large enough” of RSE also increases the computational effort and reduces the efficiency of analysis.

For boundary condition 4, the periodic boundary condition gives more reasonable estimation. It can be considered as a weak form of kinematic uniform boundary conditions, thus the estimation of effective stiffnesses is smaller than the estimation from the boundary condition 2. Indeed, the estimate for periodic boundaries is between the one calculated from static uniform boundaries and kinematic uniform boundaries which is known as universal inequality for effective moduli for elastic media. The periodic boundary conditions yield the strain energy which is among the minimum from displacement boundary conditions as well as maximum among the traction boundary conditions. Moreover, the strict periodic UC pattern is another reason for applying periodic boundary condition for UC as homogenized domain. It is also pointed out in [18] that for a not-so-strict periodic RSE geometry, the periodic boundary condition gives the best estimate among all the boundary conditions. There are ample numerical evidences that periodic boundary conditions are effective even when the microstructure is non-periodic, which is the most common situation. Thus, we adopt the periodic boundary condition to satisfy the Hill-Mandel Condition and solve homogenization problems. The periodicity of the fluctuation functions guarantees that the
effective properties are the independent of the choice of the periodic RSE. As a result, we can choose the most convenient RSE to carry out the analysis.

2.4 A Brief Introduction of the VAMUCH Theory

VAMUCH has been continuously developed as a general purpose micro mechanics code for multiphysics modeling of heterogeneous materials [122]. It has been demonstrated as an efficient and accurate method for periodic heterogeneous materials. The heterogeneous composites can be homogenized into equivalent homogeneous materials having effective properties. The VAMUCH can be used to not only to predict the complete set of fully coupled multiphysical properties but also the corresponding local fields. As the basic principles of the VAMUCH theory are used for all these applications, it is adequate to use linear elastic materials as an illustration for simplicity. Considering the periodicity as a small parameter, the variational statement is formulated over the unit cell through an asymptotic expansion of the energy functional. It is shown that the governing differential equations and periodic boundary conditions of mathematical homogenization theories can be reproduced from variational statement.

Exploiting the fact of the existence of UCs with the characteristic UC size much smaller than the characteristic wavelength of the macroscopic deformation, one can formulate the following constrained minimization problem as presented in Ref. [51]

\[
\frac{1}{2} \bar{C}_{ijkl} \bar{\epsilon}_{ij} \bar{\epsilon}_{kl} = \min_{\text{periodic } \chi_i} \frac{1}{2\Omega} \int_{\Omega} C_{ijkl} \left[ \bar{\epsilon}_{ij} + \chi_{(ij)} \right] \left[ \bar{\epsilon}_{kl} + \chi_{(kl)} \right] d\Omega, \quad (2.25)
\]

where \( \bar{\epsilon}_{ij} \) is the global strain tensor, \( \chi_i \) is the fluctuating function, \( \Omega \) is the domain occupied by the unit cell, \( C_{ijkl} \) is the fourth-order elasticity tensor which is position dependent as the UC is heterogeneous, \( \bar{C}_{ijkl} \) is the effective elasticity tensor for the UC after homogenization, \( \chi_{(ij)} = \frac{1}{2}(\chi_{i,j} + \chi_{j,i}) \) with comma denoting the partial derivative with respect to the unit cell coordinates, \( i.e., (\cdot)_{,i} = \frac{\partial (\cdot)}{\partial y_i} \). The constraint that the fluctuating function \( \chi_i \) is periodic implies that \( \chi_i \) must be equal on the corresponding edges/surfaces of the UC. The constraint is a direct consequence from the essential requirement that any choice of UC must be
representative of the entire composites.

Although the analytical form of the constrained minimization problem in Eq. (2.25) can be used directly to obtain exact solution for binary composites [123, 124], we used the FEA in VAMUCH to solve this problem to take advantage of the versatility of FEA in meshing UC with arbitrary microstructure. Introduce the following matrix notations

\[
\bar{\varepsilon} = \begin{bmatrix} \bar{\varepsilon}_{11} & 2\bar{\varepsilon}_{12} & 2\bar{\varepsilon}_{13} & 2\bar{\varepsilon}_{23} & \bar{\varepsilon}_{33} \end{bmatrix}^T.
\] (2.26)

\[
\begin{bmatrix}
\frac{\partial \chi_1}{\partial y_1} \\
\frac{\partial \chi_1}{\partial y_2} + \frac{\partial \chi_2}{\partial y_1}
\frac{\partial \chi_2}{\partial y_2} \\
\frac{\partial \chi_2}{\partial y_3} + \frac{\partial \chi_3}{\partial y_1}
\frac{\partial \chi_3}{\partial y_2} \\
\frac{\partial \chi_3}{\partial y_3}
\end{bmatrix}
= \begin{bmatrix}
\frac{\partial \chi_1}{\partial y_1} & 0 & 0 \\
\frac{\partial \chi_2}{\partial y_2} & \frac{\partial \chi_2}{\partial y_1} & 0 \\
0 & \frac{\partial \chi_3}{\partial y_2} & 0 \\
0 & 0 & \frac{\partial \chi_3}{\partial y_3}
\end{bmatrix}
\begin{bmatrix}
\chi_1 \\
\chi_2 \\
\chi_3
\end{bmatrix}
\equiv \Gamma_h \chi,
\] (2.27)

where \( \Gamma_h \) is an operator matrix and \( \chi \) is a column matrix containing the three components of the fluctuating functions. If we discretize \( \chi \) using the finite elements as

\[
\chi(x_i; y_i) = S(y_i)\mathcal{X}(x_i),
\] (2.28)

where \( S \) represents the shape functions and \( \mathcal{X} \) a column matrix of the nodal values of the fluctuating functions. Then we can convert the functional in right hand side of Eq. (2.25) into the following discretized version

\[
\Pi_\Omega = \frac{1}{2\Omega} (\mathcal{X}^TD_{EE}\mathcal{X} + 2\mathcal{X}^TD_{he}\bar{\varepsilon} + \bar{\varepsilon}^TD_{\varepsilon\varepsilon}),
\] (2.29)

where

\[
D_{EE} = \int_\Omega (\Gamma_h S)^T D (\Gamma_h S)d\Omega, \quad D_{he} = \int_\Omega (\Gamma_h S)^T D d\Omega, \quad D_{\varepsilon\varepsilon} = \int_\Omega D d\Omega,
\] (2.30)
with $D$ as the $6 \times 6$ material matrix condensed from the fourth-order elasticity tensor $C_{ijkl}$. Denoting $\mathcal{X} = X_0\mathcal{E}$, we can minimize $\Pi_\Omega$ in Eq. (2.29) as

$$D_{EE}\mathcal{X}_0 = -D_{he}. \quad (2.31)$$

There are two constraints for the fluctuating functions where the first is

$$\frac{1}{\Omega} \int_{\Omega} \chi_i d\Omega = 0, \quad (2.32)$$

which indicates $\chi_i$ have the zero volume averages over the UC. The second constraint is $\chi_i$ periodic, which implies $\chi_i$ must be equal on the corresponding edges/surfaces of the UC. It is a direct consequence from the essential requirement that any choice of UC must be representative of the entire composites. After applying these two constraints to Eq. (2.31), we reach a linear system

$$\widetilde{D}_{EE}\tilde{X}_0 = -\widetilde{D}_{he}, \quad (2.33)$$

to solve for $\tilde{X}_0$. Then we can obtain the effective stiffness matrix $\tilde{D}$ corresponding to $\tilde{C}_{ijkl}$ in Eq. (2.25) as

$$\tilde{D} = \frac{1}{\Omega} (\tilde{X}_0^T\tilde{D}_{he} + D_{ee}) \quad (2.34)$$

The effective elastic constants such as Young’s moduli, Poisson’s ratios and shear moduli can be calculated from $\tilde{D}$ if the effective material features at least orthotropic symmetry.

Recovering $X_0$ from $\tilde{X}_0$ with Eq. (2.32) and periodic condition, the local strain field can be recovered using the following formula

$$\epsilon = \mathcal{E} + \Gamma_h S \mathcal{X}_0 \mathcal{E}, \quad (2.35)$$

and the local stress field can be recovered straightforwardly as

$$\sigma = D\epsilon. \quad (2.36)$$
Clearly from the above two equations, we observe that computation of local stress and strain fields only involves some simple matrix multiplications and additions and the expensive linear system is completely avoided for the recovery procedure.

**A vibration example**

In this section, a modal analysis is used to give a comparison between the direct-FEA and homo-FEA. Here the direct-FEA stands for the direct finite element analysis of the original structure which contains a large number of unit cells. The homo-FEA means that, first the homogenization is carried on to obtain the effective properties using VAMUCH, particularly the effective stiffness matrix and effective density in this example; Second, the effective properties are applied to the whole structure, usually, the mesh of the new structure is much simpler (with much less number of more regular elements) compared with the original structure because the micro-level complexities are homogenized using VAMUCH. Through this example, we will also see some restrictions of the homogenization method. With the periodic boundary condition applied in all the three directions of unit cell, it intrinsically implies that there are a large number of unit cells along those three directions. However, the repeated unit cell in some directions may violate this assumption which makes the effective properties become an approximation and will introduce some errors compared to direct-FEA.

Let us consider a heterogeneous panel with length=10 mm, width=10 mm and thickness=1 mm shown in Fig. 2.3 which has reinforced square fiber in it. The Young’s modulus, Poisson’s ratio, and density for the fiber are 379300 MPa, 0.1, and 783 Kg/m$^3$, respectively, and for the matrix are 68300 MPa, 0.3, and 500 Kg/m$^3$.

A 2D square UC can be extracted from the original heterogeneous panel as shown in Fig. 2.4. The length of unit cell is 1 mm and volume fraction of the fiber is 40%. With VAMUCH, the effective elastic stiffnesses and effective density can be calculated in Table 2.1 where directions 2 and 3 are in-plane directions, and direction 1 is out-plane direction normal to the plane of the UC.

The direct-FEA and homo-FEA solutions are obtained in ANSYS. Two sets of meshes
Fig. 2.3: Heterogenous structure.

Fig. 2.4: Unit cell of the heterogeneous panel.

Table 2.1: Effective properties obtained using VAMUCH.

<table>
<thead>
<tr>
<th>Models</th>
<th>$\bar{E}_1$(GPa)</th>
<th>$\bar{E}_2$(GPa)</th>
<th>$\bar{E}_3$(GPa)</th>
<th>$\bar{G}_{12}$(GPa)</th>
<th>$\bar{G}_{13}$(GPa)</th>
<th>$\bar{G}_{23}$(GPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC</td>
<td>193.6</td>
<td>132.4</td>
<td>132.4</td>
<td>49.14</td>
<td>49.14</td>
<td>41.70</td>
</tr>
<tr>
<td>Models</td>
<td>$\bar{\nu}_{12}$</td>
<td>$\bar{\nu}_{13}$</td>
<td>$\bar{\nu}_{23}$</td>
<td>$\bar{\rho}$ (Kg/m$^3$)</td>
<td></td>
<td></td>
</tr>
<tr>
<td>UC</td>
<td>0.2081</td>
<td>0.2081</td>
<td>0.2548</td>
<td>613.2</td>
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<td></td>
</tr>
</tbody>
</table>
Table 2.2: Frequency comparison.

<table>
<thead>
<tr>
<th>Mode</th>
<th>Direct-FEA (Hz)</th>
<th>Homo-FEA (Hz)</th>
<th>Error</th>
</tr>
</thead>
<tbody>
<tr>
<td>Thickness=1mm</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>82223</td>
<td>88671</td>
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</tr>
<tr>
<td>8</td>
<td>121770</td>
<td>142670</td>
<td>17.16%</td>
</tr>
<tr>
<td>9</td>
<td>146770</td>
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</tr>
<tr>
<td>12</td>
<td>336300</td>
<td>386220</td>
<td>14.84%</td>
</tr>
<tr>
<td>13</td>
<td>364500</td>
<td>398920</td>
<td>9.44%</td>
</tr>
<tr>
<td>14</td>
<td>381710</td>
<td>440280</td>
<td>15.34%</td>
</tr>
<tr>
<td>15</td>
<td>395510</td>
<td>459830</td>
<td>16.26%</td>
</tr>
</tbody>
</table>

of SOLID185 element are chosen in analysis to handle two cases as shown in Fig. 2.5 and Fig. 2.6. Free-free boundary conditions are set in both cases and the modes are extracted with Block Lanczos method. Frequencies of mode 7 to mode 15 (mode 1 to mode 6 are rigid body modes and thus excluded) from two cases are listed in Table 2.2 and first three mode shapes are shown in Fig. 2.7. Modal Assurance Criterion (MAC) plot is shown in Fig. 2.8 which indicates the same mode shape pairs from mode 7 to mode 13.

From the results, the simulation between direct-FEA and homo-FEA show that the mode shapes match pretty well, though the deviation between the frequencies are noticeable. The reason is that along the thickness direction, there is no repeated unit cell which violates the periodic requirement along the thickness direction in the unit cell. As a result, the
Fig. 2.7: Mode shape comparison.
effective stiffnesses become too stiff when applying to a single layer case. For such structures, it is better to consider them as plate and follow the work in [125] to obtain the equivalent plate stiffnesses. In the direction along the length and width of the plate, the unit cell repeats 10 times which we consider as enough for periodic requirement. To further test the applicable requirement of current VAMUCH, we increase the thickness of the panel from 1 mm up to 5 mm. The same modal analysis are carried out and the results shown in Table 2.3. It is clear seen that with the increasing repeated UC along thickness direction, the difference between two approaches becomes small. Considering the number of elements in homo-FEA model is less than half of them in the direct-FEA model, this example demonstrate the accuracy and efficiency of the homogenization method, and VAMUCH application in modal analysis.
<table>
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<th>Direct-FEA (Hz)</th>
<th>Homo-FEA (Hz)</th>
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<td>251200</td>
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<td>309160</td>
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</tr>
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<td>3.00%</td>
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<td>639600</td>
<td>2.58%</td>
<td></td>
</tr>
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<td>Direct-FEA (Hz)</td>
<td>Homo-FEA (Hz)</td>
<td>Error</td>
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<td>Direct-FEA (Hz)</td>
<td>Homo-FEA (Hz)</td>
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<td>711860</td>
<td>700090</td>
<td>-1.65%</td>
<td></td>
</tr>
<tr>
<td>Mode</td>
<td>Thickness=5mm</td>
<td>Direct-FEA (Hz)</td>
<td>Homo-FEA (Hz)</td>
<td>Error</td>
</tr>
<tr>
<td>7</td>
<td>302450</td>
<td>301980</td>
<td>-0.15%</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>458400</td>
<td>459090</td>
<td>0.15%</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>552590</td>
<td>561020</td>
<td>1.53%</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>563260</td>
<td>564560</td>
<td>0.23%</td>
<td></td>
</tr>
<tr>
<td>11</td>
<td>597970</td>
<td>601380</td>
<td>0.57%</td>
<td></td>
</tr>
<tr>
<td>12</td>
<td>606750</td>
<td>610270</td>
<td>0.58%</td>
<td></td>
</tr>
<tr>
<td>13</td>
<td>658610</td>
<td>657890</td>
<td>-0.11%</td>
<td></td>
</tr>
<tr>
<td>14</td>
<td>677780</td>
<td>676880</td>
<td>-0.13%</td>
<td></td>
</tr>
<tr>
<td>15</td>
<td>711000</td>
<td>699450</td>
<td>-1.63%</td>
<td></td>
</tr>
</tbody>
</table>
Chapter 3

Construction of RSE

The first step of the homogenization analysis is to get a represent structural elements (RSE) from the original structure. Under the current framework of VAMUCH, the periodic boundary conditions are applied to the RSE. To maximize the computational efforts, the smallest RSE which is so called unit cell (UC) is used in VAMUCH. By repeating the UCs, the original material or structure can be rebuilt exactly or statistically.

For heterogeneities, the heterogeneous patterns can be categorized along 1, 2, and 3 directions. There are three types of unit cells.

- 1D UC: Materials change along the thickness direction but stay the same along the in-plane directions. Such as binary composite.

- 2D pattern: Materials change in-plane directions but stay the same along axial direction. Such as fiber reinforced composite.

- 3D pattern: Materials change along all the three directions. Such as particle reinforced composite.

According to the heterogeneous pattern, 1D, 2D and 3D UCs can be used in VAMUCH as shown in Fig. 3.1 to achieve the theoretically the maximum computational efficiency regarding to the characteristics of heterogeneities.

A good homogenization theory should provide the flexibility and the robustness to choose different UCs from the heterogeneous material. For example, in binary composite in Fig. 3.2, these three UCs are all feasible candidates, to calculate effective material properties. However, the dimension of the elements and number of total degrees of freedom are different for different choices. Within VAMUCH, these three UCs give the same homogenization
results, and the analysts have the flexibility to choose the UC to minimize the computational effort.

Following issues are addressed in this chapter:

1. Demonstrate that the different unit cells choosing from same heterogeneous material produces the same homogenized results;

2. Extend the VAMUCH ability to handle a periodic boundary condition which are not orthogonal (Fig. 3.3 Upper);

3. Extend the VAMUCH ability to handle a wavy unit cell (Fig. 3.3 Lower);

4. Handle UCs with mismatched nodes on corresponding boundaries.

3.1 Dealing with Periodic Boundary Conditions

Before solving the linear system Eq. (2.31), the periodic constraints which impose additional relationships among degrees of freedom (DOFs) in the linear system, should be
Fig. 3.2: Possible UCs for binary composite.

Fig. 3.3: Non-orthogonal periodicity and wavy UC.
enforced. Typically in literature there are three methods to deal with this type of constraints including transforming equation method, Lagrange multiplier method, and penalty function method.

### 3.1.1 Transforming equation method

Periodic constraints that relate DOFs in $\mathcal{X}_0$ can be written in the following form

$$C\mathcal{X}_0 = \{0\},$$

where $C$ contains constants which define the periodic conditions among the UC edges/surfaces. There are more DOFs in $\mathcal{X}_0$ than periodic constraints, so we can partition $\mathcal{X}_0$ into two groups which $\tilde{\mathcal{X}}_0$ are master degrees of freedom which is to be retained and $\mathcal{X}_{0c}$ is to be eliminated by periodic constraints. Then Eq. (3.1) becomes

$$\begin{bmatrix} C_r & C_c \end{bmatrix} \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \mathcal{X}_{0c} \end{bmatrix} = \{0\},$$

in which $C_c$ is square and nonsingular because the number of DOFs of $\mathcal{X}_{0c}$ is equal to the number of periodic constraints and $\mathcal{X}_{0c}$ can be solved as

$$\mathcal{X}_{0c} = -C_c^{-1}C_r\tilde{\mathcal{X}}_0,$$

so $\mathcal{X}_0$ can be expressed in the form of $\tilde{\mathcal{X}}_0$

$$\mathcal{X}_0 = \begin{bmatrix} \tilde{\mathcal{X}}_0 \\ \mathcal{X}_{0c} \end{bmatrix} = \begin{bmatrix} I \\ -C_c^{-1}C_r \end{bmatrix} \tilde{\mathcal{X}}_0,$$
where \( \mathbf{I} \) is an identity matrix. Now we put \( \mathbf{X}_0 \) back into Eq. (2.31) and multiply both sides of Eq. (2.31) the transpose of

\[
\begin{bmatrix}
\mathbf{I} \\
-\mathbf{C}^{-1}\mathbf{C}_r
\end{bmatrix}
\]

The new linear system will be

\[
\widetilde{D}_{EE} \mathbf{X}_0 = -\widetilde{D}_{h\varepsilon},
\]

(3.5)

where

\[
\widetilde{D}_{EE} = \begin{bmatrix}
\mathbf{I} \\
-\mathbf{C}^{-1}\mathbf{C}_r
\end{bmatrix}^T \mathbf{D}_{EE} \begin{bmatrix}
\mathbf{I} \\
-\mathbf{C}^{-1}\mathbf{C}_r
\end{bmatrix}
\]

and

\[
\widetilde{D}_{h\varepsilon} = \begin{bmatrix}
\mathbf{I} \\
-\mathbf{C}^{-1}\mathbf{C}_r
\end{bmatrix}^T \mathbf{D}_{h\varepsilon}.
\]

In this way the number of equations in linear system Eq. (3.5) is reduced through applying the periodic constraints.

### 3.1.2 Lagrange multiplier method

To impose periodic constraints by Lagrange multipliers, we multiply a row vector \( \lambda^T \) to Eq. (3.1) and add it into Eq. (2.29) to construct a new functional. Minimizing this new functional, we obtain the following set of algebraic equations

\[
\begin{bmatrix}
\mathbf{D}_{EE} & \mathbf{C}^T \\
\mathbf{C} & 0
\end{bmatrix} \begin{bmatrix}
\mathbf{X}_0 \\
\lambda
\end{bmatrix} = \begin{bmatrix}
-D_{h\varepsilon} \\
0
\end{bmatrix}
\]

(3.6)

and can be solved directly for \( \mathbf{X}_0 \) as the periodic constraints are already incorporated into this linear system. We note that the number of equations is increased by introducing Lagrange multipliers \( \lambda^T \), and the solving time will be much longer when there are many nodes in the UC edges/surfaces. Also special attention is needed for the linear solver as now many diagonal terms of the coefficient matrix is zero.

### 3.1.3 Penalty function method

The basics of penalty function method is to use a large number to enforce the zero identity due to periodic constraints. We will not provide details here but invite those who are interested to to find out the details in [126]. The penalty method will not increase the number of equations in the linear system, but it will alter the coefficient matrix and may
cause error due to ill-conditioning and one set of penalty functions might not work for all cases, resulting in the robustness issue of this method.

### 3.1.4 Paired nodes method

Although transforming equation method is the most straightforward method, without the disadvantage of introducing more unknowns like the Lagrange multiplier method or causing robustness issue like the penalty function method, its direct implementation requires modifying the linear system as shown in Eq. (3.5). In most time, it requires two large arrays to store the original coefficient matrix and the modified matrix, which is a serious deficiency for using this method to deal with large finite element models. To avoid this deficiency of transforming equation method, we propose to use the paired nodes method to enforce the periodic constraints in VAMUCH. As the UC is the representative building block of the material, it is natural to assume that a node on one boundary (say positive $y_2$ edge/surface) is paired to a corresponding node on the boundary on the another side of the UC (the negative $y_2$ edge/surface). By choosing to make the nodes on the positive boundary edge/surface slave to the corresponding nodes on the corresponding negative boundary edge/surface, we can implicitly and exactly incorporate the periodic constraints. The slave node numbers will be eliminated from the finite element model and replaced with the corresponding master node numbers. In this way, we also reduce the total number of unknowns in the linear system without modifying the linear system after assembly as the constraints are introduced before assembly.

The way we introduce periodic constraints through paired nodes allows the boundary of UC to be possibly defined by curved edges/surfaces, not necessarily straight as commonly used by other methods. In some applications, the flexibility to choose a UC with curved edges/surfaces instead of one with straight edges/surfaces will facilitate the mesh generation and help increase the analysis efficiency and accuracy. It is noted that this observation leads us to relax three restrictions of VAMUCH we had in previous versions of the code. Previously we require the user to choose the UC in such a way that the origin must be at the center of the UC and the boundary edges/surfaces must be straight and must be
perpendicular to each other at the junctions. The VAMUCH has been updated to relax these restrictions and the current version of VAMUCH gives the maximum freedom to the analysts in choosing UCs. The only requirement now is that the UC must be representative of the entire material which can be decided by the analyst who is the end user of the theory and the code.

### 3.1.5 UCs without paired nodes

To analyze real heterogeneous materials in practical engineering applications, it is possible that some or all of the nodes on the boundaries of a UC do not have the corresponding paired nodes. One typical situation is that the finite element mesh of a reconstructed microstructure based on \( \mu \)-CT images using some imagine processing softwares such as Simpleware™, Mimics™, etc. This happens because the end users have little control of the mesh schemes in these software tools and/or there is no area/volume at the boundary to be meshed (such as porous materials).

#### Dummy material approach

Without paired nodes on corresponding boundaries, VAMUCH cannot enforce the periodic boundary conditions in a rigorous way. To avoid this difficulty, we will add a thin layer of dummy material around the original edges/surfaces of UC (Fig. 3.4), which can be meshed to provide the needed paired nodes at the boundary of the modified UC. For the modified UC to satisfy the general requirement of being a representative building block of the original material, the properties of the added dummy material, \( C_{ijkl}^d \), must be the same as those of the homogenized effective material, \( \bar{C}_{ijkl} \). It is timely noted that this concept is similar to the well-known self-consistent scheme in micromechanics [34].

However, we do not know \( C_{ijkl}^d = \bar{C}_{ijkl} \) at the beginning of the analysis as they are the outputs of our homogenization analysis. We need to start with some initial guess for \( C_{ijkl}^d \) as the properties of the dummy material, as shown in Fig. 3.5. For simplicity, we assume that \( C_{ijkl}^{Initial} \) equal to some values located among the constituent properties. Then the effective material properties of the modified UC, \( \bar{C}_{ijkl}^i \), can be obtained using VAMUCH
Mismatched nodes case

No area at boundaries case

Fig. 3.4: Add dummy material to original UCs.

and it is usually not the same as our initial guess $C^{\text{Initial}}_{ijkl}$. We can refine our assumption for $C^d_{ijkl}$ by using $\bar{C}^i_{ijkl}$ as the material properties for the dummy material to carry out another homogenization analysis to obtain $\bar{C}^{i+1}_{ijkl}$ using VAMUCH. Such iterations will stop once the effective properties computed by VAMUCH are numerically equal to each other at adjacent steps. Thus, the requirement that the dummy material has the same properties as the effective material properties, $C^d_{ijkl} = \bar{C}_{ijkl}$, is achieved. As a rule, if the added dummy material is just a very small portion of the modified UC, only a few iterations are necessary for convergence.

Multi-point constraints approach

In paired nodes method, every slave node is paired to one master node only which can be considered as a specific example of the multi-point constraints (MPC). In MPC, one node can slave to multiple master nodes through linear combination of coefficients, which can be done by using shape functions along the boundaries. This constraints, such
as $\chi_1 - 2\chi_5 - 3\chi_6 = 0$, is often homogeneous since no constant involves in the constraint equation. The periodic boundary condition can be described by a bunch of MPC with left side containing the original degrees of freedom and right side containing the master degrees of freedom. Applying the MPC to the Eq. (2.31) equals applying the periodic boundary condition. Compared with dummy material approach, which increases the total degrees of freedom because of adding extra elements, MPC approach is recommended in the mismatched nodes case as shown in Fig. 3.4 with the advantage of high efficiency without increasing the number of elements and degrees of freedom.

Here we use a 2D 4-node quad element to illustrate how to build MPC equations. In Fig. 3.6, the negative side of $x$ and $y$ are chosen as master nodes and nodes on the positive side are slaved to the negative side. Node 4 is projected to the master side and described by the shape function formed by node 1 and node 3. In general cases, for a 2D problem, the number of master nodes can be as many as 3; In 3D problem, the number of master nodes to describe a slave node could be as many as 9 which depends on the type of element. This
Fig. 3.6: An example to apply MPC periodic condition.

