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## SYMMETRIC SOLUTIONS TO THE MAXIMALLY GAUSS-BONNET EXTENDED EINSTEIN EQUATIONS

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### ABSTRACT

The most general gravitational lagrangian which can be constructed from the curvature two-form, the vielbein one-form, and tensors invariant in the tangent space is a linear combination of dimensionally extended Euler characteristics. Several recent studies indicate that superstring lagrangians include such terms. In an arbitrary number of dimensions, with arbitrary torsion, we show that in the most general such extended theory the only static, spherically symmetric, massive solutions to the variational equations of motion contain gravitational singularities. The existence of an event horizon is proved for certain cases, and a bound on the location of singularities is found. A certain class of non-asymptotically flat solutions is found for which the metric and the torsion are not entirely determined by the field equations.

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## 1. Introduction

Consider the class of all densities of the form<sup>1</sup>

$$\mathbf{E}_k = \mathbf{R}^{ab}\mathbf{R}^{cd} \dots \mathbf{R}^{efg} \dots \mathbf{e}^h \epsilon_{abcd \dots efg \dots h} \quad (1)$$

If there are no factors of the vielbein, these densities become the total divergences whose integrals are proportional to the Euler characteristic. When there are one or more vielbein forms present, they are known as dimensionally extended Euler characteristic densities. Such densities are no longer total derivatives, and may therefore serve as lagrangian densities in gravitational field theories. Several properties suggest them strongly for this role:

1. The first order density,  $\mathbf{E}_1$ , is the usual Einstein lagrangian, while the lowest order density,  $\mathbf{E}_0$ , gives rise to a cosmological constant. Theories involving  $\mathbf{E}_k$  therefore make close contact with the usual theory of gravity.
2.  $\mathbf{E}_k$  is the only parity conserving lagrangian constructible from  $k$  factors of the curvature two-form, the vielbein one-form, and invariant tensors in the tangent space.
3. All of the densities  $\mathbf{E}_k$  are free of ghosts ([1], [2]), ie., there are no negative norm eigenstates in the quantized theory. All are divergences to lowest order in perturbation theory.

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<sup>1</sup>  $\mathbf{R}^{ab}$  is the curvature 2-form,  $\mathbf{e}^a$  is the vielbein 1-form and  $\epsilon_{a \dots b}$  is the Levi-Civita tensor.  $\mathbf{E}_k$  contains  $k$  factors of  $\mathbf{R}^{ab}$  and enough factors of the vielbein to saturate the remaining indices on the Levi-Civita tensor. The wedge product is assumed between forms.

4.  $\mathcal{L}_2$  arises as the order  $\alpha'$  correction in the low-energy expansion of string models ([1], [3], [4]). It is likely that higher order characteristics also occur in string theory.<sup>2</sup>

We therefore consider the most general lagrangian composed of extended Euler densities

$$\mathcal{L} = \sum_{k=0}^{k_m} \bar{c}_k \mathcal{L}_k, \quad (2)$$

where  $k_m$  is the integer part of  $(D-2)/2$  and the  $\bar{c}_k$  are arbitrary constants. The series terminates at  $k_m$  since  $\mathcal{L}_k$  becomes a total divergence for  $2k = D$  and vanishes for  $2k > D$ . For generality the cosmological term  $c_0$  is included.

Solutions to system with  $k_m = 2$  have been studied previously. Static, spherically symmetric solutions have been found by Wheeler [5] and by Boulware and Deser [6]. Boulware and Deser also studied the stability of the two classes of solution which arise due to the quadratic nature of this case. Cosmological solutions have been studied by Müller-Hossein [7] and by Wheeler [5].

In this work, solutions are found for arbitrary  $k_m \leq D$ , in an arbitrary finite number of dimensions. The possibility of nonvanishing torsion is included. It is shown that the only static, spherically symmetric, asymptotically flat solutions to the variational equations of motion contain gravitational singularities.

In section II below, the full lagrangian  $\mathcal{L}$  (eq.(2)) is varied and the resulting equations of motion are evaluated for the case of a static, spherically symmetric vacuum. The main results of the paper are proved in

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<sup>2</sup> Based on presence of  $\mathcal{L}_1$  and  $\mathcal{L}_2$  in string theories, Zwiebach [1] conjectured that the gravitational part of the full string lagrangian is composed entirely of dimensionally extended Euler densities. However, as Zumino [2] has pointed out, the three-graviton vertex includes a  $k^6$  term which cannot arise from these densities. Nonetheless, Romans and Warner show that partial supersymmetrization of the Lorentz Chern-Simons term requires  $\mathcal{L}_2$  in the purely gravitational sector and suggest that full supersymmetrization may involve  $\mathcal{L}_3$ ,  $\mathcal{L}_4$ ,  $\mathcal{L}_5$  and conceivably no further terms. If this is true, then these densities may form a substantial part of the string lagrangian, and an understanding of their contribution to the nature of solutions may be helpful in finding solutions to the full theory.

section III. The full asymptotically flat solution, represented as a power series, is derived and shown to be singular in all cases. The existence of an event horizon is proved for certain cases, and a bound on the location of singularities is found. In section IV, special values of the coefficients  $c_k$  are chosen for which there exist further, non-asymptotically flat solutions. Some of these solutions are shown to be geometrically free in the sense that the metric and the torsion are not entirely determined by the field equations. The generalization of this special case to any spacetime with a maximally symmetric subspace is also discussed. The final section includes a summary of these results.

