September 3, 1987 IFP-298-UNC

The Heterotic String from 4-Dimensional Geometry

James T. Wheeler

Department of Physics and Astronomy
University of North Carolina at Chapel Hill

Abstract

In any conformally invariant metric-connection theory of 4-dimensional spacetime, there are 26 ways that a body can distort. This follows from the absence of any preferred metric, and is equivalent to the ability of an observer to consider objects on any given world line to be distortion free. Within the simplest conformally invariant theory of distortions that allows metric compatible spaces as trivial solutions, we find the maximal classical solution. This solution corresponds to the bosonic sector of the heterotic string: 26 left-moving fields plus 10 right-moving fields.

1. Introduction

One of the outstanding tasks facing string theory[1] is to relate the unique predictions of the higher-dimensional theory to the results of experiments carried out in a 4-dimensional universe. There are two principal approaches to this difficulty: compactifications and the introduction of symmetries.

Compactifications, including Calabi-Yau[2], orbifolds[3], or other low-energy gravitational solutions[4], attempt to find ground states of the string system with all but 4 of the dimensions compactified. As a result, such solutions suffer from their perturbative nature. While compactifications can be produced with a typical scale that is consistent with the low energy approximation, one can have little confidence that such solutions are the ones actually chosen by the system. It seems far more likely that the compactification occurs near the Planck length -- just the point where the perturbation series diverges. Even if an approach to non-perturbative solutions can be found, one still must solve a gravitational problem exactly before looking at particle fields perturbatively.

While the inclusion of supersymmetry has led to the very successful 10-dimensional heterotic string[5], the introduction of other internal symmetries[6] has led to a different kind of uniqueness problem. There may be literally thousands of possible reductions, any of which might give plausible models. The predictive power that led one to string theory in the first place is lost.

Work in these directions is promising, but there is a deeper unanswered question posed by the success of string theory -- why should *strings* be the fundamental building blocks of the physical world?

Here we follow a line of reasoning that differs from the standard interpretation of string theory. We suppose at the outset that the world is well modeled by some 4-dimensional geometry. The problem is then to find parameters describing that geometry that obey dymanical equations which resemble the equations of motion for a string. The 'string' will then be just a mathematical description of the behavior of some of the fields required for a description of 4-dimensional spacetime. In contrast to the standard picture of string theory, the higher-dimensional space in which the string moves is a parameter space, and not the physical spacetime.

In particular, we consider a manifold $\mathcal{M}$, with fixed, torsion-free connection $\Gamma^{\mu}{}_{\alpha\beta}$. The connection is independent of any specification of metric, and may be such that there does not exist any metric for which it is the Christoffel connection. It describes the change in the components of tensor fields which propagate in spacetime. Such geometries have been proposed by a number of authors[7].

Tensors are used to describe physical objects. We take a tensor field to model a physical system well when the measurable properties of that physical system transform and propagate in the same way as the tensor field transforms and propagates on $\mathcal{M}$. For example, an infinitesimal vector may be identified with one edge of a small, freely moving cube. The components of a symmetric tensor field, $g_{\alpha\beta}(x)$, may be defined in a particular coordinate system by the following procedure:

> Let each side of a small cube be our standard of length in each of three orthogonal spatial directions and let an associated clock tick unit seconds. Taking the associated unit vectors to be mutually orthogonal (this is what defines the box to be a cube), we assign a coordinate along each direction and define a diagonal matrix $g_{\alpha\beta}(P)$ which gives the lengths of and angles between the sides.
>
> Now we allow the cube to move as determined by the connection. We assume that the cube is rigid in the sense that the partial derivatives of the lengths of the sides vanish, or equivalently, that the sides do not appear to fluctuate to a nearby comoving observer. At each point of the world-line, $x^\mu(\lambda)$, of the cube, $g_{\alpha\beta}(\lambda)$ is defined as the matrix that makes these four unit vectors orthonormal.
>
> Finally, $g_{\alpha\beta}(x)$ is an arbitrary smooth extension off of this world line to the rest of spacetime.

If two identical cubes with associated clocks are allowed to move along different trajectories then compared, they may no longer be identical. Therefore, the tensor fields determined by the above procedure will not be equal. If we had wished to use one of the tensor fields as the metric, then lengths computed with that metric would be path dependent.

The metric will be regarded as a measurement tool defined by a given observer. Note the similarity between the procedure described above and typical experimental procedure. In practice an experimenter always assumes the constancy of the size of the measuring apparatus. *This is a metric choice.* In choosing a meter rod to compare lengths, the observer disregards the possibility that the rod itself may fluctuate in length relative to other rods on distinct trajectories. While this assumption is certainly justified at macroscopic scales, it may fail at small scales where the corresponding energies are sufficient to excite the longitudinal modes of the metric.

In most cases a metric is necessary in order to produce a number capable of measurement.

In the present work this is still true, but no metric gives consistent results for measurements along different paths. What is required is some kind of average over metrics. Such an average is expected to be equivalent to quantization of the system. In the present work, an arbitrary distinction between 'classical' and 'quantum' realms will be made. Only a classical treatment will be given. Investigation of the 'quantum' aspects of this theory is still in progress.

This distinction between metric averaged and classical solutions is somewhat arbitrary. It will be seen that some results which properly depend on quantum behavior are already present in the classical version. The most notable instance of this is the appearance of the 26 dimensional character of the string solution. It is presumed, though it has not been demonstrated, that the 26 arises because of the deep connection in this theory between the classical and the quantum arenas.

In solving the classical theory, we will assume that we are free to choose any metric. This assumption is perturbatively sound if the connection differs only slightly from the Christoffel connection computed from the chosen metric. In this case, the metric averaging will produce only small deviations from the chosen metric. Since the solution found below is already perturbative, these small deviations are negligible.

Nonetheless, there are consequences of choosing a metric arbitrarily. It is impossible to write a classical lagrangian for the connection without using a metric, so there are equations of motion for the metric. These equations are reinterpreted as constraints on the connection. It is these constraints on the solutions considered that restricts them to be stringlike. In fact, the solution presented for the connection replicates the bosonic sector of the classical heterotic string. But unlike the heterotic string, the space in which the string moves is the space of physical connections and not the spacetime itself.

In the following section, we examine the space of distortions and show it to have 26 physical dimensions. In section 3 we develop some formal aspects of metric-connection theories, including a necessary and sufficient condition for metric-connection compatibility. A lagrangian is proposed in section 4, and a solution is found to the equations of motion resulting from its variation. This solution is maximal in a sense described below. In the penultimate section we relate the solution found in section 4 to the usual solution for the heterotic string, and examine the Poisson brackets in sufficient detail to see how quantization will proceed. Section 6 provides a summary and some discussion of these results.

2. The space of distortions.

If we choose a symmetric tensor field $g_{\alpha\beta}$, whether by the prescription above, or by any other means, the contractions formed with it will not be invariant quantities. The changes in apparent scalars such as the length of a vector when the vector is parallel transported around the manifold involve the covariant derivative of the metric, $Q_{\alpha\beta\mu} \equiv D_\mu g_{\alpha\beta}$ $(\alpha, \beta, \mu = 0, 1, 2, 3)$. To see this, parallel-transport a vector v^α in the direction of a vector field w^α.

$$w^\mu D_\mu v^\alpha \equiv w^\mu (\partial_\mu v^\alpha + v^\rho \Gamma^\alpha_{\rho\mu}) = 0. \qquad (2.1)$$

The change in (length)^2 of v^α is then given by:

$$w^\mu D_\mu L^2 = w^\mu D_\mu (g_{\alpha\beta} v^\alpha v^\beta) = Q_{\alpha\beta\mu} v^\alpha v^\beta w^\mu. \qquad (2.2)$$

Even though the components have been transported as constantly as possible, the vector still is altered in length. This is a generalization of the change in orientation experienced by a vector displaced around a closed path in a Riemannian space. In a non-Riemannian space the length may change as well.