Example gives us the following MPC equations

\[
\begin{align*}
\chi_2 &= \chi_1, \\
\chi_4 &= \frac{2}{3} \chi_1 + \frac{1}{3} \chi_3, \\
\chi_6 &= \chi_3, \\
\chi_8 &= \chi_7, \\
\chi_1 &= \chi_7, \\
\chi_2 &= \chi_8.
\end{align*}
\]  
(3.7)

Here \(\chi_i\) denotes fluctuation function at node \(i\). In matrix form,

\[
\begin{pmatrix}
\chi_1 \\
\chi_2 \\
\chi_3 \\
\chi_4 \\
\chi_5 \\
\chi_6 \\
\chi_7 \\
\chi_8
\end{pmatrix} =
\begin{pmatrix}
0 & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\frac{1}{3} & 0 & \frac{2}{3} \\
0 & 1 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{pmatrix}
\begin{pmatrix}
\chi_3 \\
\chi_5 \\
\chi_7
\end{pmatrix}
\]  
(3.8)
This is the periodic condition, and mater degrees of freedom are $\chi_3$, $\chi_5$, $\chi_7$. In general, this relation can be described as

$$X_0 = \mathcal{T}\tilde{X}_0,$$  \hspace{1cm} (3.9)

where $X_0$ is a vector of the original degrees of freedom, $\tilde{X}_0$ the master degrees of freedom and $\mathcal{T}$ the MPC transformation matrix. Using $\tilde{D}_{EE} = \mathcal{T}^T D_{EE} \mathcal{T}$ and $\tilde{D}_{he} = \mathcal{T}^T D_{he}$, we get the modified linear system from Eq. (2.31) to

$$\tilde{D}_{EE}\tilde{X}_0 = -\tilde{D}_{he}. \hspace{1cm} (3.10)$$

where the unknowns are reduced from $\chi_1 - \chi_7$ to $\chi_3$, $\chi_5$, $\chi_7$. Upon solving it, $\chi_1$, $\chi_2$, $\chi_4$, $\chi_6$, $\chi_8$ are recoverable from Eq. (3.8).

This method enjoys the advantage of reducing the total number of degrees of freedom and being exact, however, the main drawback is the complexity of the general constrained case and a sparse matrix storage used for $\tilde{D}_{EE}$ which can be expensive for allocating and storing these entries. Numerical problems can arise if the MPC equations are linearly dependent which needs to take care in VAMUCH code in a general way. After all, in the code, MPC imposition and assembly are carried out simultaneously to avoid handling large sparse matrices.

The VAMUCH code and ANSYS-VAMUCH User Interface [127] are upgraded to implement the new capabilities developed here. The most recent version of the code, VAMUCH 3.0.7, can analyze any heterogeneous materials as long as the analyst can identify a UC as the building block of the heterogeneous material, the very fundamental requirement of micromechanics. A few numerical examples will be used to demonstrate such a freedom.

### 3.2 Numerical Examples

Choosing a UC from a strictly periodic heterogeneous composite is a relative easy task and straightforward. Although not all composites have periodic UCs, and in reality most composites are aperiodic. Nevertheless, when an analyst decide to use micromechanics...
model to obtain effective properties, he or she already made an inherent assumption that some building block, \textit{i.e.}, the UC, exists.

### 3.2.1 Case 1, different UCs from same material

Different analysts would have different preferences to choose UCs. As long as a UC can be repeated to assemble the original macroscopic structure (except for the boundary which can be formed by a portion of the UC), it is a feasible one. The options I, II, III, IV, V and VI and their finite element models in Fig. 3.7 could be a few possible UC choices for VAMUCH analysis.

To show that VAMUCH will predict the same effective properties based on these six different UC choices, we assume the heterogeneous material in FE models is composed of two different isotropic materials with the blue phase (the inclusion) having Young’s modulus and Poisson’s ratio equal to 379.3 GPa and 0.1 respectively and the purple phase (the matrix) having Young’s modulus and Poisson’s ratio equal to 68.3 GPa and 0.3 respectively. Volume fraction of the inclusion is 0.264. The effective properties calculated from these six
Table 3.1: Effective properties using six different UCs ($\bar{E}, \bar{G}$: GPa).

<table>
<thead>
<tr>
<th>UCs</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\nu_{12}$</th>
<th>$\nu_{13}$</th>
<th>$\nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC1</td>
<td>151.15</td>
<td>102.16</td>
<td>102.16</td>
<td>38.95</td>
<td>38.95</td>
<td>35.46</td>
<td>0.2381</td>
<td>0.2381</td>
<td>0.3130</td>
</tr>
<tr>
<td>UC2</td>
<td>151.15</td>
<td>102.06</td>
<td>102.06</td>
<td>38.93</td>
<td>38.93</td>
<td>35.43</td>
<td>0.2381</td>
<td>0.2381</td>
<td>0.3124</td>
</tr>
<tr>
<td>UC3</td>
<td>151.15</td>
<td>102.17</td>
<td>102.17</td>
<td>38.95</td>
<td>38.95</td>
<td>35.46</td>
<td>0.2381</td>
<td>0.2381</td>
<td>0.3129</td>
</tr>
<tr>
<td>UC4</td>
<td>151.09</td>
<td>102.07</td>
<td>101.82</td>
<td>38.93</td>
<td>38.89</td>
<td>35.43</td>
<td>0.2381</td>
<td>0.2382</td>
<td>0.3131</td>
</tr>
<tr>
<td>UC5</td>
<td>151.09</td>
<td>102.06</td>
<td>102.06</td>
<td>38.92</td>
<td>38.83</td>
<td>35.42</td>
<td>0.2381</td>
<td>0.2381</td>
<td>0.3122</td>
</tr>
<tr>
<td>UC6</td>
<td>151.09</td>
<td>102.07</td>
<td>101.90</td>
<td>38.88</td>
<td>38.93</td>
<td>35.42</td>
<td>0.2381</td>
<td>0.2381</td>
<td>0.3121</td>
</tr>
</tbody>
</table>

UC choices are listed in Table 3.5 where directions 2 and 3 are in-plane directions, and direction 1 is out-plane direction normal to the plane of the UCs. The almost identical results, except some minor differences due to numerical noises, clearly demonstrate that VAMUCH will predict the same effective properties which are not dependent on the UC choices, which is expected as we are modeling the same heterogeneous materials. The fifth and sixth UC choices of curved edges also demonstrate that the edges of UC are not necessarily to be restricted to be straight.

### 3.2.2 Case 2, wavy UCs

The second example is used to further demonstrate that VAMUCH provides the same prediction for both effective properties and local fields no matter whether the UC has straight edges or curved edges. For this purpose, we choose a wavy composite studied in the literature by Khatam and Pindera [128]. The wavy composite is composed of hard (purple) and soft (blue) layers (see the sketch of Fig. 3.8). For comparison purpose, both constituents are isotropic with Young’s modulus $E_h = 70$ GPa and Poisson’s ratio $\nu_h = 0.22$ for hard layers, and $E_s = 7$ GPa and Poisson’s ratio $\nu_s = 0.3$ for soft layers. Both layers are of sinusoidal wavy shape and the amplitude-to-wavelength ratio is 0.05 with volume fraction of hard layers is 0.4. It can be observed that there are at least two ways to choose the unit cell: a rectangular unit cell or a wavy unit cell as shown in Fig. 3.9 and Fig. 3.10 respectively. The effective material properties predicted by VAMUCH using these two different UCs are listed in Table 3.2 along with the results predicted by FVDAM, a
micromechanics code introduced in [128]. These results show that the effective properties predicted by VAMUCH are not dependent on whether you choose a UC with straight edge or curved edge. One can also observe a good agreement for all the predictions between VAMUCH and FVDAM.

To verify that VAMUCH also predicts the same local fields such as stresses for these two UC choices, we carry out a recovery procedure of VAMUCH using the same macroscopic loading for both UCs. The local stress of $\sigma_{11}$ is plotted in both cases in Fig. 3.11 and Fig. 3.12. The $\sigma_{11}$ along $y_3 = 0$ are compared in Fig. 3.12 and the results demonstrate again that the local fields are independent of UC choices.

For small amplitude-to-wavelength ratio, the mesh patterns between the rectangular unit cell and wavy unit cell are almost the same, the total number of elements are the
Fig. 3.10: Wavy unit cell.

Table 3.2: Effective material properties of wavy composites (small amplitude-to-wavelength ratio).

<table>
<thead>
<tr>
<th>Models</th>
<th>$E_{33}/E_s$</th>
<th>$E_{11}/E_s$</th>
<th>$E_{22}/E_s$</th>
<th>$G_{13}/G_s$</th>
<th>$G_{23}/G_s$</th>
<th>$G_{12}/G_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAMUCH (rect UC)</td>
<td>4.6037</td>
<td>3.8939</td>
<td>1.8459</td>
<td>4.7156</td>
<td>1.6160</td>
<td>1.7002</td>
</tr>
<tr>
<td>FVDAM (rect UC)</td>
<td>4.6037</td>
<td>3.8907</td>
<td>1.8446</td>
<td>4.7140</td>
<td>1.6195</td>
<td>1.7040</td>
</tr>
<tr>
<td>VAMUCH (wavy UC)</td>
<td>4.6037</td>
<td>3.8939</td>
<td>1.8459</td>
<td>4.7156</td>
<td>1.6160</td>
<td>1.7001</td>
</tr>
<tr>
<td>FVDAM (wavy UC)</td>
<td>4.6037</td>
<td>3.8909</td>
<td>1.8451</td>
<td>4.7140</td>
<td>1.6178</td>
<td>1.7023</td>
</tr>
</tbody>
</table>

Fig. 3.11: $\sigma_{11}$ of rectangular UC.

Fig. 3.12: $\sigma_{11}$ of wavy UC.
same and the convergence of the effective moduli with respect to the number of elements are almost the same as shown in the previous example. The advantage of choosing a wavy unit cell to obtain effective moduli can be more clearly seen in a large amplitude-to-wavelength ratio example such as that shown in Fig. 3.14. The FEA model of smallest unit cell of a rectangular shape from this material shows in Fig. 3.15 which needs many more elements than the FEA model of smallest wavy unit cell as shown in Fig. 3.16 to reach the convergence. Choosing a wavy UC in this case reduces the number of areas to be meshed and improves the flexibility to control elements quality, as a result, wavy UC provides more flexibility to the curved structure, such as curved fins in a heat exchanger. The effective properties of these two UCs are compared in Table 3.3, where hard and soft material properties are taken to be $E_h = 379.3$ GPa, $\nu_h = 0.1$, and $E_s = 68.3$ GPa, $\nu_s = 0.3$ respectively.

The following example illustrates the effects of curved surface. As shown in Fig. 3.17, two different UCs are extracted from two types of heat exchanger. The straight fin UC is on the left side and the curved surface UC is on the right side. The cross section geometry in $y_1$-$y_2$ plane are the same and the length along $y_3$ direction are the same. The Young's
Fig. 3.14: Cross section of periodic wavy structure.

Fig. 3.15: Rectangular unit cell.

Fig. 3.16: Wavy unit cell.
Table 3.3: Effective material properties of wavy composites (large amplitude-to-wavelength ratio).

<table>
<thead>
<tr>
<th>Models</th>
<th>$E_{33}/E_s$</th>
<th>$E_{11}/E_s$</th>
<th>$E_{22}/E_s$</th>
<th>$G_{13}/G_s$</th>
<th>$G_{23}/G_s$</th>
<th>$G_{12}/G_s$</th>
</tr>
</thead>
<tbody>
<tr>
<td>VAMUCH (rect UC)</td>
<td>1.9312</td>
<td>1.3720</td>
<td>1.3271</td>
<td>1.6772</td>
<td>1.5020</td>
<td>1.5192</td>
</tr>
<tr>
<td>VAMUCH (wavy UC)</td>
<td>1.9311</td>
<td>1.3775</td>
<td>1.3296</td>
<td>1.6815</td>
<td>1.5043</td>
<td>1.5254</td>
</tr>
</tbody>
</table>

Fig. 3.17: Wavy unit cell.

modulus and Poisson’s ratio are set to be 205.440 GPa and 0.2780. The effective stiffness matrix are shown in Table. 3.4, from which the shear $G_{12}$ resistance increases from 22 MPa to 187 MPa and the elastic $E_{33}$ reduces from 221.344 GPa to 162.554 GPa.

3.2.3 Case 3, UCs without paired nodes

To illustrate that the self-consistent scheme can provide a good approximation of the exact solution with adding small layer of dummy media, we carry out a 2-D homogenization problem as a verification. For simplicity, we assume the heterogeneous material is composed of two different isotropic materials with the blue phase (fiber) (Table. 3.5), which occupies 40% volume fraction (VF) of total volume, having Young’s modulus, Poisson’s ratio and density equal to 379.3 GPa, 0.1, and 1840 Kg/m$^3$, respectively, and those of the purple phase (matrix) having 68.3 GPa, 0.3, and 840 Kg/m$^3$, respectively. Clearly, we can solve this homogenization problem exactly routinely by VAMUCH. To compare the result with the self-consistent scheme, the red phase (dummy media) is added around the original unit cell. The initial values of Young’s modulus, Poisson’s ratio and density for dummy phase are only required to be fall between the hard phase and soft phase, thus set to 100 GPa,
Table 3.4: Comparison between stiffness matrix of straight and curved UCs.

<table>
<thead>
<tr>
<th></th>
<th>Straight UC (MPa)</th>
<th>Curved UC (MPa)</th>
</tr>
</thead>
<tbody>
<tr>
<td>130550 0 1477 0 0 36704</td>
<td>137647 0 2247 0 0 37024</td>
<td></td>
</tr>
<tr>
<td>0 22 0 0 0 0</td>
<td>0 187 0 0 0 0</td>
<td></td>
</tr>
<tr>
<td>1477 0 72263 0 0 20500</td>
<td>2247 0 68238 0 0 5121</td>
<td></td>
</tr>
<tr>
<td>0 0 0 56117 0 0</td>
<td>0 0 0 56988 0 0</td>
<td></td>
</tr>
<tr>
<td>0 0 0 0 26507 0</td>
<td>0 0 0 21764 0</td>
<td></td>
</tr>
<tr>
<td>36704 0 20500 0 0 221344</td>
<td>37024 0 5121 0 0 162554</td>
<td></td>
</tr>
</tbody>
</table>

0.2, and 1000 Kg/m³, respectively. For the three dummy cases, the fractions of dummy length to the original length are 0.05, 0.1 and 0.2 respectively. As a result, dummy volume fractions (DVF) range from 0.0093 to 0.306. The effective properties can be approximate as orthotropic material and the comparisons are listed in Table. 3.5, where \( \bar{\rho} \) denotes effective density, \( \bar{E} \) effective Young’s modulus, \( \bar{G} \) effective shear modulus, \( \bar{\nu} \) effective Poisson’s ratio, directions 2 and 3 in-plane directions, and direction 1 out-plane direction. The analysis for cases with dummy material runs until the convergence is achieved and the tolerance is set to be \( 10^{-10} \). From the results listed in Table. 3.5, it is observed that self-consistent scheme of VAMUCH will: (1) provide exact effective density as the original UC; (2) approximate the exact effective properties of the original UC (the case DVF=0) when DVF getting smaller. The reason of the distinction is from the change of periodic boundaries. As the dummy material adds to original UC, the periodic edges/surfaces are pushed out to the dummy edges/surfaces, thus the solution of fluctuation functions \( \tilde{X}_0 \) in Eq. (2.33) are slightly changed which leads to the changing of effective properties. However, this method is an accurate and efficient way to obtain the effective properties from RVE without paired nodes along edges/surfaces under the current framework of VAMUCH.

The next example deals with a UC without paired nodes as shown in upper figure of
Table 3.5: Effective properties with different DVF ($\bar{\rho}$: Kg/m$^3$; $\bar{E}$, $\bar{G}$: GPa).

<table>
<thead>
<tr>
<th>DVF</th>
<th>$\bar{\rho}$</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\bar{\nu}_{12}$</th>
<th>$\bar{\nu}_{13}$</th>
<th>$\bar{\nu}_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1239.36</td>
<td>193.35</td>
<td>125.81</td>
<td>125.81</td>
<td>484.38</td>
<td>484.38</td>
<td>432.28</td>
<td>0.20887</td>
<td>0.20887</td>
<td>0.28882</td>
</tr>
<tr>
<td>0.093</td>
<td>1239.36</td>
<td>193.35</td>
<td>126.67</td>
<td>127.21</td>
<td>483.74</td>
<td>483.74</td>
<td>421.98</td>
<td>0.20893</td>
<td>0.20893</td>
<td>0.28064</td>
</tr>
<tr>
<td>0.174</td>
<td>1239.36</td>
<td>193.35</td>
<td>126.67</td>
<td>127.21</td>
<td>483.74</td>
<td>483.74</td>
<td>421.98</td>
<td>0.20893</td>
<td>0.20893</td>
<td>0.28064</td>
</tr>
<tr>
<td>0.306</td>
<td>1239.36</td>
<td>193.35</td>
<td>126.67</td>
<td>127.21</td>
<td>483.74</td>
<td>483.74</td>
<td>421.98</td>
<td>0.20893</td>
<td>0.20893</td>
<td>0.28064</td>
</tr>
</tbody>
</table>

Fig. 3.4. For numerical analysis, we set the Young’s modulus, Poisson’s ratio and density of the blue phase equal to 379.3 GPa, 0.1, and 1840 Kg/m$^3$, respectively, and those of the purple phase equal to 68.3 GPa, 0.3, and 840 Kg/m$^3$, respectively. To start the iteration process, we give an initial guess to the dummy material with Young’s modulus, Poisson’s ratio and density equal to 100 GPa, 0.2, and 1000 Kg/m$^3$ which are located between the material properties of two phases. The diagonal terms of effective stiffness matrix and the effective density ($\bar{\rho}$) of each iteration are listed in Table 3.6. The convergence trend of three diagonal terms $C_{22}, C_{44}, C_{55}$ and the densities are shown in Fig. 3.18 and Fig. 3.19, respectively, for illustrative purpose. The rest of the quantities have similar convergence trends. We can observe that the effective properties converge rapidly which validates the idea of using the self-consistent scheme in VAMUCH to deal with UC without paired nodes.

In the last example, it is a 1mm × 1mm × 1 mm three-dimensional UC cut from Silicon Carbide (SiC) foam, with nominal density around 18%, which has an extraordinary thermal shock resistant qualities. The porous microstructure is accurately captured by X-ray mi-
Table 3.6: Convergence of effective properties and effective density of a UC without paired
nodes ($\bar{C}$: GPa).

<table>
<thead>
<tr>
<th>Iterations</th>
<th>$C_{11}$</th>
<th>$C_{22}$</th>
<th>$C_{33}$</th>
<th>$C_{44}$</th>
<th>$C_{55}$</th>
<th>$C_{66}$</th>
<th>$\bar{\rho}$ (Kg/m$^3$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1.287</td>
<td>0.362</td>
<td>1.094</td>
<td>0.357</td>
<td>0.337</td>
<td>1.083</td>
<td>981.387</td>
</tr>
<tr>
<td>2</td>
<td>1.319</td>
<td>0.352</td>
<td>1.092</td>
<td>0.347</td>
<td>0.326</td>
<td>1.080</td>
<td>978.156</td>
</tr>
<tr>
<td>3</td>
<td>1.325</td>
<td>0.350</td>
<td>1.092</td>
<td>0.345</td>
<td>0.324</td>
<td>1.080</td>
<td>977.596</td>
</tr>
<tr>
<td>4</td>
<td>1.326</td>
<td>0.350</td>
<td>1.091</td>
<td>0.345</td>
<td>0.323</td>
<td>1.079</td>
<td>977.498</td>
</tr>
<tr>
<td>5</td>
<td>1.326</td>
<td>0.350</td>
<td>1.091</td>
<td>0.345</td>
<td>0.323</td>
<td>1.079</td>
<td>977.482</td>
</tr>
<tr>
<td>6</td>
<td>1.326</td>
<td>0.350</td>
<td>1.091</td>
<td>0.345</td>
<td>0.323</td>
<td>1.079</td>
<td>977.478</td>
</tr>
</tbody>
</table>

Fig. 3.18: Convergence of terms from the effective stiffness matrix.
Fig. 3.19: Convergence of density.

Fig. 3.20: 3D foam UC with dummy material.
crotomography. Then the mesh is prepared by Simpleware™ with a total of 39,268 nodes and 114,439 elements. To create paired nodes, we add a dummy material to the original UC shown in Fig. 3.20 by ANSYS-VAMUCH User Interface automatically. After the modification, the number of nodes and elements increases to 43,485 and 138,299 respectively. The SiC having Young’s modulus, Poisson’s ratio, and density equal to 448 GPa, 0.14, and 3210 Kg/m$^3$, respectively, from [129]. The effective properties are successfully evaluated according to the self-consistent scheme using VAMUCH. The results feature the same rapid convergence. In this relatively large model, An out-of-core solver is used in VAMUCH and the total 10 iterations took around 160 seconds in a computing server with AMD Opteron Processor 6174 2.2 GHz and 128 GB RAM. It can be observed that VAMUCH is an efficient way to handle such problems. The diagonal terms of effective stiffness matrix and densities in each iteration are shown in Table 3.7. For recovery purpose, by providing global displacements and their gradients, the displacement, strain and stress fields of the UC can be recovered using VAMUCH. Here we use ANSYS to visualize the element stress result $\sigma_{33}$.
Table 3.7: Effective properties of SiC foam (\(\tilde{C}:\) GPa).

<table>
<thead>
<tr>
<th>Iterations</th>
<th>(C_{11})</th>
<th>(C_{22})</th>
<th>(C_{33})</th>
<th>(C_{44})</th>
<th>(C_{55})</th>
<th>(C_{66})</th>
<th>(\bar{\rho}) (Kg/m(^3))</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>13.01</td>
<td>4.32</td>
<td>12.11</td>
<td>4.98</td>
<td>4.18</td>
<td>10.73</td>
<td>601.19</td>
</tr>
<tr>
<td>2</td>
<td>8.85</td>
<td>2.82</td>
<td>7.75</td>
<td>3.41</td>
<td>2.74</td>
<td>7.12</td>
<td>589.72</td>
</tr>
<tr>
<td>3</td>
<td>7.57</td>
<td>2.34</td>
<td>6.33</td>
<td>2.87</td>
<td>2.27</td>
<td>6.09</td>
<td>589.06</td>
</tr>
<tr>
<td>4</td>
<td>7.09</td>
<td>2.14</td>
<td>5.75</td>
<td>2.65</td>
<td>2.09</td>
<td>5.71</td>
<td>589.02</td>
</tr>
<tr>
<td>5</td>
<td>6.88</td>
<td>2.06</td>
<td>5.49</td>
<td>2.55</td>
<td>2.00</td>
<td>5.54</td>
<td>589.02</td>
</tr>
<tr>
<td>6</td>
<td>6.78</td>
<td>2.01</td>
<td>5.37</td>
<td>2.50</td>
<td>1.96</td>
<td>5.47</td>
<td>589.02</td>
</tr>
<tr>
<td>7</td>
<td>6.74</td>
<td>2.00</td>
<td>5.31</td>
<td>2.48</td>
<td>1.94</td>
<td>5.44</td>
<td>589.02</td>
</tr>
<tr>
<td>8</td>
<td>6.73</td>
<td>1.99</td>
<td>5.28</td>
<td>2.47</td>
<td>1.93</td>
<td>5.43</td>
<td>589.02</td>
</tr>
<tr>
<td>9</td>
<td>6.72</td>
<td>1.98</td>
<td>5.27</td>
<td>2.47</td>
<td>1.93</td>
<td>5.42</td>
<td>589.02</td>
</tr>
<tr>
<td>10</td>
<td>6.71</td>
<td>1.98</td>
<td>5.26</td>
<td>2.47</td>
<td>1.93</td>
<td>5.42</td>
<td>589.02</td>
</tr>
</tbody>
</table>

obtained from VAMUCH in the condition of the global strain \(\epsilon_{33} = 0.009\) in Fig. 3.21.
Chapter 4

Bounds for Effective Properties of Random Cell Structure

The random cell structure, such as Voronoi tessellation [17], is used to mimic polycrystals, cells, organs and etc. Though the present theory, the details of which will be presented in the next section, can handle general complex random heterogeneous materials. For simplicity, we take a two phases, two-dimensional (2D) fiber reinforced material represented as the random cell structure shown in Fig. 4.1 as an example. As the random variable, the position of inclusions in different cells can vary independently. Considering all the possible positions in the random cell structure, it is obvious that the mean position is at the center of square as shown in I of Fig. 4.1. The variance measures the distance between possible center of inclusions and the mean position. When the variance is equal to zero, the random structure degenerates into a deterministic regularly packed periodic unit cell and the upper and lower bounds of effective properties should be the same and coincide with that of the deterministic unit cell. With the increasing of variance, one will naturally expect that the gap between the upper bound and lower bound increases.

However, the theories of rigorous bounds in the literature cannot capture this phenomena because those bounds are based on $n$-point correlation functions which cannot capture the effects of positional variances. Furthermore, the assumption of the absence of any long-range order [30] in $n$-point correlation functions always violates the periodic assumption in analysis. Using $n$-point correlation functions to describe microstructures would miss some critical information of randomness due to the ad hoc assumptions, such as statistical homogeneity and the isotropy, which exclude the possibility to model anisotropic random materials. Hence, a new microstructure descriptor other than $n$-point correlation functions is required in computing the bounds of the random cell structures.

The statistical description of the microstructure is commonly named stochastic unit
cell (SUC). The SUC has to include a very large number of trial fields (possible microstates) to cover enough diversity of heterogeneities, so that it becomes statistically homogeneous and ergodic. The SUC can be obtained experimentally or numerically. The algorithm for generating random fields has been studied in the literature, such as Random Sequential Absorption Algorithm (RSA) [21], Monte Carlo simulation [28], and image reconstruction technique [29], etc. Because it is hard to determine how large is enough to satisfy ergodic assumption, a large number of trial fields (see the sketch in Fig. 4.1) are usually used to obtain an accurate description of the random heterogeneous material.