## II. Static, Spherically Symmetric Equations of Motion

We wish to vary (see also Zumino, [2]) both the vielbein  $e^a$  and the connection  $\omega_a{}^b$  in the action integral of the lagrangian density

$$\mathbf{E} = \sum_{k=0}^{k_m} \bar{c}_k \mathbf{R}^{ab} \dots \mathbf{R}^{cdefeg} \dots e^h \epsilon_{ab\dots cd fg\dots h} \quad (3)$$

Since  $\mathbf{R}^{ab} = d\omega_a{}^b + \omega_a{}^c \omega_c{}^b$ , the connection variation of  $\mathbf{E}$ ,  $\delta_\omega \mathbf{E}$ , gives

$$\delta_\omega \mathbf{E} = \sum \bar{c}_k k (d\delta\omega^{ab} + 2\omega^{am}\delta\omega_m{}^b) \mathbf{R}^{cd} \dots \mathbf{R}^{efeg} \dots e^h \epsilon_{abcd\dots efg\dots h}$$

We integrate the first term by parts, while the identity (A1.1) derived in Appendix 1 is applied to the second term. Use of the Bianchi identity  $D\mathbf{R}^{cd} = 0$  then reduces the connection variation to

$$\delta_\omega \mathbf{E} = \sum (-)^k c_k k \delta\omega^{ab} \mathbf{T}^c \mathbf{R}^{cd} \dots \mathbf{R}^{efeg} \dots e^h \epsilon_{abcd\dots efg\dots h}$$

where  $c_k \equiv \bar{c}_k (-)^k (D-2k)$ , and  $\mathbf{T}^a \equiv D e^a$  is the torsion 2-form. This expression contains  $k-1$  factors of  $\mathbf{R}^{ab}$  and  $D-2k-1$  factors of  $e^a$ .

The vielbein variation,  $\delta_e \mathbf{E}$ , of  $\mathbf{E}$  is straightforward since  $\mathbf{R}^{ab}$  is independent of the vielbein.

Vanishing of the variation of the action therefore implies

$$0 = \sum C_k (-)^k \delta e^a \mathbf{R}^{bc} \dots \mathbf{R}^{def} \dots e^g \epsilon_{abcdef\dots g} \quad (4)$$

$$0 = \sum C_k (-)^k \delta \omega^{ab} \mathbf{T}^c \mathbf{R}^{de} \dots \mathbf{R}^{fgh} \dots e^i \epsilon_{abcde\dots fgh\dots i} \quad (5)$$

Equations (4) and (5) are the full field equations for the system.

Next, equations (4) and (5) are evaluated for a static, spherically symmetric spacetime. Latin indices on forms are tangent space indices, while greek indices refer to coordinates on the manifold, with the exception that  $dx^0 = dt$  and  $dx^r = dr$  refer to the manifold. Indices from the beginning of either alphabet (a,b, ... or  $\alpha, \beta, \dots$ ) run over values  $1, 2, \dots, D$ , while indices from the middle of either alphabet (i,j,k, ... or  $\mu, \nu, \dots$ ) other than 0 and r take only the values  $1, 2, \dots, d$ , where  $d = D-2$  is the dimension of the maximally symmetric subspace.

For a static, spherically symmetric spacetime, the metric tensor may be written in the form

$$g_{\alpha\beta} = \begin{pmatrix} -g^2 & & \\ & h^{-2} & \\ & & r^2(\delta_{\mu\nu} + x_\mu x_\nu / (1-x^2)) \end{pmatrix} \quad (6)$$

where  $x^2 = x^\mu x_\mu$  and  $x^\mu = x_\mu$ . The components of the vielbein one-form are therefore

$$e^0 = g dt \quad (6a)$$

$$e^r = h^{-1} dr \quad (6b)$$

$$e^i = \delta_\mu^i \{ r dx^\mu - r(1 + (1-x^2)^{-1/2} x^\mu x_\nu dx^\nu / x^2) \}. \quad (6c)$$