$Q_{\alpha\beta\mu}$ may always be divided into a trace part, $A_\mu \equiv \frac{1}{4} g^{\alpha\beta} Q_{\alpha\beta\mu}$ and a traceless part $S_{\alpha\beta\mu}$:

$$Q_{\alpha\beta\mu} = g_{\alpha\beta} A_\mu + S_{\alpha\beta\mu}. \qquad (2.3)$$

These parts reflect the two types of possible change in spacetime scale: volume changing *scalings* and volume preserving *distortions.* Volume changing spacetimes are precisely those first discussed by Weyl[8] and later by others[9], for which $S_{\alpha\beta\mu} = 0$ and

$$D_\mu g_{\alpha\beta} = A_\mu g_{\alpha\beta}. \qquad (2.4)$$

To see that these describe scalings, consider the covariant derivative of the volume element:

$$D_\mu \sqrt{-g} = \frac{1}{2} g^{\alpha\beta} Q_{\alpha\beta\mu} \sqrt{-g} = 2 A_\mu \sqrt{-g}\ . \qquad (2.5)$$

The change in volume is fully described by the vector field A_μ, here called the Weyl potential. We will not follow Weyl's interpretation of A_μ as the electromagnetic potential. In the particular model we consider, A_μ will be a pure gradient.

The main focus of this paper is the distortions $S_{\alpha\beta\mu}$: volume-preserving covariant changes of the metric. We will show:

1. In any conformally invariant theory of distortions, $S_{\alpha\beta\mu}$ has a maximum of 26 independent components. This follows from the absence of any preferred metric. It is equivalent to the ability of any observer to consider objects on his or her world line to be distortion free.

2. Within the simplest conformally invariant theory of distortions that allows metric compatible spaces as trivial solutions, we find a classical solution that realizes this maximum, and which is equivalent to the right-moving sector of a bosonic string in 26-dimensions plus the left-moving sector of a bosonic string in 10-dimensions.

The remainder of this section will be concerned with the proof of claim 1. We begin by studying the dependence of $S_{\alpha\beta\mu}$ on the conformal factor. Let

$$\hat{g}_{\alpha\beta} = e^\phi g_{\alpha\beta}. \qquad (2.6)$$

Then

$$\hat{Q}_{\alpha\beta\mu} = e^\phi Q_{\alpha\beta\mu} + \hat{g}_{\alpha\beta} \phi_{,\mu}. \qquad (2.7)$$

which according to eq. (2.3) may be decomposed into

$$\hat{A}_\mu = A_\mu + \phi_{,\mu} \qquad (2.8)$$

and

$$\hat{S}_{\alpha\beta\mu} = e^\phi S_{\alpha\beta\mu}. \qquad (2.9)$$

The Weyl potential is shifted to $A_\mu + \phi_{,\mu}$ just like an electromagnetic gauge transformation. Historically, this is one reason that A_μ has been mistaken for the electromagnetic potential.

The distortion potential is scaled, so that its norm is indeterminate:

$$\hat{S}^{\alpha\beta\mu}\hat{S}_{\alpha\beta\mu} = e^{-\Phi} S^{\alpha\beta\mu} S_{\alpha\beta\mu} \tag{2.10}$$

Since $Q_{\alpha\beta\mu}$ has 40 independent components, $S_{\alpha\beta\mu}$ has 36. The conformal invariance of the theory therefore means that the components of $S_{\alpha\beta\mu}$ form a projective space of dimension 35, equivalent to a 35 dimensional pseudo-sphere.

Now consider the dependence of $S_{\alpha\beta\mu}$ on the remaining part of $g_{\alpha\beta}$. For a fixed, torsion free connection, $\Gamma^{\mu}{}_{\alpha\beta}$, consider the covariant derivative of the Minkowski metric $\eta_{\alpha\beta}$ = diag(-1, 1, 1, 1). This matrix can be forced to be a tensor field by choosing a set of coordinates, defining $\eta_{\alpha\beta}$ to be diag(-1,1,1,1) at each point in those coordinates, then declaring the components to transform as a tensor under coordinate changes. We have

$$\bar{Q}_{\alpha\beta\mu} \equiv D_{\mu}\eta_{\alpha\beta} = \eta_{\rho\sigma} N^{\rho\sigma}{}_{\alpha\beta\mu} \tag{2.11}$$

where

$$N^{\rho\sigma}{}_{\alpha\beta\mu} \equiv \delta^{(\rho}_{\alpha}\Gamma^{\sigma)}_{\beta\mu} + \delta^{(\rho}_{\beta}\Gamma^{\sigma)}_{\alpha\mu} \tag{2.12}$$

depends only on the connection. For any other metric $g_{\alpha\beta}$ we find

$$Q_{\alpha\beta\mu} \equiv D_{\mu}g_{\alpha\beta} = \partial_{\mu}g_{\alpha\beta} + g_{\rho\sigma} N^{\rho\sigma}{}_{\alpha\beta\mu}. \tag{2.13}$$

It will be convenient below to represent a symmetric pair of indices ($\alpha\beta$) by a single latin index, a,b,... running from 1 to 10. With this notation (2.13) becomes:

$$Q_{a\mu} \equiv D_{\mu}g_{a} = \partial_{\mu}g_{a} + g_{b} N^{b}{}_{a\mu}. \tag{2.13'}$$

Changing the metric does not change the connection, but only the division of the connection into compatible (Christoffel) and incompatible ($Q_{\alpha\beta\mu}$) parts:

$$\Gamma_{\alpha\beta\mu} = \frac{1}{2}(\partial_{\mu}g_{\alpha\beta} + \partial_{\beta}g_{\alpha\mu} - \partial_{\alpha}g_{\mu\beta}) - \frac{1}{2}(Q_{\alpha\beta\mu} + Q_{\alpha\mu\beta} - Q_{\beta\mu\alpha}) \tag{2.14}$$

Compare the relationship between eq.(2.11) and eq.(2.13) to a gauge transformation. If $g_{\alpha\beta}$ in the second term of eq. (2.13) was $\eta_{\alpha\beta}$, then our freedom to choose the metric would make it a gauge field for $Q_{\alpha\beta\mu}$. If the metric is perturbatively Minkowski and $Q_{\alpha\beta\mu}$ is small, then to lowest order the relation is exactly that of a gauge transformation. Even with the extra factor of the metric multiplying $N^{\rho\sigma}{}_{\alpha\beta\mu}$, we can still use the choice of metric to reduce the freedom in $Q_{\alpha\beta\mu}$. The question is, how much of $Q_{\alpha\beta\mu}$ is metric-gauge dependent? There are 10 degrees of freedom in $g_{\alpha\beta}$. Is it possible to reduce the number of independent components of $Q_{\alpha\beta\mu}$ by 10?

We conclude this section with a demonstration that the answer is yes. In fact, the metric described in the introduction is one metric-gauge choice that maximally reduces $Q_{\alpha\beta\mu}$. We can always choose a metric defined locally to coincide with a small cube and a clock. Since distortion is then defined relative to the cube, such a metric will be distortion free along the observer's world line. What we have is a *principle of relativity of distortion* . It is, simply, that *an observer can always choose a metric-gauge that is distortion free along any given world line* . There is no such thing as absolute distortion, so the observer in question has only to define the metric by the size and shape of objects in an appropriate local, covariantly transported neighborhood.