4.1 Mathematical Formulation

We follow Ref. [130] to formulate the theory needed for VAMUCH to predict upper and lower bounds for the effective elastic properties of random heterogeneous materials which can be described as a random cell structure.
For each realization of the random heterogeneous materials, the homogenization problem becomes deterministic and the effective material properties can be obtained by minimizing the following functional

$$\Pi_{\Omega}^\omega = \frac{1}{2\Omega} \int_\Omega C_{ijkl}^\omega \left[ \varepsilon_{ij} + \frac{1}{2}(\chi_{i,j}^\omega + \chi_{j,i}^\omega) \right] \left[ \varepsilon_{kl} + \frac{1}{2}(\chi_{k,l}^\omega + \chi_{l,k}^\omega) \right] d\Omega,$$

(4.1)

subject to the periodic boundary conditions. $\Pi_{\Omega}^\omega$ stands for strain energy in a possible realization; $C_{ijkl}^\omega$ stands for pointwise fourth-order elasticity tensor of a possible realization (the Latin indices run through 1, 2, 3); $\chi_i^\omega$ stands for fluctuation function of a possible realization; $\bar{\varepsilon}_{ij}$ stands for macroscopic strain tensor; After minimizing $\Pi_{\Omega}^\omega$ with respect to $\chi_i^\omega$, we can obtain the effective property for each specific realization $\omega$ as

$$\frac{1}{2} C_{ijkl}^{(\text{eff})} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} = \min_{\text{periodic } \chi_i} \Pi_{\Omega}^\omega,$$

(4.2)

where $C_{ijkl}^{(\text{eff})}$ is the effective fourth-order elasticity tensor of a possible realization. The expectation of the effective properties of all the realizations of the random microstructure, $\bar{C}_{ijkl}^{(\text{eff})}$, can be obtained as

$$\bar{C}_{ijkl}^{(\text{eff})} = M C_{ijkl}^{(\text{eff})},$$

(4.3)

where $\bar{C}_{ijkl}^{(\text{eff})}$ is expectation of $C_{ijkl}^{(\text{eff})}$. We can conclude from Eqs. (4.2) and (4.3) the following

$$\frac{1}{2} C_{ijkl}^{(\text{eff})} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} = \frac{1}{2} M C_{ijkl}^{(\text{eff})} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} = M \min_{\text{periodic } \chi_i} \Pi_{\Omega}^\omega = \min_{\text{periodic } \chi_i} M \Pi_{\Omega}^\omega.$$

(4.4)

where the last equality uses the fact that the order of minimization and expectation can be changed according to Ref. [130]. If we chose trial fields for $\chi_i$ that remain the same for each realization, we have

$$\frac{1}{2} C_{ijkl}^{(\text{eff})} \bar{\varepsilon}_{ij} \bar{\varepsilon}_{kl} \leq \min_{\text{periodic } \chi_i} \frac{1}{2\Omega} \int_\Omega M C_{ijkl}^\omega \left[ \bar{\varepsilon}_{ij} + \frac{1}{2}(\chi_{i,j} + \chi_{j,i}) \right] \left[ \bar{\varepsilon}_{kl} + \frac{1}{2}(\chi_{k,l} + \chi_{l,k}) \right] d\Omega.$$

(4.5)
Note the order of $M$ and the integration operation can also be switched. As it will be shown in the next section, $MC_{ijkl}^\omega$, which is the descriptor of microstructure can be straightforwardly evaluated at every point within the microstructure, which implies that the right-hand side of the inequality can be routinely solved by VAMUCH. The computation essentially homogenizes a heterogeneous material with point-wise different constituent properties. Let us denote the effective properties as $\hat{C}_{ijkl}^{(\text{eff})}$, then we have

$$C_{ijkl}^{(\text{eff})} \varepsilon_{ij} \varepsilon_{kl} \leq \hat{C}_{ijkl}^{(\text{eff})} \varepsilon_{ij} \varepsilon_{kl},$$

which implies that the minimization problem of the right hand of the inequality in Eq. (4.5) effectively computes an upper bound for the effective properties of the random heterogeneous material which is statistically described by all the microstructural realizations which we have used to compute $MC_{ijkl}^\omega$. Note the inequality in Eq. (4.6) should be interpreted in the sense of a fourth-order tensor, and it is not necessary hold for each component of the effective properties. For example, $\hat{C}_{1233}^{(\text{eff})}$ is not necessarily equal or smaller than $\hat{C}_{1233}^{(\text{eff})}$. The lower bound of effective material properties can be obtained from a variational statement dual to Eq. (4.1). In each realization, $\omega$, we define the strain energy density as

$$L = \frac{1}{2} C_{ijkl}^\omega \varepsilon_{ij}^\omega \varepsilon_{kl}^\omega,$$

where $\varepsilon_{ij}^\omega$ is local strain tensor of a possible realization. The Young-Fenchel transformation of the functional $L$ in Eq. (4.7) is

$$L^* = \frac{1}{2} (C_{ijkl}^\omega)^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega,$$

where $\sigma_{ij}^\omega$ refers to local stress tensor of a possible realization. The Young-Fenchel transformation of $L^*$ can be denoted as

$$L^{**} = \max \sigma_{ij}^\omega \left[ \sigma_{ij}^\omega \varepsilon_{ij}^\omega - \frac{1}{2} (C_{ijkl}^\omega)^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega \right].$$
Because functional $L$ is convex and continuous, $L = L^{**}$ holds according to Ref. [111]. Hence minimization of variational problem respect to $\chi_i$ in Eq. (4.2) can be rewritten as

$$
\frac{1}{2} C_{ijkl}^{(\text{eff})} \epsilon_{ij} \tilde{\epsilon}_{kl} = \min_{\text{periodic } \chi_i} \frac{1}{\Omega} \int_{\Omega} \max_{\sigma_{ij}^\omega} \left[ \sigma_{ij}^\omega (\epsilon_{ij} + \chi_{i,j}) - \frac{1}{2} (C_{ijkl}^\omega)^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega \right] d\Omega. \quad (4.10)
$$

According to inequality $\max_{v \in N} \min_{u \in M} \Phi(u, v) \leq \min_{u \in M} \max_{v \in N} \Phi(u, v)$ in Ref. [111], for the realization $\omega$, we have

$$
\frac{1}{2} C_{ijkl}^{(\text{eff})} \epsilon_{ij} \tilde{\epsilon}_{kl} \geq \max_{\sigma_{ij}^\omega} \min_{\text{periodic } \chi_i} \frac{1}{\Omega} \int_{\Omega} \left[ \sigma_{ij}^\omega (\epsilon_{ij} + \chi_{i,j}) - \frac{1}{2} (C_{ijkl}^\omega)^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega \right] d\Omega. \quad (4.11)
$$

Choose the stress field to be statically admissible for the cell problem, implying,

$$
\sigma_{ij,j} = 0 \quad \text{and} \quad \sigma_{ij}^\omega n_j \text{ being periodic along the boundaries.} \quad (4.12)
$$

Thus, the $\chi_i^\omega$ related term can be dropped for the maximization of the minimization of the integral in the right hand side of Eq. (4.11). Then, this inequality can be simplified to be

$$
\frac{1}{2} C_{ijkl}^{(\text{eff})} \epsilon_{ij} \tilde{\epsilon}_{kl} \geq \max_{\sigma_{ij}^\omega} \frac{1}{\Omega} \int_{\Omega} \left[ \sigma_{ij}^\omega \epsilon_{ij} - \frac{1}{2} M (C_{ijkl}^{\omega})^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega \right] d\Omega. \quad (4.13)
$$

Like what we did in the upper bound calculation, we can apply the expectation calculation to both side, such that

$$
\frac{1}{2} C_{ijkl}^{(\text{eff})} \epsilon_{ij} \tilde{\epsilon}_{kl} \geq \max_{\sigma_{ij} \in (4.12)} \frac{1}{\Omega} \int_{\Omega} M \left[ \sigma_{ij}^\omega \epsilon_{ij} - \frac{1}{2} (C_{ijkl}^{\omega})^{-1} \sigma_{ij}^\omega \sigma_{kl}^\omega \right] d\Omega. \quad (4.14)
$$

If we choose an admissible stress field which is also deterministic, it will obey the following equations

$$
\sigma_{ij,j} = 0 \quad \text{and} \quad \sigma_{ij} n_j \text{ being periodic along the boundaries.} \quad (4.15)
$$

Then

$$
\frac{1}{2} C_{ijkl}^{(\text{eff})} \epsilon_{ij} \tilde{\epsilon}_{kl} \geq \max_{\sigma_{ij} \in (4.15)} \frac{1}{\Omega} \int_{\Omega} \left[ \sigma_{ij} \epsilon_{ij} - \frac{1}{2} M (C_{ijkl}^{\omega})^{-1} \sigma_{ij} \sigma_{kl} \right] d\Omega. \quad (4.16)
$$
The constraints in Eq. (4.15) can be taken care of by introducing Lagrange’s multipliers \( \lambda_i \), and rewriting Eq. (4.16) as a minimax problem such that

\[
\frac{1}{2} C_{ijkl}^{\text{eff}} \epsilon_{ij} \epsilon_{kl} \geq \max_{\sigma_{ij}} \min_{\text{periodic} \lambda_i} \frac{1}{\Omega} \int_{\Omega} \left[ \sigma_{ij} \epsilon_{ij} + \frac{1}{2} \sigma_{ij} (\lambda_{i,j} + \lambda_{j,i}) - \frac{1}{2} M(C_{ijkl}^{\omega})^{-1} \sigma_{ij} \sigma_{kl} \right] d\Omega.
\]  

(4.17)

As shown in Ref. [111], the order of min and max can be changed. Maximization over \( \sigma_{ij} \) of the functional is conducted explicitly at each point of \( \Omega \)

\[
\max_{\sigma_{ij}} \left[ \sigma_{ij} \epsilon_{ij} + \frac{1}{2} \sigma_{ij} (\lambda_{i,j} + \lambda_{j,i}) - \frac{1}{2} M(C_{ijkl}^{\omega})^{-1} \sigma_{ij} \sigma_{kl} \right] = \frac{1}{2} \left( M(C_{ijkl}^{\omega})^{-1} \right)^{-1} \left[ \epsilon_{ij} + \frac{1}{2} (\lambda_{i,j} + \lambda_{j,i}) \right] \left[ \epsilon_{kl} + \frac{1}{2} (\lambda_{k,l} + \lambda_{l,k}) \right].
\]  

(4.18)

Thus the low estimate takes the form

\[
\frac{1}{2} C_{ijkl}^{\text{eff}} \epsilon_{ij} \epsilon_{kl} \geq \min_{\text{periodic} \lambda_i} \frac{1}{2\Omega} \int_{\Omega} \left( M(C_{ijkl}^{\omega})^{-1} \right)^{-1} \left[ \epsilon_{ij} + \frac{1}{2} (\lambda_{i,j} + \lambda_{j,i}) \right] \left[ \epsilon_{kl} + \frac{1}{2} (\lambda_{k,l} + \lambda_{l,k}) \right] d\Omega.
\]  

(4.19)

The minimization problem of right hand side can be solved routinely by VAMUCH with evaluating \( \left( M(C_{ijkl}^{\omega})^{-1} \right)^{-1} \) at each Gaussian point. The solution is denoted as \( \check{C}_{ijkl}^{\text{eff}} \epsilon_{ij} \epsilon_{kl} \), which implies,

\[
\check{C}_{ijkl}^{\text{eff}} \epsilon_{ij} \epsilon_{kl} \geq \check{C}_{ijkl}^{\text{eff}} \epsilon_{ij} \epsilon_{kl}.
\]  

(4.20)

4.2 Simulation Procedure

To obtain bounds for random cell structures, we need to firstly generate a large number of possible realizations of SUC samples using revised RSA method [21]. The second step is to calculate the point-wise expectation material properties for the SUC. For the upper bound, we need to calculate \( MC_{ijkl}^{\omega} \), for the lower bound, we need to calculate \( \left( M\left( C_{ijkl}^{\omega} \right)^{-1} \right)^{-1} \).

Finally, the expectation material properties are used as inputs for VAMUCH to obtain the effective material properties of the SUC which are the corresponding upper bound and lower bound.

The randomness of the microstructure is incorporated in the second step of obtaining
the expectation material properties of the SUC. We used Monte Carlo simulation for this purpose. The required properties at each Gaussian integration point of the microstructure are calculated by averaging all the possible realizations. In the literature, \( n \)-point correlation function method \([131,132]\) is also widely used and a brief introduction is needed to compare these two microstructure descriptors.

One-point probability function has the meaning of volume fraction and is given as known parameters in numerical simulation. The standard two-point correlation function is defined as

\[
S_2^{(ij)}(x_1, x_2) = \langle \zeta^{(i)}(x_1) \zeta^{(j)}(x_2) \rangle, \tag{4.21}
\]

where \( \zeta^{(i)}(x) \) is a random variable defined by

\[
\zeta^{(i)}(x) = \begin{cases} 
1 & x \in \Omega_i \\
0 & \text{otherwise}
\end{cases} \tag{4.22}
\]

with \( \Omega_i \) denoting the domain occupied by phase \( i \).

This function is the probability of finding two points \( x_1 \) and \( x_2 \) in phase \( i \) and \( j \) respectively. A fast way of obtaining \( S_2^{(ij)}(i, j = 1, 2) \) in two-dimensional 2-phase problems is by forming a “sampling template” \([131]\). A large number of sampling templates (say, 1000) has to be used in each possible realization of microstructure to obtain convergent value of \( S_2 \). If considering multi-phase (more than 2) problems, the possible combinations of two-point correlation functions increase dramatically.

With the increasing order of \( n \)-point correlation functions obtained from the microstructure, one can finally rebuild the exact same microstructure compared with the original one from which we obtained the \( n \)-point correlation functions. It is noted that \( n \)-point correlation functions require that the microstructure is statistical homogeneous and statistical isotropic.

Compared with \( n \)-point correlation function to describe the characteristics of random microstructure, the advantages of the present approach using expectation material properties to describe the random material are: (1) the implementation is not restricted to
Fig. 4.2: Two-point correlation functions.

statistically homogeneous or isotropic microstructure; (2) it is more general and can handle three-dimensional and multi-phase microstructures; (3) the present approach requires the least information for homogenization; (4) the present approach provides a natural link between homogenization of deterministic heterogeneous materials and random heterogeneous materials. Take two-point correlation functions for example. We plot the correlation function by changing the variance of possible positions of the inclusions in Fig. 4.2. Each correlation function is averaged from 50 possible realizations. It shows that the two-point correlations are nearly the same for different variances. In other words, it is impossible for two-point as well as $n$-point correlation functions to characterize such random variances.

We take a two-phase porous material (shown in Fig. 4.1 with a circular hole as the inclusion) to demonstrate how the expectation material properties are obtained. We assume the matrix material is isotropic with Young’s modulus $E_1 = 1$ (for simplicity), and the pore is a circle of radius $R$ with Young’s modulus $E_2 = 0$, and the cell is a square of the $1 \times 1$ size. The material properties in the cell can be described as the function of the $y$ and $r$

$$a(y, r) = 1 - \varphi(y - r),$$

(4.23)
here, \( r \) is the center of the pores and \( \varphi(y) \) is the characteristic function satisfying

\[
\varphi(y) = \begin{cases} 
1 & |y_i| \leq R \\
0 & |y_i| > R 
\end{cases}
\] (4.24)

The admissible values of the center of the pores are inside a small square as

\[
|r_i| \leq \frac{1}{2} - R, \quad i = 1, 2
\] (4.25)

\( R \), which is directly related with the volume fraction, is a given parameter that does not exceed 1/2. Since there is no other constraint of this problem, the admissible values of the center are assumed to be equiprobable. Given the radius \( R \) of the pore, the expectation material properties in the cell can be obtained analytically or by Monte Carlo simulations. Two cases \((R = 0.29; R = 0.33)\) were studied. Analytical result can be found in example 2 of Ref. [130]. 100 Monte Carlo simulations were done in each case and the results of expectation material properties are shown in Fig. 4.3. In Fig. 4.4, the expectation Young’s modulus along \( y_1 = 0 \) are compared with analytical solution through 100 simulations and 500 simulations respectively. As expected, the results agree with each other pretty well even when the number of simulations is relatively small. To enable VAMUCH to predict the bounds for random cell structures, we just need to input the geometry of SUC with the point-wise expectation material properties. In the next section, we will use this enhanced capability of VAMUCH to analyze a few examples and compare with other results available in the literature.

4.3 Evaluation of Bounds

In this section, bounds obtained through VAMUCH with SUC described using expectation material properties will be compared with bounds from literature at high contrast properties, and the degeneration result from random heterogeneous material to deterministic heterogeneous materials will be also shown. First, the bounds of fiber reinforce composites (modeled with 2D SUC) will be compared with Hashin-Shtrikman bounds and third-order
Fig. 4.3: Expectation Young’s modulus of SUC ($R = 0.29$; $R = 0.33$).

Fig. 4.4: Expectation comparison ($R = 0.29$; $R = 0.33$).
bounds. Secondly, the bounding results of trabecular bone (modeled with 3D SUC) will also be compared with other theories and experimental results. The reason of no comparison with Voigt (arithmetic mean) bounds and Reuss (harmonic mean) bounds is that it was already shown that these bounds are the worst although simplest in literature [133].

In the fiber reinforced composites, both constituents are isotropic with $E = 5.32$ GPa and $\nu = 0.365$ for the matrix, and $E = 572.5$ GPa and $\nu = 0.2$ for the fiber. The contrast of Young’s modulus between fiber and matrix is relatively large to amplify the differences of different approaches. The fibers are of circular shape and arranged in a square array. 500 Monte Carlo simulations are used to create the SUC. Based on Ref. [134] and Ref. [135], Hashin-Shtrikman bounds and third-order bounds for transversely isotropic materials were calculated and compared with bounds obtained using VAMUCH based on the present theory for different fiber volume fractions. Since the present theory assumes periodic random heterogeneous materials, the bounds are denoted as periodic upper and periodic lower in the plots. Although there are five independent effective properties for transversely isotropic materials, the trends for upper and lower bounds are similar, thus, we only plot the longitudinal shear modulus and transverse shear modulus in Fig. 4.5 and Fig. 4.6. These plots show that the gap between lower and upper bounds predicted by the present theory is apparently smaller than other two types of bounds for materials with high volume fractions. Some samples of effective properties from a certain realization of the random material for each volume fraction have been added in those figures to act as our numerical experiments which are located within the bounds as expected.

To test the capability of VAMUCH to model 3D random heterogeneous materials, we model the trabecular bone as a two-phase material consisting of bone tissue (hard phase) and soft tissue (soft phase). Both phases are assumed linear elastic and isotropic. Bone tissues are assigned Young’s modulus 13.0 GPa and Poisson’s ratio of 0.3 while Young’s modulus of soft tissue is given as 1.3 KPa with the same Poisson’s ratio 0.3 according to Ref. [17]. The soft tissue is modeled as spheres randomly distributed in cubic unit cell and 5000 Monte Carlo simulations are used to create calculate the point-wise expectation
Fig. 4.5: Bounds comparison of longitudinal shear modulus.

Fig. 4.6: Bounds comparison of transverse shear modulus.
material properties for the SUC. Note here we provide a simplistic model for the soft tissues as they usually contain many small holes distributed in the hard tissue. We used spheres to represent those small holes and the boundary of unit cell can be imagined as cutting a cubic volume as a sample of the soft tissue. However, if we want to provide a more accurate representation of the trabecular bone, we can use some image reconstruction software, such as Simpleware™ to generate more realistic microstructures. Ryan and Williams [136] used tensile testing experiments on single trabeculae, and found a trabecular tissue modulus of 0.4-3.6 GPa with the volume fraction of bone tissues estimated at 20%. From Ref. [5] and Chapter 21 in Ref. [12], Hashin-Shtrikman bounds and third-order bounds were calculated and compared with new bounds predicted with the present theory for different volume fractions of bone tissue, which are shown in Fig. 4.7. The experimental results by Ryan and Williams [136] are also marked in the plot. It is shown that the lower bounds of those three methods are almost the same. For the upper bound, the third-order upper bound shows the tightest result but it conflicts with experimental results. The VAMUCH upper bound shows improvement over the Hashin-Shtrikman upper bound as it provides a lower estimate of the upper bound. In other words the gap between VAMUCH upper and lower bounds is smaller than that of the Hashin-Shtrikman bounds.

The unique advantage of the proposed approach is that it can predict the bounds changing with respect to the changing of variance, in other words, it can calculate the bounds of effective properties with considering randomness effect. Fig. 4.8, Fig. 4.9 and Fig. 4.10 demonstrated that when variance decreases or in other words the material is less random, the upper bound and lower bound are getting closer and when variance increases or in other words the material is more random, the discrepancy of bounds increases. This prediction can be very useful in designing and optimization of heterogeneous materials, to provide some guidance for position tolerance under quality control process.

4.4 Summary

Then VAMUCH can be used to homogenize random cell structure and obtain corresponding lower bounds and upper bounds. Because the trial points used in obtaining
Fig. 4.7: Bounds comparison of Young’s modulus.

Fig. 4.8: Convergence of transverse Young’s modulus $E_2$. 
Fig. 4.9: Convergence of longitudinal shear modulus $G_{12}$.

Fig. 4.10: Convergence of longitudinal Poisson’s ratio $\nu_{12}$. 
expectation material properties are much less than getting three-point correlation functions, the present approach is more efficient than the third-order bounds. The 2D and 3D random heterogeneous material examples we analyzed also show that the new approach is more effective in predicting bounds than existing methods.

The present approach can naturally bridge the effective material properties between the deterministic heterogeneous materials and the random heterogeneous materials. The gap between the lower bound and the upper bound predicted by the present approach decreases when the variance decreases and both bounds converges to the effective properties of the mean microstructure when the variance is zero. Such a capability cannot be found in any other methods in the literature.

We note that this new approach is also effective for predicting bounds for the multiple phases and anisotropic microstructures because the expectation of material properties can be easily calculated by Monte Carlo method and the SUC can be straightforwardly generated. Unlike nth-order bounds method, the empirical equations to calculate integral parameters are different in different situations, the new approach is systematically derived and remains the same when analyzing different SUCs based on different random microstructures, which makes it general-purpose. The present theory can handle other types of randomness such as shape of the fiber, dimension of the fiber, fiber waviness, fiber misalignment, etc, as long as we can obtain the expectation material properties with all the admissible microstructures provided in all those situations. The main challenge is to provide a statistical description to correctly represent the randomness and how to achieve it will be different case by case.
Chapter 5
Homogenization of Corrugated Plate

The common practice in design and analyses of the corrugated structures is to use an equivalent orthotropic plate, which is possible if the period of corrugation is much smaller than the size of the structure, as shown in Fig. 5.1. One needs first to obtain the equivalent plate stiffnesses based on analysis of a single period of the corrugation, commonly called unit cell in the literature on micromechanics. These constants are then used in the plate analysis to obtain the overall behavior. For the failure analysis, it is important to know also the local stress and strain fields. To our knowledge, the latter issue has not been addressed yet.

Generally speaking, the slope of corrugations can be categorized as the continuity (1-5 in Fig. 1.5) and the discontinuity (6 in Fig. 1.5). In the continuous case, the slope is described by a unique function along the unit cell. However, in the discontinuous case, piecewise constants are obtained by the derivative of the height with respect to the span. In the following sections, we will solve two cases one after another and see how the slope plays in the solving process. At last, the numerical solution will be obtained through finite element approach. Since the variational form of shell energy contains the second order terms of fluctuation functions, C1 continuity elements are required in the numerical solutions.

5.1 Homogenization of Continuous Corrugated Structures

5.1.1 Shell formulation of corrugated structures

The thin-walled corrugated structure can be accurately described by the shell theory. We choose a Cartesian coordinate system $x_i$ with basic vectors $\hat{e}_i$. Latin indices run through the values 1, 2, and 3; Greek indices assume values 1 and 2, and summation is conducted
over repeated indices except where explicitly indicated. The position vector of the shell mid-surface can be considered as a function of coordinates $x_1$ and $x_2$

$$
\mathbf{r}(x_1, x_2) = x_1 \hat{e}_1 + x_2 \hat{e}_2 + x_3 \hat{e}_3.
$$

(5.1)

If there are corrugations along both $x$ and $y$ directions, $x_3$ is a function of both coordinates $x_1$ and $x_2$ (This case is formulated using FEM in Appendix B). Herein, we restrict our consideration to the case of periodic corrugations in one direction, $x$, as in Fig. 5.2. The tangent vectors $\mathbf{a}_\alpha$ of the shell surface can be obtained by differentiating the position vector with respect to $x_\alpha$, $\mathbf{a}_\alpha = \partial \mathbf{r} / \partial x_\alpha$, so that

$$
\mathbf{a}_1 = \hat{e}_1 + \varphi(x) \hat{e}_3, \quad \mathbf{a}_2 = \hat{e}_2,
$$

(5.2)

with

$$
\varphi(x) = \frac{dx_3(x)}{dx}.
$$

(5.3)
Fig. 5.2: Shell geometry and unit cell.

For brevity uses, we also write \( a_\alpha = r^i_\alpha \hat{e}_i \), which implies

\[
\begin{align*}
    r^1_1 &= 1, & r^2_1 &= 0, & r^3_1 &= \varphi(x), & r^1_2 &= 0, & r^2_2 &= 1, & r^3_2 &= 0. \\
\end{align*}
\]

(5.4)

The metric tensor of the shell surface, \( a_{\alpha\beta} \), defined as

\[
    a_{\alpha\beta} = a_\alpha \cdot a_\beta,
\]

that is

\[
    a_{11} = 1 + \varphi^2, \quad a_{12} = 0, \quad a_{22} = 1, \quad a = \det \|a_{\alpha\beta}\| = 1 + \varphi^2. \quad (5.6)
\]

The contravariant components of the surface metric tensor \( a^{\alpha\beta} \) are the components of the inverse matrix to the matrix \( \|a_{\alpha\beta}\| \), i.e. \( a^{\alpha\beta} a_{\gamma\beta} = \delta_{\alpha\gamma} \), \( \delta_{\alpha\gamma} \) being the Kronecker symbol.
We have from Eq. (5.6)

\[
a^{11} = \frac{1}{1 + \varphi^2}, \quad a^{12} = 0, \quad a^{22} = 1. \tag{5.7}
\]

The normal vector of the shell mid-surface is

\[
\hat{n} = \frac{a_1 \times a_2}{|a_1 \times a_2|} = -\frac{\varphi}{\sqrt{a}} \hat{e}_1 + \frac{1}{\sqrt{a}} \hat{e}_3, \tag{5.8}
\]
or in terms of the components,

\[
n_1 = -\frac{\varphi}{\sqrt{a}}, \quad n_2 = 0, \quad n_3 = \frac{1}{\sqrt{a}}. \tag{5.9}
\]

The curvature tensor, or the so-called second quadratic form, of the shell mid-surface is defined as

\[
b_{\alpha\beta} = \frac{\partial a_\alpha}{\partial x_\beta} \cdot \hat{n}. \tag{5.10}
\]

Hence, we have

\[
b_{11} = \frac{1}{\sqrt{a}} \frac{d\varphi}{dx}, \quad b_{12} = b_{22} = 0, \tag{5.11}
\]

\[
b_1^1 = \frac{1}{a^{3/2}} \frac{d\varphi}{dx}, \quad b_1^2 = b_2^1 = b_2^2 = 0,
\]

where \(b_\beta^\gamma = a^{\alpha\gamma}b_{\gamma\beta}\).