In the same coordinate system the static, spherically symmetric torsion two form will be (see Appendix 2):

$$\mathbf{T}^0 = T_{or}{}^0(r) dt e^r \quad (7a)$$

$$\mathbf{T}^r = T_{or}{}^r(r) dt e^r \quad (7b)$$

$$\mathbf{T}^i = (T_o(r) dt + T_r(r) dr) e^i \quad (7c)$$

where

$$T_o = T_{o\mu}{}^\mu(r) \quad (7d)$$

$$T_r = T_{r\mu}{}^\mu(r) \quad (7e)$$

Finally, the nonvanishing curvature two-forms are given by

$$\mathbf{R}^{ij} = (1/2)R_{\mu\nu}{}^{ij} dx^\mu dx^\nu = -F(r)e^i e^j \quad (8a)$$

$$\mathbf{R}^{oi} = R_{o\mu}{}^{oi} dt dx^\mu = (g^{-1}g'h^2/r)e^o e^i \quad (8b)$$

$$\mathbf{R}^{ri} = R_{r\mu}{}^{ri} dr dx^\mu = (hh'/r)e^r e^i \quad (8c)$$

$$\mathbf{R}^{or} = R_{or}{}^{or} dt dr = hg^{-1}(hg')'e^o e^r. \quad (8d)$$

where

$$F(r) = (1 - h^2)/r^2 \quad (8e)$$

For later reference we also note here that the scalar curvature when  $g = h$  is given in terms of  $F$  by

$$R = -[(d^2+2)F + (d+4)rF'/2 + r^2F''] \quad (8f)$$

The only components of  $\delta_e \mathbf{E}$  which preserve the symmetry are  $\delta e^o$  and  $\delta e^r$ . Consider first the  $\delta e^o$  component. Note that  $\delta e^o$  is proportional to  $dt$ . Therefore the 2-forms  $\mathbf{R}^{oi}$  and  $\mathbf{R}^{or}$  cannot occur in  $\delta_e \mathbf{E}$ , since they each are also proportional to  $dt$ . Moreover,  $\mathbf{R}^{ir}$  can occur only once since it is proportional to  $dr$ . The variation therefore has only two terms, for any value of  $k$ :

$$0 = \sum c_k (-)^k \mathbf{R}^{ij} \dots e^k [-2k \mathbf{R}^{ir} e^m + (D-2k-1) \mathbf{R}^{lm} e^r] dt \epsilon_{ij\dots klmro}. \quad (9)$$

The components of the volume form may be written as

$$e \epsilon^{\alpha\beta \dots \gamma} dx^1 dx^2 \dots dx^D = e \epsilon^{ij\dots klmro} dV, \quad (10)$$

where  $e$  is the determinant of the vielbein. Substituting the expression in equation (10) and the expressions for the curvature from (8) into equation (9), then integrating over the maximally symmetric subspace, we find

$$0 = \sum c_k r^D [F(r)]^{k-1} [kr(r^2F)' + (D-2k-1)r^2F]. \quad (11)$$

Note that, for each value of  $k$ , the  $r$ -dependent terms form a total derivative. Integrating term by term, with  $s$  an integration constant, gives

$$sr^{-(D-1)} = \sum c_k F^k \equiv P(F). \quad (12)$$

The variation with respect to  $\vartheta$ , after integration over the subspace and subtraction of equation (11), yields an expression proportional to  $g'/g - h'/h$ :

$$0 = \sum c_k r^D P'(F) [g'/g - h'/h]. \quad (13a)$$

where the prime on  $P$  denotes differentiation with respect to  $F$  and  $h$  may be regarded as a function of  $F$  given by

$$h^2 = 1 - r^2 F(r). \quad (13b)$$

Next, we impose the static, spherical symmetry on equation (5). We eliminate terms which vanish due to the particular structure of the curvature, torsion and vielbein forms given in equations (6), (7) and (8), and use the components of the volume element as before. Only four of the variations --  $\delta\omega_r^{or}$ ,  $\delta\omega_o^{or}$ ,  $\delta\omega_\mu^{oi}$  and  $\delta\omega_\mu^{ir}$  -- give nontrivial equations. The  $\delta\omega_\mu^{oi}$  and  $\delta\omega_\mu^{ir}$  equations may be further simplified by using the  $\delta\omega_r^{or}$  and  $\delta\omega_o^{or}$  equations. The final results, together with equations (12) and (13) from the vielbein variation, give the full set of equations to be satisfied by a static, spherically symmetric solution.

We collect here the full set of variational equations:

$$sr^{-(D-1)} = P(F) \quad (14a)$$

$$0 = P'(F) [g'/g - h'/h] \quad (14b)$$

$$0 = P'(F) T_o \quad (14c)$$

$$0 = P'(F) T_r \quad (14d)$$

$$0 = P'(F) T_{or}{}^r - P''(F) T_o r F' \quad (14e)$$

$$0 = P'(F) T_{or}{}^o - 2P''(F) T_r [g^{-1}g'h^2/r - F] \quad (14f)$$

These are the static, spherically symmetric field equations arising from the lagrangian in equation (3).

The parameter  $s$ , as defined in equation (12), measures the mass of the gravitating system, since equations (8e) and (14a) imply that it is the

coefficient of the gravitational potential at infinity. In the following section, asymptotically flat solutions to equations (14a-14f) when  $s \neq 0$  are found, and shown to be singular. When  $s = 0$  the nature of the solutions depends crucially on the polynomial  $P(F)$  defined in equation (12). Subject to certain special conditions, solutions other than flat space are possible. These exceptional solutions are discussed in detail in section IV.