Notice that these observer-metrics are not the only ones that reduce the freedom in $Q_{\alpha\beta\mu}$. In the third section we will make use of a metric-gauge in which $\partial_{\mu}Q_{\alpha\beta}{}^{\mu} = 0$. It is a simple exercise to show perturbatively that there exists a metric for which this condition holds.

<u>Theorem 1</u>: There exists a metric-gauge which reduces the number of independent distortions to 26.

For proof we need only calculate explicitly the metric described above. Let the metric be given by the Minkowski metric on an initial spacelike hypersurface. If it remains covariantly constant along some timelike path $x^{\mu}(\lambda)$ then we have:

$$g_{\alpha\beta} = \eta_{\alpha\beta} \tag{2.15}$$

on the initial spacelike hypersurface and

$$u^{\mu} D_{\mu}\eta_{\alpha\beta} = 0. \tag{2.16}$$

where $u^{\mu} = dx^{\mu}/d\lambda$ is tangent to the path.

Let the given world line $x^{\mu}(\lambda)$ be smoothly extended to a congruence of timelike curves $x^{\mu}(\lambda; \xi^{i})$ so that the tangent vector u^{μ} is smoothly extended to a timelike divergence-free vector field $v^{\mu}(\lambda; \xi^{i}) = dx^{\mu}(\lambda; \xi^{i})/d\lambda$. Here the ξ^{i}, i=1,2,3 parameterize the λ = constant hypersurfaces in $\mathcal{M}$. We assume that we restrict our consideration to some region of $\mathcal{M}$ where such an extension is possible. We will have the metric we need once we solve the equation

$$v^\mu D_\mu g_{\alpha\beta} = v^\mu Q_{\alpha\beta\mu} = 0 \qquad (2.17)$$

with the initial condition, eq.(2.15). In terms of $N^a{}_{b\mu}$ this becomes:

$$v^\mu [\partial_\mu g_a - g_b N^b{}_{a\mu}] = 0. \qquad (2.18)$$

Letting $N^a{}_{b\mu}v^\mu = N^a{}_b$ we may formally integrate along the congruence to find

$$g_a = \eta_a + \int g_b N^b{}_a \, d\lambda \qquad (2.19)$$

which is then iterated to produce the series expansion

$$g_a = \eta_a + \int \eta_b N^b{}_a \, d\lambda + \int \eta_b N^b{}_c d\lambda \int^\lambda N^c{}_a \, d\lambda' + \ \ldots \qquad (2.20)$$

The tensor-valued integrands pose no problem since we have chosen an orthonormal coordinate system. If we denote "time ordering" in the parameter λ by Λ, this may be written as:

$$g_{\alpha\beta} = \Lambda \exp[\int N^b{}_a]\eta_b \qquad (2.21)$$

Eq.(2.17) now provides 10 constraints on $Q_{\alpha\beta\mu}$. From the decomposition in equation (2.3) we see that the Weyl and distortion potentials now satisfy

$$v^\mu A_\mu = 0 \qquad (2.22)$$

and

$$v^\mu S_{\alpha\beta\mu} = 0. \qquad (2.23)$$

Since the trace of eq.(2.23) vanishes by eq.(2.3), eq.(2.23) provides 9 constraints, reducing $S_{\alpha\beta\mu}$ to 26 arbitrary functions.

3. Formal Developments.

We wish to construct a conformally invariant Lagrangian for the 26 allowed distortions. While $Q_{\alpha\beta\mu}$ is a convenient tensor for discussing length change for a given metric and establishing

the number of true distortions, it is not approppropriate for formulating a theory in which the metric is a gauge field. What we need is a metric-invariant field strength. A nonzero value for this field strength should measure true distortions. Conversely, its vanishing will indicate the existence of a connection-compatible metric.

We begin this section by developing metric-invariant formalism. The section continues with a proof of the necessary and sufficient condition for the existence of a connection-compatible metric. Finally, a convenient metric-independent criterion for the existence of nontrivial distortion is presented. All of the results of this section hold in an arbitrary number of dimensions, d.

First, define the covariant derivative 1-form

$$\mathbf{D} \equiv dx^\mu D_\mu \qquad (3.1)$$

where the action of D_μ on a vector is defined by eq.(2.1). Greek indices run from 1 to d, while latin indices run from 1 to $\frac{d(d+1)}{2}$. Forms, denoted by boldface type, are always assumed to be multiplied using the wedge product. The action of **D** on covariant, symmetric second rank tensors, written as $\frac{d(d+1)}{2}$-dimensional vectors, is given by

$$\mathbf{D}H_a = \mathbf{d}H_a - H_b\mathbf{N}^b{}_a, \qquad (3.2)$$

where

$$\mathbf{N}^a{}_b \equiv N^a{}_{b\mu}\, dx^\mu \qquad (3.3)$$

and **d** is the usual exterior derivative. For contravariant tensors the minus sign in eq.(3.2) is replaced by a plus and the sum is on the other index of $\mathbf{N}^a{}_b$. If we require

$$\mathbf{D}^2 g_a = - g_b \mathbf{M}^b{}_a \qquad (3.4)$$

where

$$\mathbf{M}^a{}_b \equiv \frac{1}{2} M^a{}_{b\mu\nu}\, dx^\mu\, dx^\nu \qquad (3.5)$$

then

$$\mathbf{M}^a{}_b = \mathbf{d}\mathbf{N}^a{}_b - \mathbf{N}^a{}_c\mathbf{N}^c{}_b \qquad (3.6)$$

$\mathbf{M}^a{}_b$ obeys the Bianchi identity:

$$DM^a{}_b = 0. \tag{3.7}$$

It is a simple matter to show that the components of $M^a{}_b$ are given in terms of the usual curvature tensor constructed from $\Gamma^\alpha{}_{\beta\nu}$ by

$$M^{\alpha\beta}{}_{\mu\nu\rho\sigma} = 2\delta^{(\alpha}{}_{(\mu}R^{\beta)}{}_{\nu)\rho\sigma}\,. \tag{3.8}$$

This is manifestly metric-invariant.

Scaling and distortion parts of $M^a{}_b$ may be defined as before. We find that a gauge-invariant form of A_μ in d-dimensions is given by:

$$N^a{}_a = -\frac{d(d+1)}{2}\,[A - \frac{1}{d}\,d\ln\sqrt{-g}\,] \tag{3.9}$$

with $A = dx^\mu A_\mu$. The corresponding field strength is

$$M^a{}_a = -\frac{d(d+1)}{2}\,dA = -\frac{d(d+1)}{2}\,F\,. \tag{3.10}$$

The tracefree part of $N^a{}_b$ will depend only on the distortions. Therefore, let

$$S^a{}_b \equiv -N^a{}_b + \frac{2}{d(d+1)}\,\delta^a{}_b\,N^c{}_c\,. \tag{3.11}$$

For any particular metric $g_{\alpha\beta}$, $Q_{\alpha\beta\mu}$ may be written as a 1-form:

$$Q_a = D\,g_a \tag{3.12}$$

The distortion tensor of section 2 is then given by:

$$S_a \equiv Q_a - \frac{1}{d}\,g_a g^b Q_b \tag{3.13}$$

$S^a{}_b$ is related to S_a by:

$$g_b S^b{}_a = -S_a + dg_a - \frac{1}{d}\,g_a\,d\ln\sqrt{-g} \tag{3.14}$$

We shall now prove:

Theorem 2: The necessary and sufficient condition for the metric g_b to be connection-compatible is:

$$g_b M^b{}_a = 0. \tag{3.15}$$

Proof: The necessity of the condition eq.(3.15) is trivial, for if g_a is connection-compatible then

$$Q_a = D\,g_a = 0 \tag{3.16}$$

immediately implies

$$0 = D^2 g_a = -g_b M^b{}_a \tag{3.17}$$

The sufficiency of eq.(3.15) follows by explicit construction of the compatible metric. The metric we need is just that given by eq.(2.20), only now the expression must hold for arbitrary vectors v^α. To begin, note that $g_a(P)$ is an arbitrary symmetric matrix at the point P. Choose locally Lorentz coordinates so that $g_a(P) = \eta_a$. Because of the metric freedom, we can simply declare these coordinates to be lorentzian throughout the entire coordinate patch. Then our integrals are well-defined locally, and we can write:

$$g_a(\lambda) = \eta_a(0) + \int_0^\lambda g_b(\lambda')N^b{}_a(\lambda')\,d\lambda' \tag{3.18}$$

which when iterated leads to eqs.(2.20) and (2.21). What we require is that when eq.(3.18) is integrated around an arbitrary closed path, the metric is unchanged from its original value. Setting $g_a(\lambda) = g_a(0)$ in eq.(3.18) for such a path gives:

$$\oint_C N_a{}^b(\lambda')\,g_b(\lambda')\,d\lambda' = 0. \tag{3.19}$$

We may now apply Stoke's theorem. Letting $u^\alpha = dx^\alpha/d\lambda$, and remembering the definition of $N_a{}^b$, we find:

$$\iint_S [(N_a{}^b{}_\mu g_b)_{,\nu} - (N_a{}^b{}_\nu g_b)_{,\mu}]\,dx^\mu dx^\nu = 0. \tag{3.20}$$

The integrand must vanish since S is an arbitrary surface bounded by C. Because we require eq.(3.18) to give g_a along any path, we may use it to calculate $g_{b,\mu}$ in eq.(3.20). This gives

$$g_b(\lambda)[N^b{}_{a\mu,\nu} - N^b{}_{a\mu,\nu} + N^b{}_{c\mu} N^c{}_{a\nu} - N^b{}_{c\nu} N^c{}_{a\mu}] = 0. \qquad (3.21)$$

This is precisely $g_b M^b{}_a = 0$ where $M^b{}_a$ is given in terms of $N^b{}_{a\mu}$ by eq.(3.6).

We have shown that $g_b M^b{}_a = 0$ is a necessary and sufficient condition for the existence of a connection-compatible metric. If we do not require the metric to be invertible then this condition is equivalent to the vanishing of the curvature tensor. To see this, choose $\frac{d(d+1)}{2}$ independent basis vectors $g_a{}^i$ $(i = 1, 2, \ldots, \frac{d(d+1)}{2})$ spanning the space of symmetric matrices. If $g_b{}^i M^b{}_a = 0$ for each i then $M^b{}_a$ must vanish, and with it, $R^\alpha{}_{\beta\mu\upsilon}$. However, if we restrict consideration to invertible matrices as candidate metrics, there may be nontrivial solutions. In order to decide whether or not a given spacetime is connection-compatible we might have to compute $g_b M^b{}_a$ for every metric.

We can devise a simple test for nontrivial distortion that avoids this problem for the distortions that we wish to consider. If

$$\det [M^b{}_{a\mu\nu}] \neq 0 \qquad (3.22)$$

for any pair $[\mu\nu]$, the corresponding $M^a{}_{b\mu\nu}$ may be inverted, forcing $g_a = 0$. We then have nontrivial distortion. The determinant in eq.(3.22) refers to the $\frac{d(d+1)}{2} \times \frac{d(d+1)}{2}$ matrix labeled by the indices a and b. Note that if $\det[M^a{}_{b\mu\nu}]$ <u>does</u> vanish for all $\frac{d(d-1)}{2}$ values of $[\mu\nu]$ then a solution of eq.(3.15) for nonzero g_a may exist for nonvanishing $M^a{}_{b\mu\nu}$. Such an $M^a{}_{b\mu\nu}$ might not correspond to nontrivial distortion.

While this test does not provide all of the conditions necessary (26 in 4-dimensions) for the existence of a compatible metric it does provide a relatively simple set of sufficient conditions, enough for our purposes. To check that a given nonzero solution for the distortions is nontrivial, we need only verify that eq.(3.22) holds for some value of $[\mu\nu]$.

4. Lagrangian and Solutions.

We now return to 4-dimensions and consider lagrangian densities $\mathcal{L}$ which may be constructed purely from the metric-invariant field strength 2-form, $M^a{}_b$, its dual, $^*M^a{}_b$, and the metric g_a. It is desirable for the field equations to have trivial solutions since we know spacetime to be well modeled macroscopically by metric compatible spaces. We therefore require $\mathcal{L}$ to be quadratic in $M^a{}_b$. There are two natural choices:

$$\mathcal{L}_P = M^a{}_b M^b{}_a\,, \qquad (4.1)$$

$$\mathcal{L}_W = M^a{}_b\, {}^*M^b{}_a\,. \qquad (4.2)$$

The first of these is the Pontryagin characteristic density. It is not conformally invariant, violates parity, and gives a topological invariant when integrated. Its variation gives a pure divergence plus a term which vanishes identically due to the Bianchi identity, eq.(3.7).

$\mathcal{L}_W$ on the other hand, is conformally invariant, and does not violate parity. Its variation leads to Maxwell-like field equations.

We therefore use $\mathcal{L}_W$. Notice that it differs from the Weyl lagrangian[7],[8] in two respects. First, the square of the trace of $M^a{}_b$ is subtracted from the usual Weyl term. Secondly, and most importantly, the upper index on $M^a{}_b$ is contracted with the lower index on $^*M^a{}_b$ and vice-versa. This avoids unnecessary use of the metric, resulting in a simplification of the field equations. It also changes the conjugate momentum, which turns out to be crucial for correct quantization of the string solutions.

Notice that $\mathcal{L}_W$ depends on the metric. This dependence is necessary. Since all of the components of the distortion tensor are not independent we should not be able to solve independently for all 36 components of $S_{\alpha\beta\mu}$. Variation of the lagrangian with respect to the metric gives 10 equations. When these equations are regarded as constraints on the distortion tensor, and not equations for the metric, only the requisite 26 components of $S_{\alpha\beta\mu}$ remain independent.

The variation of $\mathcal{L}_W$ with respect to $N^a{}_{b\mu}$ and $g_{\mu\nu}$ straightforwardly leads to the following equations of motion:

$$D_\nu M^a{}_b{}^{\mu\nu} - 2A_\nu M^a{}_b{}^{\mu\nu} = 0. \qquad (4.3)$$

$$g^{\mu\nu} M^a{}_{b\mu\alpha} M^b{}_{a\nu\beta} - \frac{1}{4} g_{\alpha\beta} M^a{}_{b\mu\nu} M^b{}_a{}^{\mu\nu} = 0. \qquad (4.4)$$

The local conformal invariance of the lagrangian insures that the trace of eq. (4.4) is automatically satisfied. While $N^a{}_{b\mu}$ has been varied to produce eq.(4.3), varying either $Q_{\mu\alpha\beta}$ or $\Gamma^\alpha{}_{\beta\mu}$ leads to an equivalent result.

Note that $N^a{}_{b\mu}$ depends on the entire torsion-free connection and therefore contains all 40 degrees of freedom of $Q_{\alpha\beta\mu}$. We will seek a solution for all 40 components, separating out scaling and distortion parts only after a solution to the equations of motion is found.

