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Variation of the linear biconformal action

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Abstract

We find the field equations of biconformal space in a basis adapted to Lagrangian submanifolds on which the restriction of the Killing metric is non-degenerate.

1 Introduction

Starting with the conformal symmetries of Euclidean space, we have shown in [1] how construct a manifold where time manifests as a part of the geometry. The result is based on a theorem by Spencer and Wheeler [2] detailing when this is possible. In addition, though there is no matter present in the geometries studied in [1], geometric terms analogous to dark energy and dark matter appear in the Einstein tensor.

Specifically, the quotient of the conformal group of Euclidean n-space by its Weyl subgroup results in a biconformal geometry possessing many of the properties of relativistic phase space, including both a natural symplectic form and non-degenerate Killing metric. The general solution for this homogeneous space possesses orthogonal Lagrangian submanifolds, with the induced metric and the spin connection on the submanifolds necessarily Lorentzian, despite the Euclidean starting point.

An action functional linear in the Cartan curvatures of the conformal group was developed in [8]. Though an explicit gravitational theory is presented in [1], the main results of that work apply to the homogeneous space. It is of great interest to extend this study to nontrivial gravitational solutions. In this work, we vary the conformal connection fields to find the field equations. The variation is carried out in a basis adapted to the Lagrangian submanifolds.

We begin in the next Section with a description of the biconformal gauging. Section 3 contains the variation and the final Section a summary of the field equations.

2 Biconformal gravity

We review the construction of biconformal gravity theory.

2.1 General properties of the conformal group

We begin with the conformal group, C, of compactified $\mathbb{R}^n$. The one-point compactification at infinity allows a global definition of inversion, with translations of the point at infinity defining the special conformal transformation. For pseudo-Euclidean spaces, $\mathbb{R}^{p,q}$, (of dimension $n = p + q$ and signature $s = p - q$; all parts of this construction work for any $(p, q)$) the compactification requires a null cone at infinity in addition at the point at infinity. Then $C$ has a real linear representation in $n + 2$ dimensions, $\mathbb{V}^{n+2}$ (alternatively we could choose the complex representation $\mathbb{C}^{[n+2]/2}$ for $Spin(p + 1, q + 1)$). The isotropy subgroup of $\mathbb{R}^n$ is the subgroup of rotations, $SO(p, q)$, together with dilatations. We call this subgroup the homogeneous Weyl group, $W$ and require our fibers to contain it, leaving only three allowed subgroups for a quotient: $W$ itself; the inhomogeneous Weyl group, $ZW$, found by appending the translations; and $W$ together with special
conformal transformations, isomorphic to $\mathcal{W}$. The quotient of the conformal group by either inhomogeneous Weyl group, called the auxiliary gauging, leads naturally to Weyl gravity (see [4]). We concern ourselves with the only other meaningful conformal quotient, the biconformal gauging, generalizing the principal $\mathcal{W}$-bundle formed by the quotient of the conformal group by its Weyl subgroup.

The Maurer-Cartan structure equations, follow immediately from a knowledge of the structure constants of the Lie algebra. In addition to the $n(n-1)/2$ generators $M^\alpha_\beta$ of $SO(p,q)$ and $n$ translational generators $P_\alpha$, there are $n$ generators of translations of a point at infinity (special conformal transformations, or co-translations) $K_\alpha$, and a single dilatational generator $D$. Dual to these, we have the connections $\xi^\alpha_\beta, \chi^\alpha, \pi_\alpha, \delta$, respectively. Substituting the structure constants into the Maurer-Cartan dual form of the Lie algebra [5] gives

\begin{align}
\text{d}\xi^\alpha_\beta &= \xi^\mu_\beta \wedge \xi^\alpha_\mu + 2\Delta^\alpha_\mu_\beta \pi_\mu \wedge \chi^\nu \\
\text{d}\chi^\alpha &= \chi^\beta \wedge \xi^\alpha_\beta + \delta \wedge \chi^\alpha \\
\text{d}\pi_\alpha &= \xi^\beta_\alpha \wedge \pi_\beta - \delta \wedge \pi_\alpha \\
\text{d}\delta &= \chi^\alpha \wedge \pi_\alpha
\end{align}

where $\Delta^\alpha_\mu_\beta \equiv 1/2 \left( \delta^\alpha_\mu \delta^\beta_\mu - \delta^\alpha_\mu \delta^\beta_\mu \right)$ antisymmetrizes $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ tensors with respect to the original $(p,q)$ metric,

$\delta_{\mu\nu} = \text{diag}(1,\ldots,1,-1,\ldots,-1)$

These equations, which are the same regardless of the gauging chosen, describe the Cartan connection on the conformal group manifold. Before proceeding to the quotient, we note that the conformal group has a nondegenerate Killing form, $K_{AB} \equiv \text{tr}(G_A G_B) = c^C_{AD} c^D_{BC} = \begin{pmatrix} \Delta^a_{db} & 0 \\ 0 & \delta^a_b \\ \delta^a_b & 0 \end{pmatrix}$

This provides a metric on the conformal Lie algebra. When restricted to $\mathcal{M}_0^{2n} \equiv \mathcal{C}/\mathcal{W}$, it remains nondegenerate.