### III. Asymptotically Flat Solutions: a Singularity Theorem

This section is devoted to the proof of the following results:

- A. The only massive ( $s \neq 0$ ) asymptotically flat solution to equations (14a-14f) for any  $\lambda$  given by equation (3) has vanishing torsion and has a metric given by equation (6) with

$$g^2 = h^2 = 1 - Fr^2 \quad (15)$$

where  $F$  is a solution of the polynomial equation

$$sr^{-(D-1)} = P(F) = \sum c_k F^k. \quad (16)$$

- B. A solution of equation (16) for  $F$  always exists in a neighborhood of  $r = \infty$ , for at least one value of the sign of  $s$ .
- C. The extension of the asymptotic solution for  $F$  increases monotonically with decreasing  $r$  until it terminates at small  $r$  in one of two ways. Either
- 1) There is a curvature singularity at the origin surrounded by exactly one event horizon, or
  - 2) There is a curvature singularity at some finite value of  $r$  with at most one event horizon, and possibly none.
- D. If the extension of (C) terminates at some nonzero value  $r = r_0 > 0$ , then  $r_0$  is bounded by

$$r_0 \leq (\alpha_M)^{M/(D-1)},$$

where  $\alpha_M$  is the largest coefficient in the normalized series expansion of  $F$  (see equation (20') below).



### Proof of IIIA.

$P'(F)$  can vanish for all  $r$  only if  $F$  is independent of  $r$ . But this would contradict equation (14a), since  $s \neq 0$ . Therefore  $P' \neq 0$  and it follows immediately from equations (14b-14f) that

$$g'/g - h'/h = 0 \quad (17)$$

and

$$T_0 = T_r = T_{or}r = T_{or}^0 = 0. \quad (18)$$

The first of these equations implies that, up to a constant rescaling of the time coordinate,  $g = h$ , while the second expresses the complete vanishing of the torsion. The only equation remaining to be satisfied after the conditions (17) and (18) are imposed is equation (16) with  $s \neq 0$ .

### Proof of IIIB.

Noting that a neighborhood about  $r = \infty$  corresponds to a neighborhood about  $F = 0$ , we expand  $F$  as a power series in inverse powers of  $r$ . Experimentation with a general series for  $F$  quickly leads to the conclusion that only certain powers of  $r$  will enter the final solution. In particular, let  $M$  be the smallest integer such that  $c_M \neq 0$ , and let  $y$  be given by the positive  $M^{\text{th}}$  root of

$$y^M \equiv r^{-(D-1)}. \quad (19)$$

Then any series solution for  $F \approx 0$  is of the form

$$F(y) = \sum_{n=1}^{\infty} a_n y^n \quad (20)$$

where the  $a_n$  are constant coefficients to be determined in terms of the lagrangian coefficients  $c_k$ . The restriction to  $n \geq 1$  in equation (20) is the necessary and sufficient condition for asymptotic flatness of the metric. Notice that the solution is asymptotically flat if and only if the cosmological constant  $c_0$  vanishes.

Rather than writing the explicit expressions for the  $a_n$  in terms of the  $c_k$ , we will demonstrate the existence of solutions quite generally. Substituting equation (20) into equation (16) and writing  $r$  in terms of  $y$  gives

$$y = + [P(F(y))/s]^{1/M} \quad (21)$$

The form of equation (21) makes it clear that the function  $F(y)$  defined in equation (20) is the inverse of the function  $[P(x)/s]^{1/M}$ . The series for  $F(y)$  will converge to a finite value everywhere that  $[P(x)/s]^{1/M}$  is well-defined and invertible. It follows from equations (15), (19) and (20) that  $h^2$  is well-defined everywhere that the series for  $F(y)$  converges.

Now consider the invertibility of  $[P(x)/s]^{1/M}$ . The solution of physical interest is the region of small, positive  $x$  ( $x > 0$  insures attractive gravity). In this region  $P(x)$  is given approximately by

$$P(x) \approx c_M x^M / s.$$

This gives a positive solution for  $F$  as long as  $c_M/s > 0$ , for then

$$[P(x)/s]^{1/M} \approx (c_M/s)^{1/M} x.$$

This root always exists and is positive if

$$\text{sgn}(s) = \text{sgn}(c_M).$$

With this choice  $[P(x)/s]^{1/M}$  is always invertible, since  $c_M \neq 0$  by definition. Therefore, an asymptotic solution always exists.

### Proof of IIC.

We first demonstrate that this solution always terminates in a curvature singularity. Then we study the existence of an event horizon.