The Christoffel symbols can be found from the equation

\[
\Gamma^\gamma_{\alpha\beta} = \frac{1}{2} a^{\gamma\delta} \left( \frac{\partial a_{\alpha\delta}}{\partial x_\beta} + \frac{\partial a_{\beta\delta}}{\partial x_\alpha} - \frac{\partial a_{\alpha\beta}}{\partial x_\delta} \right). \tag{5.12}
\]

Using Eq. (5.6), we obtain that all components of \(\Gamma^\gamma_{\alpha\beta}\) vanish except

\[
\Gamma_1^{11} = \frac{1}{2a} \frac{da}{dx} = \frac{1}{2} \frac{d\ln a}{dx} \tag{5.13}
\]

In derivations of the equivalent plate stiffnesses, the coupling between extension and
bending is conveniently neglected for the independent evaluation of equivalent extension
and bending stiffnesses. Since coupling does exist, such a neglection put the validity of
Eqs. (1.18) and (1.26) in question to reexamine the problem. Besides of coupling coeffi-
cients, we aim to recover also the local stresses and strains. To this end we employ the
variational asymptotic method [110]. It allows one to construct the equivalent plate model
for corrugated structures without invoking any ad hoc assumptions.

According to the general theory of periodic structures [137,138] (see also [112] chapter
17), the functions describing the behavior of the shell should be considered as functions
of the cell coordinate \(X\), and slow coordinates \(x\), and \(y\). To formulate our results for
stiffnesses, we need to set up the necessary notations. Let \(x\) be the Cartesian coordinate in
the corrugation direction and \(\varepsilon\) the period of corrugation (Fig. 5.2). We denote by

\[
X = \frac{x}{\varepsilon},
\]

the dimensionless “cell coordinate”. Within a cell, \(X\) changes between \(-\frac{1}{2}\) and \(\frac{1}{2}\). For
any parameter, \(f\), changing within a cell, \(\langle f \rangle\) means the average of the cell,

\[
\langle f \rangle \equiv \int_{-\frac{1}{2}}^{\frac{1}{2}} f(X) dX.
\]

The shape of the corrugation is described by the dependence of the vertical position of
the mid-surface of the corrugated plate, \(x_3\), on \(X\) (Fig. 5.2)

\[
x_3 = x_3(X).
\]

This function is a periodic function with the period unity. Without loss of generality, one
can set

\[
\langle x_3 \rangle = 0,
\]
by shifting the observer’s frame in the vertical direction. The magnitude of \( x_3(X) \) is assumed to be of order \( \epsilon \),

\[
x_3 = \epsilon \phi(X),
\]

(5.18)

so that the derivative

\[
\varphi = \frac{dx_3(x)}{dx} = \frac{d\phi(X)}{dX},
\]

(5.19)

is finite. Due to periodicity of \( \phi \),

\[
\langle \varphi \rangle = 0
\]

(5.20)

Since

\[
\langle \phi \rangle = 0,
\]

(5.21)

function \( \phi(X) \) takes both positive and negative values. Two cases should be distinguished: symmetric corrugations and asymmetric ones. For symmetric corrugations, \( \phi(X) \) is an antisymmetric function of \( X \),

\[
-\phi(X) = \phi(-X),
\]

(5.22)

and due to periodicity of \( \phi(X) \),

\[
\phi(1/2) = 0.
\]

(5.23)

Derivative \( \varphi = d\phi/dX \) is an even function, and so is \( a = 1 + \varphi^2 \). Therefore, \( \phi \sqrt{a} \) is an odd function and

\[
\langle \phi \sqrt{a} \rangle = 0.
\]

(5.24)

For asymmetric corrugations, function \( \phi(X) \) is not antisymmetric. Therefore, in general, \( \phi(1/2) \neq 0 \), and \( \langle \phi \sqrt{a} \rangle \neq 0 \). For shallow corrugation an additional small parameter, the magnitude of \( \phi \), appears. The lengths along the corrugated plate, \( S \), and along the effective plates, \( x \), are linked by the factor \( \sqrt{a} \)

\[
ds = \sqrt{a} dx.
\]

(5.25)
The total length of the corrugated plate inside a cell, $S$, and the cell size over effective plate, $\varepsilon$, are related as

$$S/\varepsilon = \langle \sqrt{a} \rangle .$$  (5.26)

All the geometric characteristics we just introduced are functions of $X$ only, e.g.

$$x_3 = \varepsilon \phi(X), \quad \varphi(X) = \frac{d\phi(X)}{dX},$$

$$b_{11} = \frac{1}{\varepsilon\sqrt{a}} \frac{d\varphi}{dX}, \quad b_{11}' = \frac{1}{\varepsilon a^{3/2}} \frac{d\varphi}{dX}, \quad \Gamma_{11}^1 = \frac{1}{2\varepsilon} \frac{d\ln a}{dX}.$$  (5.27)

Let $u_i(X,x,y)$ be the components of the displacement vector. The extension strain measures $\gamma_{\alpha\beta}$ and bending measures $\rho_{\alpha\beta}$ are expressed in terms of $u_i$ as follows [112]

$$2\gamma_{\alpha\beta} = r_{i\alpha} \frac{\partial u_i}{\partial x_\beta} + r_{i\beta} \frac{\partial u_i}{\partial x_\alpha},$$

$$2\rho_{\alpha\beta} = \frac{\partial}{\partial x_\beta} \left( n_i \frac{\partial u_i}{\partial x_\alpha} \right) + \frac{\partial}{\partial x_\alpha} \left( n_i \frac{\partial u_i}{\partial x_\beta} \right) - 2\Gamma_{\alpha\gamma}^\gamma n_i \frac{\partial u_i}{\partial x_\gamma} + \theta \left( e_{\alpha\beta} b_{\gamma\gamma} + e_{\gamma\gamma} b_{\alpha\alpha} \right).$$  (5.28)

where $e_{\alpha\beta}$ denotes surface Levi-Civita tensor ($e_{11} = e_{22} = 0, e_{12} = -e_{21} = \sqrt{a}$). $\theta$ is the angle of rotation of the surface around the normal vector

$$\theta = \frac{1}{2\sqrt{a}} \left( r_{1i} \frac{\partial u_i}{\partial x_2} - r_{2i} \frac{\partial u_i}{\partial x_1} \right).$$  (5.29)

Note that $u^i = u_i$ because $u_i$ are the displacement components in the Cartesian coordinate systems $\hat{e}_i$. While $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are tensor components in surface coordinates, and, therefore the components with upper indices acquire additional metric factors. Because $X$ is related with $x = x_1$ according to Eq. (5.14), the derivative of the displacement field with respect to $x_1$ can be expressed as

$$\frac{\partial u_i}{\partial x_1} = \frac{\partial u_i}{\partial x} = \frac{\partial u_i}{\partial X} \frac{\partial X}{\partial x} |_{x=\text{const}} + \frac{\partial u_i}{\partial x} |_{X=\text{const}} = \frac{1}{\varepsilon} u'_i + u_{i,1},$$  (5.30)

with $u'_i = \frac{\partial u_i}{\partial x} |_{x=\text{const}}$ and $u_{i,1} = \frac{\partial u_i}{\partial x} |_{X=\text{const}}$. We also denote $u_{i,2} = \frac{\partial u_i}{\partial x_2}$ and $\frac{\partial u_i}{\partial y}$. 
The elastic behavior of the shell is governed by its strain energy density which is given by the following expression

\[
\Phi = \mu h \left( \sigma \left( a^{\alpha \beta} \gamma_{\alpha \beta} \right)^2 + a^{\alpha \beta} a^{\gamma \delta} \gamma_{\alpha \gamma} \gamma_{\beta \delta} \right) + \frac{\mu h^3}{12} \left( \sigma \left( a^{\alpha \beta} \rho_{\alpha \beta} \right)^2 + a^{\alpha \beta} a^{\gamma \delta} \rho_{\alpha \gamma} \rho_{\beta \delta} \right),
\]

(5.31)

Here in Eq. (5.31) \( \mu = E/(1 + \nu) \) is the shear modulus, \( \nu \) the Poisson’s ratio, and \( \sigma = \nu/(1 - \nu) \). The first part is the extension energy and second part the bending energy. The strain energy of the unit cell can be written as

\[
J = \left\langle \Phi \sqrt{a} \right\rangle = \left\langle \mu h \sqrt{a} \left( \sigma \left( \frac{1}{a} \gamma_{11} + \gamma_{22} \right)^2 + \frac{1}{a^2} \gamma_{11}^2 + \frac{2}{a} \gamma_{12} + \gamma_{22} \right) \right\rangle + \left\langle \frac{\mu h^3}{12} \sqrt{a} \left( \sigma \left( \frac{1}{a} \rho_{11} + \rho_{22} \right)^2 + \frac{1}{a^2} \rho_{11}^2 + \frac{2}{a} \rho_{12} + \rho_{22} \right) \right\rangle
\]

(5.32)

with \( \nu = \sigma/(1 + \sigma) \). Here the material parameters \( \mu, \sigma \) and the shell thickness \( h \) could be functions of \( X \), but for simplicity, we assume that they are constant.

### 5.1.2 Asymptotic analysis of the shell strain energy

To model the corrugated structure by an equivalent plate, we start from the assumption that the shell displacements can be expressed in the form

\[
u_{\alpha}(X, x, y) = v_{\alpha}(x, y) - x_3(X)v_{3, \alpha} + \varepsilon \psi_{\alpha}(X, x, y),
\]

(5.33)

\[
u_3(X, x, y) = v_3(x, y) + \varepsilon \psi_3(X, x, y).
\]

In fact, this is a short cut, and Eq. (5.33) can be derived by the variational asymptotic method [112], chapter 17.2. In Eq. (5.33), \( v_i \) have the meaning of the effective plate displ-
placements, and \( \psi_i \) are some functions which are periodic in \( X \). Without loss of generality, we can define \( v_i \) as the average of \( u_i \) over the cell

\[
v_i(x, y) = \langle u_i(X, x, y) \rangle.
\] (5.34)

Then, obviously,

\[
\langle \psi_i(X, x, y) \rangle = 0.
\] (5.35)

Then the derivation proceeds as follows for shell strains. We have

\[
\gamma_{11} = v_{1,1} - x_3 v_{3,11} + \psi'_1 + \varphi \psi'_3 + \varepsilon (\psi_{1,1} + \varphi \psi_{3,1}),
\]

\[
2\gamma_{12} = v_{2,1} - 2x_3 v_{3,12} + \psi'_2 + \varepsilon (\psi_{1,2} + \psi_{2,1} + \varphi \psi_{3,2}),
\]

\[
\gamma_{22} = v_{2,2} - x_3 v_{3,22} + \varepsilon \psi_{2,2},
\]

\[
\rho_{11} = \frac{1}{\varepsilon} U'_1 - \frac{1}{2\varepsilon} (\ln a)' U_1 + U_{1,1} = \sqrt{a} \left( \frac{U_1}{\sqrt{a}} \right)' + U_{1,1},
\]

\[
2\rho_{12} = U_{1,2} + U_{2,1} + \frac{1}{\varepsilon} U'_2 + \frac{\varphi'}{\varepsilon a^{3/2}} \sqrt{a} \theta,
\]

\[
\rho_{22} = U_{2,2},
\]

with

\[
U_1 = n_1 (v_{1,1} - x_3 v_{3,11} + \psi'_1) + n_3 \psi'_3 + \sqrt{a} v_{3,1} + \varepsilon (n_1 \psi_{1,1} + n_3 \psi_{3,1}),
\] (5.37)

\[
U_2 = n_1 (v_{1,2} - x_3 v_{3,12}) + n_3 v_{3,2} + \varepsilon (n_1 \psi_{1,2} + n_3 \psi_{3,2}),
\] (5.38)

and rotation \( \theta \),

\[
2\sqrt{a} \theta = v_{1,2} - v_{2,1} + 2\varphi v_{3,2} - \psi'_2 + \varepsilon (\psi_{1,2} - \psi_{2,1} + \varphi \psi_{3,2}).
\]
The leading terms of the extension strain measures in Eq. (5.36) are
\[
\begin{align*}
\gamma_{11}^0 &= v_{1,1} - x_3 v_{3,11} + \psi_1' + \varphi \psi_3', \\
2\gamma_{12}^0 &= v_{1,2} + v_{2,1} - 2x_3 v_{3,12} + \psi_2', \\
\gamma_{22}^0 &= v_{2,2} - x_3 v_{3,22}.
\end{align*}
\tag{5.39}
\]

Considering
\[
\begin{align*}
n_3 - n_1 \varphi &= \sqrt{a}, \\
n_1' &= -\frac{\varphi'}{a^{3/2}}, \\
n_3' &= -\frac{\varphi' \varphi}{a^{3/2}}.
\end{align*}
\tag{5.40}
\]
The bending strains contributing to the leading terms of the bending energy Eq. (5.36) are
\[
\begin{align*}
\rho_{11}^0 &= \sqrt{a} \varepsilon \left(\psi_3' - \frac{\varphi'}{a} \gamma_{11}^0\right)' + \sqrt{a} v_{3,11}, \\
2\rho_{12}^0 &= 2\sqrt{a} v_{3,12} - \frac{\varphi'}{2a^{3/2}} (2\gamma_{12}^0), \\
\rho_{22}^0 &= \frac{1}{\sqrt{a}} v_{3,22}.
\end{align*}
\tag{5.41}
\]
Thus, \(\gamma_{22}^0, \rho_{22}^0\) are constants respect with \(\psi_1, 2\gamma_{12}^0, 2\rho_{12}^0\) relate to \(\psi_2\) only, and \(\gamma_{11}^0, \rho_{11}^0\) relate to \(\psi_1, \psi_3\).

Let us focus on solving \(\psi_2\) first. The leading strain energy in Eq. (5.32) related with \(\psi_2\) is
\[
J_2 = \left\langle \mu h \frac{1}{2\sqrt{a}} \left( (2\gamma_{12}^0)^2 + \frac{h^2}{12} (2\rho_{12}^0)^2 \right) \right\rangle.
\tag{5.42}
\]
We need to minimize \(2\gamma_{12}^0, 2\rho_{12}^0\) in Eq. (5.42) over periodic functions \(\psi_2(X)\) subject to the constraints Eq. (5.35). The constraints can be taken care of by introducing the Lagrange multipliers. The corresponding Euler-Lagrange equation is
\[
\left( \frac{1}{\sqrt{a}} 2\gamma_{12}^0 - \frac{h^2}{12} 2\rho_{12}^0 \frac{\varphi'}{2a^{3/2}} \right)' - \lambda_2 = 0.
\tag{5.43}
\]
along with boundary conditions

\[ [\psi_2] = 0, \quad \left[ \frac{1}{\sqrt{a}} 2\gamma_{12}^0 - \frac{h^2}{12} 2\rho_{12}^0 \frac{\varphi'}{2\varepsilon a^{3/2}} \right] = 0. \tag{5.44} \]

The second condition in Eq. (5.44) leads to \( \lambda_2 = 0 \). Hence

\[ \frac{1}{\sqrt{a}} 2\gamma_{12}^0 - \frac{h^2}{12} 2\rho_{12}^0 \frac{\varphi'}{2\varepsilon a^{3/2}} = c_2. \tag{5.45} \]

Thus

\[ 2\gamma_{12}^0 = \frac{\sqrt{ac_2} + \frac{h^2 \varphi' \nu_{3,12}}{12\varepsilon a}}{1 + \frac{\varphi'^2 h^2}{48 \varepsilon^2 a^3}}, \tag{5.46} \]

\[ v_{1,2} + v_{2,1} - 2x_3 v_{3,12} + \psi_2' = \frac{\sqrt{ac_2} + \frac{h^2 \varphi' \nu_{3,12}}{12\varepsilon a}}{1 + \frac{\varphi'^2 h^2}{48 \varepsilon^2 a^3}}. \tag{5.47} \]

Integrating Eq. (5.47) over the cell length, we obtain the constant \( c_2 \)

\[ v_{1,2} + v_{2,1} = \left( \frac{\sqrt{a}}{1 + \frac{\varphi'^2 h^2}{48 \varepsilon^2 a^3}} \right) c_2 + \left( \frac{h^2 \varphi' \nu_{3,12}}{12\varepsilon a} \right) v_{3,12}, \tag{5.48} \]

\[ c_2 = \alpha_1 (v_{1,2} + v_{2,1}) - \alpha_2 v_{3,12}. \tag{5.49} \]

where \( \alpha_1 = 1 / \left( \frac{\sqrt{a}}{1 + \frac{\varphi'^2 h^2}{48 \varepsilon^2 a^3}} \right) \) and \( \alpha_2 = \alpha_1 \left( \frac{h^2 \varphi' \nu_{3,12}}{12\varepsilon a} \right) \). With \( c_2 \) and the first condition in Eq. (5.44), \( \psi_2 \) can be solved explicitly, but for the equivalent plate stiffnesses, the solution of \( c_2 \) is enough at this stage.

The leading strain energy in Eq. (5.32) related with \( \psi_1 \) and \( \psi_3 \) is

\[ J_1 = \left( \mu h \sqrt{a} (1 + \sigma) \left( \frac{\gamma_{11}^0}{a} + \nu \gamma_{22}^0 \right)^2 + \frac{\mu h^3}{12} \sqrt{a} (1 + \sigma) \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right)^2 \right). \tag{5.50} \]
Similarly, we use Lagrange multiplier to take care of the constraints of $\psi_1$ and $\psi_3$ in Eq. (5.35). The corresponding Euler-Lagrange equations are

$$\frac{1}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) + \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{\varphi}{a}' - \lambda_1 = 0,$$

$$\frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) - \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{1}{a}' - \lambda_3 = 0. \quad (5.51)$$

along with boundary conditions

$$[\psi_1] = 0, \quad [\psi_1'] = 0, \quad \left[ \frac{1}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) + \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{\varphi}{a} \right] = 0,$$

$$[\psi_3] = 0, \quad [\psi_3'] = 0, \quad \left[ \frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) - \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{1}{a} \right] = 0, \quad (5.52)$$

$$\left[ \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right] = 0.$$

The third and sixth conditions in Eq. (5.52) leads to $\lambda_1 = \lambda_3 = 0$. Hence

$$\frac{1}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) + \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{\varphi}{a} = c_1, \quad (5.53)$$

$$\frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_{11}}{a} + \nu \gamma_{22} \right) - \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{1}{a} = c_3. \quad (5.54)$$

Integrate ($\varphi \times (5.53) - (5.54)$) over the cell length with considering the seventh conditions in Eq. (5.52) conclude

$$c_3 = 0. \quad (5.55)$$

Then Eqs. (5.53) and (5.54) can be simplified as

$$\left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right)' = c_1 \frac{12 \varphi \varepsilon}{h^2}, \quad (5.56)$$

$$\left( \frac{\gamma_{11}^0}{a} + \nu \gamma_{22}^0 \right) = \frac{c_1}{\sqrt{a}}. \quad (5.57)$$
Integrate Eq. (5.56)
\[
\left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) = c_1 \frac{12 x_3}{h^2} + c_4, \tag{5.58}
\]

Rewriting Eq. (5.58) considering Eq. (5.41)
\[
\left( \psi'_3 - \frac{\varphi}{a} \gamma_{11}^0 \right)' = \varepsilon \left( c_1 \frac{12}{h^2} x_3 \sqrt{a} + c_4 \sqrt{a} - (v_{3,11} + \nu v_{3,22}) \right). \tag{5.59}
\]

Integrate over the cell length with the fact \[ \psi'_3 - \frac{\varphi}{a} \gamma_{11}^0 = 0, \] \( c_4 \) is
\[
\frac{1}{\langle \sqrt{a} \rangle} (v_{3,11} + \nu v_{3,22}) - \frac{12}{h^2} c_1 \frac{\langle x_3 \sqrt{a} \rangle}{\langle \sqrt{a} \rangle}. \tag{5.60}
\]

Integrate Eq. (5.59) considering \( c_4 \),
\[
\psi'_3 - \frac{\varphi}{a} \gamma_{11}^0 = -\frac{12 \varepsilon^2}{h^2} c_1 A + \varepsilon \left( \frac{\int_0^X \sqrt{a} dY}{\langle \sqrt{a} \rangle} - X \right) (v_{3,11} + \nu v_{3,22}) + c_5. \tag{5.61}
\]

with
\[
A(X) = -\int_0^X \sqrt{a} \phi(Y) dY + \frac{\langle \sqrt{a} \phi \rangle}{\langle \sqrt{a} \rangle} \int_0^X \sqrt{a} dY, \tag{5.62}
\]

\( \varphi \times (5.57) + (5.61) \) gives
\[
\psi'_3 + \nu \varphi \gamma_{22}^0 = \frac{c_1 \varphi}{\sqrt{a}} - \frac{12 \varepsilon^2}{h^2} c_1 A + \varepsilon \left( \frac{\int_0^X \sqrt{a} dY}{\langle \sqrt{a} \rangle} - X \right) (v_{3,11} + \nu v_{3,22}) + c_5, \tag{5.63}
\]

Integrating over the cell length, we obtain
\[
c_5 = -c_1 \left( \frac{\varphi}{\sqrt{a}} \right) + \frac{12 \varepsilon^2}{h^2} c_1 \langle A \rangle - \varepsilon \frac{\langle \int_0^X \sqrt{a} dY \rangle}{\langle \sqrt{a} \rangle} (v_{3,11} + \nu v_{3,22}), \tag{5.64}
\]

Here, notice \( \langle \varphi x_3 \rangle = 0 \). Substitute \( c_5 \) into Eq. (5.63)
\[
\psi'_3 = -\nu \varphi \gamma_{22}^0 + c_1 \left( \frac{\varphi}{\sqrt{a}} - \left( \frac{\varphi}{\sqrt{a}} \right) \right) - \frac{12 \varepsilon^2}{h^2} c_1 (A - \langle A \rangle)
\]
\[
+ \varepsilon \left( \frac{\int_0^X \sqrt{a} dY}{\langle \sqrt{a} \rangle} - X \right) (v_{3,11} + \nu v_{3,22}), \tag{5.65}
\]
Rewrite Eq. (5.57) as

\[ v_{1,1} - x_3 v_{3,11} + \psi_1' + \varphi \psi_3' = c_1 \sqrt{a} - \nu a \gamma_{22}^0, \]  

(5.66)

Substitute Eq. (5.65) into Eq. (5.66) and integrate over the cell length

\[ v_{1,1} + \nu v_{2,2} = \frac{12 \varepsilon^2}{h^2} c_1 \langle \varphi A \rangle + \varepsilon B (v_{3,11} + \nu v_{3,22}) + c_1 \left\langle \frac{1}{\sqrt{a}} \right\rangle, \]  

(5.67)

with the constant \( B \)

\[ B = \left( \langle \varphi X \rangle - \frac{\langle \varphi \int_0^X \sqrt{a} dY \rangle}{\langle \sqrt{a} \rangle} \right) = \frac{\langle \sqrt{a} \rangle \int_{-\frac{1}{2}}^{\frac{1}{2}} X d\phi - \int_{-\frac{1}{2}}^{\frac{1}{2}} \int_0^X \sqrt{a} dY d\phi}{\langle \sqrt{a} \rangle} \]  

(5.68)

Thus

\[ c_1 = \frac{\varepsilon B (v_{3,11} + \nu v_{3,22}) - (v_{1,1} + \nu v_{2,2})}{C}, \]  

(5.69)

where

\[ C = -12 \langle \varphi A \rangle \frac{\varepsilon^2}{h^2} - \left\langle \frac{1}{\sqrt{a}} \right\rangle. \]  

(5.70)

5.1.3 Equivalent plate energy

Now, everything is ready to compute the equivalent plate energy. It is convenient to split the strain energy in Eq. (5.32) into three part. \( J_1 \) is associated with energy in Eq. (5.50), \( J_2 \) with energy in Eq. (5.42), and \( J_3 \) with energy

\[ J_3 = \left\langle \mu h \sqrt{a} (1 + \nu) \left( \gamma_{22}^0 \right)^2 + \frac{\mu h^3}{12} \sqrt{a} (1 + \nu) (\rho_{22}^0)^2 \right\rangle. \]  

(5.71)

Let us compute \( J_1 \) first. Using Eq. (5.57) and Eq. (5.58)

\[ J_1 = \left\langle \mu h \sqrt{a} (1 + \sigma) \left( \frac{c_1}{\sqrt{a}} \right)^2 + \frac{\mu h^3}{12} \sqrt{a} (1 + \sigma) \left( \frac{c_1}{h^2} + c_4 \right)^2 \right\rangle. \]  

(5.72)
Substituting Eq. (5.60) and Eq. (5.69) into Eq. (5.72), $J_1$ becomes

$$J_1 = \left\langle \mu h \left( \frac{1}{\sqrt{a}} (1 + \sigma) \left( \frac{\varepsilon B (v_{3,11} + \nu v_{3,22}) - (v_{1,1} + \nu v_{2,2})}{C} \right)^2 \right. \right.$$  
\[ \left. + \frac{\mu h^3}{12} \frac{1}{\sqrt{a}} (1 + \sigma) \left( - \frac{(v_{1,1} + \nu v_{2,2}) 12}{h^2} (x_3 - \varepsilon B) \right) \right. \]
\[ \left. + \left( \frac{1}{\langle \sqrt{a} \rangle} + \frac{12 \varepsilon B}{h^2 C} (x_3 - \varepsilon B) \right) (v_{3,11} + \nu v_{3,22}) \right)^2 \rangle \right. $$  
\[ = (v_{1,1} + \nu v_{2,2})^2 \mu (1 + \sigma) \frac{1}{C^2} \left( \frac{1}{\langle \sqrt{a} \rangle} + \frac{12}{h} \varepsilon^2 \langle \varphi A \rangle \right) \]
\[ + (v_{3,11} + \nu v_{3,22})^2 \mu h (1 + \sigma) \left( \frac{\varepsilon^2 B^2}{C^2} \left( \frac{1}{\langle \sqrt{a} \rangle} \right) + \frac{h^2}{12} \left( \frac{12^2 \varepsilon^4 B^2}{h^4 C^2} (\varphi A) + \frac{1}{\langle \sqrt{a} \rangle} \right) \right) \]
\[ - (v_{1,1} + \nu v_{2,2}) (v_{3,11} + \nu v_{3,22}) \mu h (1 + \sigma) \left( \frac{2 \varepsilon B}{C^2} \left( \frac{1}{\langle \sqrt{a} \rangle} \right) + \frac{24 B \varepsilon^3}{h^2 C^2} (\varphi A) \right). \]

Note

$$\langle \sqrt{a} (x_3 - B \varepsilon)^2 \rangle = \left\langle \sqrt{a} x_3^2 \right\rangle - \left\langle \frac{\sqrt{a} x_3^2}{\sqrt{a}} \right\rangle = \varepsilon^2 \langle \varphi A \rangle, \quad \left\langle \sqrt{a} (x_3 - \varepsilon B) \right\rangle = 0. \tag{5.74}$$

Rewriting Eq. (5.42)

$$J_2 = \frac{\mu h}{2} \left\langle \frac{1}{\sqrt{a}} \left( (2 \gamma_{12})^2 + \frac{h^2}{12} \left( 2 \sqrt{a} v_{3,12} - \frac{\varphi'}{2 \varepsilon a^{3/2}} 2 \gamma_{12}^2 \right) \right) \right\rangle. \tag{5.75}$$