Finally, we note that the conformal group is invariant under inversion. Within the Lie algebra, this manifests itself as the interchange between the translations and special conformal transformations $P_\alpha \leftrightarrow \delta_{\alpha\beta} K^\beta$ along with the interchange of conformal weights, $D \rightarrow -D$. The corresponding transformation of the connection forms, $\chi^\alpha \leftrightarrow \delta^\alpha_\beta \pi_\beta, \delta \rightarrow -\delta$, is easily seen to leave eqs.(1)-(4) invariant. This symmetry leads to complex and Kähler structures.

### 2.2 The homogeneous quotient $\mathcal{C}/\mathcal{W}$

In the conformal group, translations and special conformal transformations are related by inversion. Indeed, a special conformal transformation is a translation centered at the point at infinity instead of the origin. Because the biconformal gauging maintains the symmetry between translations and special conformal transformations, it is useful to name the corresponding connection forms and curvatures to reflect this. Therefore, the biconformal basis will be described as the solder form and the co-solder form, and the corresponding curvatures as the torsion and co-torsion. When we speak of “torsion-free biconformal space” we do not imply that the co-torsion (Cartan curvature of the co-solder form) vanishes. In phase space interpretations, the solder form is taken to span the cotangent spaces of the spacetime manifold, while the co-solder form is taken to span the cotangent spaces of the momentum space. The opposite convention is equally valid.

Unlike other quotient manifolds arising in conformal gaugings, the biconformal quotient manifold possesses natural invariant structures arising from the underlying groups. The first is the restriction of the Killing
metric, which is non-degenerate,
\[
\begin{pmatrix}
\Delta_{ab}^{ac} \\
\delta_{a}^{b}
\end{pmatrix}
\begin{pmatrix}
0 & \delta_{a}^{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}
\begin{pmatrix}
1 \\
1
\end{pmatrix}
\bigg|_{\mathcal{M}^{(2n)}} = \begin{pmatrix}
0 & \delta_{a}^{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}_{2n \times 2n}.
\]

This gives an inner product for the basis,
\[
\left[ \langle \chi^{\alpha}, \chi^{\beta} \rangle, \langle \pi_{\alpha}, \chi^{\beta} \rangle \right] = 0 \\
\left[ \langle \pi_{\alpha}, \pi_{\beta} \rangle \right] = \begin{pmatrix}
0 & \delta_{a}^{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}
\]
\( (5) \)

This metric remains unchanged by the generalization to curved base manifolds, when we replace \((\chi^{\alpha}, \pi_{\alpha}) \rightarrow (\omega^{\alpha}, \omega_{\alpha})\) below.

The second natural invariant property is the generic presence of a symplectic form. The original fiber bundle always has this, because the structure equation, eq.(4), shows that \(\chi^{\alpha} \wedge \pi_{\alpha}\) is exact hence closed, \(d\omega = 0\), while it is clear that the two-form product is non-degenerate because \((\chi^{\alpha}, \pi_{\alpha})\) together span \(\mathcal{M}^{(2n)}\). Moreover, the symplectic form is canonical,
\[
[\Omega]_{AB} = \begin{pmatrix}
0 & \delta_{a}^{b} \\
-\delta_{a}^{b} & 0
\end{pmatrix}
\]
so that \(\chi^{\alpha}\) and \(\pi_{\alpha}\) are canonically conjugate, in the sense that they form a canonical basis for the symplectic form \(\Omega\). The symplectic form persists for the 2-form, \(\omega^{\alpha} \wedge \omega_{\alpha} + \Omega\) (see below), as long as it is non-degenerate, so curved biconformal spaces are generically symplectic.

Next, we consider the effect of inversion symmetry. As a \((1,1)\) tensor, the basis interchange takes the form
\[
I_{B}^{A} \chi^{B} = \begin{pmatrix}
0 & \delta_{a}^{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}
\begin{pmatrix}
\chi^{\mu} \\
\pi_{\nu}
\end{pmatrix}
= \begin{pmatrix}
\delta^{\alpha \mu} \pi_{\nu} \\
\delta^{\beta \mu} \chi^{\nu}
\end{pmatrix}
\]

In order to interchange conformal weights, \(I_{B}^{A}\) must anticommute with the conformal weight operator, which is given by
\[
W_{B}^{A} \chi^{B} = \begin{pmatrix}
\delta_{a}^{b} & 0 \\
0 & -\delta_{a}^{b}
\end{pmatrix}
\begin{pmatrix}
\chi^{\mu} \\
\pi_{\nu}
\end{pmatrix}
= \begin{pmatrix}
+\chi^{\alpha} \\
-\pi_{\beta}
\end{pmatrix}
\]