The solution (20) may be extended until  $[P(x)/s]^{1/M}$  fails to exist or fails to be invertible. This can happen in one of two ways:

1. If  $P(x)/s$  changes sign, then  $[P/s]^{1/M}$  may fail to exist. This is signaled by  $P = 0$ .

2. For any  $M$ ,  $[P/s]^{1/M}$  fails to be invertible if  $P' = 0$ .

However, since  $P(0) = 0$  and  $P'(x) > 0$  for  $0 < x \ll 1$ , condition 2 always occurs before condition 1, and we need only examine the zeros of  $P'$ . In general, there can be zeros of  $P'(x)$  for real  $x > 0$ , and the metric will become singular at those points. By expanding  $P(x)$  about the smallest positive  $x_0$  such that  $P'(x_0) = 0$ , and computing the corresponding approximation to  $F(y)$ , it can be shown using equation (8f) that the curvature is also singular at  $x_0$ .

However, if all of the zeros of  $P'(x)$  occur with  $x < 0$  or  $x$  imaginary, then the solution may be extended all the way to  $x = \infty$  ( $r = 0$ ). If this happens then we are guaranteed not to have a curvature singularity at any finite value of  $r$ . This follows from the horizon theorem proved in [5], together with the easily proved fact that  $F$  is monotonically increasing as long as  $P \neq 0$ .

We now show that even if there is no singularity at any finite value of  $r$  that there must be one at the origin. Suppose  $F$  is nonsingular for all positive  $r$ . Define  $N \leq k_m \leq (D-2)/2$  as the largest integer for which  $c_N \neq 0$ . Then (16) is a polynomial equation for  $F$  of degree  $N$ . Multiplying both sides of equation (16) by  $r^{D-1}$  and taking the limit as  $r \rightarrow 0$  gives

$$s = c_N F^N r^{D-1} + O(r^{D-1} F^{N+1}).$$

Clearly  $F$  must diverge at  $r = 0$ . Moreover, the strength of the singularity of the metric is given by

$$h^2 \approx -[s/c_N]^{1/N} r^{-(D-2N-1)/N}$$

with the obvious consistency condition that

$$\text{sgn}(c_N) = \text{sgn}(c_M).$$

It is easy to show using equation (8f) that the scalar curvature is also singular.

Now consider the existence of an event horizon. If the only singularity occurs at  $r = 0$ , then since  $h^2 \rightarrow -\infty$  as  $r \rightarrow 0$  and since  $h^2 \rightarrow +1$  as  $r \rightarrow \infty$ , there must be some finite value  $r_0$  of the radial coordinate at which  $h$  (and hence  $g$ ) vanish. This condition, together with the

monotonicity of  $F$ , guarantees the existence of a single event horizon, located at  $r_0$ .

A singularity at a nonzero value  $r = r_0$  may be naked, since it is possible that  $P'(r_0) = 0$  when  $r_0^2 F < 1$ . If  $r_0^2 F < 1$ , there is exactly one horizon, by the monotonicity of  $F$  and the theorem of reference [5].

### Proof of IID.

It is possible to estimate the position of the onset of singularities in the solution (20). Given values for the  $a_n$  in terms of the  $c_n$  the series will converge for all  $y$  within the radius of convergence of the series. Since a solution exists near  $y = 0$ , the  $a_n$  must be bounded. Using  $a_M$  to set the scale, equation (20) may be rewritten in dimensionless variables as

$$(a_M)^{2/(D-3)} F(\bar{y}) \equiv \sum (\alpha_n y)^n \leq \sum \bar{y}^n \quad (20')$$

where

$$\alpha_n \equiv [a_M^{2/(D-3)} a_n]^{1/n}$$

and  $\bar{y}$  is  $y$  times the maximum,  $\alpha_M$ , of the  $\alpha_n$ . This means that the solution is nonsingular at least for  $\bar{y} < 1$ , or

$$r > (\alpha_M)^{M/(D-1)}.$$

For typical string theories,  $M = 1$  and  $D = 10$  or  $26$ . The scale  $\alpha_M$  will be on the order of the usual gravitational radius if the theory reduces to the Einstein theory, since quadratic and higher order terms have coefficients on the order of a power of the Planck length.

This concludes the proof of the claims stated at the beginning of the section. We end the section with an example. For  $c_1 \neq 0$ , we list explicitly the values of the first few coefficients  $a_n$  in terms of the  $c_k$ . Expanding  $F$  in powers of  $r^{-(D-1)}$  we find

$$\begin{aligned} a_1 &= s/c_1 \\ a_2 &= -(1/c_1)[c_2 a_0^2] \\ a_3 &= -(1/c_1)[2c_2 a_0 a_1 + c_3 a_0^3] \end{aligned}$$

$$a_4 = -(1/c_1)[c_2 a_1^2 + 2c_2 a_0 a_2 + 3c_3 a_0^2 a_1 + c_4 a_0^4]$$

$$a_5 = -(1/c_1)[2c_2(a_0 a_3 + a_1 a_2) + 3c_3(a_0 a_1^2 + a_0^2 a_2) + 4c_4 a_0^3 a_1 + c_5 a_0^5]$$

... etc.

All of the  $a_n$  with  $n > N$  are determined in terms of the  $a_n$  with  $n < N$ . Note that the only conditions on the existence of this solution are that the cosmological constant vanish, and that  $c_1 \neq 0$ . When  $c_1 = 0$ , but  $c_2 \neq 0$ , the expansion must be in powers of  $r^{-(D-1)/2}$ , and so on, as described above.