Substituting Eq. (5.47) and Eq. (5.49) into Eq. (5.75),

$$J_2 = (v_{1,2} + v_{2,1})^2 \left\langle \frac{\mu h a_1^2}{2} \left( \frac{\sqrt{a}}{1 + \frac{\varepsilon^2 h^2}{4 \varepsilon a^4}} \right) \right\rangle \]
\[ + \frac{v_{3,12}^2}{2} \mu h \left\langle \frac{\sqrt{a} h^2}{3} - \frac{1}{\sqrt{a}} \left( \frac{h^4 \varepsilon^2}{12 \varepsilon^4 a^2} - \frac{a_1^2}{2} \right) \right\rangle \]
\[ - (v_{1,2} + v_{2,1}) v_{3,12} \mu h a_1 a_2 \left\langle \frac{\sqrt{a}}{1 + \frac{\varepsilon^2 h^2}{4 \varepsilon a^4}} \right\rangle. \tag{5.76}$$
Substituting $\gamma_{22}^0$ in Eq. (5.39) and $\rho_{22}^0$ in Eq. (5.41) into Eq. (5.71),

$$J_3 = v_{2,2}^2 \mu h(1 + \nu) \langle \sqrt{a} \rangle + v_{3,22}^2 \mu h(1 + \nu) \left( \varepsilon^2 \langle \sqrt{a} \phi^2 \rangle + \frac{h^2}{12} \langle \frac{1}{\sqrt{a}} \rangle \right)$$

$$- v_{2,2} v_{3,22} 2 \mu h(1 + \nu) \varepsilon \langle \sqrt{a} \phi \rangle.$$  

If we set

$$\epsilon_{xx} = v_{1,1}, \quad 2\epsilon_{xy} = v_{1,2} + v_{2,1}, \quad \epsilon_{yy} = v_{2,2},$$

$$\kappa_{xx} = -v_{3,11}, \quad \kappa_{xy} = -v_{3,12}, \quad \kappa_{yy} = -v_{3,22},$$

in

$$J = \frac{1}{2} \begin{bmatrix} \epsilon_{xx} \\ 2\epsilon_{xy} \\ \epsilon_{yy} \\ \kappa_{xx} \\ 2\kappa_{xy} \\ \kappa_{yy} \end{bmatrix} = \begin{bmatrix} A_{11} & 0 & A_{13} & B_{11} & 0 & B_{13} \\ 0 & A_{22} & 0 & 0 & B_{22} & 0 \\ A_{13} & 0 & A_{33} & B_{13} & 0 & B_{33} \\ B_{11} & 0 & B_{13} & D_{11} & 0 & D_{13} \\ 0 & B_{22} & 0 & 0 & D_{22} & 0 \\ B_{13} & 0 & B_{33} & D_{13} & 0 & D_{33} \end{bmatrix} \begin{bmatrix} \epsilon_{xx} \\ 2\epsilon_{xy} \\ \epsilon_{yy} \\ \kappa_{xx} \\ 2\kappa_{xy} \\ \kappa_{yy} \end{bmatrix},$$

(5.79)

We obtain the following relations for the equivalent plate stiffnesses

$$A_{11} = \frac{E}{1 - \nu^2} \frac{12 \varepsilon^2 \langle \varphi A \rangle}{h C^2} + \frac{E h}{1 - \nu^2} \left( \frac{1}{\sqrt{a}} \right) \left( \frac{1}{C^2} \right), \quad A_{13} = \nu A_{11},$$

$$A_{22} = \mu h a_1^2 \left( \frac{\sqrt{a}}{1 + \frac{\varphi^2}{h^2} \frac{a^2}{\varepsilon^2}} \right), \quad A_{33} = E h \langle \sqrt{a} \rangle + \nu^2 A_{11},$$

$$B_{11} = \frac{E}{1 - \nu^2} \frac{12 \varepsilon^3 \langle \varphi A \rangle}{h C^2} B + \frac{E h}{1 - \nu^2} \left( \frac{1}{\sqrt{a}} \right) \left( \frac{1}{C^2} \right) B \varepsilon, \quad B_{13} = \nu B_{11},$$

$$B_{22} = \mu h a_1 a_2 \left( \frac{\sqrt{a}}{1 + \frac{\varphi^2}{h^2} \frac{a^2}{\varepsilon^2}} \right), \quad B_{33} = E h \varepsilon \langle \sqrt{a} \phi \rangle + \nu^2 B_{11},$$

$$D_{11} = \frac{E h^3}{12(1 - \nu^2)} \left( \frac{12 \varepsilon^4 B^2}{h C^2} \langle \varphi A \rangle + \frac{1}{\langle \sqrt{a} \rangle} \right) + \frac{E h}{1 - \nu^2} \frac{\varepsilon^2 B^2}{C^2} \left( \frac{1}{\sqrt{a}} \right), \quad D_{13} = \nu D_{11},$$

$$D_{22} = \frac{\mu h}{4} \left( \frac{\sqrt{a}}{3} \right) \left( \frac{h^4 \varphi^2}{12 \varepsilon^2 a^2} - a a_2^2 \right), \quad D_{33} = E h \varepsilon^2 \langle \phi^2 \sqrt{a} \rangle + \frac{E h^3}{12} \left( \frac{1}{\sqrt{a}} \right) + \nu^2 D_{11}.$$  

(5.80)
In the case of $\varphi = 0$ where the corrugated plate degenerates to the flat plate.

$$\sqrt{a} = 1, \quad x_3 = 0, \quad A = 0, \quad B = 0, \quad C = -1, \quad \alpha_1 = 1, \quad \alpha_2 = 0.$$  \hfill (5.81)

Thus

$$A_{11} = \frac{Eh}{1 - \nu^2}, \quad A_{13} = \nu A_{11}, \quad A_{22} = \frac{Eh}{2(1 + \nu)}, \quad A_{33} = \frac{Eh}{1 - \nu^2},$$

$$B_{11} = 0, \quad B_{13} = 0, \quad B_{22} = 0, \quad B_{33} = 0 \hfill (5.82)$$

$$D_{11} = \frac{Eh^3}{12(1 - \nu^2)}, \quad D_{13} = \nu D_{11}, \quad D_{22} = \frac{Eh^3}{4(1 + \nu)}, \quad D_{33} = \frac{Eh^3}{12(1 - \nu^2)}.$$

which generates the plate stiffnesses as it should be.

If we take the advantage as $h/\varepsilon \ll 1$ as a shallow corrugated plate.

$$C^2 \approx 12^2 \langle \varphi A \rangle^2 \frac{\varepsilon^4}{h^4}, \quad \alpha_1 \approx \frac{1}{\langle \sqrt{a} \rangle}, \quad \alpha_2 \approx 0.$$  \hfill (5.83)

The leading terms of equivalent plate stiffnesses become

$$A_{11} = \frac{Eh^3}{12(1 - \nu^2)} \varepsilon^2 \langle \varphi A \rangle, \quad A_{13} = \nu A_{11}, \quad A_{22} = \frac{\mu h}{\langle \sqrt{a} \rangle}, \quad A_{33} = Eh \langle \sqrt{a} \rangle,$$

$$B_{11} = \frac{Eh^3 B}{12(1 - \nu^2)} \varepsilon \langle \varphi A \rangle, \quad B_{13} = \nu B_{11}, \quad B_{22} = 0, \quad B_{33} = Eh \langle x_3 \sqrt{a} \rangle,$$

$$D_{11} = \frac{Eh^3}{12(1 - \nu^2)} \left( \frac{B^2}{\langle \varphi A \rangle} + \frac{1}{\langle \sqrt{a} \rangle} \right), \quad D_{13} = \nu D_{11}, \quad D_{22} = \frac{\mu h^3}{12} \langle \sqrt{a} \rangle, \quad D_{33} = Eh \langle x_3^2 \sqrt{a} \rangle.$$  \hfill (5.84)

### 5.1.4 Recovery relations

The equivalent plate stiffnesses constants can be used as inputs to carry out a plate analysis, either analytically or numerically, to predict the plate displacement field ($v_i$) and strain field ($\epsilon_{xx} , 2\epsilon_{xy} , \epsilon_{yy} , \kappa_{xx} , 2\kappa_{xy} , \kappa_{yy}$). This information can be used first to recovery the displacement field in the original corrugated shell using Eq. (5.33). Usually it is more critical to know the strain field within the original corrugated shell which can be obtained
from Eq. (5.39) and Eq. (5.41) as

\[ \gamma_{11} = c_1 \sqrt{a} - \nu a (\epsilon_{yy} + x_3 \kappa_{yy}), \]

\[ 2 \gamma_{12} = \frac{\sqrt{a} (2 \alpha_1 \epsilon_{xy} + \alpha_2 \kappa_{xy}) - \frac{h^2 \varphi' \kappa_{yy}}{12a}}{1 + \frac{\varphi'^2 h^2}{48 \kappa_{yy}^2}}, \]

\[ \gamma_{22} = \epsilon_{yy} + x_3 \kappa_{yy}, \]

\[ \rho_{11} = -a \left( \frac{(\kappa_{xx} + \nu \kappa_{yy})}{\sqrt{a}} \right) + \frac{12}{h^2 c_1} \left( \frac{x_3 \sqrt{a}}{\sqrt{a}} - x_3 \right) - \frac{1}{\sqrt{a}} \varphi' \kappa_{yy}, \]

\[ 2 \rho_{12} = -2 \sqrt{a} \kappa_{xy} - \frac{1}{\varepsilon a^{3/2}} \epsilon_{xy}, \]

\[ \rho_{22} = -\frac{1}{\sqrt{a}} \kappa_{yy}. \]  

The stress resultants can be recovered using the constitutive relations corresponding to the strain energy in Eq. (5.31), which can be used to further recover the three-dimensional stresses based on the relations of the starting shell theory and the three-dimensional elasticity theory.

### 5.2 Homogenization of Piecewise Straight Corrugated Structures

In this section, we derive the analytical solution when the slope \( \varphi \) behaves as piecewise constants function respect to \( x \) as shown in Fig. 5.3. Still, we restrict our consideration to two pieces corrugated structures to keep the solving process as simple as possible, however, the solutions can be applied to more than two pieces cases, such as trapezoidal case. We follow the exactly same procedure as we did in continuous case in previous section though new constraints at the discontinuity point need to be introduced to solve the problem.

#### 5.2.1 Shell formulation of corrugated structures

Considering \( \varphi \) is a piecewise constant, \( a = 1 + \varphi^2 \) becomes a piecewise constant too. \( \varphi' = 0 \) as well as \( a' = 0 \) lead to new values of Eq. (5.11), such that

\[ b_{11} = b_{12} = b_{22} = b_1^1 = b_2^1 = b_1^2 = b_2^2 = 0. \]  

(5.86)
and all components of Christoffel’s symbols are vanished.

With the superscript with parenthesis denoting the segment, $x_3(X)$ can be decomposed (Fig. 5.4)

$$
\begin{align*}
  x_3^{(1)}(X) &= \varphi^{(1)} \left( \varepsilon X - \frac{P - \frac{\varepsilon}{2}}{2} \right) & \frac{-1}{2} \leq X \leq \frac{P}{\varepsilon} \\
  x_3^{(2)}(X) &= \varphi^{(2)} \left( \varepsilon X - \frac{P + \frac{\varepsilon}{2}}{2} \right) & \frac{P}{\varepsilon} < X \leq \frac{1}{2}
\end{align*}
$$

(5.87)

The extension and bending strain measures are calculated following Eq. (5.28). The leading terms of extension energy remain the same as Eq. (5.39) and the leading term of bending
strains become

\[
\rho_{11}^0 = \frac{1}{\varepsilon \sqrt{a}} \psi_3'' - \frac{\varphi}{\varepsilon \sqrt{a}} \psi_1'' + v_{3,11} \frac{\varphi^2 + a}{\sqrt{a}},
\]
\[
2\rho_{12}^0 = 2\sqrt{a} v_{3,12},
\]
\[
\rho_{22}^0 = \frac{1}{\sqrt{a}} v_{3,22}.
\]

(5.88)

By observing Eqs. (5.39) and (5.88), \(\gamma_{22}^0, \rho_{22}^0, 2\rho_{12}^0\) do not contain terms \(\psi_i\), \(2\gamma_{12}^0\) relate to \(\psi_2\) only, and \(\gamma_{11}^0, \rho_{11}^0\) relate to \(\psi_1, \psi_3\). We also split the total strain energy into three parts, where \(J_1\) corresponds to \(\psi_1, \psi_3, J_2\) to \(\psi_2\), and \(J_3\) contains the constant part.

5.2.2 Asymptotic analysis of the shell strain energy

Let us focus on solving \(\psi_2\) first. Minimize \(2\gamma_{12}^0\) in Eq. (5.42) over periodic functions \(\psi_2(X)\) subject to the constraints Eq. (5.35). The constraints can be taken care of by introducing the Lagrange multipliers. The corresponding Euler-Lagrange equation is

\[
\left( \frac{1}{\sqrt{a}} 2\gamma_{12}^0 \right)' - \lambda_2 = 0,
\]

(5.89)

along with boundary conditions

\[
[\psi_2] = 0, \quad [\frac{1}{\sqrt{a}} 2\gamma_{12}^0] = 0.
\]

(5.90)

The floor square brackets denoting the difference between the end values in the whole domain, for example \([\psi_i] = 0\) denotes two conditions, one is for periodic boundary condition \(\psi_i^{(2)} (\frac{1}{2}) - \psi_i^{(1)} (-\frac{1}{2}) = 0\) and another is for continuity \(\psi_i^{(1)} (\frac{p}{\varepsilon}) - \psi_i^{(2)} (\frac{p}{\varepsilon}) = 0\). The second condition in Eq. (5.90) leads to \(\lambda_2 = 0\) and two integral constants from Eq. (5.89) \(c_2^{(1)} = c_2^{(2)}\). Hence

\[
\frac{1}{\sqrt{a}} 2\gamma_{12}^0 = c_2.
\]

(5.91)

Thus

\[
2\gamma_{12}^0 = \sqrt{a} c_2,
\]

(5.92)
$$v_{1,2} + v_{2,1} - 2x_3v_{3,12} + \psi_2' = \sqrt{ac_2}.$$  \hspace{1cm} (5.93)

Integrating Eq. (5.93) over the cell length, we obtain the constant $c_2$

$$v_{1,2} + v_{2,1} = \langle \sqrt{a} \rangle c_2,$$  \hspace{1cm} (5.94)

$$c_2 = \frac{(v_{1,2} + v_{2,1})}{\langle \sqrt{a} \rangle}.$$  \hspace{1cm} (5.95)

With $c_2$ and the first condition in Eq. (5.90), $\psi_2$ can be solved explicitly, but for the equivalent plate stiffnesses, the solution of $c_2$ is enough at this stage.

Similarly, we use Lagrange multiplier to take care of the constraints of $\psi_1$ and $\psi_3$ in Eq. (5.35). The corresponding Euler-Lagrange equations are

$$\left( \frac{1}{\sqrt{a}} \left( \frac{\gamma_1^0}{a} + \nu \gamma_2^0 \right) + \frac{h^2}{12\varepsilon} \left( \frac{\rho_1^0}{a} + \nu \rho_2^0 \right) \frac{\varphi}{a} \right)' - \lambda_1 = 0,$$

$$\left( \frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_1^0}{a} + \nu \gamma_2^0 \right) - \frac{h^2}{12\varepsilon} \left( \frac{\rho_1^0}{a} + \nu \rho_2^0 \right) \frac{1}{a} \right)' - \lambda_3 = 0.$$  \hspace{1cm} (5.96)

along with boundary conditions

$$[\psi_1] = 0, \quad \left[ \frac{1}{\sqrt{a}} \left( \frac{\gamma_1^0}{a} + \nu \gamma_2^0 \right) + \frac{h^2}{12\varepsilon} \left( \frac{\rho_1^0}{a} + \nu \rho_2^0 \right) \frac{\varphi}{a} \right] = 0,$$

$$[\psi_3] = 0, \quad \left[ \frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_1^0}{a} + \nu \gamma_2^0 \right) - \frac{h^2}{12\varepsilon} \left( \frac{\rho_1^0}{a} + \nu \rho_2^0 \right) \frac{1}{a} \right] = 0,$$

$$\left[ \left( \frac{\rho_1^0}{a} + \nu \rho_2^0 \right) \left( \frac{\varphi}{a} \delta \psi'_1 - \frac{1}{a} \delta \psi'_3 \right) \right] = 0.$$  \hspace{1cm} (5.97)

To evaluate the fifth condition in Eq. (5.97), we need the geometry relation of the angle changing between the two segment as shown in Fig. 5.5. As we formulate the corrugated plate under classical plate theory (Kirchhoff-Love plate theory), no shear effect lead to the changing of the angle and $\Delta \alpha = 0$. 
The angle changing between two segments is defined by

\[
\frac{\mathbf{K}^{(1)} + \frac{d\mathbf{u}^{(1)}}{ds^{(1)}}}{|\mathbf{K}^{(1)} + \frac{d\mathbf{u}^{(1)}}{ds^{(1)}}|} \cdot \frac{\mathbf{K}^{(2)} + \frac{d\mathbf{u}^{(2)}}{ds^{(2)}}}{|\mathbf{K}^{(2)} + \frac{d\mathbf{u}^{(2)}}{ds^{(2)}}|} = \cos(\alpha + \Delta \alpha). \tag{5.98}
\]

where \(s^{(1)}\) and \(s^{(2)}\) are arc lengths measuring from the conjunction for segment (1) and segment (2), respectively. For small displacement with neglecting high order terms, Eq. (5.98) is rewritten as

\[
\left( \mathbf{K}^{(1)} \cdot \frac{d\mathbf{u}^{(2)}}{ds^{(2)}} + \mathbf{K}^{(2)} \cdot \frac{d\mathbf{u}^{(1)}}{ds^{(1)}} \right) - \cos \alpha \left( \mathbf{K}^{(1)} \cdot \frac{d\mathbf{u}^{(1)}}{ds^{(1)}} + \mathbf{K}^{(2)} \cdot \frac{d\mathbf{u}^{(2)}}{ds^{(2)}} \right) = -\Delta \alpha \sin \alpha. \tag{5.99}
\]

Here

\[
\begin{align*}
\mathbf{K}^{(1)} &= -\frac{1}{\sqrt{a^{(1)}}} \hat{e}_1 - \frac{\varphi^{(1)}}{\sqrt{a^{(1)}}} \hat{e}_3, \\
\mathbf{K}^{(2)} &= \frac{1}{\sqrt{a^{(2)}}} \hat{e}_1 + \frac{\varphi^{(2)}}{\sqrt{a^{(2)}}} \hat{e}_3, \\
\frac{d\mathbf{u}^{(1)}}{ds^{(1)}} &= \frac{du^{(1)}}{dx_i} K_i^{(1)} \hat{e}_1 + \frac{du^{(1)}}{dx_i} K_i^{(1)} \hat{e}_2 + \frac{du^{(3)}}{dx_i} K_i^{(1)} \hat{e}_3, \\
\frac{d\mathbf{u}^{(2)}}{ds^{(2)}} &= \frac{du^{(2)}}{dx_i} K_i^{(2)} \hat{e}_1 + \frac{du^{(2)}}{dx_i} K_i^{(2)} \hat{e}_2 + \frac{du^{(3)}}{dx_i} K_i^{(2)} \hat{e}_3. 
\end{align*}
\tag{5.100}
\]

Plug Eq. (5.33) in Eq. (5.100) then into Eq. (5.99), gives the edge condition

\[
\left( \frac{\varphi^{(1)}}{a^{(1)}} - \frac{\varphi^{(2)}}{a^{(2)}} \right) (v_{1,1} - x_3 v_{3,11}) + \psi^{(1)} \frac{\varphi^{(1)} 1}{a^{(1)}} - \psi^{(2)} \frac{\varphi^{(2)} 1}{a^{(2)}} - \psi^{(3)} \frac{1}{a^{(1)}} + \psi^{(3)} \frac{1}{a^{(2)}} = 0. \tag{5.101}
\]
Revising fifth condition at discontinuous points in Eq. (5.97) with Eq. (5.101) gives

$$\left\lfloor \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right\rfloor = 0.$$  \hspace{1cm} (5.102)

The second and fourth conditions in Eq. (5.97) leads to $\lambda_1 = \lambda_3 = 0$ and integral constants $c_1^{(1)} = c_1^{(2)}$, $c_3^{(1)} = c_3^{(2)}$. Hence

$$\frac{1}{\sqrt{a}} \left( \frac{\gamma_{11}^0}{a} + \nu \gamma_{22}^0 \right) + \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{\varphi}{a} = c_1, \hspace{1cm} (5.103)$$

$$\frac{\varphi}{\sqrt{a}} \left( \frac{\gamma_{11}^0}{a} + \nu \gamma_{22}^0 \right) - \frac{h^2}{12 \varepsilon} \left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{1}{a} = c_3. \hspace{1cm} (5.104)$$

Integrate $(\varphi \times (5.103) - (5.104))$ over the cell length with considering the condition in Eq. (5.102) conclude

$$c_3 = 0.$$ \hspace{1cm} (5.105)

Then Eqs. (5.103) and (5.104) can be simplified as

$$\left( \frac{\rho_{11}^0}{a} + \nu \rho_{22}^0 \right) \frac{\varphi}{a} = c_1 \frac{12 \varphi \varepsilon}{h^2}, \hspace{1cm} (5.106)$$

$$\left( \frac{\gamma_{11}^0}{a} + \nu \gamma_{22}^0 \right) = \frac{c_1}{\sqrt{a}}. \hspace{1cm} (5.107)$$

Integrate Eq. (5.106)

$$\left\{ \begin{array}{l}
\left( \frac{\rho_{11}^{(1)}}{a^{(1)}} + \nu \rho_{22}^0 \right) = c_1 \frac{12 x_3^{(1)}}{h^2} + c_4^{(1)} \hspace{1cm} - \frac{1}{2} \leq X \leq \frac{P}{\varepsilon} \\
\left( \frac{\rho_{11}^{(2)}}{a^{(2)}} + \nu \rho_{22}^0 \right) = c_1 \frac{12 x_3^{(2)}}{h^2} + c_4^{(2)} \hspace{1cm} \frac{P}{\varepsilon} < X \leq \frac{1}{2}
\end{array} \right.$$  \hspace{1cm} (5.108)
Rewriting Eq. (5.108) considering Eq. (5.88)

\[
\begin{align*}
\left\{ \begin{array}{l}
\psi_3^{(1)} - \frac{\varphi^{(1)}}{a^{(1)}} \gamma_{11}^{(1)} = \varepsilon \left( c_1 \frac{12}{h^2} x_3^{(1)} \sqrt{a^{(1)}} + c_4^{(1)} \sqrt{a^{(1)}} - (v_{3,11} + \nu v_{3,22}) \right) \quad -\frac{1}{2} \leq X \leq \frac{P}{\varepsilon} \\
\psi_3^{(2)} - \frac{\varphi^{(2)}}{a^{(2)}} \gamma_{11}^{(2)} = \varepsilon \left( c_1 \frac{12}{h^2} x_3^{(2)} \sqrt{a^{(2)}} + c_4^{(2)} \sqrt{a^{(2)}} - (v_{3,11} + \nu v_{3,22}) \right) \quad \frac{P}{\varepsilon} < X \leq \frac{1}{2}
\end{array} \right.
\end{align*}
\]

Integrating over the cell length with the edge condition Eq. (5.101), gives \( |\psi_3 - \frac{\varepsilon}{\varepsilon} \gamma_{11}^{(0)}| = 0 \).

Evaluate Eq. (5.108) at discontinuous points with condition Eq. (5.102). Thus, \( c_4^{(1)} = c_4^{(2)} \), and

\[
\begin{align*}
c_4 = \frac{1}{\langle \sqrt{a} \rangle} (v_{3,11} + \nu v_{3,22}).
\end{align*}
\]

Note here \( \langle x_3 \sqrt{a} \rangle = 0 \).