Also note that eq.(4.4), from the metric variation, is not trivially solved by choosing any special metric. It is solved identically if $M^a{}_{b\mu\nu}$ vanishes, for any metric. This supports the claim that this set of equations be regarded as constraints on $M^a{}_{b\mu\nu}$ rather than equations for the metric.

In finding a solution to this system, we will make 3 assumptions:

1. We will assume that we are free to choose any metric which might reasonably arise as the macroscopic limit of a small-scale metric-averaged theory. In particular, it must be possible to find solutions with a conformally Minkowski metric, since we know (up to scale!) that the Mikowski metric is well approximated at large scales. It remains to be shown elsewhere that this metric actually can arise as the limit of some averaging process.

2. Suppose that in orthonormal coordinates, $\int N^a{}_{b\mu}dx^\mu << f$ along all timelike paths, for each value of a and b, and for some function f. Then a perturbation expansion is justified since the distortions can be made small even for long times by an appropriate conformal transformation (See eq.(2.21)). Keeping only terms quadratic or lower in $N^a{}_{b\mu}$ in the lagrangian, we find that eq.(4.4) remains the same, with $M^a{}_{b\mu\nu} = 2N^a{}_{b[\mu,\nu]}$, while eq. (4.3) becomes:

$$\partial_\nu N^a{}_b{}^{[\mu,\nu]} = 0. \tag{4.5}$$

3. The only further assumption that will be made is that we will seek solutions which keep as many as possible of the components of $N^a{}_{b\mu}$ independent. Such solutions will be called maximal. While there may be many non-maximal solutions, only this further property is required to lead us to strings. Only stringlike solutions have the property of decoupling all components of the distortion tensor from one another.

There is an important subtlety in the combination of conditions 1. and 3. above. It will do us no good to find a supposedly maximal solution having 26 independent functions if a change of

metric can remove some of those functions without adding others. One way to insure that our metric choice allows a truly maximal solution is to also require there to be some vector field w^μ such that $Q_{a\mu}w^\mu = 0$. That way we know by construction that $\eta_{\alpha\beta}$ is a 'good' choice of metric, in the sense that it is some observer's local reference frame. We will assume that the vector field w^μ is non-spacelike.

We now specify $g_{\alpha\beta} = \eta_{\alpha\beta}$ and substitute for $N^a{}_{b\mu}(x)$ in terms of its Fourier transform $N^a{}_{b\mu}(k)$. Eq.(4.5) then reads

$$k_\mu(N^a{}_b{}^\mu k^\nu - N^a{}_b{}^\nu k^\mu) = 0. \tag{4.6}$$

If we impose the condition $\partial_\mu N^a{}_b{}^\mu = 0$ in order to reduce eq.(4.6) to the wave equation, then we have already used many of the degrees of freedom of $N^a{}_{b\mu}$. However, if we do not, then $k^2 \neq 0$ and we can solve for the transform of $N^a{}_{b\mu}$ in terms of its divergence and k_μ:

$$N^a{}_{b\mu} = \frac{1}{k^2}(N^a{}_{b\nu}k^\nu)k_\mu = 0. \tag{4.7}$$

This decomposition reduces $N^a{}_{b\mu}$ to at most 14 independent functions. We therefore take $N^a{}_{b\mu}$ to be divergence free on the final index and k_μ to be null, and seek to satisfy the constraint equation.

Consider Fourier expansion of the first term of eq.(4.4):

$$\int d^4k\ \delta(k^2)\int d^4m\ \delta(m^2)\ [N^a{}_{b\mu}k_\nu - N^a{}_{b\nu}k_\mu][N^b{}_a{}^\mu m^\nu - N^b{}_a{}^\nu m^\mu]\ e^{i(k+m)\cdot x}. \tag{4.8}$$

There will be nontrivial constraints introduced unless this can be substantially simplified. We must make use of the constraint we already have,

$$k_\mu N^b{}_a{}^\mu(k^\nu) = 0, \tag{4.9}$$

but the expression (4.8) includes instead terms like

$$k_\mu N^b{}_a{}^\mu(m^\nu) = 0, \tag{4.10}$$

Also, there is a term containing

$$N^a{}_{b\mu}(k^\sigma)\,N^b{}_{a\nu}(m^\rho)\,k\cdot m \tag{4.11}$$

which will give 10 new relations among the $N^a{}_{b\mu}$ unless $k_\mu m^\mu = 0$. Only if k^μ is parallel to m^μ will (4.10) and (4.11) vanish without constraining $N^a{}_{b\mu}$. It is this reduction of the allowed momentum space required by the metric-constraint equation that forces this solution to depend on one coordinate instead of all four, making the solution stringlike.

We can now proceed to solve eq.(4.5). The restriction of the momentum space can achieved by letting each component of $N^a{}_{b\mu}$ depend only on the amplitude of a single uniform wave. Let $\varphi(x^\mu)$ be any function with nonvanishing gradient, and let

$$N^a{}_{b\mu} = N^a{}_{b\mu}(\varphi). \tag{4.12}$$

Then $\varphi(x^\mu)$ must satisfy

$$\partial_\nu \partial^\nu \varphi = \partial_\nu \varphi \partial^\nu \varphi = 0, \tag{4.13}$$

and $N^a{}_{b\mu}$ is restricted by

$$N^a{}_{b\mu}{}' \varphi^{,\mu} = 0, \tag{4.14}$$

where the prime (') denotes differentiation with respect to φ. Substituting into eq.(4.4) now requires only one condition:

$$N^a{}_{b\mu}{}' N^b{}_a{}^{\mu}{}' = 0. \tag{4.15}$$

Finally, the conformal factor may be chosen so that:

$$N^a{}_{b\mu} N^b{}_a{}^\mu = 1. \tag{4.16}$$

Now consider the restriction of the solution above to each of the separate pieces of $N^a{}_{b\mu}$. Eqs.(4.14) and (4.15) together imply that $N^a{}_{a\mu}$ is the gradient of $A(\varphi)$, where A is an arbitrary function of φ. Therefore, $M^a{}_{a\mu\nu} = -10(A_{\mu,\nu} - A_{\nu,\mu}) = 0$. In particular, we can identify A with φ, since φ was arbitrary to begin with. Therefore, $N^a{}_{b\mu}$ reduces essentially to $S^a{}_{b\mu}$, which now satisfies:

$$S^a{}_{b\mu}{}' n^\mu = 0 \qquad S^a{}_{a\mu} = 0 \qquad S^a{}_{b\mu} S^b{}_a{}^\mu = 1 \tag{4.17}$$

and

$$S^a{}_{b\mu}{}' S^b{}_a{}^{\mu}{}' = 0, \tag{4.18}$$

It is easy to show that the field strength $M^a{}_{b\mu\nu}$ calculated from this solution satisfies eq.(3.22). These therefore represent true distortions.

Eqs.(4.17) leave 26 of the components of $S^a{}_{b\mu}$ arbitrary. Each component satisfies the 4-dimensional wave equation, and the components are together subject to the constraint, eq.(4.18). This is easily recognized as being related to the form of the solution to the classical string equations in the critical number of dimensions, once it is realized that the quadratic constraint (4.18) corresponds to the usual orthonormality condition on the string coordinates.