#### IV. Geometrically Free Solutions

When  $s = 0$ , the polynomial  $P(F)$  must vanish for all  $F$ . This can happen only if

$$F(r) = \alpha = \text{constant}$$

so that

$$h^2 = 1 - \alpha r^2.$$

Unless  $\alpha = 0$ , such solutions are not asymptotically flat. Notice that the particular choice  $h^2 = 1 - \alpha r^2$  gives the metric in equation (6) maximal symmetry on  $D-1$  rather than only  $D-2$  spatial dimensions. The constant  $\alpha$  is seen to be the inverse of the square of the radius of curvature of the maximally symmetric subspace.

We now examine the field equations with  $s = 0$  and  $F = \alpha$ . Equations (14) now take the form:

$$0 = P(\alpha) \tag{14a'}$$

$$0 = P'(\alpha) [g'/g - h'/h] \tag{14b'}$$

$$0 = P'(\alpha) T_0 \tag{14c'}$$

$$0 = P'(\alpha) T_r \tag{14d'}$$

$$0 = P'(\alpha) T_{or} \tag{14e'}$$

$$0 = P'(\alpha) T_{or}{}^0 - 2P''(\alpha) T_r [g^{-1}g'h^2/r - \alpha] \tag{14f'}$$

If  $P'(\alpha) \neq 0$  then equations (15) and (16) follow as before and spacetime is torsion free with  $g^2 = h^2 = 1 - \alpha r^2$ . In particular, for  $\alpha = 0$ , we have flat space while for  $\alpha > 0$  ( $\alpha < 0$ ) we have (anti-)deSitter solutions. However, suppose

$$P'(\alpha) = 0. \quad (22)$$

Then equations (14b', c', d' and e') are satisfied for arbitrary values of  $g$ ,  $T_0$ ,  $T_r$  and  $T_{or}{}^r$ . The remaining equation requires one of the three conditions

$$\begin{aligned} T_r &= 0 \\ g &= \text{constant} \\ P''(\alpha) &= 0. \end{aligned} \quad (23)$$

Regardless of which of these holds, there remain components of the metric or torsion which are not determined by the field equations. All of the solutions of this type require particular relations among the coefficients  $c_k$ . As soon as any geometric quantity -- metric component  $g$  or torsion component  $T_0$ ,  $T_r$ ,  $T_{or}{}^0$  or  $T_{or}{}^r$  -- is left arbitrary, it becomes necessary to specify at least one of the coefficients  $c_k$  in terms of the rest. Solutions with one or more geometric variables undetermined by the variational equations will be called *geometrically free*.

Let us examine the geometrically free solutions. This occurs when  $P'$ , and possibly also  $P''$ , vanish at  $\alpha$ . Let  $r$  be scaled so that  $\alpha = 1$ . Then each of the conditions on  $P$  determines one of the coefficients  $c_k$  in terms of the rest. Therefore, for  $D < 6$ , no solutions of this type exist, while for  $D = 6$ , a cosmological constant is required. In particular, the Einstein theory does not have free geometry solutions since equations (14a') and (22) together require  $c_0 = c_1 = 0$ .  $D = 8$  is the smallest dimension for which all three conditions (14a'), (22) and (23) can hold simultaneously.

To better understand the nature of geometrically free solutions we return to the original class of lagrangians and consider the subclass which in even dimensions may be written in the form:

$$\mathcal{L} = (R^{ab} - \alpha_1 e^a e^b)(R^{cd} - \alpha_2 e^c e^d) \dots (R^{ef} - \alpha_{D/2} e^e e^f) \epsilon_{abcdfg\dots h} \quad (24)$$

where the  $\alpha_i$  are constants related implicitly to the  $\bar{c}_k$  by

$$\sum \bar{c}_k x^k = \Pi (x - \alpha_i).$$

Odd dimensions are treated by inserting an extra vielbein one-form in  $\mathcal{L}$ , contracted onto the extra index on the Levi-Civita symbol. All lagrangians of the form of equation (24) admit geometrically free solutions as long as at least two of the  $\alpha_i$  (say, for  $i = 1, 2$ ) are equal. For then every term in the variation of  $\mathcal{L}$  contains at least one factor  $(R^{ab} - \alpha_i e^a e^b)$ . Suppose for simplicity that all of the  $\alpha_i$  are equal. Then whenever the spacetime under consideration has a metric tensor which is block diagonal of the form

$$g_{ab} = \begin{pmatrix} g_{\mu\nu}(x) & 0 \\ 0 & s_{ij}(y) \end{pmatrix} \quad (25)$$

where  $s_{ij}$  is the metric of a maximally symmetric subspace with curvature 2-form given by

$$R_{ij} = (\alpha/2) e^m e^n (\delta_m^j \delta_n^i - \delta_m^i \delta_n^j) = -\alpha e^i e^j,$$

the field equations will be identically satisfied, whatever the form of  $g_{\mu\nu}(x)$ . Notice that  $g_{\mu\nu}$  could equally well be a function of  $y^2 = y_i y^i$  as in equations (14). The important thing is that the curvature tensor should be block diagonal, vanishing unless all four of its indices take values in the maximally symmetric subspace or all four take values in the free space. This is sufficient, given the presence of the Levi-Civita tensor, to guarantee that one the factors  $(R^{ab} - \alpha e^a e^b)$  always vanishes.