Integrate Eq. (5.109) considering \( c_4 \). Similarly, \( c_5^{(1)} = c_5^{(2)} \) when using edge condition Eq. (5.101). It gives

\[
\begin{align*}
\psi_3' - \frac{\varphi}{a} \gamma_{11}^{(0)} = -\frac{12\varepsilon}{h^2} c_1 A + \varepsilon \left( \frac{\int_{X}^{P} \sqrt{a}dY}{\langle \sqrt{a} \rangle} - X \right) (v_{3,11} + \nu v_{3,22}) + c_5.
\end{align*}
\]

with

\[
\begin{align*}
A(X) = -\int_{X}^{P} \sqrt{a}x_3dY.
\end{align*}
\]

\( \varphi \times (5.107) + (5.111) \) gives

\[
\begin{align*}
\psi_3' + \nu \varphi \gamma_{22}^{(0)} = \frac{c_1 \varphi}{\sqrt{a}} - \frac{12\varepsilon}{h^2} c_1 A + \varepsilon \left( \frac{\int_{X}^{P} \sqrt{a}dY}{\langle \sqrt{a} \rangle} - X \right) (v_{3,11} + \nu v_{3,22}) + c_5,
\end{align*}
\]

Integrating over the cell length, we obtain

\[
\begin{align*}
c_5 = -c_1 \left( \frac{\varphi}{\sqrt{a}} \right) + \frac{12\varepsilon}{h^2} c_1 \langle A \rangle - \varepsilon \frac{\langle \int_{X}^{P} \sqrt{a}dY \rangle}{\langle \sqrt{a} \rangle} (v_{3,11} + \nu v_{3,22}).
\end{align*}
\]
Here, notice $\langle \psi'_3 \rangle = 0$ and $\langle \varphi x_3 \rangle = 0$. Substitute $c_5$ into Eq. (5.113)

$$\psi'_3 = -\nu \varphi \gamma_{22}^0 + c_1 \left( \frac{\varphi}{\sqrt{a}} - \left\langle \frac{\varphi}{\sqrt{a}} \right\rangle \right) - \frac{12\varepsilon}{h^2} c_1(A - \langle A \rangle)$$

$$+ \varepsilon \left( \frac{\int_{X}^{Y} \sqrt{a} dY - \left\langle \int_{X}^{Y} \sqrt{a} dY \right\rangle}{\left\langle \sqrt{a} \right\rangle} - X \right) (v_{3,11} + \nu v_{3,22}),$$

(5.115)

Rewrite Eq. (5.107) as

$$v_{1,1} - x_3 v_{3,11} + \psi'_1 + \varphi \psi'_3 = c_1 \sqrt{a} - \nu a \gamma_{22}^0.$$  

(5.116)

Substitute Eq. (5.115) into Eq. (5.116) and integrate over the cell length

$$v_{1,1} + \nu v_{2,2} = \frac{12\varepsilon}{h^2} c_1 \langle \varphi A \rangle + c_1 \left\langle \frac{1}{\sqrt{a}} \right\rangle,$$

(5.117)

Thus

$$c_1 = \frac{(v_{1,1} + \nu v_{2,2})}{C},$$

(5.118)

where

$$C = 12 \langle \varphi A \rangle \frac{\varepsilon}{h^2} + \left\langle \frac{1}{\sqrt{a}} \right\rangle.$$  

(5.119)

$J_3$ is not a function of $\psi_1$, such that

$$J_3 = \left\langle \mu h \sqrt{a} (1 + \nu) \gamma_{22}^0 \right\rangle^2 + \frac{\mu h^3}{12} \sqrt{a} (1 + \nu) \langle \rho_{22}^0 \rangle^2.$$  

(5.120)

### 5.2.3 Equivalent plate energy

Let us compute $J_1$ first. Using Eq. (5.107) and Eq. (5.108)

$$J_1 = \left\langle \mu h \sqrt{a} (1 + \sigma) \left( \frac{c_1}{\sqrt{a}} \right)^2 + \frac{\mu h^3}{12} \sqrt{a} (1 + \sigma) \left( c_1 \frac{12 x_3}{h^2} + c_4 \right)^2 \right\rangle.$$  

(5.121)
Substituting Eqs. (5.110) and (5.118) into Eq. (5.121) gives

\[ J_1 = (v_{1,1} + \nu v_{2,2})^2 \mu (1 + \sigma) \frac{1}{C^2} \left( h \left( \frac{1}{\sqrt{a}} \right) + \frac{12}{h} \varepsilon(\varphi A) \right) + (v_{3,11} + \nu v_{3,22})^2 \mu h (1 + \sigma) \left( \frac{1}{\sqrt{a}} \right). \]  

(5.122)

Rewrite Eq. (5.42) as

\[ J_2 = \frac{\mu h}{2} \left( \frac{1}{\sqrt{a}} \right)^2 \left( \frac{2\gamma_0^0}{12} + \frac{h^2}{12} (2\sqrt{\alpha v_{3,12}})^2 \right). \]  

(5.123)

Substituting Eqs. (5.93) and (5.95) into Eq. (5.123) gives

\[ J_2 = (v_{1,2} + v_{2,1})^2 \frac{\mu h}{2} \frac{1}{\sqrt{a}} + v_{3,12}^2 \frac{\mu h^3}{6} \left( \frac{1}{\sqrt{a}} \right). \]  

(5.124)

Substituting \( \gamma_{22}^0 \) in Eq. (5.39) and \( \rho_{22}^0 \) in Eq. (5.88) into Eq. (5.120) gives

\[ J_3 = v_{2,2}^2 \mu h (1 + \nu) \left( \frac{1}{\sqrt{a}} \right) + v_{3,22}^2 \mu h (1 + \nu) \left( \frac{1}{\sqrt{a} x_3^2} \right) + \frac{h^2}{12} \left( \frac{1}{\sqrt{a}} \right). \]  

(5.125)

Rewriting the total strain energy into matrix form, we obtain the following relations for the equivalent plate stiffnesses

\[ A_{11} = \frac{E}{1 - \nu^2} \frac{12 \varepsilon(\varphi A)}{h C^2} + \frac{E h}{1 - \nu^2} \left( \frac{1}{\sqrt{a}} \right) \frac{1}{C^2}, \quad A_{13} = \nu A_{11}, \]

\[ A_{22} = \frac{\mu h}{\sqrt{a}}, \quad A_{33} = E h \left( \frac{1}{\sqrt{a}} \right) + \nu^2 A_{11}, \]

\[ B_{11} = B_{13} = B_{22} = B_{33} = 0 \]

(5.126)

\[ D_{11} = \frac{E h^3}{12(1 - \nu^2)} \frac{1}{\sqrt{a}}, \quad D_{13} = \nu D_{11}, \]

\[ D_{22} = \frac{\mu h^3}{12} \frac{1}{\sqrt{a}}, \quad D_{33} = E h \left( x_3^2 \sqrt{a} \right) + \frac{E h^3}{12} \left( \frac{1}{\sqrt{a}} \right) + \nu^2 D_{11}. \]

In piecewise straight case, we can conclude the leading parts of equivalent bending rigidities have the same forms from Huber [72] which derived from free body diagram. It is also noticed that, without coupling terms, the effective stiffnesses in piecewise continuous
case are exactly the same with continuous slope case in Eqs. (5.80). With adding more discontinuity points in the corrugations, these equations can be extended to more types of corrugations, such as trapezoidal corrugated plate showed in the example parts.

5.2.4 Recovery relations

Similarly, the local strain fields using Eqs. (5.39) and (5.88), are calculated as

\[
\gamma_{11}^0 = c_1 \sqrt{a} - \nu a (\epsilon_{yy} + x_3 \kappa_{yy}),
\]
\[
2 \gamma_{12}^0 = 2 \sqrt{a} \epsilon_{xy},
\]
\[
\gamma_{22}^0 = \epsilon_{yy} + x_3 \kappa_{yy},
\]
\[
\rho_{11}^0 = - a \left( \frac{(\kappa_{xx} + \nu \kappa_{yy})}{\sqrt{a}} - \frac{12}{h^2} c_1 x_3 - \frac{1}{\sqrt{a}} \nu \kappa_{yy} \right),
\]
\[
2 \rho_{12}^0 = - 2 \sqrt{a} \kappa_{xy},
\]
\[
\rho_{22}^0 = - \frac{1}{\sqrt{a}} \kappa_{yy}.
\]

The force and moment resultants \( \mathbf{N} \) can be recovered by using the constitutive relations corresponding to the strain fields

\[
\mathbf{N} = \mathbf{D} \mathbf{\Upsilon}
\]

where \( \mathbf{N} = \begin{bmatrix} N_{xx} & N_{xy} & N_{yy} & M_{xx} & M_{xy} & M_{yy} \end{bmatrix}^T \) and \( \mathbf{D}, \mathbf{\Upsilon} \) are defined in next section.

5.3 FEA Solution

In this section, we use finite element approach to solve \( \psi_i \) in Eqs. (5.39) and (5.41). Using Eq. (5.78), we have

\[
\gamma_{11}^0 = \epsilon_{xx} + x_3 \kappa_{xx} + \psi_1' + \varphi \psi_3',
\]
\[
2 \gamma_{12}^0 = 2 \epsilon_{xy} + 2 x_3 \kappa_{3,12} + \psi_2',
\]
\[
\gamma_{22}^0 = \epsilon_{yy} + x_3 \kappa_{yy},
\]

(5.129)
\[ \rho_{11}^0 = -\frac{\varphi'(1-\varphi^2)}{\varepsilon a^{3/2}} \epsilon_{xx} - \left( \frac{\varphi^2 + a}{\varepsilon a^{3/2}} + \frac{\varphi' x_3 (1-\varphi^2)}{\varepsilon a^{3/2}} \right) \kappa_{xx} \]
\[ - \frac{\varphi'(1-\varphi^2)}{\varepsilon a^{3/2}} \psi_1' - \frac{\varphi}{\varepsilon \sqrt{a}} \psi_3'' - \frac{2 \varphi \varphi'}{\varepsilon a^{3/2}} \psi_3' + \frac{1}{\varepsilon \sqrt{a}} \psi_3'', \]
\[ 2 \rho_{12}^0 = -\frac{\varphi'}{2 \varepsilon a^{3/2}} 2 \epsilon_{xy} - \left( \sqrt{a} + \frac{x_3 \varphi'}{2 \varepsilon a^{3/2}} \right) 2 \kappa_{xy} - \frac{\varphi'}{2 \varepsilon a^{3/2}} \psi_2', \]
\[ \rho_{22}^0 = -\frac{1}{\sqrt{a}} \kappa_{yy}. \]

Write in matrix form,
\[ \Upsilon = \Gamma_\epsilon \epsilon + \Gamma_h \Psi. \] (5.131)

where

\[ \Upsilon = \begin{bmatrix} \gamma_{11}^0 & 2 \gamma_{12}^0 & \gamma_{22}^0 & \rho_{11}^0 & 2 \rho_{12}^0 & \rho_{22}^0 \end{bmatrix}^T, \]
\[ \Gamma_\epsilon = \begin{bmatrix} 1 & 0 & 0 & x_3 & 0 & 0 \\
0 & 1 & 0 & 0 & x_3 & 0 \\
0 & 0 & 1 & 0 & 0 & x_3 \\
-\frac{\varphi'(1-\varphi^2)}{\varepsilon a^{3/2}} & 0 & 0 & -\left( \frac{\varphi^2 + a}{\varepsilon a^{3/2}} + \frac{\varphi' x_3 (1-\varphi^2)}{\varepsilon a^{3/2}} \right) & 0 & 0 \\
0 & -\frac{\varphi'}{2 \varepsilon a^{3/2}} & 0 & 0 & -\left( \sqrt{a} + \frac{x_3 \varphi'}{2 \varepsilon a^{3/2}} \right) & 0 \\
0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{a}} \end{bmatrix}, \]
\[ \epsilon = \begin{bmatrix} \epsilon_{xx} & 2 \epsilon_{xy} & \epsilon_{yy} & \kappa_{xx} & 2 \kappa_{xy} & \kappa_{yy} \end{bmatrix}^T, \]
\[ \Gamma_h = \begin{bmatrix} \frac{\partial}{\partial X} & 0 & \varphi \frac{\partial}{\partial X} \\
0 & \frac{\partial}{\partial X} & 0 \\
0 & 0 & 0 \\
-\left( \frac{\varphi'(1-\varphi^2)}{\varepsilon a^{3/2}} \frac{\partial}{\partial X} + \frac{\varphi}{\varepsilon \sqrt{a}} \frac{\partial^2}{\partial X^2} \right) & 0 & -\frac{2 \varphi \varphi'}{\varepsilon a^{3/2}} \frac{\partial}{\partial X} + \frac{1}{\varepsilon \sqrt{a}} \frac{\partial^2}{\partial X^2} \end{bmatrix}, \]
\[ \Psi = \begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix}^T. \]
Eq. (5.32) can be rewritten as

\[ J = \frac{1}{2} \langle \mathbf{Y}^T \mathbf{D} \mathbf{Y} \rangle \]  

(5.132)

where

\[
\mathbf{D} = \begin{bmatrix}
\frac{E_h}{(1-\nu^2)a^{3/2}} & 0 & \frac{E_h\nu}{(1-\nu^2)a^{3/2}} & 0 & 0 & 0 \\
0 & \frac{E_h}{2(1+\nu)a^{3/2}} & 0 & 0 & 0 & 0 \\
\frac{E_h\nu}{(1-\nu^2)a^{3/2}} & 0 & a^{1/2} & \frac{E_h}{1-\nu^2} & 0 & 0 \\
0 & 0 & 0 & \frac{E_h^3}{12(1-\nu^2)a^{3/2}} & 0 & 0 \\
0 & 0 & 0 & \frac{E_h^3\nu}{12(1-\nu^2)a^{3/2}} & 0 & a^{1/2} \frac{E_h^3\nu}{12(1-\nu^2)} \\
0 & 0 & 0 & 0 & \frac{E_h^3}{12(1-\nu^2)a^{3/2}} & 0
\end{bmatrix} \]  

(5.133)

The cross-section of corrugated plate can be discretized by 3-node curve element with the node number shown in Fig. 5.6. The reason to choose 3-node instead of 2-node element is to describe \( \psi' \) existing in Eqs. (5.130), which is the double derivative respect with \( x_3(X) \). Since the second derivative respect with \( \psi_1, \psi_3 \) exist in the fourth and fifth equations in Eqs. (5.130), the continuous conditions are required for the \( \psi'_1, \psi'_3 \) which results Hermite element shape function for \( \psi_1, \psi_3 \).

If we discretize \( \mathbf{Psi} \) using the finite elements as

\[ \mathbf{Psi} = S\mathbf{Psi}, \]  

(5.134)
where $S$ representing the shape functions and $Ψ$ denoting the nodal values of fluctuation functions. The full sets of shape functions are obtained in Appendix A.

Plugging Eq. (5.134) into Eq. (5.131) then into Eq. (5.132), we get

$$J = \frac{1}{2}(Ψ^T D_{EE} Ψ + 2Ψ^T D_{he} ε + ε^T D_{ee} ε) \quad (5.135)$$

where

$$D_{EE} = \langle (Γ_h S)^T D (Γ_h S) \rangle, \quad D_{he} = \langle (Γ_h S)^T D Γ_ε \rangle, \quad D_{ee} = \langle Γ_ε^T D Γ_ε \rangle. \quad (5.136)$$

Denoting $Ψ = Ψ_0 ε$, we can minimize $J$ in Eq. (5.135) along with the periodic constraints to obtain the following linear system

$$D_{EE} Ψ_0 = -D_{he}, \quad (5.137)$$

to solve for $Ψ_0$. Then we can obtain the effective stiffness matrix $\overline{ABD}$ as

$$\overline{ABD} = X_0^T D_{he} + D_{ee}. \quad (5.138)$$

Recovery of displacement fields can be obtained from Eq. (5.33) and strain fields from Eq. (5.131). The FEA can be also applied to a two-way corrugated plate embedded with 2D quadratic element, however, containing 108 degrees of freedoms (9-node element) for each element is a drawback compared with 3D brick element which has 24 degrees of freedoms (8-node element) per element. The interested reader is referred to Appendix B for detailed equations.

### 5.4 Validation Examples

In this section, three shapes of corrugation are studied. The first one is a sinusoidal corrugation which represents the symmetric case with no coupling effects ($B_{11} = B_{13} = B_{22} = B_{33} = 0$). The second is a exponential-sinusoidal corrugation which is an example of
Table 5.1: Equivalent plate stiffnesses of sinusoidal corrugation.

<table>
<thead>
<tr>
<th></th>
<th>Eqs. (1.18)(1.26)</th>
<th>Xia et al. [139]</th>
<th>VAPAS</th>
<th>Analytical (Eq. (5.80))</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$ (N/m)</td>
<td>53805</td>
<td>47613</td>
<td>48152</td>
<td>47613</td>
</tr>
<tr>
<td>$A_{13}$ (N/m)</td>
<td>7927</td>
<td>9523</td>
<td>9630</td>
<td>9523</td>
</tr>
<tr>
<td>$A_{22}$ (N/m)</td>
<td>$5.0113 \times 10^7$</td>
<td>$5.0113 \times 10^7$</td>
<td>5.0097 $\times 10^7$</td>
<td>$5.0113 \times 10^7$</td>
</tr>
<tr>
<td>$A_{33}$ (N/m)</td>
<td>$1.8708 \times 10^8$</td>
<td>$1.8708 \times 10^8$</td>
<td>$1.8692 \times 10^8$</td>
<td>$1.8708 \times 10^8$</td>
</tr>
<tr>
<td>$D_{11}$ (N·m)</td>
<td>261.004</td>
<td>261.004</td>
<td>263.972</td>
<td>261.004</td>
</tr>
<tr>
<td>$D_{13}$ (N·m)</td>
<td>52.20</td>
<td>52.20</td>
<td>52.95</td>
<td>52.20</td>
</tr>
<tr>
<td>$D_{22}$ (N·m)</td>
<td>136.29</td>
<td>162.39</td>
<td>163.38</td>
<td>162.39</td>
</tr>
<tr>
<td>$D_{33}$ (N·m)</td>
<td>1025270</td>
<td>1068260</td>
<td>1022874</td>
<td>1025540</td>
</tr>
</tbody>
</table>

The nonsymmetric corrugations thus exhibiting coupling effects, and the last is a trapezoidal corrugated plate composed by piecewise straight components.

5.4.1 Sinusoidal shape

The mid-surface of sinusoidal shape,

$$\phi(X) = \frac{T}{\varepsilon} \sin(2\pi X),$$

is characterized by one parameter, $T$, the rise of the corrugation (Fig. 1.7). From the definition of $\varphi(X)$ (Eq. (5.19)),

$$\varphi(X) = \frac{2\pi T}{\varepsilon} \cos(2\pi X).$$

For numerical values we choose $\varepsilon = 0.64$ m, $T = 0.11$ m, $h = 0.005$ m and material properties are taken to be $E = 30$ GPa, $\nu = 0.2$, $\rho = 7830$ kg/m$^3$.

The equivalent plate stiffnesses obtained using different approaches are listed in Table 5.1. VAPAS is a 3D elasticity numerical code introduced in [125] for equivalent plate modeling of panels with microstructures. Corrugated structures can be considered as a special case of such panels and the results obtained can be used as benchmark for the present study. For the case under consideration, $\alpha_2 = 0$ and $\langle \phi \sqrt{a} \rangle = 0$, thus there is no extension-bending coupling. It is seen from Table 5.1 that the results obtained by the analytical
Table 5.2: Convergence study of FEA of sinusoidal corrugation.

<table>
<thead>
<tr>
<th></th>
<th>5 elements</th>
<th>20 elements</th>
<th>50 elements</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$ (N/m)</td>
<td>56232</td>
<td>47724</td>
<td>47630</td>
</tr>
<tr>
<td>$A_{13}$ (N/m)</td>
<td>10741</td>
<td>9572</td>
<td>9530</td>
</tr>
<tr>
<td>$A_{22}$ (N/m)</td>
<td>$5.0748 \times 10^7$</td>
<td>$5.0114 \times 10^7$</td>
<td>$5.0113 \times 10^7$</td>
</tr>
<tr>
<td>$A_{33}$ (N/m)</td>
<td>$1.86984 \times 10^8$</td>
<td>$1.8708 \times 10^8$</td>
<td>$1.8708 \times 10^8$</td>
</tr>
<tr>
<td>$D_{11}$ (N·m)</td>
<td>260.778</td>
<td>260.946</td>
<td>260.994</td>
</tr>
<tr>
<td>$D_{13}$ (N·m)</td>
<td>49.26</td>
<td>52.04</td>
<td>52.17</td>
</tr>
<tr>
<td>$D_{22}$ (N·m)</td>
<td>1211.35</td>
<td>166.94</td>
<td>162.51</td>
</tr>
<tr>
<td>$D_{33}$ (N·m)</td>
<td>1020320</td>
<td>1025520</td>
<td>1025540</td>
</tr>
</tbody>
</table>

approach are very close to those predicted by VAPAS and Xia et al. [139]. However, the differences between the present approach and the usual one for $A_{11}$, $A_{13}$, $D_{22}$ in Eqs. (1.18) and (1.26) are noticeable. Convergence study of FEA results from VAMUCH using 3-node curve element are listed in Table 5.2 with varying the number of elements. Formula (1.27) gives $A_{11} = 8666$ N/m, which is also well off the correct result.

5.4.2 Exponential-sinusoidal shape

In the second example, a non-symmetric corrugated shape is chosen to show the coupling effects. We use an exponential-sinusoidal function with unit cell length $\varepsilon = 1$ m,

$$
\phi(X) = \eta \left( e^{\sin(2\pi X)} - \left< e^{\sin(2\pi X)} \right> \right),
$$

as sketched in Fig. 5.7. An additive constant is added to satisfy Eq. (5.17). We choose thickness $h = 0.005$ m and material properties $E = 30$ GPa, $\nu = 0.2$. The trend of $B_{11}$, $B_{22}$, and $B_{33}$ as a function of $\eta$ are shown in Fig. 5.8, Fig. 5.9, and Fig. 5.10, respectively. Equivalent plate stiffnesses obtained by different approaches are listed for comparison in Table 5.3. Since the corrugation is not symmetric, the rise of the corrugation $T$ in Eq. (1.26) is measured as half of the total swing. Apparently, the extension-bending coupling, particularly the coupling coefficient $B_{33}$ between $\varepsilon_{yy}$ and $\kappa_{yy}$, is not negligible comparing to other stiffnesses terms as $\eta$ grows larger. For the other stiffness constants, the four sets of results have
Fig. 5.7: Shapes of nonsymmetric corrugations for different values of parameter $\eta$.

Fig. 5.8: $B_{11}$ as a function of $\eta$.

a fair agreement except for $A_{22}$, $A_{33}$ for which the present approach, VAPAS and Xia et al. have a better agreement than the results in Eqs. (1.18) and (1.26). It is also noticeable that the value of $B_{22}$ obtained from VAPAS [125] is much larger than present approach, the reason of which remains to be examined.

5.4.3 Trapezoidal shape

This example is originally taken from Samanta and Mukhopadhyay [140] and cited in Xia et al. [139]. The parameters describe the profile of the mid-surface of this trapezoidal corrugation (Fig. 1.9) are: $\varepsilon = 0.1016$ m, $T = 0.0127$ m, $h = 0.00635$ m, $\theta = 45^\circ$ and material properties are taken to be isotropic $E = 21$ GPa, $\nu = 0.3$. 
Fig. 5.9: \( B_{22} \) as a function of \( \eta \).

Fig. 5.10: \( B_{33} \) as a function of \( \eta \).
Table 5.3: Equivalent plate stiffnesses of exponential-sinusoidal corrugation ($\eta = 0.1$).

<table>
<thead>
<tr>
<th></th>
<th>Eqs. (1.18)(1.26)</th>
<th>Xia et al. [139]</th>
<th>VAPAS</th>
<th>Analytical (Eq. (5.80))</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$ (N/m)</td>
<td>47139.4</td>
<td>43765.5</td>
<td>46366.1</td>
<td>43911.9</td>
</tr>
<tr>
<td>$A_{13}$ (N/m)</td>
<td>9427.89</td>
<td>8753.09</td>
<td>9273.22</td>
<td>8782.2</td>
</tr>
<tr>
<td>$A_{22}$ (N/m)</td>
<td>$2.74 \times 10^4$</td>
<td>$5.49 \times 10^4$</td>
<td>$5.48 \times 10^4$</td>
<td>$5.49 \times 10^4$</td>
</tr>
<tr>
<td>$A_{33}$ (N/m)</td>
<td>$3.42 \times 10^8$</td>
<td>$1.71 \times 10^8$</td>
<td>$1.71 \times 10^8$</td>
<td>$1.71 \times 10^8$</td>
</tr>
<tr>
<td>$B_{11}$ (N)</td>
<td>N/A</td>
<td>N/A</td>
<td>225.98</td>
<td>204.26</td>
</tr>
<tr>
<td>$B_{13}$ (N)</td>
<td>N/A</td>
<td>N/A</td>
<td>45.20</td>
<td>40.85</td>
</tr>
<tr>
<td>$B_{22}$ (N)</td>
<td>N/A</td>
<td>N/A</td>
<td>10829</td>
<td>0.005</td>
</tr>
<tr>
<td>$B_{33}$ (N)</td>
<td>N/A</td>
<td>N/A</td>
<td>817802</td>
<td>794841</td>
</tr>
<tr>
<td>$D_{11}$ (N·m)</td>
<td>285.76</td>
<td>285.76</td>
<td>263.972</td>
<td>286.71</td>
</tr>
<tr>
<td>$D_{13}$ (N·m)</td>
<td>57.15</td>
<td>57.15</td>
<td>52.95</td>
<td>57.34</td>
</tr>
<tr>
<td>$D_{22}$ (N·m)</td>
<td>119.75</td>
<td>148.33</td>
<td>163.38</td>
<td>148.33</td>
</tr>
<tr>
<td>$D_{33}$ (N·m)</td>
<td>$1.12 \times 10^6$</td>
<td>$1.16 \times 10^6$</td>
<td>$1.02 \times 10^6$</td>
<td>$1.12 \times 10^6$</td>
</tr>
</tbody>
</table>

Table 5.4: Equivalent plate stiffnesses of trapezoidal corrugation.

<table>
<thead>
<tr>
<th></th>
<th>Xia et al. [139]</th>
<th>VAPAS</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_{11}$ (MN/m)</td>
<td>4.289</td>
<td>4.118</td>
<td>4.150</td>
</tr>
<tr>
<td>$A_{13}$ (MN/m)</td>
<td>1.287</td>
<td>1.235</td>
<td>1.245</td>
</tr>
<tr>
<td>$A_{22}$ (MN/m)</td>
<td>42.489</td>
<td>43.297</td>
<td>42.489</td>
</tr>
<tr>
<td>$A_{33}$ (MN/m)</td>
<td>161.354</td>
<td>161.338</td>
<td>161.479</td>
</tr>
<tr>
<td>$D_{11}$ (N·m)</td>
<td>407.917</td>
<td>414.865</td>
<td>407.917</td>
</tr>
<tr>
<td>$D_{13}$ (N·m)</td>
<td>122.375</td>
<td>124.844</td>
<td>122.375</td>
</tr>
<tr>
<td>$D_{22}$ (N·m)</td>
<td>208.032</td>
<td>210.328</td>
<td>208.033</td>
</tr>
<tr>
<td>$D_{33}$ (N·m)</td>
<td>16824</td>
<td>16588</td>
<td>16251</td>
</tr>
</tbody>
</table>

The equivalent plate stiffnesses obtained using different approaches are listed in Table 5.4. VAPAS is a 3D elasticity numerical code introduced in Ref. [125] for equivalent plate modeling of panels with microstructures. Corrugated structures can be considered as a special case of such panels and the results obtained can be used as benchmark for the present study. It is seen from Table 5.1 that the results obtained by the present approach have a good agreement with those predicted by VAPAS and Xia et al.

To validate the equivalent plate stiffnesses, a square trapezoidal corrugated plate with 9 corrugations is subjected to a uniformly distributed load of 100 Pa in ANSYS. Element SURF154 is overlaid onto element SHELL181 of the corrugated area to enforce the load.
Table 5.5: Deflection of simply supported trapezoidal corrugated plate at geometric center.

<table>
<thead>
<tr>
<th></th>
<th>Xia et al.</th>
<th>VAPAS</th>
<th>Present</th>
</tr>
</thead>
<tbody>
<tr>
<td>Deflection (×10^{-4} m)</td>
<td>-0.575</td>
<td>-0.583</td>
<td>-0.599</td>
</tr>
<tr>
<td>Error to ANSYS (%)</td>
<td>-3.10</td>
<td>-1.85</td>
<td>0.84</td>
</tr>
</tbody>
</table>

directions. To get rid of rigid body movements, besides constraining out of plane movements of four edges, the displacements along four edges were under constraint simultaneously. The analytical equation of the deflection surface is in Eq. 5.142

\[
w(x, y) = \frac{16p_0}{\pi^6} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{\sin[m\pi x/r] \sin[n\pi y/s]}{mn} \left( \frac{D_{33}m^4}{p^4} + 2\left( \frac{D_{13}+2D_{22}}{r^4s^4} \right) mn^2 + \frac{D_{33}n^4}{s^4} \right).
\]  

(5.142)

where \(p_0\) is the pressure, \(r, s\) the length and width of the whole corrugated plate. The deflections \(w\) along the center lines of the corrugated plate obtained by different methods are shown in Fig. 5.12. For this case, the predictions from all the equivalent plate stiffnesses have a good agreement with ANSYS results (Fig. 5.11). Deflection at geometric center point are also compared in Table 5.5 and slightly better prediction of the present theory can be noticed.
Fig. 5.12: Deflections along the center line show the best agreement between current method and ANSYS.