This is the case: we easily check that \([I, W]_{B}^{A} = I_{C}^{A} W_{B}^{C} + W_{C}^{A} I_{B}^{C} = 0\). The commutator gives a new object,
\[
J_{B}^{A} = [I, W]_{B}^{A} = \begin{pmatrix}
0 & -\delta_{a}^{b} \\
\delta_{a}^{b} & 0
\end{pmatrix}
\]

Squaring, \(J_{B}^{A} J_{B}^{C} = -\delta_{B}^{A}\), we see that \(J_{B}^{A}\) provides an almost complex structure. That the almost complex structure is integrable follows immediately in this (global) basis by the obvious vanishing of the Nijenhuis tensor,
\[
N_{BC}^{A} = J_{C}^{D} \partial_{D} J_{B}^{A} - J_{C}^{D} \partial_{D} J_{B}^{A} - J_{D}^{A} \left( \partial_{C} J_{B}^{D} - \partial_{B} J_{C}^{D} \right) = 0
\]

Next, using the symplectic form to define the compatible metric
\[
g(u, v) = \Omega(u, Jv)
\]
we find that in this basis \(g = \begin{pmatrix}
\delta_{a}^{b} & 0 \\
0 & \delta^{a b}
\end{pmatrix}\), and we check the remaining compatibility conditions of the triple \((g, J, \Omega)\),
\[
\omega(u, v) = g(Ju, v) \\
J(u) = (\phi_{g})^{-1}(\phi_{\omega}(u))
\]

3
where $\phi_\omega$ and $\phi_g$ are defined by
\[
\phi_\omega (u) = \omega (u, \cdot) \\
\phi_g (u) = g (u, \cdot)
\]
These are easily checked to be satisfied, showing that that $\mathcal{M}_0^{(2n)}$ is a Kähler manifold. Notice, however, that the metric of the Kähler manifold is not the restricted Killing metric which we use in the following considerations. Kähler manifolds play a role in string theory and geometric quantization, and we plan to study this relationship further.

Finally, a surprising result emerges if we require $\mathcal{M}_0^{(2n)}$ to match our usual expectations for a relativistic phase space. To make the connection to phase space clear, the precise requirements were studied in [2]. There it was required that the flat biconformal gauging of $SO (p, q)$ in any dimension $n = p + q$ have Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have non-degenerate n-dim restrictions of the metric. This is found to be possible only if the original space is Euclidean or signature zero ($p \in \{0, \frac{n}{2}, n\}$). The signature of the Lagrangian submanifolds is severely limited ($p \rightarrow p + 1$), leading in the two Euclidean cases to Lorentzian configuration space, and hence the origin of time. For the case of flat, 8-dim biconformal space we paraphrase the the following theorem from [2]:

Flat n-dim biconformal space is a metric phase space with the following properties:

1. There exist Lagrangian submanifolds orthogonal with respect to the 2n-dim biconformal (Killing) metric
2. The restriction of the Killing metric to each Lagrangian submanifold is non-degenerate

if and only if the initial n-dim space we gauge is Euclidean or signature zero. In the Euclidean case the resulting configuration sub-manifold is necessarily Lorentzian [2].

Thus, it is possible to impose the conditions necessary to make biconformal space a metric phase space only in a restricted subclass of cases, and in 4-dim or Euclidean cases the configuration space metric must be Lorentzian. In [1], it was shown that coordinates $(u^\alpha, s_\beta)$ exist such that the metric takes the form
\[
k_{\alpha\beta} = \frac{1}{2} v^2 \left( \delta_{\alpha\beta} - \frac{2}{v^2} \delta^{\alpha\mu} \delta^{\beta\nu} v_\mu v_\nu \right)
\]
where $\delta_{\alpha\beta}$ is the Euclidean metric and $v^2 = \delta^{\alpha\beta} s_\alpha s_\beta$. The signature changing character is easily seen.

The Euclidean gauge theory necessarily possesses a special vector, $v = \omega - \frac{1}{2} \eta_{ab} d_{bc}$, built from the Weyl vector and the conformal gauge of the metric [1]. It is preferable to write $k_{\alpha\beta}$ in terms of the components of this vector, $u^\alpha$ and $v_\mu$ defined above. In the $(u^\alpha, s_\beta)$ coordinates, $v_\alpha = -k_\beta s_\alpha$ and $u^\alpha = \beta k_{\alpha\beta} s_\beta$, so we find
\[
k_{\alpha\beta} = \frac{1}{2} v^2 \left( \delta_{\alpha\beta} - \frac{2}{v^2} v_\alpha v_\beta \right)
\]
with inverse $k^{\alpha\beta} = \frac{v^2}{v^2} \left( \delta^{\alpha\beta} - \frac{2}{v^2} \delta^{\alpha\mu} \delta^{\beta\nu} v_\mu v_\nu \right)$. This shows that the vector $v_\alpha$ determines the timelike directions. This vector gives the time direction on both the orthogonal Lagrangian submanifolds, making them necessarily Lorentzian. The full manifold retains its original symmetry.