It is easy to see that the conditions (14a'), (22) and (23) are equivalent to demanding equality of three or more of the  $\alpha_k$  in equation (24), so that the solutions with those conditions are special cases of this more general approach. It is also simple to confirm from the form of the lagrangian in equation (24), that there are no geometrically free solutions when  $D = 4$ .

## Summary

In an arbitrary number of dimensions, with arbitrary torsion, we have found static, spherically symmetric solutions to the most general gravitational theory in which the lagrangian is composed entirely of dimensionally extended Euler forms (eq. 3). All of the asymptotically flat solutions with nonvanishing mass parameter ( $s \neq 0$ ) have curvature singularities. All of the remaining solutions ( $s = 0$ ) have metrics with additional symmetry. These remaining solutions include flat space, de-Sitter and anti-de-Sitter spaces, and a special class we have called geometrically free solutions.

For the singular ( $s \neq 0$ , asymptotically flat) solutions the following results hold:

- A. The solution is unique and has vanishing torsion and a metric given by equation (6) with

$$g^2 = h^2 = 1 - Fr^2$$

where  $F$  is a solution of the polynomial equation

$$sr^{-(D-1)} = P(F) = \sum c_k F^k$$

- B. A solution of the equation for  $F$  always exists in a neighborhood of  $r = \infty$ , for at least one value of the sign of  $s$ .
- C. The extension of the asymptotic solution for  $F$  increases monotonically with decreasing  $r$  until it terminates at small  $r$  in one of two ways. Either
- 1) There is a curvature singularity at the origin surrounded by exactly one event horizon, or
  - 2) There is a curvature singularity at some finite value of  $r$  with at most one event horizon, and possibly none.
- D. If the extension of (C) terminates at some nonzero value  $r = r_0 > 0$ , then  $r_0$  is bounded by

$$r_0 \leq (\alpha \eta)^{M/(D-1)}.$$



where  $\alpha_M$  is the largest coefficient in the normalized series expansion of  $F$ .

Notice that the possibility of naked singularities is expected, since the quadratic and higher terms in the lagrangian can always be thought of as source terms for the usual Einstein theory and suitable choices of the coefficients can always make the resulting effective energy density negative. However, these solutions are qualitatively different from negative mass Schwarzschild solutions, since they all have negative gravitational potential ( $-r^2F < 0$ ) at infinity.

For  $s = 0$ , possible solutions include flat, deSitter and anti-deSitter spacetimes. In addition, there are special values of the coefficients  $c_k$  for which there exist geometrically free solutions, characterized by the existence of a maximally symmetric subspace and a curvature tensor which is block diagonal.

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## Appendix 1: Matrix product manipulations

We prove here a matrix identity used in section II to reduce the variational equations. The proof is a generalization of one given by Zumino [8] for even dimensions. Let  $M_i^a$ ,  $i = 1, 2, \dots, n$  be a collection of vectors and  $N^{ab}$  an antisymmetric matrix, and form the product

$$A = M_1^a M_2^b \dots M_n^c \epsilon_{ab\dots c}$$

Now consider the effect of the rotation produced by  $\exp(\lambda N^{ab})$ . Define

$$M_i^a(\lambda) = [\exp(\lambda N)]^a_c M_i^c(0)$$

Since  $A$  is a scalar, there will be no net effect of the rotation, so

$$A(0) = A(\lambda) = M_1^a(\lambda) \dots M_n^b(\lambda) [\exp(-\lambda N)]^c_a \dots [\exp(-\lambda N)]^d_b \epsilon_{c\dots d}$$

But  $\epsilon_{ef\dots gh}$  is rotationally invariant, so the group elements acting on it have no net effect, leaving

$$A(0) = A(\lambda) = M_1^a(\lambda) \dots M_n^b(\lambda) \epsilon_{a\dots b}$$

The derivative of  $A$  with respect to  $\lambda$  therefore vanishes. Explicitly taking the derivative, then setting  $\lambda = 0$  gives the identity

$$[N^a_d M_1^d M_2^b \dots M_n^c + M_1^a N^b_d M_2^d \dots M_n^c + \dots + M_1^a M_2^b \dots N^c_d M_n^d] \epsilon_{ab\dots c} = 0 \quad (A1.1)$$

The identity still holds if some or all of the  $M$ 's are differential forms, as long as appropriate signs are inserted. Also, any pair of vectors may be replaced by an antisymmetric matrix with the inclusion of a factor of two. For example, if  $M_1^a$  and  $M_2^b$  are replaced by  $M^{ab}$  then equation (A1.1) is easily shown to reduce to

$$[2N^a_e M^{eb} M_3^c \dots M_n^c + M^{ab} N^c_e M_3^e \dots M_n^d + \dots + M^{ab} M_3^c \dots N^d_e M_n^e] \epsilon_{abc\dots d} = 0$$

The relationship holds in any number of dimensions.