Before drawing out the exact correspondence with the string, we must apply the maximum criterion to the condition

$$Q_{a\mu} w^\mu = 0 \tag{4.19}$$

where w^μ is non-spacelike. Notice that this condition already holds for the solution above, eqs.(4.12) - (4.16). If we let $\varphi^{,\mu}$ be a constant null vector then integration of eq.(4.14) leads to

$$N^a{}_{b\mu} + h^a{}_{b\mu}(\bar{\varphi}, y, z) \tag{4.20}$$

so that if we take $h^a{}_{b\mu} = 0$ then

$$N^a{}_{b\mu} \varphi^{,\mu} = 0, \tag{4.21}$$

and it follows that

$$Q_{b\mu} \varphi^{,\mu} = \eta_a N^a{}_{b\mu} \varphi^{,\mu} = 0. \tag{4.22}$$

However, this choice is not maximal. To find the maximal solution, notice that $M^a{}_{b\mu\nu}$ and the equations of motion are invariant to lowest order under changes in $N^a{}_{b\mu}$ of the form:

$$N^a{}_{b\mu} \longrightarrow \hat{N}^a{}_{b\mu} \equiv N^a{}_{b\mu} + h^a{}_{b,\mu}. \tag{4.23}$$

If $h^a{}_b$ is of the form $\delta^{(\alpha}{}_{(\beta} h^{\mu)}{}_{\nu)}$ then this corresponds to a metric gauge transformation. The particular gauge choices of our solution:

$$\partial^\mu \hat{N}^a{}_{b\mu} = 0 \tag{4.24}$$

and

$$\partial_\mu \partial^\mu \hat{N}^a{}_{b\mu} = 0 \tag{4.25}$$

are also preserved as long as $\partial_\alpha \partial^\alpha h^a{}_b = 0$. With our choice of $\eta_{\mu\nu}$ as the metric, we have given up our freedom to remove such an additional term. Therefore we have to consider what form of

h^a_b satisfies eq.(4.19) while introducing the maximal number of new fields.

For simplicity let φ to be a simple null coordinate, $\varphi = x + t$, and let $\bar{\varphi} = x - t$. Then in general the solution is of the form:

$$\hat{N}^a{}_{b\mu} = N^a{}_{b\mu}(\varphi) + h^a{}_{b,\mu}(\varphi,\bar{\varphi}, y, z) \tag{4.26}$$

Let us assume that the vector field w^μ is constant. Then it may be decomposed as:

$$w^\mu = \alpha\varphi^{,\mu} + \beta\bar{\varphi}^{,\mu} + s^\mu \tag{4.27}$$

where α and β are constants and s^μ is a constant vector lying in the yz-plane. After separation of variables, and assuming eq.(4.21), eq.(4.19) becomes the pair of equations:

$$\beta\bar{\varphi}^{,\mu}Q_{b\mu}(\varphi) + s^\mu Q_{b\mu}(\varphi) + 2\beta\frac{\partial h_b}{\partial\varphi} = 0 \tag{4.28}$$

$$2\alpha\frac{\partial h_b}{\partial\varphi} + h_{b,\mu}s^\mu = 0 \tag{4.29}$$

Eq.(4.28) implies

$$h_b = \int[\beta(Q_{b0} + Q_{b1}) + s^\mu\ Q_{b\mu}]\ d\varphi\ +\ \bar{h}_b(\bar{\varphi}, y, z). \tag{4.30}$$

There are no further constraints on h_b if α and s^μ are taken to vanish, so that

$$\bar{\varphi}^{\mu}\hat{Q}_{b\mu} = 0 \tag{4.31}$$

Here $\hat{Q}_{b\mu} \equiv \eta_a \hat{N}^a{}_{b\mu} = Q_{b\mu}(\varphi) + h_{b,\mu}$. Taking $\beta = 0$ instead of $\alpha = 0$ forces

$$\frac{\partial h_b}{\partial\bar{\varphi}} = 0, \tag{4.32}$$

which is not as general.

Finally, h_b must solve the wave equation:

$$-\frac{\partial^2 h_b}{\partial\varphi\partial\bar{\varphi}} + \frac{\partial^2 h_b}{\partial y^2} + \frac{\partial^2 h_b}{\partial z^2} = 0. \tag{4.33}$$

The first term vanishes leaving an equation with only exponentially divergent solutions. To maintain finiteness at infinity, h_b must be a function of $\bar{\varphi}$ alone. Since we desire h_b to be in the form of a metric-gauge transformation, this determines h^a_b entirely. The solution for $\hat{N}^a{}_{b\mu}$ may be written as:

$$\hat{N}^a{}_{b\mu} = N^a{}_{b\mu}(\varphi) + \partial_\mu h^a{}_b(\bar{\varphi}) - \partial_\mu\int(N^a{}_{b0} + N^a{}_{b1})d\varphi \tag{4.34}$$

and satisfies eq.(4.31).

The result is that, assuming reasonable boundary conditions, *the maximal allowed form of* $\hat{N}^a{}_{b\mu}$ *has 10 right-moving fields and 26 left-moving fields on a 2-dimensional subspace. This is precisely the bosonic sector of the heterotic string.*

5. Comparison with the heterotic string.

Let us make the comparison with the string explicit. We begin with a brief review of the bosonic string. The standard bosonic string is described[1] by 26 functions X^A $(A = 1, 2, \ldots, 26)$ of two variables, $\xi^i = (\sigma,\tau)$ satisfying:

$$\partial_i\partial^i X^A = 0. \tag{5.1}$$

We also have the orthonormality condition:

$$(\partial_\sigma X^A + \partial_\tau X^A)^2 = 0. \tag{5.2}$$

Requiring the usual end conditions $X'^A(\sigma=0, \tau) = X'^A(\sigma=\pi, \tau) = 0$, the wave equation restricts X^A to the form:

$$X^A(\sigma,\tau) = X_0{}^A + a^A\tau + \sum_{n=1}^{n=\infty}(a_{-n}{}^A e^{in\tau} + a_n{}^A e^{-in\tau})\cos n\sigma \tag{5.3}$$

The orthonormality condition (5.2) becomes a function of $(\sigma + \tau)$ only:

$$[\sum_{n=-\infty}^{n=\infty} n\, a_n{}^A e^{-in(\sigma+\tau)}]^2 = 0. \tag{5.4}$$

This is the form of the constraint regardless of the boundary conditions.

The combination $\partial_\tau X^A + \partial_\sigma X^A$ is the Fubini-Veniziano [10] momentum $P^A(z)$, evaluated on the unit circle, $z = e^{i(\sigma+\tau)}$. Its conjugate momentum $Q^A(z)$, when evaluated on the unit circle, is also a function of $(\sigma+\tau)$ only.

$$Q^A(z) = \sum_{n=-\infty}^{n=\infty} i a_n^A z^{-n} \tag{5.5}$$

$$P^A(z) = iz\frac{dQ^A}{dz} = \sum_{n=-\infty}^{n=\infty} n\, a_n^A z^{-n} \tag{5.6}$$

These variables are conjugate when the bosonic string is quantized and the constraint of eq.(5.2) may be expressed in terms of the operators

$$L_n = -\frac{1}{2}\int\frac{dz}{2\pi i z} z^n :P(z)^2: \tag{5.8}$$

where : : denotes normal ordering.

The heterotic string is composed of the left-moving sector of a bosonic string, compactified to 10-dimensions then combined with the right-moving sector of a 10-dimensional superstring. To simplify the comparison, we will examine only the relationship between the left-moving sector of $N^b{}_{a\mu}$ and the left-moving sector of the bosonic string. The right-moving sectors are similarly related. The left-moving sector is completely characterized by the Fubini-Veneziano variables Q^A and P^A. The right-moving sector has its own set of variables built from $\partial_\tau X^A - \partial_\sigma X^A$.