Thus, the structures of the conformal group give rise to a natural direction of time. The situation is reminiscent of previous studies. In 1979, Stelle and West introduced a special vector field to choose the local symmetry of the MacDowell-Mansouri theory. The vector breaks the de Sitter symmetry, eliminating the need for the Wigner-Inönu contraction. Recently, Westman and Zlosnik[9] have looked in depth at both the de Sitter and anti-de Sitter cases using a class of actions which extend that of Stelle and West by including derivative terms for the vector field and therefore lead to dynamical symmetry breaking. In [10, 11] and Einstein-Aether theory [12], there is also a special vector field introduced into the action by hand that can make the Lorentzian metric Euclidean. These approaches are distinct from that of the biconformal approach, where the vector necessary for specifying the timelike direction occurs naturally from the underlying group structure.
2.3 Gauging

By gauge theory, we typically understand a theory (i.e., the specification of an action functional) which is invariant under a local symmetry group – the gauge symmetry. Thus, there may be many gauge theories having the same gauge group. However, gauge theories having the same gauge group share a common structure: the underlying principal fiber bundle in which the base manifold is spacetime or some other world manifold and the fibers are copies of the gauge group. Such a principal fiber bundle is most simply constructed as the quotient of a larger group by the symmetry group. Constructed in this way, we have immediate access to relevant tensor fields: any group invariant tensors, the curvatures of the bundle, and the vectors of the group representation. Then any functional built invariantly from these tensors is a gauge theory. For example, the quotient of the Poincaré group by its Lorentz subgroup may be generalized to a principal fiber bundle with Lorentz group fibers and a general base manifold having arbitrary Riemannian curvature. Identifying the curvature, solder form, Lorentz metric, and Levi-Civita tensor as tensors with the vectors of the group representation. Then any functional built invariantly from these tensors is a gauge theory. For these reasons, we will define a gauging to be the fiber bundle of a specific quotient, along with the identification of its associated tensors. A gauge theory remains the specification of an action functional invariant on this bundle.

The biconformal gauging was first considered by Ivanov and Niederle [6]. They considered a curvature-quadratic action, with the extra four dimensions restricted to the minimum necessary for consistency with conformal symmetry. The full 2n-dimensional geometry was first studied in [7], where tensorial constraints on the full higher-dimensional curvatures were shown to reduce the geometry to general relativity. It was soon realized that the dimensionless volume form permits an action linear in the curvatures [8], and the resulting field equations were found to reduce to the system described in [7].

In the fiber bundle produced by the quotient \( \mathcal{C}/\mathcal{W} \), the one-forms \( (\xi^\alpha, \delta) \) span the \( \mathcal{W} \)-fibers, with \( (\chi^\alpha, \pi_\alpha) \) spanning the remaining 2n independent directions of the homogeneous base manifold. We now generalize the connection (and the manifold, if desired) by replacing \( (\xi^\alpha, \chi^\alpha, \pi_\alpha, \delta) \) → \( (\omega^\alpha, e^\alpha, \omega_\alpha, \omega) \) in the Maurer-Cartan equations, eqs.(1-4) to give the Cartan curvatures in terms of the new connection forms,

\[
\begin{align*}
\text{d}\omega^\alpha_\beta &= \omega^\mu_\beta \wedge \omega^\alpha_\mu + 2\Delta^\alpha_\mu \omega_\mu \wedge \omega^\nu + \Omega^\alpha_\beta \\
\text{d}e^\alpha &= e^\beta \wedge \omega^\alpha_\beta + \omega \wedge e^\alpha + T^\alpha \\
\text{d}\omega_\alpha &= \omega^\beta_\alpha \wedge \omega_\beta - \omega \wedge \omega_\alpha + S_\alpha \\
\omega &= \omega^\alpha_\wedge \omega_\alpha + \Omega 
\end{align*}
\]

Equations eq.(7-10) give the curvature two-forms in terms of the connection forms. We have therefore constructed a 2n-dimensional manifold based on the conformal group with local \( \mathcal{W} \) symmetry. We require integrability of these equations and horizontality of the curvatures.