## Appendix 2: Static, Spherically Symmetric Torsion

A symmetry in a spacetime is a consequence of the existence of a Killing vector field which at each point gives the direction of the symmetry. For a tensor field to possess the given symmetry then means that its Lie derivative in the direction of the Killing field vanishes. For the torsion this gives

$$\mathcal{L}_\xi T_{abc} = 0 = \xi^d{}_{,a} T_{dbc} + \xi^d{}_{,b} T_{adc} + \xi^d{}_{,c} T_{abd} + \xi^d T_{abc,d} \quad (\text{A2.1})$$

where a comma denotes a partial derivative and all indices (a, ...) are world indices taking values 1, 2, ..., D. Indices from the center of the alphabet (i, ...) will take only values 1, 2, ..., d = D-2, with the remaining indices denoted by o and r. A semicolon will be used to denote covariant differentiation.

First, consider the effect of the Killing fields in the time direction and in the coordinate directions of the maximally symmetric subspace. Given a value for  $T_{abc}$  at any location in spacetime, these fields may be used to find the value of  $T_{abc}$  at any other point on the orbit of one of these Killing fields by Lie transport. This includes all points of spacetime with the same radial coordinate r, so  $T_{abc}$  can be a function of r only.

Now consider the remaining (rotational) Killing fields of the maximally symmetric subspace. Following Weinberg [9], we consider a point P, at which  $\xi_i$ , but not  $\xi_{i;j}$ , vanishes. Then at P

$$\xi_i(P) = 0$$

gives

$$0 = \xi_{i;j}(P) + \xi_{j;i}(P) = \xi_{i,j}(P) + \xi_{j,i}(P)$$

and

$$\xi^i{}_{;j}(P) = g^{ik}(P)\xi_{k;j}(P)$$

These relations imply that  $\xi_{i;j}(P)$  is an arbitrary antisymmetric matrix. We may therefore write equation (A2.1) at P as

$$0 = \delta^{[i} T^{j]}{}_{bc} + \delta^{[i} T^{j]}{}_{ac} + \delta^{[i} T^{j]}{}_{ab} \quad (\text{A2.2})$$

When a, b and c all take values on the maximally symmetric subspace, it is easy to show from equation (A2.2) that  $T_{ijk} = 0$  except when D = 5. The

exception arises because when  $D = 5$  the maximally symmetric subspace admits the rank three totally antisymmetric invariant tensor field  $\epsilon_{ijk}$ . This allows the possibility  $T_{ijk} = T(r)\epsilon_{ijk}$ .

Most of the remaining components of the torsion can also be determined from (A2.2). For example, we compute  $T_{oij}$  as follows. Let  $a = k$ ,  $b = o$  and  $c = i$  in (A2.2). Then

$$0 = \delta^{[i}_k T^{j]}_{o1} + \delta^{[i}_1 T_{ko}^{j]} \quad (\text{A2.3})$$

Contraction on  $i$  and  $k$  yields, after lowering indices

$$0 = (d-1)T_{joi} + T_{ioj} - g_{ji} T_{io}^j.$$

The antisymmetric part of  $T_{oij}$  must therefore vanish unless  $d = 2$ , when it may be proportional to  $\epsilon_{ij}$ . The part symmetric on  $i$  and  $j$  is given by:

$$T_{\alpha(ij)} = (1/d) g_{ij} T_{\alpha k}^k \equiv (1/d) g_{ij} T_\alpha \quad (\text{A2.4})$$

where  $T_\alpha$  is an arbitrary function of  $r$ . With  $T_{oij} = T_{\alpha(ij)}$  as given by (A2.4), equation (A2.3) is identically satisfied.

The full static, spherically symmetric torsion tensor can be shown in this way to have components

$$T_{ijk} = 0 \quad (\text{A2.5a})$$

$$T_{ijr} = T_{ij0} = 0 \quad (\text{A2.5b})$$

$$T_{ioo} = T_{irr} = T_{ior} = T_{roi} = T_{iro} = 0 \quad (\text{A2.5c})$$

$$T_{oij} = -T_{ioj} = (1/d) g_{ij} T_o \quad (\text{A2.6a})$$

$$T_{rij} = -T_{irj} = (1/d) g_{ij} T_r \quad (\text{A2.6b})$$

$$T_r = T_r(r) \quad T_o = T_o(r) \quad (\text{A2.7})$$

$$T_{ror} = T_{ror}(r) \quad T_{oro} = T_{oro}(r) \quad (\text{A2.8})$$

Equations (A2.5) through (A2.8) give the full consequences of static, spherical symmetry for the torsion.

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