First, identify φ with $\sigma+\tau$. Then it seems natural to identify the independent components of $N^b{}_{a\mu}$ with Q^A, but this is not quite right. Because of the staggering of contracted indices in the lagrangian, the cannonical momentum conjugate to $N^a{}_{b\mu}$ is:

$$\Pi_a{}^b{}_\mu = -N^b{}_{a\mu,0} + N^b{}_{a0,\mu} \tag{5.9}$$

so that the Poisson brackets become:

$$\{N^a{}_{b\mu}, \Pi_d{}^c{}_\nu\} = \delta^a{}_d\delta^c{}_b\eta_{\mu\nu}. \tag{5.10}$$

It is the crossing of the indices that yields the positive definite Kroniker δ's here, instead of products of $\eta_{\alpha\beta}$. This means that the quantization will proceed without introducing extra ghosts. But it also means that the conjugate pairs are

$$(N^a{}_{b\mu};\ \Pi_a{}^b{}_\mu) \tag{5.11}$$

instead of

$$(N^a{}_{b\mu};\ \Pi_b{}^a{}_\mu). \tag{5.12}$$

Since the Fubini-Veneziano pairs are $(Q^A; P^A)$ we must identify the various Q^A with sums or differences of the $N^a{}_{b\mu}$, for example:

$$Q^A \Leftrightarrow \frac{1}{\sqrt{2}}(N^a{}_{b\mu} \pm N^b{}_{a\mu});\quad P^A \Leftrightarrow \frac{1}{\sqrt{2}}(\Pi^a{}_{b\mu} \pm \Pi^b{}_{a\mu}). \tag{5.13}$$

The actual numbering of the variables is of course unimportant, as long as only independent $N^a{}_{b\mu}$ and $\Pi^a{}_{b\mu}$ are included. One possible identification is detailed in the Appendix.

It is also important to notice that the negative brackets that arise in eq.(5.10) because of the remaining $\eta_{\mu\nu}$ are dependent because of the gauge condition, eq.(4.21) or eq.(4.31). The requirement that w^μ in eq.(4.31) be non-spacelike is sufficient to guarantee that $N^a{}_{b0}$ can be expressed in terms of $N^a{}_{bi}$:

$$N^a{}_{b0} = \frac{w^i}{w^0} N^a{}_{bi} \tag{5.14}$$

thereby expressing the potential negative norm states in terms of positive norm states.

We end this section with some comments on the interpretation of the solution. This is to be regarded as a theory of distortions in 4-dimensional spacetime. The fact that it can be interpreted as the heterotic string follows from the existence of 26 distortion fields and 10 metric-gauge fields. The space in which this 'string' moves is simply the solution space of these fields, and not the actual spacetime in which we live. The solution for the 'motion' of the string in the space of physical distortions gives us the values of these fields, and therefore gives us a solution for the 4-dimensional spacetime that these fields describe.

This interpretation provides a neat separation of the left and right movers of the heterotic string. In the standard interpretation of the string the 'heteroticity' is achieved by pasting together the left-moving modes of a bosonic string with the right-moving modes from a superstring. While this produces quite satisfactory results and is perfectly reasonable mathematically, it is conceptually a bit bizarre. In the present model, however, the left-movers are intrinsically related to the connection while the right-moving fields are linked to the metric. Presumably, the supersymmetry of the heterotic string can be achieved here by supersymmetrizing the metric only. The connection may remain unaltered, so that only the 10 metric-gauge fields have superpartners.

6. Summary and Observations.

We regard motion in spacetime as a fundamental physical property, and the measurement of that motion to be derivative. The real influence of nature is to determine the dynamics of fields (hence, the connection). The measurement of those fields must be formulated in a way that is not only independent of how we choose coordinates, but of how we choose to measure in a deeper sense -- it must be independent of certain choices of metric.

It may seem at first odd to speak of a freedom to choose the metric when there is plentiful macroscopic evidence for metric-connection compatibility. Compatibility leads us to a preferred metric. But this is like choosing a static coordinate system on the rotating earth -- it makes sense only so long as the effects of the choice are unobservable. Ultimately, we have no reason to assume that a preferred microscopic metric exists.

If spacetime has no compatible metric at small scales, then objects following different paths will distort relative to one another. In such a world, the freedom to select the metric arbitrarily is manifest. No choice will give universal correspondence with measurements. The best that one can do is to choose a metric that is covariantly constant along one's own world line. This is the *principal of relativity of distortion*. It parallels the statement that one can always work in the rest frame in special relativity, or that one can always pick a local Lorentz frame in general relativity. The principal implies that there are no more than 26 independent components to the distortion tensor in any conformally invariant theory.

If we adopt the requirement that metric compatible spaces should be trivial solutions to our theory, then we can easily recover the metric of macroscopic spacetime as an emergent property. Any metric-connection compatible spacetime will be an exact solution to the theory. This requirement leads us to write the quadratic lagrangian of eq.(4.2). At large scales, to the degree that noncompatible small-scale oscillations can average out, there will be a reasonably effective average measure on the spacetime. This is the same phenomenon that occurs with the averaging out of any net electromagnetic charge, or the modeling of the universe as homogeneous.

The maximal solution to the field equations resulting from eq.(4.2) is equivalent to the bosonic sector of the heterotic string, that is, 26 left-moving fields and 10 right-moving fields. This model has the advantage over standard string models of not requiring compactification. It already directly describes a 4-dimensional spacetime.

The correct quantization of the distortions follows from their relationship to the string. Linear combinations of the $S^a{}_{b\mu}$ and of the $S'^a{}_{b\mu}$ correspond to the conjugate variables $Q^A(e^{i\phi})$ and $P^A(e^{i\phi})$, and obey cannonical commutation relations. A full development of the quantization of this system will be presented in a subsequent paper.

Finally, we remark about the critical dimension. The number of independent distortions depends on the dimension of the original spacetime. In d-dimensions, the number of independent components of $S^a{}_{b\mu}$ is $d(d^2-3)/2$. If 26 degrees of freedom are really required for consistent quantization of this system, then the quantization will only work in 4-dimensions. Therefore we have two possibilities:

1. The theory of distortions will produce consistent quantum theories with critical number different from 26 when correctly quantized in other dimensions. Such theories need not be string theories.

2. The quantization is inconsistent in other dimensions.

The first possibility predicts a new class of consistent quantum theories if we can find the correct means of quantizing them. If the second possibility holds instead, it provides a strong argument that spacetime must be 4 dimensional. Either outcome will therefore prove interesting.

I would like to thank Harry Braden, Frank Cuiper, Bernard Kay, Yukio Kikuchi, Caren Marzban, Jack Ng, Marcello Ubriaco, Bernard Whiting, and Jim York for discussions. This research was supported in part by U. S. Department of Energy grant number DE-FG05-85ER-40219.

Appendix

Here a specific mapping between distortion and string fields is presented in detail. The geometric variables $N^a{}_{b\mu}$ are defned by

$$N^{\alpha\beta}{}_{\mu\nu\rho} \equiv 2\delta^{(\alpha}{}_{(\mu}\Gamma^{\beta)}{}_{\nu)\rho}. \qquad (A.1)$$

and will be taken to satisfy the constraints:

$$0 = \eta_a N^a{}_{b\mu}\varphi^{,\mu} = (-N^1{}_{b\mu} + N^5{}_{b\mu} + N^8{}_{b\mu} + N^{10}{}_{b\mu})\varphi^{-,\mu} \qquad (A.2)$$

$$0 = \eta^b Q_{b\mu} = \eta_a\eta^b N^a{}_{b\mu} = (-N^1{}_{1\mu} + N^5{}_{5\mu} + N^8{}_{8\mu} + N^{10}{}_{10\mu}) \qquad (A.3)$$

Here we have adopted the following index convention for the 10-dimensional indices.