Horizontality prevents the curvatures from depending on either the spin connection \( \omega^\alpha_\beta \) or the Weyl vector \( \omega \), but they now still depend on the 2n non-vertical forms, \( (\omega^\alpha, \omega_\alpha) \). This means there are far more components than for an n-dim Riemannian geometry. For example,

\[
\Omega^\alpha_\beta = \frac{1}{2} \Omega^\alpha_\beta_{\mu\nu} \omega^\mu \wedge \omega^\nu + \Omega^\alpha_\beta_{\mu} \omega_\mu \wedge \omega^\nu + \frac{1}{2} \Omega^\alpha_\beta_{\mu\nu} \omega_\mu \wedge \omega_\nu
\]

The coefficients of the pure terms, \( \Omega^\alpha_\beta_{\mu\nu} \) and \( \Omega^\alpha_\beta_{\mu\nu} \), each have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry, while the cross-term coefficients \( \Omega^\alpha_\beta_{\mu} \) have more, being asymmetric on the final two indices.

Notice that the spin connection, \( \xi^\alpha_\beta \), is antisymmetric with respect to the original \( (p, q) \) metric, \( \delta_{\alpha\beta} \), in the sense that

\[
\xi^\alpha_\beta = -\delta^\alpha \mu \delta_{\beta\nu} \xi^\nu_\mu
\]
It is crucial to note that $\omega^{\alpha}_{\beta}$ retains this property, $\omega^\alpha_{\beta} = -\delta^\alpha_\mu \delta^\beta_\nu \omega^\mu_\nu$. This expresses metric compatibility with the $SO(p,q)$-covariant derivative, since it implies

$$D \delta_{\alpha \beta} \equiv d \delta_{\alpha \beta} - \delta_{\mu \beta} \omega^\mu_\alpha - \delta_{\alpha \mu} \omega^\mu_\beta = 0$$

Therefore, the curved generalization has a connection which is compatible with a locally $(p,q)$-metric. This relationship is general. If $\kappa_{\alpha \beta}$ is any metric, its compatible spin connection will satisfy $\omega^\alpha_{\beta} = -\kappa^\alpha_\mu \kappa^\beta_\nu \omega^\mu_\nu$.

Since we also have local scale symmetry, the full covariant derivative we use will also include a Weyl vector term. Integrability of the modified structure equations is guaranteed by a Bianchi identity for each of the curvatures,

$$0 = D\Omega^\alpha_\beta + 2\Delta^\alpha_\beta (\Omega^\mu_\mu \omega^\nu_\mu - \omega^\alpha_\mu \Omega^\nu_\mu ) \quad (11)$$
$$0 = DT^\alpha - \epsilon^\beta \wedge \Omega^\alpha_\beta + \Omega \wedge e^\alpha \quad (12)$$
$$0 = DS_\alpha + \Omega^\alpha_\beta \wedge \omega^\beta_\alpha - \omega^\alpha_\beta \wedge \Omega \quad (13)$$
$$0 = D\Omega + T^\alpha \wedge \omega^\alpha - \omega^\alpha \wedge S_\alpha \quad (14)$$

where $D$ is the Weyl covariant derivative,

$$D\Omega^\alpha_\beta = d\Omega^\alpha_\beta + \Omega^\mu_\mu \wedge \omega^\alpha_\mu - \omega^\alpha_\mu \wedge \Omega^\nu_\mu$$
$$DT^\alpha = dT^\alpha + T^\beta \wedge \omega^\beta_\alpha - \omega \wedge T^\alpha$$
$$DS_\alpha = dS_\alpha - \omega^\beta_\alpha \wedge S_\beta + S_\alpha \wedge \omega$$
$$D\Omega = d\Omega$$

Notice that our development to this point was based solely on group quotients and generalization of the resulting principal fiber bundle. We have arrived at the form of the curvatures in terms of the Cartan connection, and Bianchi identities required for integrability, thereby describing certain classes of geometry with local symmetry. Within the biconformal quotient, the demand for orthogonal Lagrangian submanifolds with non-degenerate $n$-dim restrictions of the Killing metric leads to the selection of certain Lorentzian submanifolds.

We are guided in the choice of action functional by the example of general relativity. In Riemannin geometries we may write the Einstein-Hilbert action and proceed. More systematically, however, we may write the most general, even-parity action linear in the curvature and torsion. This turns out to be the tetradic Palatini action, $S_P = \int R^{ab} \wedge e^c \wedge e^d \epsilon_{abcd}$. A full variation of the connection, $(\delta e^b, \delta \omega^a_\mu)$, implies vanishing torsion in addition to the Einstein equation. The latter, more robust approach is what we follow for conformal gravity theories.

It is generally of interest to build the simplest class of actions possible, and we use the following criteria:

1. The pure-gravity action should be built from the available curvature tensor(s) and other tensors which occur in the geometric construction.

2. The action should be of lowest possible order $\geq 1$ in the curvatures.

3. The action should be of even parity.

These are of sufficient generality not to bias our choice and sufficiently constraining to give a reasonable class of theories.