The analytical solution of the $\sigma_{xx}$ is calculated as [141]

$$
\sigma_{xx}(x, y) = x^3 \frac{16p_0}{b^2\pi^4} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{(m^2Q_{11} + n^2Q_{12}) \sin[m\pi x/r] \sin[n\pi y/s]}{mn \left( \frac{D_{11}m^4}{r^4} + 2\frac{(D_{13}+2D_{22})m^2n^2}{r^2s^2} + \frac{D_{33}n^4}{s^4} \right)}.
$$

(5.143)

where $Q_{11} = \frac{E}{1-\nu^2}$ and $Q_{12} = \frac{E\nu}{1-\nu^2}$. $\sigma_{xx}$ from ANSYS is shown in Fig. 5.13 and the comparison of $\sigma_{xx}$ between the ANSYS and results based on current theory is shown in Fig. 5.14.

5.5 Summary

The variational asymptotic method has been used to construct an equivalent plate model for both continuous and piecewise straight corrugated structures. The theory handles general corrugation shape as long as the shell thickness is small in comparison to the size of the corrugation. The present theory not only presents a complete set of effective plate stiffnesses but also the complete set of recovery relations to obtain the local fields within the corrugated shell. The numerical approach using FEA is also developed to obtain the equivalent material properties and does the recovery for continuous case. In comparison to
Fig. 5.13: $\sigma_{xx}$ of a trapezoidal corrugated plate calculated in ANSYS.

Fig. 5.14: $\sigma_{xx}$ along the center line show the good agreement between current method and ANSYS.
other approaches in the literature for equivalent plate modeling of corrugated structures, the new points of this work are:

1. A complete set of analytical formulas for stiffnesses of the equivalent plate including extension-bending coupling stiffnesses are obtained (piecewise straight case do not involve these terms). These formulas are valid for any corrugated shell with corrugations along one direction.

2. The complete set of the displacement, strain, and stress fields within the original corrugated shell in terms of the equivalent plate behavior can be recovered.

The difference of the present approach is demonstrated through a few examples. For the corrugated structures studied here, the cell problem can be investigate analytically. For more general corrugated structures, the study of the cell problem can be done only numerically.
Chapter 6

SERDF for Thin-walled RSE Homogenization

High porosity materials or structures, whose porosity (the volume fraction of voids) exceeds 90%, become more and more important in industry due to their light-weighted characteristics and capability of providing designated strength and functionalities. One typical application is to use the SiC foam in thermal protection system (TPS) as an efficient thermal barrier. The size of 1mm $\times$ 1mm $\times$ 1mm SiC foam in Fig. 6.1 takes around 1.5 million solid elements and a typical TPS block has the volume 750mm $\times$ 500mm $\times$ 150mm, which means it is an impossible mission to analysis the TPS block with considering the foam microstructure in a direct FEA approach. Another example is a core of a heat exchanger (in Fig. 6.2) with hundreds and thousands of fins and large empty space among them which can be also categorized as a high porosity structure. Homogenization is the only choice in analysis related with such materials or structures. The idea is to replace the original materials or structures with effective material properties (such as in Fig. 6.2), which are obtained from the unit cell (UC), and use these effective properties in the structural analysis for overall behavior. After obtaining the overall behavior, the micromechanical analysis over the UC can be used to recover the local displacement, strain and stress fields.

In most cases, high porosity materials/structures contain many thin members the thickness of which is much smaller than the other two dimensions. If one uses 3D brick elements for the computation, a large number of brick elements are needed to satisfy the element quality test. However, the number of elements can be largely reduced by using shell elements. Enabling VAMUCH to use shell elements for homogenization of thin-walled materials and structures is addressed in following sections.
Fig. 6.1: 1mm × 1mm × 1mm SiC foam (courtesy AFRL).

Fig. 6.2: Homogenization of the core from a heat exchanger.
6.1 Shell Element of Relative Degree of Freedom

Although there are many types of shell elements, we chose the shell element of relative degree freedom (SERDF) for its simplicity and its easy connection with 3D brick elements. The key idea of SERDF is to introduce the relative degrees of freedom to reduce the stiffness difference between the in-plane direction and out-of-plane direction. As a result, the numerical difficulties in solving the linear equations will be avoided. The rest of the procedures are exactly the same as the original VAMUCH. First, the strain energy is constructed. Then, the VAM is used to minimize the energy and solve for the fluctuating functions. After solving the fluctuating functions, one obtains the effective properties. The local displacement, strain and stress fields can be also recovered.

Porous materials/structures usually contain thin-walled members which has such a characteristic that one dimension is much smaller than the other two. If 3D brick elements are used in the analysis, usually a very fine mesh is needed to avoid large aspect ratios. The large aspect ratio element creates numerical difficulties because the stiffness along thickness direction is much smaller than the others. During assembling, the small numbers could be truncated (because of finite machine precision) and result in ill-conditioning of the coefficient matrix. To conquer this problem, shell elements of relative degree of freedom (SERDF) is introduced to replace brick elements and avoid the numerical difficulties. In fact, SERDF is a special solid element with only two nodes along thickness direction [142, 143]. It is equivalent to assume that the line along the thickness direction remains straight in the deformed shape. In SERDF, the degrees of freedom of these two nodes along the thickness direction are changed so that one nodal value describes the corresponding value of the middle surface, and another nodal value is the relative nodal values of these two nodes. Fundamentally speaking, SERDF just performs a change of variable from the 3D elements, the underpinning theory is still the 3D theory. SERDF is a type of $C_0$ element, and only requires the nodal values to be continuous.

The transformation from brick elements to SERDF is shown in Fig. 6.3. The hexahedron serendipity elements and theirs shape functions are used to formulate SERDF. The
Fig. 6.3: Upper: 8-node brick element to 8-node SERDF; Lower: 16-node brick element to 16-node SERDF.

8-node SERDF will be constructed step by step and the similar procedures can be applied to build 16-node SERDF. It is noted that there are only two nodes along the thickness direction in both cases.

The interpolation functions of 8-node hexahedron serendipity element are

\[ S_i = \frac{1}{8} (1 + \xi_i \xi)(1 + \eta_i \eta)(1 + \zeta_i \zeta), \]  

(6.1)

where \( \xi_i, \eta_i, \zeta_i \) denote the natural coordinates of node \( i \). Then the nodal coordinates \((x, y, z)\) and fluctuating function \((\chi_1, \chi_2, \chi_3)\) can be expressed as

\[
\begin{align*}
    x &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)x_i, \\
    y &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)y_i, \\
    z &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)z_i, \\
    \chi_1 &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)\chi_{1i}, \\
    \chi_2 &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)\chi_{2i}, \\
    \chi_3 &= \sum_{i=1}^{8} S_i(\xi, \eta, \zeta)\chi_{3i}.
\end{align*}
\]  

(6.2)

Redefine the nodal coordinates in the SERDF in such a way that the last four element coordinates respectively equal to that of the middle surface in hexahedron element and the
first four coordinates denote the relative distance between two surfaces as shown in Fig. 6.3. We get
\[ x'_i = \frac{1}{2}(x_i - x_{i+4}), \quad y'_i = \frac{1}{2}(y_i - y_{i+4}), \quad z'_i = \frac{1}{2}(z_i - z_{i+4}), \]
\[ x'_{i+4} = \frac{1}{2}(x_i + x_{i+4}), \quad y'_{i+4} = \frac{1}{2}(y_i + y_{i+4}), \quad z'_{i+4} = \frac{1}{2}(z_i + z_{i+4}), \]  
(6.3)
and we also modify the fluctuating functions correspondingly such that
\[ \mathcal{X}'_{1i} = \frac{1}{2}(\mathcal{X}_i - \mathcal{X}_{1i+4}), \quad \mathcal{X}'_{2i} = \frac{1}{2}(\mathcal{X}_i - \mathcal{X}_{2i+4}), \quad \mathcal{X}'_{3i} = \frac{1}{2}(\mathcal{X}_i - \mathcal{X}_{3i+4}), \]
\[ \mathcal{X}'_{1i+4} = \frac{1}{2}(\mathcal{X}_i + \mathcal{X}_{1i+4}), \quad \mathcal{X}'_{2i+4} = \frac{1}{2}(\mathcal{X}_i + \mathcal{X}_{2i+4}), \quad \mathcal{X}'_{3i+4} = \frac{1}{2}(\mathcal{X}_i + \mathcal{X}_{3i+4}), \]  
(6.4)
where \( i = 1, 2, 3, 4 \). From Eqs. (6.3) and (6.4), the nodal coordinates and fluctuating functions corresponding to the original hexahedron element can be recovered from those in SERDF as
\[ x_i = (x'_i + x'_{i+4}), \quad y_i = (y'_i + y'_{i+4}), \quad z_i = (z'_i + z'_{i+4}), \]
\[ x_{i+4} = (-x'_i + x'_{i+4}), \quad y_{i+4} = (-y'_i + y'_{i+4}), \quad z_{i+4} = (-z'_i + z'_{i+4}), \]
\[ \mathcal{X}_{1i} = (\mathcal{X}'_{1i} + \mathcal{X}'_{1i+4}), \quad \mathcal{X}_{2i} = (\mathcal{X}'_{2i} + \mathcal{X}'_{2i+4}), \quad \mathcal{X}_{3i} = (\mathcal{X}'_{3i} + \mathcal{X}'_{3i+4}), \]
\[ \mathcal{X}_{1i+4} = (-\mathcal{X}'_{1i} + \mathcal{X}'_{1i+4}), \quad \mathcal{X}_{2i+4} = (-\mathcal{X}'_{2i} + \mathcal{X}'_{2i+4}), \quad \mathcal{X}_{3i+4} = (-\mathcal{X}'_{3i} + \mathcal{X}'_{3i+4}), \]  
(6.5)
where \( i = 1, 2, 3, 4 \). Substituting Eq.(6.5) back into Eq.(6.2), the new shape function \( S'_i \) of SERDF can be found as
\[ S'_i = (S_i - S_{i+4}), \quad S'_{i+4} = (S_i + S_{i+4}), \]  
(6.6)
where Eq.(6.2) is rewritten as
\[ x = \sum_{i=1}^{8} S'_{ix'_i}, \quad y = \sum_{i=1}^{8} S'_{iy'_i}, \quad z = \sum_{i=1}^{8} S'_{iz'_i}, \]
\[ \chi_1 = \sum_{i=1}^{8} S'_{i\mathcal{X}'_{1i}}, \quad \chi_2 = \sum_{i=1}^{8} S'_{i\mathcal{X}'_{2i}}, \quad \chi_3 = \sum_{i=1}^{8} S'_{i\mathcal{X}'_{3i}}. \]  
(6.7)
With the new set of shape functions and nodal values, we can routinely solve the homogenization problem following Eqs. (2.29)-(2.31). It is noted that the changes only applied to nodal values and shape functions. The constraint Eq. (2.32) and periodic boundary conditions of $X_i$ need to modify accordingly based on different situations.

Multi-point constraints (MPC) should be used to enforce the continuity if different thin members are joining each other such as the case of heat exchanger in Fig. 6.2. A typical situation is shown in Fig. 6.4. Element II connects the vertical elements (sketched using red lines) and horizontal elements (sketched using green lines). If we take the middle surface of horizontal elements (shown in green) and assign it to the element II the new nodal values based on Eq. (6.3) will be

$$x'_1 = \frac{1}{2} (x_1 - x_5), \quad x'_2 = \frac{1}{2} (x_2 - x_6), \quad x'_3 = \frac{1}{2} (x_3 - x_7), \quad x'_4 = \frac{1}{2} (x_4 - x_8),$$
$$x'_5 = \frac{1}{2} (x_1 + x_3), \quad x'_6 = \frac{1}{2} (x_2 + x_6), \quad x'_7 = \frac{1}{2} (x_3 + x_7), \quad x'_8 = \frac{1}{2} (x_4 + x_8).$$

(6.8)

The $y'$, $z'$ coordinates and fluctuating functions relations have exactly same equations and are not listed here. In contrast, if we take the middle surface of vertical elements (shown in red), the relative coordinates become

$$x'_1 = \frac{1}{2} (x_1 - x_2), \quad x'_4 = \frac{1}{2} (x_4 - x_3), \quad x'_5 = \frac{1}{2} (x_5 - x_6), \quad x'_8 = \frac{1}{2} (x_8 - x_7),$$
$$x'_2 = \frac{1}{2} (x_1 + x_2), \quad x'_3 = \frac{1}{2} (x_4 + x_3), \quad x'_6 = \frac{1}{2} (x_5 + x_6), \quad x'_7 = \frac{1}{2} (x_8 + x_7).$$

(6.9)

Solve Eqs. (6.8) for $x_1$-$x_8$ and put them in Eqs. (6.9). The relations become

$$x'_1 = \left[x'_5 - (x'_2 + x'_6)\right], \quad x'_4 = \left[x'_8 - (x'_3 + x'_7)\right], \quad x'_5 = \left[-x'_1 - (x'_6 - x'_2)\right],$$
$$x'_2 = \left[(x'_1 + x'_3) + x'_6\right], \quad x'_3 = \left[(x'_4 + x'_8) + x'_7\right], \quad x'_6 = \left[(x'_5 - x'_1) - x'_2\right],$$
$$x'_8 = \left[-x'_4 - (x'_7 - x'_3)\right], \quad x'_7 = \left[(x'_8 - x'_4) - x'_3\right].$$

(6.10)

These equations are not linearly independent and can be further simplified as

$$x'_1 = -x'_6, \quad x'_2 = x'_5, \quad x'_3 = x'_8, \quad x'_4 = -x'_7.$$

(6.11)
By providing additional relationships during assembling in Eqs. (6.11) as multi-point constraints, the number of independent degrees of freedom is reduced. The degrees of freedom on nodes 5-8 are condensed to nodes 1-4. The element II serves as a connecting element between the vertical and horizontal SERDFs. This way, we can correctly handle the joining of two thin-members.

The linear equations Eq. (2.31) to solve fluctuating functions can be modified by the multi-point constraints in the UC, such as Eq. (6.10), then the new system can be formed as

$$D_{EE} \bar{X}_0 = -D_{hc} \implies \bar{D}_{EE} \bar{X}_0 = -\bar{D}_{hc}. \quad (6.12)$$

Submitted to the equation solver, the new $\bar{X}_0$ can be returned and the original $X_0$ can also be recovered. To apply this method, each MPC equations, such as Eqs. (6.11), are taken to form a new set of $\bar{X}_0$ by removing all the slave degrees of freedom form $X_0$. Since we only deal with homogeneous constraints (e.g., $x'_1 + x'_6 = 0$), a linear transformation constraint matrix $T$ is formed as

$$X_0 = T \bar{X}_0. \quad (6.13)$$
Substituting Eq. (6.13) to Eq. (6.12), we obtain

\[
\widehat{D}_{EE} = \mathcal{T}^T D_{EE} \mathcal{T}, \quad \widehat{D}_{he} = \mathcal{T}^T D_{he}
\]  

(6.14)

This procedure yields a set of modified variables, which are in terms of the new nodal values. Meanwhile, the active degrees of freedom in system have been effectively eliminated. The periodic boundary conditions can also be handled similarly using multi-point constraints as described in Chapter 3.

Note, it is not efficient to enforce the constraints after assembly. In VAMUCH, we apply multi-point constraints right after the element matrices are computed. The related constraints need to be prescribed in the input data and users decide which degrees of freedom are to be treated as slaves.

### 6.2 Numerical Examples

In this section, several examples are presented to validate the applicability, accuracy, and efficiency of the new shell element, SERDF, of VAMUCH. The examples include predicting the effective material properties of a binary composite, a hollow frame, and a heat exchanger core. Some of the results are compared with the 3D brick elements in VAMUCH. The results obtained using different UCs are also compared with each other. When meshing with SERDF, it needs to keep in mind that only one shell element is allowed along the thickness direction of the thin member.

In the first example, a homogenization problem of a binary composite is studied. The existing VAMUCH result (using brick element of good quality) is used as a benchmark to validate the SERDF result. Furthermore, the thickness of the unit cell is reduced gradually and the advantage of SERDF could be seen (when aspect ratio of the brick element becomes so large). Choosing a UC from a strictly periodic heterogeneous composite is a relative easy task and straightforward. In binary composite, we assume that two in-plane dimensions are infinite large and the materials is changing repeatedly along another direction. The UC1 in Fig. 6.5 represents a material block cut from the binary composite with \( l \) as the
length, $w$ the width and $t$ the thickness. Since the material does not change along the $l$ and $t$ direction, those length should not effect the effective properties in homogenization. In extreme cases, when $l$ or $t$ shrinks to zero, UC1 reduced to a 2D UC. When $l$ and $t$ both shrink to zero at the same time, it will become a 1D UC (Fig. 3.2). Those results can be found in the benchmark examples in VAMUCH. In this paper, we focus on the validation of SERDF which correspond to the situation of 3D UC and $t$ will be varied to a very small number to calculate effective material properties.

We assume that the binary composite is composed of two different isotropic materials with the red phase having Young’s modulus and Poisson’s ratio equal to 50 GPa and 0.3, respectively, and the blue phase having Young’s modulus and Poisson’s ratio equal to 30 GPa and 0.2, respectively. Volume fraction of the red phase is 0.5 and $l \times w \times t = 2 \text{ mm} \times 4 \text{ mm} \times 1 \text{ mm}$. The effective properties calculated for UC1 with two brick elements and two SERDFs are listed in Table 6.1 where direction 1 is along $l$, direction 2 is along $w$, and direction 3 is along $t$. The effective results are exactly the same, which is expected as we are modeling the same materials, this result is also considered as true effective properties in the following
Table 6.1: Effective properties using brick elements and SERDF of UC1 ($E$: GPa, $G$: GPa).

<table>
<thead>
<tr>
<th>Models</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\nu_{12}$</th>
<th>$\nu_{13}$</th>
<th>$\nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brick Element</td>
<td>40.10</td>
<td>37.52</td>
<td>40.10</td>
<td>15.15</td>
<td>15.87</td>
<td>15.15</td>
<td>0.2498</td>
<td>0.2637</td>
<td>0.2337</td>
</tr>
<tr>
<td>SERDF</td>
<td>40.10</td>
<td>37.52</td>
<td>40.10</td>
<td>15.15</td>
<td>15.87</td>
<td>15.15</td>
<td>0.2498</td>
<td>0.2637</td>
<td>0.2337</td>
</tr>
</tbody>
</table>

Table 6.2: Effective properties using brick elements and SERDF of UC2 ($E$: GPa, $G$: GPa).

<table>
<thead>
<tr>
<th>Models</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\nu_{12}$</th>
<th>$\nu_{13}$</th>
<th>$\nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brick Element</td>
<td>40.29</td>
<td>40.29</td>
<td>40.29</td>
<td>15.87</td>
<td>15.87</td>
<td>15.87</td>
<td>0.2698</td>
<td>0.2698</td>
<td>0.2698</td>
</tr>
<tr>
<td>SERDF</td>
<td>40.10</td>
<td>37.52</td>
<td>40.10</td>
<td>15.15</td>
<td>15.87</td>
<td>15.15</td>
<td>0.2498</td>
<td>0.2637</td>
<td>0.2337</td>
</tr>
</tbody>
</table>

comparisons. Though SERDF belongs to shell element, the reason it can also model the ‘thick’ 3D problem is because for this special case the fluctuating functions are not varying along the thickness direction, thus the assumption of SERDF that fluctuating functions vary linearly along the thickness is valid.

The advantage of SERDF will appear when we reduce the thickness $t$. As explained before, along $t$ direction, the material does not change which means no matter how to choose the thickness of a UC, it still a feasible UC to represent binary composite. By reducing $t$, we get UC2 and further get UC3. Here in UC2 we use the $t \times w \times t = 2\text{mm} \times 4\text{mm} \times 1 \times 10^{-10}$ mm and the effective properties are listed in Table 6.2. The result from SERDF stays the same while brick element gives wrong results compared with the true effective properties. The numerical difficulties come from the ill-conditioning of the coefficient matrix. Further reducing the thickness $t$ to $1 \times 10^{-15}$ mm, the double precision digits in the code can not handle the case for brick element but SERDF can still give exact results as shown in Table 6.3. The $C_{11}$ from the first component of stiffness matrix using both types of elements is depicted in Fig. 6.6 regarding the the changing of thickness. It is clear that when the thickness is about $1 \times 10^{-9}$ mm, the brick element has difficulties to calculate the accurate effective properties while SERDF presents good consistent results even if the thickness is extremely small.

Next let us consider two UCs of a hollow frame depicted in Fig. 6.7. The Young’s
Table 6.3: Effective properties using brick elements and SERDF of UC3 (\(E\): GPa, \(G\): GPa).

<table>
<thead>
<tr>
<th>Models</th>
<th>(E_1)</th>
<th>(E_2)</th>
<th>(E_3)</th>
<th>(G_{12})</th>
<th>(G_{13})</th>
<th>(G_{23})</th>
<th>(\nu_{12})</th>
<th>(\nu_{13})</th>
<th>(\nu_{23})</th>
</tr>
</thead>
<tbody>
<tr>
<td>Brick Element</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
<td>N/A</td>
</tr>
<tr>
<td>SERDF</td>
<td>40.10</td>
<td>37.52</td>
<td>40.10</td>
<td>15.15</td>
<td>15.87</td>
<td>15.15</td>
<td>0.2498</td>
<td>0.2637</td>
<td>0.2337</td>
</tr>
</tbody>
</table>

Fig. 6.6: The relation between effective stiffness matrix \(C_{11}\) and thickness \(t\).
Table 6.4: Effective properties of the hollow frame ($E$: GPa, $G$: GPa).

<table>
<thead>
<tr>
<th>Models</th>
<th>$E_1$</th>
<th>$E_2$</th>
<th>$E_3$</th>
<th>$G_{12}$</th>
<th>$G_{13}$</th>
<th>$G_{23}$</th>
<th>$\nu_{12}$</th>
<th>$\nu_{13}$</th>
<th>$\nu_{23}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>UC1-SERDF (5 elements)</td>
<td>1.866</td>
<td>1.525</td>
<td>3.071</td>
<td>0.012</td>
<td>0.651</td>
<td>0.523</td>
<td>-0.047</td>
<td>0.213</td>
<td>0.174</td>
</tr>
<tr>
<td>UC1-brick (5 elements)</td>
<td>1.866</td>
<td>1.525</td>
<td>3.071</td>
<td>0.012</td>
<td>0.651</td>
<td>0.523</td>
<td>-0.047</td>
<td>0.213</td>
<td>0.174</td>
</tr>
<tr>
<td>UC2-SERDF (8 elements)</td>
<td>1.985</td>
<td>1.642</td>
<td>3.071</td>
<td>0.257</td>
<td>0.719</td>
<td>0.595</td>
<td>0.014</td>
<td>0.263</td>
<td>0.187</td>
</tr>
<tr>
<td>UC2-brick (8 elements)</td>
<td>1.985</td>
<td>1.642</td>
<td>3.071</td>
<td>0.257</td>
<td>0.719</td>
<td>0.595</td>
<td>0.014</td>
<td>0.263</td>
<td>0.187</td>
</tr>
<tr>
<td>UC1-SERDF (37 elements)</td>
<td>1.847</td>
<td>1.505</td>
<td>3.068</td>
<td>0.001</td>
<td>0.647</td>
<td>0.519</td>
<td>-0.055</td>
<td>0.211</td>
<td>0.172</td>
</tr>
<tr>
<td>UC1-brick (37 elements)</td>
<td>1.847</td>
<td>1.505</td>
<td>3.071</td>
<td>0.001</td>
<td>0.647</td>
<td>0.519</td>
<td>-0.055</td>
<td>0.211</td>
<td>0.172</td>
</tr>
<tr>
<td>UC2-SERDF (72 elements)</td>
<td>1.847</td>
<td>1.505</td>
<td>3.071</td>
<td>0.004</td>
<td>0.649</td>
<td>0.521</td>
<td>-0.054</td>
<td>0.211</td>
<td>0.172</td>
</tr>
<tr>
<td>UC2-brick (72 elements)</td>
<td>1.847</td>
<td>1.505</td>
<td>3.071</td>
<td>0.004</td>
<td>0.649</td>
<td>0.521</td>
<td>-0.054</td>
<td>0.211</td>
<td>0.172</td>
</tr>
</tbody>
</table>

modulus and Poisson’s ratio of the strut are 69 GPa and 0.35, respectively. The strut width (n) has 0.01 m. Each UC of the hollow frame is of width (p) 0.5 m, height (q) 0.4 m, and depth (not shown) 0.5 m. The only difference between the UCs is that the geometric centers are differently selected. The porosity can be calculated as 95.55%. Two feasible UCs including a cross (UC1) and a hollow square (UC2) are meshed using brick elements and SERDF to obtain the effective properties. First, we used five elements to mesh UC1 and eight elements to mesh UC2. Then we used more elements (see Fig. 6.8) to check the convergence of effective properties compared with solutions generated by 3D brick meshing. With the increasing of number of elements, the effective properties between UC1 and UC2 are getting closer (Table 6.4). The reason is that UCs are chosen from the same porous structure, and they should represent its effective material properties. The $E_1$ of UC1 from brick element and SERDF are compared with its convergent value and plotted in Fig. 6.9, from which reveals that the SERDF has good convergence trend as well as brick element. The $G_{12}$ contribute the smallest part in strain energy and it exhibits slower convergence compared with other effective material properties, such as in Table 6.4. When the meshes are further refined, the convergence between UC1 and UC2 can be found in Fig. 6.10.

The last example is to calculate effective properties of the core of a heat exchanger. The heat exchanger serves as an important component in the air conditioning unit of airplanes and they mainly contain air to air fins as shown in Fig. 6.2, which is known as the core.
Fig. 6.7: Two possible unit cells of a hollow frame.

UC1: 11 elements  
UC2: 18 elements

Fig. 6.8: One of the meshes of UC1 and UC2.

UC1: 11 elements  
UC2: 18 elements
Fig. 6.9: Fast convergence behavior of $E_1$ from UC1 compared between brick element and SERDF.

Fig. 6.10: Convergence behavior of $G_{12}$ from UC1 and UC2.
Fig. 6.11: Unit cell for a heat exchanger with box-shaped channels.

The core is composed of alternating stack of layers in which pass the hot air and cold air, respectively. They are separated by sheets between them. The fins of the core form box-shaped channels along the air path. Since the core contains hundreds of thousands of fins, direct FEA of this core is impossible. The core can be considered as a structure by periodic repeating along three directions of a basic UC (see Fig. 6.11). Clearly this UC is formed by several thin members and we can naturally mesh it using SERDF elements. In this example we have $a = 0.8925$ mm, $b = 1.5355$ mm, $t = 0.025$ mm, $h = 0.071$ mm, and Young’s modulus and Poisson’s ratio equal to 205.441 GPa and 0.278, respectively. The stiffness matrix of effective properties obtained in VAMUCH can be found in Table 6.5.

6.3 Summary

SERDF element has been implemented in VAMUCH to carry out micromechanical analysis of high porosity RSEs formed of thin members. Several examples have been used
Table 6.5: Effective properties of the box-shaped core (MPa).