For a - ($\alpha\beta$):				
	1 - 00	2 - 01	3 - 02	4 - 03
	5 - 11	6 - 12	7 - 13	8 - 22
	9 - 23	10 - 33		

The constraints suggest the variables

$$K_{b\mu} \equiv \eta_a N^a{}_{b\mu} \qquad (A.4)$$

with $K_{b0} = K_{b1}$ and $K_{1\mu} = K_{5\mu} + K_{8\mu} + K_{10\mu}$. This choice leads to the following combinations of the $N^a{}_{b\mu}$ as independent variables:

$$\begin{array}{ccc} -N^1{}_{2i} + N^5{}_{2i} & -N^1{}_{3i} + N^8{}_{3i} & N^5{}_{5i} \\ -N^1{}_{4i} + N^{10}{}_{4i} & N^5{}_{6i} + N^8{}_{6i} & N^8{}_{8i} \\ N^5{}_{7i} + N^{10}{}_{7i} & N^8{}_{9i} + N^{10}{}_{9i} & N^{10}{}_{10i} \end{array} \qquad (A.5)$$

where i = 1, 2, 3. The problem with this set is that the conjugate variables are not included, since for example the conjugate of $N^1{}_{2i}$ is $\Pi_1{}^2{}_i = \partial_t N^2{}_{1i}$. Fortunately, there are relations among the $N^1{}_{2i}$, such as $N^5{}_{2i} = N^2{}_{1i}$ of exactly the form needed so that we can instead choose:

$$\begin{array}{ccc} -N^1{}_{2i} + N^2{}_{1i} & -N^1{}_{3i} + N^3{}_{1i} & N^5{}_{5i} \\ -N^1{}_{4i} + N^4{}_{1i} & N^5{}_{6i} + N^6{}_{5i} & N^8{}_{8i} \\ N^5{}_{7i} + N^7{}_{5i} & N^8{}_{9i} + N^9{}_{8i} & N^{10}{}_{10i} \end{array} \qquad (A.6)$$

as a set of 27 variables which are independent modulo conformal transformation. These variables may be identified with the Fubini-Veneziano variables Q^A since if we let, eg.,

$$Q^1 \equiv \frac{1}{\sqrt{2}}\,(N^5{}_{62} + N^6{}_{52}) \qquad (A.7)$$

then the conjugate momentum is

$$P^1 = \frac{1}{\sqrt{2}}\,(\dot{N}^6{}_{52} + \dot{N}^5{}_{62}) = \frac{1}{\sqrt{2}}\,(\dot{N}^5{}_{62} + \dot{N}^6{}_{52}) = \dot{Q}^1, \qquad (A.8)$$

where the dot denotes a time derivative. The asymmetry in the pairings has been overcome. For the difference combinations we still have $\{Q^1, P^1\} = 1$, but now $P^1 = -\dot{Q}^1$. While $P = -\dot{Q}$ is usually an indication of a non-positive definite metric, the crossing of the indices preserves positive definite Poisson brackets.

The Minkowski bracket of bosonic string theory

$$\{Q^A, P^B\} = \eta^{AB} \qquad (A.9)$$

can be matched by taking $N^5{}_{60} + N^6{}_{50}$ instead of $N^5{}_{61} + N^6{}_{51}$ (or any other change of a final '1' index to a final '0' index) as an independent variable. This makes use of the η_{00} term in eq.(5.10) to produce one negative bracket.

References

[1] Green, M. B., Schwarz, J. H., Witten, E., Superstring Theory, Vol. 1, Cambridge University Press, Cambridge, (1987).

Scherk, J., 'An introduction to the theory of dual models and strings', Rev. Mod. Phys. **47**, (1975) 123.

[2] Candelas, P., Horowitz, G., Strominger, A., Witten, E., 'Vacuum configurations for superstrings', Nuc. Phys. **B258**, (1985)46.

[3] Dixon, L., Harvey, J., Vafa, C. and Witten, E., 'Strings on orbifolds', Nuc. Phys. **B261** (1985) 678.

[4] Freund, P. G. O. and Oh, P., 'Cosmological solutions with "ten into four" compactification', Nuc. Phys. **B255** (1985) 688.

Freund, P. G. O., Oh, P. and Wheeler, J. T., 'String-induced space compactification', Nuc. Phys. **B246** (1984) 371.

Lüst, D., 'Compactification of ten-dimensional superstring theories over Ricci flat coset spaces', Nuc. Phys., (1986).

[5] Gross, D., Harvey, J.A., Martinec, E. and Rohm, R., 'Heterotic string theory (I). The free heterotic string' Nuc. Phys. **B256**(1985) 253.

Gross, D., Harvey, J.A., Martinec, E. and Rohm, R., 'Heterotic string theory (II). The interacting heterotic string' Nuc. Phys. **B267**(1986) 75.

[6] Englert, F., Nicolai, H. and Schellekens, A., 'Superstrings from 26 dimensions', Nuc. Phys. **B274** (1986) 315.

Kawai, T., Lewellen, D. C. and Tye, H., Phys. Rev. Lett. **57** (1987).

Lerche, W., Lust, D., Schellekens, A. N., 'Chiral Four-dimensional Heterotic Strings from Self-dual Lattices', CERN-TH 4590/86, Nov. 1986.

Narain, K. S., 'New heterotic string theories in uncompactified dimensions < 10', Phys. Lett. **169B** (1986) 41.

[7] Dell, J., Metric and Connection in Einstein and Yang-Mills Theory, Phd Dissertation, University of Maryland, 1981.

Smolin, L., Nuc. Phys. B160, 253-268 (1979).

Wheeler, J. T., 26-dimensional String from 4-dimensional Geometry, preprint IFP-286-UNC, University of North Carolina at Chapel Hill, 1987.

[8] Weyl, H., The Principles of Relativity, Chapter IX, Dover Publications Inc., New York, 1922; originally: Sitz. Ber. preuss. Akad. Wiss. (1918) 465.

Weyl, H., Space Time Matter, Chapts II & IV, Dover Publications Inc., New York, 1952. Originally: Weyl, H., Raum-Zeit-Materie, (3rd Ed.,1920), Chapts II & IV, §§34+35, p.242,et seq.

[9] Audretsch, J., Gahler, F. and Straumann, N., Wave Fields in Weyl Spaces and Conditions for the Existence of a Preferred Pseudo-Riemannian Structure, Commun. Math. Phys. 95, 41-51 (1984).

Dirac, P. A. M., Proc. Royal Society A333 (1973), 403.

Einstein, A., S.B. preuss. Akad. Wiss. (1918) 478, including Weyl's reply.

Pauli, W., "Relativitätstheorie", Encykl. D. Math. Wiss. 2 (1921), 759. Translation to English: Pauli, W. , Theory of Relativity, G. Field (transl.), pp.104-285, Dover Publications, Inc., New York, 1981. (Originally, Pergammon Press, 1958).

Pauli, W., Phys. Zeits, 20, p 457, (1919).

Utiyama, R., On Weyl's Gauge Field, Prog. Theor. Physics (Japan) **50**, #6,(1973) 2080.

Utiyama, R., On Weyl's Gauge Field II, Prog. Theor. Physics (Japan) **53**, #2,(1975) 565.

[10] Fubini, S. and Veneziano, G., 'Level structure of dual-resonance models', Nuovo Cim. **64A**, (1969) 811.

Fubini, S. and Veneziano, G., 'Duality in operator formalism', Nuovo Cim. **67A**, (1970)