Notice that if we perform an infinitesimal conformal transformation to the curvatures, $(\Omega'_{\alpha \beta}, \Omega^{\alpha}, \Omega_{\beta}, \Omega)$, they all mix with one another, since the conformal curvature is really a single Lie-algebra-valued two form. However, the generalization to a curved manifold breaks the non-vertical symmetries, allowing these different components to become independent tensors under the remaining Weyl group. Thus, to find the available tensors, we apply an infinitesimal transformation of the fiber symmetry. Tensors are those objects which
transform linearly and homogeneously under these transformations. Note that this is not symmetry breaking, but rather simply part of the construction of the manifold and local symmetry.

The biconformal gauging, based on $\mathcal{C}/\mathcal{W}$, has tensorial basis forms $(\omega^a, \omega_a)$. Moreover, each of the component curvatures, $(\Omega^a, \Omega, \Omega_\beta, \Omega)$, becomes an independent tensor under the Weyl group. The volume form $\epsilon^{\alpha \beta \gamma \ldots} \omega^\alpha \wedge \omega^\beta \wedge \ldots \wedge \omega^\gamma \wedge \omega^\sigma \wedge \ldots \wedge \omega^\lambda$ has zero conformal weight. Since both $\Omega^a$ and $\Omega$ also have zero conformal weight, there exists a curvature-linear action in any dimension [8]. The most general linear case is

$$S = \int (\alpha \Omega^a_\beta + \beta \Omega \delta^a_\beta + \gamma \omega^a \wedge \omega^\beta) \wedge e^{\beta \rho \ldots \sigma} \omega^\rho \wedge \ldots \wedge \omega^\sigma \wedge \omega^\rho \wedge \ldots \wedge \omega^\sigma \wedge \omega^\rho \wedge \ldots \wedge \omega^\sigma$$

(15)

We now have three important properties of biconformal gravity that arise because of the doubled dimension: (1) the non-degenerate conformal Killing metric induces a non-degenerate metric on the manifold, (2) the dilatational structure equation generically gives a symplectic form, and (3) there exists a Weyl symmetric action functional linear in the curvature, valid in any dimension.

There are a number of known results following from the linear action. In [8] torsion-constrained solutions are found which are consistent with scale-invariant general relativity. Subsequent work along the same lines shows that the torsion-free solutions are determined by the spacetime solder form, and reduce to describe spaces conformal to Ricci-flat spacetimes on the corresponding spacetime submanifold. A supersymmetric version is presented in [13], and studies of Hamiltonian dynamics [16, 15] and quantum dynamics [16] support the idea that the models describe some type of relativistic phase space determined by the configuration space solution.

In an orthonormal frame field, $(e^a, f_a)$, adapted to the Lagrangian submanifolds, two new tensor fields emerge. The first, $v = v_a e^a + u^a f_a$ is a combination of the Weyl vector with the scale factor on the metric, and determines the timelike directions on the submanifolds. The second, $\beta^a_b = \mu^a_b + \rho^a_b = \mu_{bc} e^c + \rho^a_b f^c$, comes from the components of the spin connection, and is symmetric with respect to the new metric, $\beta^a_b = \eta^{ac} \eta_{bd} \beta^d_c$. Though this field is part of the original spin connection, it transforms homogeneously under local Lorentz transformations and local dilatations. A complete discussion is given in [1].

We now proceed to write the action, eq.(15), in this adapted basis, then vary it to find the field gravitational equations.

## 3 Variation

In the adapted basis, with the curvatures included, the Cartan structure equations become

$$d\omega^a_b = \omega^e_b \omega^a_e + \Delta^a_{de} (f_c + h_{c,f} e^f) (e^d - h^{df} f_c) + \Omega^a_b$$

$$de^a = e^e \omega^a_e + \omega e^a + \frac{1}{2} D h^{ae} (f_e + h_{c,e} e^d) + T^a$$

$$df_a = \omega^b_a f_b - \omega f_a - \frac{1}{2} D h_{ab} (e^b - h^{bc} f_c) + S_a$$

$$d\omega = e^a f_a + \Omega$$

and the action is immediately seen to be

$$S = \int (\alpha \Omega^e_b + \beta \delta^e_b \Omega + \gamma \epsilon^a b \Omega) \delta_{ac \ldots} \epsilon_{\ldots} f^e \epsilon^d \epsilon^f \ldots e^d f c \ldots f f$$

We are able to separate the Levi-Civita tensor into two pieces because we demand that $e^a, f_b$ span submanifolds.