<table>
<thead>
<tr>
<th></th>
<th>10509.39</th>
<th>0</th>
<th>606.82</th>
<th>0</th>
<th>0</th>
<th>2239.73</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>14.90</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>606.82</td>
<td>0</td>
<td>3768.73</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>606.82</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>2787.57</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>14.90</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>2239.73</td>
<td>0</td>
<td>606.82</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>10509.39</td>
</tr>
</tbody>
</table>

To demonstrate the use, accuracy and efficiency of SERDF as an efficient alternative to 3D brick elements for homogenization of these materials/structures. Using the SERDF, the aspect ratio of the thickness with respect to the in-plane dimension for each element can be extremely small. As a result, the total number of elements used to mesh high porosity materials/structures can be significantly reduced compared with using brick elements without losing accuracy.
Chapter 7
Conclusions and Recommendations

The current research presents a series of problems related with homogenization of real engineering structures and materials based on the framework of the variational-asymptotic method. The focus is on the unit cell analysis to predict the effective properties and also recover the displacement, strain, and stress fields. This work is an extension of previous research conducted on VAMUCH, a general-purpose micromechanics code for analyzing heterogeneous materials with complex microstructure. This chapter reviews the main accomplishments and lists recommendations for future work.

7.1 Accomplishments

The rules need to be followed when choosing a unit cell from a structure in the original VAMUCH has been eliminated. The requiring of rectangular or cuboid with paired nodes of the unit cell to correctly build periodic boundary conditions has been relaxed to give more freedoms to the end user of choosing a convenient unit cell. As long as the analyst can identify a unit cell as the building block of the heterogeneous material, whether it has curved boundaries or unpaired nodes, the updated VAMUCH can describe the periodic boundary conditions and carry out the homogenization analysis. This greatly expands the possible applications of homogenization theory in general engineering problems.

In a random cell structure, a stochastic unit cell is developed to use the possible microstates to calculate the expectational material properties of each material points in the stochastic unit cell through Monte Carlo simulation. Based on VAM, the upper bound for the SUC is derived by minimizing the strain energy of all the possible realizations and the lower bound is solved by evaluating stress field through Young-Fenchel transformation. The theory is also implemented in VAMUCH which gives a narrower estimate between upper
and lower bounds compared with the third-order bounds. Considering the much less computational efforts than the third-order bounds, this new method accounts for the effect and efficiency in analyzing a randomly distributed heterogeneous problem. The unique of this approach, calculating the bounds of effective properties with considering randomness effect, has the potential to provide some guidance for position tolerance under quality control process in the composite industry.

Then VAM is applied to obtain the effective stiffnesses of the Kirchhoff-Love plate theory for corrugated plates. The analytical solutions are obtained for the continuous slope case and piecewise constant slope case with removing most of the ad hoc assumptions in the literature. The coupling stiffnesses between the extension and bending are firstly reported not necessarily zero and this will have an impact on the homogeneous-model behavior. Though, the additional discontinuity boundary conditions and different geometry descriptions are applied in piecewise constant slope case, the equivalent stiffnesses shows that it can be condensed from the continuous slope results by setting $\varphi' = 0$. In addition, the numerical approach for continuous slope case has been developed in VAMUCH using a 3-node C1 continuous curve element. Both the analytical equations and numerical tools can be used by analyst to speed up modeling of corrugated structures.

At last, a newly developed type of element has been implemented in VAMUCH to adapt the characteristics of a high porosity, thin-walled problem. This element is called the shell element of relative degree freedom (SERDF). The key idea of SERDF is to introduce the relative degrees of freedom to reduce the stiffness difference between the in-plane direction and out-of-plane direction. As a result, the numerical difficulties in solving the linear system in thin-walled problems will be avoided. The efficiency and accuracy of SERDF have been proved as an alternative to 3D brick elements for homogenization of these materials/structures. Using the SERDF, the aspect ratio of the thickness with respect to the in-plane dimension for each element can be extremely small. As a result, the total number of elements used to mesh high porosity materials/structures can be significantly reduced compared with using brick elements without losing of accuracy.
7.2 Recommendations for Future Work

As aforementioned, the current research is focused on solving several problems related with homogenization of real engineering structures and materials. Although the updated VAMUCH has been armed with more powerful capabilities, some aspects are still worth to be studied and improved.

7.2.1 Choosing of a RSE

The study on the RSE is mainly focused on applying the periodic boundary conditions in the finite element model in a general engineering problem. Though this objective has been successfully achieved and proved in several examples, some issues are still remaining for future study:

1. The mesh density of a UC greatly influences the effective properties. Automatic remeshing strategies based on the gradient of fluctuation functions could be developed and save the work load of the analyst.

2. Since the convergence study is always required as a proof to verify the accuracy of effective properties in a RSE, meshfree methods is an option to give more freedom in homogenization process. Meanwhile the periodic boundary conditions can be applied more easily.

3. The homogenization problem requires the satisfaction of Hill-Mandel condition, and the periodic boundary condition is one option to meet this requirement. Some weak forms with mixed boundary conditions can also satisfy Hill-Mandel condition without reinforced boundary conditions, by which could eliminate the boundary condition difficulties in solving the problem.

7.2.2 Homogenization of a random heterogeneous material

It is a paradox that choosing a RSE to represent a whole structure and this RSE can not represent all the characteristics of the structure. This is inherent in random heterogeneous
material, though the upper and lower bounds are the efficient way to fork the expectation of the effective properties, there are some requirements constantly being asked.

1. Numerical experiments of statistically obtaining effective properties with a large group of RSEs could be an option, however, an automatic framework and the computing efficiency require further studies.

2. Obtaining the mean value and standard deviations of homogenized properties requires some break through in the theory of probability description.

7.2.3 Homogenization of a corrugated plate

The present homogenization theory only treats the corrugation structures under the classical plate model, a curve element has been developed to obtain the numerical result in one-direction corrugation case. Some recommendations are listed as following:

1. The transverse shear deformation needs to be considered in the Mindlin-Reissner models for corrugated plate in the relative thick thickness cases. This also provides some advantages of building C0 shell element in FEA.

2. Another important application of corrugated plate is morphing wings. Since the wings are made by composite materials, it is very important to extend this approach to homogenize of composite corrugated structures.

3. Corrugated sandwich panels are widely used in industry. Obtaining its equivalent stiffnesses as a plate is also important. It requires involving the two small parameters: (1) the size of unit cell; (2) the thickness of the panels.

4. Since the commercial FEA software does not accept ABD stiffness matrix as an input to do a complex analysis and analytical solutions only available for a small potion of simple problems, an FEA solver, utilizing the ABD matrix directly and other boundary conditions, is required to maximize the potential of the current approach.

5. The SERDF can be used to mesh the corrugated UC and applied to VAPAS [125] to obtain the equivalent stiffnesses as a plate model. In this way, the shell element
is used to obtain both 3D stiffness matrix (in VAMUCH) and plate stiffnesses (in VAPAS), which provides the analyst convenience to choose the appropriate theory in the global analysis.
References


Appendices
Appendix A

Shape Functions for 3-Node $C^1$ Curve Element

Below we list the shape functions for 3-node $C^1$ curve element. The natural coor-
dination is assumed to be $\xi$ and varies from -1 to 1. Discretization relation is

$$\begin{bmatrix} \psi_1 & \psi_2 & \psi_3 \end{bmatrix}^T = S \Psi,$$

(A.1)

and

$$\psi_2 = S_1 \Psi_1^1 + S_2 \Psi_2^2 + S_3 \Psi_3^3,$$

(A.2)

Here $\Psi_i^j$ denotes the node value $\Psi_i$ at node $j$. For $\psi_2$ the Lagrange shape functions are

$$S_1 = \frac{1}{2} \xi (\xi - 1) \quad S_2 = \frac{1}{2} \xi (\xi + 1) \quad S_3 = 1 - \xi^2.$$

(A.3)

$\psi_1$ and $\psi_3$ use same sets of shape functions, such as $\psi_1$,

$$\psi_1 = S_4 \Psi_1^1 + S_5 \Psi_2^2 + S_6 \Psi_3^3 + S_7 \Psi_1^{1'} + S_8 \Psi_2^{2'} + S_9 \Psi_3^{3'}.$$

(A.4)

The Hermite shape functions $S_4$ to $S_6$ are solved based on

$$S_{(i-3)}(\xi_j) = \delta_{(i-3)j}, \quad \frac{dS_{(i-3)}(\xi)}{d\xi}_{\xi_j} |_{\xi_j} = 0,$$

(A.5)

$S_7$ to $S_9$ are based on

$$S_{(i-6)}(\xi_j) = 0, \quad \frac{dS_{(i-6)}(\xi)}{d\xi}_{\xi_j} |_{\xi_j} = \delta_{(i-6)j},$$

(A.6)
where the $\delta_{ij}$ denotes Kronecker delta. The Hermite shape functions are

\begin{align*}
S_1 &= \frac{3\xi^5}{4} - \frac{\xi^4}{2} - \frac{5\xi^3}{4} + \xi^2, \\
S_5 &= -\left(\frac{3\xi^5}{4}\right) - \frac{\xi^4}{2} + \frac{5\xi^3}{4} + \xi^2, \\
S_6 &= \xi^4 - 2\xi^2 + 1, \\
S_7 &= \frac{\xi^5}{4} - \frac{\xi^4}{4} - \frac{\xi^3}{4} + \frac{\xi^2}{4}, \\
S_8 &= \frac{\xi^5}{4} + \frac{\xi^4}{4} - \frac{\xi^3}{4} - \frac{\xi^2}{4}, \\
S_9 &= \xi^5 - 2\xi^3 + \xi.
\end{align*}
Appendix B

FEA for Corrugated Structures Varies in Two Directions

The $x_3(X_1, X_2)$ varies in both $X_1$ and $X_2$ directions,

$$x_3 = \varepsilon \phi(X_1, X_2). \quad (B.1)$$

The coefficients of tangent vectors are described as

$$r_1^1 = 1, \quad r_1^2 = 0, \quad r_1^3 = \varphi_1, \quad r_2^1 = 0, \quad r_2^2 = 1, \quad r_2^3 = \varphi_2. \quad (B.2)$$

with

$$\varphi_1(X_1, X_2) = \frac{d\phi}{dX_1}, \quad \varphi_2(X_1, X_2) = \frac{d\phi}{dX_2}. \quad (B.3)$$

and then

$$a_{11} = 1 + \varphi_1^2, \quad a_{12} = \varphi_1 \varphi_2, \quad a_{22} = 1 + \varphi_2^2, \quad a = \det \|a_{\alpha\beta}\| = 1 + \varphi_1^2 + \varphi_2^2. \quad (B.4)$$

The contravariant components of metric tensor are

$$a^{11} = \frac{1 + \varphi_2^2}{a}, \quad a^{12} = -\frac{\varphi_1 \varphi_2}{a}, \quad a^{22} = \frac{1 + \varphi_1^2}{a}. \quad (B.5)$$

The normal vector of the shell mid-surface is:

$$n_1 = -\frac{\varphi_1}{\sqrt{a}}, \quad n_2 = -\frac{\varphi_2}{\sqrt{a}}, \quad n_3 = \frac{1}{\sqrt{a}}. \quad (B.6)$$
The curvature tensor and their contravariants are calculated as

\[ b_{11} = \frac{1}{\sqrt{a}} \frac{\partial \varphi_1}{\partial x_1}, \quad b_{12} = \frac{1}{\sqrt{a}} \frac{\partial \varphi_1}{\partial x_2} = b_{21} = \frac{1}{\sqrt{a}} \frac{\partial \varphi_2}{\partial x_1}, \quad b_{22} = \frac{1}{\sqrt{a}} \frac{\partial \varphi_2}{\partial x_2}, \]

\[ b_1^1 = \frac{1 + \varphi_2^2}{a^{3/2}} \frac{\partial \varphi_1}{\partial x_1} - \varphi_1 \varphi_2 \frac{\partial \varphi_2}{\partial x_1}, \quad b_1^2 = \frac{1 + \varphi_1^2}{a^{3/2}} \frac{\partial \varphi_2}{\partial x_1} - \varphi_1 \varphi_2 \frac{\partial \varphi_1}{\partial x_1}, \]

\[ b_2^1 = \frac{1 + \varphi_2^2}{a^{3/2}} \frac{\partial \varphi_1}{\partial x_2} - \varphi_1 \varphi_2 \frac{\partial \varphi_2}{\partial x_2}, \quad b_2^2 = \frac{1 + \varphi_1^2}{a^{3/2}} \frac{\partial \varphi_2}{\partial x_2} - \varphi_1 \varphi_2 \frac{\partial \varphi_1}{\partial x_2}. \]  

(B.7)

The Christoffel symbols can be found as

\[ \Gamma^1_{11} = \frac{\varphi_1}{a} \frac{\partial^2 \varphi_1}{\partial x_1^2}, \quad \Gamma^2_{11} = \frac{\varphi_2}{a} \frac{\partial^2 \varphi_1}{\partial x_1^2}, \]

\[ \Gamma^1_{21} = \Gamma^2_{12} = \frac{\varphi_1}{a} \frac{\partial^2 \varphi_1}{\partial x_2^2}, \quad \Gamma^2_{21} = \Gamma^1_{12} = \frac{\varphi_2}{a} \frac{\partial^2 \varphi_1}{\partial x_2^2}, \]

\[ \Gamma^1_{22} = \frac{\varphi_1}{a} \frac{\partial^2 \varphi_2}{\partial x_2^2}, \quad \Gamma^2_{22} = \frac{\varphi_2}{a} \frac{\partial^2 \varphi_2}{\partial x_2^2}. \]  

(B.8)

The leading terms of the extension strain measures are

\[ \gamma^0_{11} = v_{1,1} - x_3 v_{3,11} + \frac{1}{\varepsilon} \frac{\partial \psi_1}{\partial X_1} + \frac{\varphi_1}{\varepsilon} \frac{\partial \psi_3}{\partial X_1}, \]

\[ 2\gamma^0_{12} = v_{1,2} + v_{2,1} - 2 x_3 v_{3,12} + \frac{1}{\varepsilon} \frac{\partial \psi_1}{\partial X_2} + \frac{\varphi_1}{\varepsilon} \frac{\partial \psi_2}{\partial X_1} + \frac{\varphi_1}{\varepsilon} \frac{\partial \psi_3}{\partial X_1}, \]

\[ \gamma^0_{22} = v_{2,2} - x_3 v_{3,22} + \frac{1}{\varepsilon} \frac{\partial \psi_2}{\partial X_2} + \frac{\varphi_1}{\varepsilon} \frac{\partial \psi_3}{\partial X_2}. \]  

(B.9)

The bending strain measures are

\[ \rho_{11} = U_{1,1} + \frac{1}{\varepsilon} \frac{\partial U_1}{\partial X_1} - \Gamma^1_{11} U_1 - \Gamma^2_{11} U_2 - \sqrt{a} \theta b_1^2, \]

\[ 2\rho_{12} = U_{1,2} + U_{2,1} + \frac{1}{\varepsilon} \left( \frac{\partial U_1}{\partial X_2} + \frac{\partial U_2}{\partial X_1} \right) - 2 \Gamma^1_{12} U_1 - 2 \Gamma^2_{12} U_2 + \sqrt{a} \theta (b_1^2 - b_2^2), \]

\[ \rho_{22} = U_{2,2} + \frac{1}{\varepsilon} \frac{\partial U_2}{\partial X_2} - \Gamma^1_{22} U_1 - \Gamma^2_{22} U_2 + \sqrt{a} \theta b_1^2, \]  

(B.10)
with

\[ U_1 = n_1 \left( v_{1,1} - x_3 v_{3,11} + \frac{\partial \psi_1}{\partial X_1} - \varphi_1 v_{3,1} \right) + n_2 \left( v_{2,1} - x_3 v_{3,21} + \frac{\partial \psi_2}{\partial X_1} - \varphi_1 v_{3,2} \right) + n_3 \left( v_{3,1} + \frac{\partial \psi_3}{\partial X_1} \right) + \varepsilon (n_1 \psi_{1,1} + n_2 \psi_{2,1} + n_3 \psi_{3,1}), \tag{B.11} \]

\[ U_2 = n_1 \left( v_{1,2} - x_3 v_{3,12} + \frac{\partial \psi_1}{\partial X_2} - \varphi_2 v_{3,1} \right) + n_2 \left( v_{2,2} - x_3 v_{3,22} + \frac{\partial \psi_2}{\partial X_2} - \varphi_2 v_{3,2} \right) + n_3 \left( v_{3,2} + \frac{\partial \psi_3}{\partial X_2} \right) + \varepsilon (n_1 \psi_{1,2} + n_2 \psi_{2,2} + n_3 \psi_{3,2}), \]

and rotation θ

\[ 2\sqrt{\alpha} \theta = v_{1,2} - v_{2,1} + 2\varphi_1 v_{3,2} - 2\varphi_2 v_{3,1} + \frac{\partial \psi_1}{\partial X_2} - \frac{\partial \psi_2}{\partial X_1} + \varphi_1 \frac{\partial \psi_3}{\partial X_2} - \varphi_2 \frac{\partial \psi_3}{\partial X_1} + \varepsilon (\psi_{1,2} - \psi_{2,1} + \varphi_1 \psi_{3,2} - \varphi_2 \psi_{3,1}). \tag{B.12} \]

The leading terms of the bending strain become

\[ \rho_{11}^0 = l_{11} v_{1,1} + l_{12} (v_{1,2} + v_{2,1}) + l_{22} v_{2,2} + \left( 1 + 2\varphi_1^2 \right) x_3 l_{11} + \left( \frac{\varphi_1 \varphi_2}{\sqrt{\alpha}} - x_3 l_{12} \right) 2v_{3,12} - x_3 l_{22} v_{3,22} \tag{B.13} \]

\[ + \left( l_{11} \frac{\partial}{\partial X_1} + l_{12} \frac{\partial}{\partial X_2} + \frac{n_1}{\varepsilon} \frac{\partial^2}{\partial X_1^2} \right) \psi_1 + \left( l_{12} \frac{\partial}{\partial X_1} + l_{22} \frac{\partial}{\partial X_2} + \frac{n_2}{\varepsilon} \frac{\partial^2}{\partial X_1^2} \right) \psi_2 \]

\[ 2\rho_{12}^0 = m_{11} v_{1,1} + m_{12} (v_{1,2} + v_{2,1}) + m_{22} v_{2,2} + \left( \frac{2\varphi_1 \varphi_2}{\sqrt{\alpha}} - x_3 m_{11} \right) v_{3,11} + \left( \sqrt{\alpha} - x_3 m_{12} \right) 2v_{3,12} + \left( \frac{2\varphi_1 \varphi_2}{\sqrt{\alpha}} - x_3 m_{22} \right) v_{3,22} \]

\[ + \left( m_{11} \frac{\partial}{\partial X_1} + m_{12} \frac{\partial}{\partial X_2} + \frac{2n_1}{\varepsilon} \frac{\partial^2}{\partial X_1 \partial X_2} \right) \psi_1 + \left( m_{12} \frac{\partial}{\partial X_1} + m_{22} \frac{\partial}{\partial X_2} + \frac{2n_2}{\varepsilon} \frac{\partial^2}{\partial X_1 \partial X_2} \right) \psi_2 \]

\[ + \left( m_{31} \frac{\partial}{\partial X_1} + m_{32} \frac{\partial}{\partial X_2} + \frac{2n_3}{\varepsilon} \frac{\partial^2}{\partial X_1 \partial X_2} \right) \psi_3, \tag{B.14} \]
\[ \rho^0_{22} = n_{11} v_{1,1} + n_{12} (v_{1,2} + v_{2,1}) + n_{22} v_{2,2} \]
\[ - x_3 n_{11} v_{3,11} + \left( \frac{\varphi_1 \varphi_2}{\sqrt{a}} - x_3 n_{12} \right) 2 v_{3,12} + \left( \frac{1 + 2 \varphi_2^2}{\sqrt{a}} - x_3 n_{22} \right) v_{3,22} \]
\[ + \left( n_{11} \frac{\partial}{\partial X_1} + n_{12} \frac{\partial}{\partial X_2} + \frac{n_1}{\varepsilon} \frac{\partial^2}{\partial X_2^2} \right) \psi_1 + \left( n_{12} \frac{\partial}{\partial X_1} + n_{22} \frac{\partial}{\partial X_2} + \frac{n_2}{\varepsilon} \frac{\partial^2}{\partial X_2^2} \right) \psi_2 \]
\[ + \left( n_{31} \frac{\partial}{\partial X_1} + n_{32} \frac{\partial}{\partial X_2} + \frac{n_3}{\varepsilon} \frac{\partial^2}{\partial X_2^2} \right) \psi_3, \]
(B.15)

where

\[ l_{11} = \frac{1}{\varepsilon} \frac{\partial n_1}{\partial X_1} - \Gamma_{11} n_1, \quad l_{12} = \frac{3 \varphi_1 \varphi_2 \frac{\partial \varphi_1}{\partial X_1} - (1 + \varphi_1^2) \frac{\partial \varphi_2}{\partial X_1}}{2 \varepsilon a^{3/2}}, \quad l_{22} = -\Gamma_{11} n_2, \]
(B.16)

\[ l_{31} = \frac{1}{\varepsilon} \frac{\partial n_3}{\partial X_1} - \Gamma_{11} n_3 + \frac{1}{2} \delta_{12} \varphi_2, \quad l_{32} = -\Gamma_{11} n_3 - \frac{1}{2} \delta_{12} \varphi_1, \]

and

\[ m_{11} = \frac{1}{\varepsilon} \frac{\partial n_1}{\partial X_2} - 2 \Gamma_{12} n_1, \quad m_{12} = \frac{6 \varphi_1 \varphi_2 \frac{\partial \varphi_1}{\partial X_2} - (1 + \varphi_1^2) \frac{\partial \varphi_2}{\partial X_2} - (1 + \varphi_2^2) \frac{\partial \varphi_1}{\partial X_1}}{2 \varepsilon a^{3/2}}, \]
\[ m_{22} = \frac{1}{\varepsilon} \frac{\partial n_2}{\partial X_1} - 2 \Gamma_{12} n_2, \quad m_{31} = \frac{1}{\varepsilon} \frac{\partial n_3}{\partial X_1} - 2 \Gamma_{12} n_3 - \frac{\varphi_2}{2} (b_1^2 - b_2^2), \]
(B.17)

\[ m_{32} = \frac{1}{\varepsilon} \frac{\partial n_3}{\partial X_2} - 2 \Gamma_{12} n_3 + \frac{\varphi_1}{2} (b_1^2 - b_2^2), \]

and

\[ n_{11} = -\Gamma_{22} n_1, \quad n_{12} = \frac{3 \varphi_1 \varphi_2 \frac{\partial \varphi_1}{\partial X_1} - (1 + \varphi_2^2) \frac{\partial \varphi_1}{\partial X_1}}{2 \varepsilon a^{3/2}}, \quad n_{22} = \frac{1}{\varepsilon} \frac{\partial n_2}{\partial X_2} - \Gamma_{22} n_2, \]
(B.18)

\[ n_{31} = -\Gamma_{22} n_3 - \frac{1}{2} \delta_{12} \varphi_2, \quad n_{32} = \frac{1}{\varepsilon} \frac{\partial n_3}{\partial X_2} - \Gamma_{22} n_3 + \frac{1}{2} \delta_{12} \varphi_1. \]

In matrix form of strain measures,

\[ \Psi = \Gamma \varepsilon + \Gamma_h \Psi. \]
(B.19)
where

\[
\mathbf{Y} = \begin{bmatrix}
\gamma_{11}^0 & 2\gamma_{12}^0 & \gamma_{22}^0 & \rho_{11}^0 & 2\rho_{12}^0 & \rho_{22}^0
\end{bmatrix}^T,
\]

\[
\Gamma_{\epsilon} = \begin{bmatrix}
l_{11} & l_{12} & l_{22} & -\frac{1+2\varphi_{12}^2}{\sqrt{a}} + x_3 l_{11} & -\frac{2\varphi_{12}^2}{\sqrt{a}} + x_3 l_{12} & l_{22} x_3 \\
m_{11} & m_{12} & m_{22} & -\frac{2\varphi_{12}^2}{\sqrt{a}} + x_3 m_{11} & -\frac{2\varphi_{12}^2}{\sqrt{a}} + x_3 m_{12} & x_3 m_{22} \\
n_{11} & n_{12} & n_{22} & -\frac{2\varphi_{12}^2}{\sqrt{a}} + x_3 n_{11} & -\frac{2\varphi_{12}^2}{\sqrt{a}} + x_3 n_{12} & x_3 n_{22}
\end{bmatrix},
\]

\[
\epsilon = \begin{bmatrix}
\epsilon_{xx} & 2\epsilon_{xy} & \epsilon_{yy} & 2\kappa_{xx} & \kappa_{yy}
\end{bmatrix}^T,
\]

\[
\Gamma_h = \begin{bmatrix}
\frac{\partial}{\partial X_1} & 0 & \varphi_1 \frac{\partial}{\partial X_1} \\
\frac{\partial}{\partial X_2} & \frac{\partial}{\partial X_1} & \varphi_2 \frac{\partial}{\partial X_1} \\
0 & \frac{\partial}{\partial X_2} & \varphi_2 \frac{\partial}{\partial X_2}
\end{bmatrix},
\]

\[
\Psi = \begin{bmatrix}
\psi_1 & \psi_2 & \psi_3
\end{bmatrix}^T.
\]

Discretize \( \Psi \) using the finite elements as

\[
\Psi = S \Psi,
\]

where \( S \) representing the shape functions and \( \Psi \) denoting the nodal values of fluctuation functions. Each node has 12 degrees of freedom, which are

\[
\psi_1, \frac{\partial \psi_1}{\partial \xi}, \frac{\partial \psi_1}{\partial \eta}, \frac{\partial^2 \psi_1}{\partial \xi \partial \eta}, \psi_2, \frac{\partial \psi_2}{\partial \xi}, \frac{\partial \psi_2}{\partial \eta}, \frac{\partial^2 \psi_2}{\partial \xi \partial \eta}, \psi_3, \frac{\partial \psi_3}{\partial \xi}, \frac{\partial \psi_3}{\partial \eta}, \frac{\partial^2 \psi_3}{\partial \xi \partial \eta}.
\]
here $\xi$ and $\eta$ are natural coordinates. Hermite shape functions for 9-node quadratic elements are required for interpolation.

Following Eq. (5.135), the rest of procedures are solving fluctuation functions and obtaining equivalent plate stiffnesses. The local displacement, strain and stress fields can be also recovered.
Vita

About the author

Zheng Ye was born in Zhuzhou, Hunan Province, China, on December 1, 1980. He received his B.S. degree from Beijing University of Aeronautics and Astronautics majoring in Engine Design. After one year’s work, he went back to school and received his M.S. degree from Beijing University of Aeronautics and Astronautics majoring in Propulsion Engineering in May, 2007. He worked as a field engineer in West-east Pipeline Project in the western provinces of China. In 2008, he came to Mechanical and Aerospace Department at Utah State University to pursue his Ph.D. degree. Most of the work done in his study is presented here for the partial requirements of Doctor of Philosophy Degree.

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