### 3.1 The variation of the Weyl vector

We begin with the variation of the Weyl vector, $\omega$,

$$0 = \delta S$$
\[
\delta_1 = (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega_h \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right) + (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega e^{e} \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right)
\]
\[
+ (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega e^{e} \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right)
\]
\[
\delta_2 = (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega_h \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right) + (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega_h \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right)
\]
\[
+ (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega_h \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right)
\]
\[
+ (n-1) \beta \int A_j \omega^b_{kl} \left( \varepsilon_{ac...e} \varepsilon^{af...g} \omega_h \varepsilon^{e} \varepsilon^{f} \ldots \varepsilon^{f} \varepsilon_{f} \ldots \varepsilon_{f} \varepsilon_{g} \right)
\]
\[
= (-1)^n (n-1) \beta \int B^j \omega_h^k \left( \varepsilon_{ahd...e} e^k e^d \ldots e^{e\alpha f...g} f_j f_f \ldots f_g + \varepsilon_{ac...d} e^d e^e \ldots e^{d\alpha f...g} f_j f_h f_f \ldots f_g \right) \\
+ (n-1) \beta \int A_j \omega^k \left( \varepsilon_{ahd...e} e^d e^e \ldots e^f f_j f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^d e^e \ldots e^d f_j f_h f_f \ldots f_g \right)
\]

Define the volume forms,

\[
e^a \ldots e^b = \varepsilon^{a...b} \Phi^n \\
f_a \ldots f_b = \varepsilon_{a...b} \Phi^n \\
\Phi^n = \Phi^n \Phi^n
\]

Then

\[
\delta_2 = (-1)^n (n-1) \beta \int B^j \omega_h^k \left( \varepsilon_{ahd...e} e^k e^d \ldots e^{e\alpha f...g} f_j f_f \ldots f_g + \varepsilon_{ac...d} e^d e^e \ldots e^{d\alpha f...g} f_j f_h f_f \ldots f_g \right) \\
+ (-1)^n (n-1) \beta \int A_j \omega^k \left( \varepsilon_{ahd...e} e^d e^e \ldots e^f f_j f_f \ldots f_g - \varepsilon_{ac...d} e^d e^e \ldots e^d f_j f_h f_f \ldots f_g \right)
\]

\[
= (-1)^n (n-1) \beta \int B^j \omega_h^k \left( \varepsilon_{ahd...e} e^k e^d \ldots e^{e\alpha f...g} e_j f_f \ldots f_g + \varepsilon_{ac...d} e^d e^e \ldots e^{d\alpha f...g} e_j f_h f_f \ldots f_g \right) \\
+ (-1)^n (n-1) \beta \int A_j \omega^k \left( \varepsilon_{ahd...e} e^d e^e \ldots e^f f_j f_f \ldots f_g - \varepsilon_{ac...d} e^d e^e \ldots e^d f_j f_h f_f \ldots f_g \right)
\]

\[
= (-1)^n (n-1) \beta \int B^j \omega_h^k \left( \varepsilon_{ahd...e} e^k e^d \ldots e^{e\alpha f...g} e_j f_f \ldots f_g + \varepsilon_{ac...d} e^d e^e \ldots e^{d\alpha f...g} e_j f_h f_f \ldots f_g \right) \\
+ (-1)^n (n-1) \beta \int A_j \omega^k \left( \varepsilon_{ahd...e} e^d e^e \ldots e^f f_j f_f \ldots f_g - \varepsilon_{ac...d} e^d e^e \ldots e^d f_j f_h f_f \ldots f_g \right)
\]

\[
= (-1)^n (n-1) ! \beta \int B^j \left( \omega_j^k - \omega_j^h + \omega_j^k \right) \Phi^n \\
+ (-1)^n (n-1) ! \beta \int A_j \left( \omega_j^k - \omega_j^h + \omega_j^h \right) \Phi^n
\]

\[
= 0
\]

This leaves the field equation in terms of the torsion and co-torsion, as expected.

\[
0 = (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right)
\]

Expanding the variation gives two equations,

\[
0 = (n-1) \beta e^j \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right) \\
0 = (n-1) \beta f_f \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right)
\]

For the first,

\[
0 = (n-1) \beta e^j \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right) \\
= (n-1) \beta \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right)
\]

\[
= (n-1) \beta \left( \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + (-1)^{n-1} \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right) \\
= (n-1) \beta \left( (-1)^n \varepsilon_{ac...d} e^{a\alpha f...g} \tilde{T}^e d \ldots e^f f_f \ldots f_g + \varepsilon_{ac...d} e^{a\alpha f...g} e^e \ldots e^d \tilde{S}_e f_f \ldots f_g \right) \\
\]
\[ 0 = (n-1) \beta f_j \left( \varepsilon \epsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) = (n-1) \beta \left( \left( -1 \right)^n \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) \]

so that
\[
\beta \Phi_n^{\beta} = \left( \left( -1 \right)^n \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right)
\]

and similarly, for the second,
\[
0 = (n-1) \beta f_j \left( \varepsilon \epsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) = (n-1) \beta \left( \left( -1 \right)^n \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) \]

so that
\[
\beta \Phi_n^{\beta} = \left( \left( -1 \right)^n \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right)
\]

### 3.1.1 Check

Assume the tensorial character, so from the action
\[ S = \int \left( \alpha \Omega_t^\alpha + \beta \xi^\alpha \Omega + \gamma \epsilon^\alpha \epsilon_{t}^\alpha \right) \varepsilon_{ac...d}^{\alpha e...g} \epsilon^{e...d} \epsilon_{c...d} f_{e...d} f_{j} \]

we have
\[
0 = \delta S = \beta \int d \left( \delta \omega \right) \varepsilon_{ac...d}^{\alpha e...g} \epsilon^{e...d} \epsilon_{c...d} f_{e...d} f_{j}
\]

so with \( \delta \omega = A_k e^k + B^k f_k \)
\[
0 = (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) = (n-1) \beta \int A_k \left( \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right)
\]

and similarly, for the second,
\[
0 = (n-1) \beta \int \delta \omega \left( \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right) = (n-1) \beta \int B^k \left( \varepsilon_{ac...d}^{\alpha e...g} \Phi_n^{\beta} \right)
\]
\[ + (n-1) \beta \int B^k \left( (-1)^n \varepsilon_{acg...f} e^a \epsilon^b \epsilon^c \epsilon^d f_a f_b f_c f_d \right) \]

\[ = (n-1) \beta \int A_k \left( (-1)^n \varepsilon_{acg...f} T^c_{mn} e^a \epsilon^b \epsilon^c \epsilon^d f_m f_n f_c f_d \right) \]

\[ + (n-1) \beta \int B^k \left( (-1)^n \varepsilon_{acg...f} \frac{1}{2} T^c_{mn} e^a \epsilon^b \epsilon^c \epsilon^d f_k f_m f_n f_c f_d \right) \]

Therefore, the field equations are

\[ 0 = (n-1) \beta \left( (-1)^n \varepsilon_{acg...f} T^c_{mn} e^a \epsilon^b \epsilon^c \epsilon^d f_m f_n f_c f_d + (-1)^{n-1} \varepsilon_{acg...f} \frac{1}{2} \tilde{S}_e \right) \]

These check.

### 3.1.2 Check

Check again:

\[ S = \int (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a f_b) \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \]

\[ 0 = \delta \int (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a f_b) \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \]

\[ = \int \beta \delta^a_b \frac{d}{d \omega} \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \]

\[ = \int \beta \varepsilon_{acg...f} \left( \alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a f_b \right) \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \]

\[ = \int \beta \delta^a_b \frac{d}{d \omega} \left( \gamma e^a f_b \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \right) \]

\[ = \int \beta \delta^a_b \left( \gamma e^a f_b \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \right) \]

\[ = \int \beta \delta^a_b \left( \gamma e^a f_b \varepsilon_{acg...f} e^b \epsilon^c \epsilon^d f_a f_c f_d \right) \]

Then

\[ 0 = (n-1) \frac{1}{2} (n-1) \beta \varepsilon_{acg...f} T^c_{mn} e^a \epsilon^b \epsilon^c \epsilon^d f_m f_n f_a f_c f_d \]

\[ 0 = (n-1) \frac{1}{2} (n-1) \beta \varepsilon_{acg...f} T^c_{mn} e^a \epsilon^b \epsilon^c \epsilon^d f_m f_n f_a f_c f_d \]
For the first,

\[
0 = (-1)^n (n-1) \beta \epsilon_{ae \ldots f} T^{cm} \epsilon_{mnk \ldots d} \epsilon_{me \ldots f} + (-1)^{n-1} \frac{1}{2} (n-1) \beta \epsilon_{ae \ldots f} \epsilon_{mnk \ldots d} S_{e mn} \epsilon_{kc \ldots d} \epsilon_{mnk \ldots f}
\]

\[
= (-1)^n (n-1)! (n-1) \beta \epsilon_{ae \ldots f} T^{cm} \epsilon_{mnk \ldots d} \epsilon_{me \ldots f} + (-1)^{n-1} \frac{1}{2} (n-1) (n-2) \beta \epsilon_{ae \ldots f} \epsilon_{mnk \ldots d} S_{e mn}
\]

\[
= (-1)^n (n-1)! (n-1) (n-2) \beta \left( \delta_{m n}^{c k} - \delta_{m n}^{c k} \right) T^{cm} \epsilon_{mnk \ldots d} \epsilon_{me \ldots f} + (-1)^{n-1} (n-1)! (n-1) \beta S_{e kc}
\]

so that

\[
\beta \left( T^{km} m - T^{nk} n - S_{e kc} \right) = 0
\]

For the second,

\[
0 = (-1)^n \frac{1}{2} (n-1) \beta \epsilon_{ae \ldots f} T^{cm} \epsilon_{mnk \ldots d} \epsilon_{ke \ldots f} + (-1)^n (n-1) \beta \epsilon_{ae \ldots f} \epsilon_{mnk \ldots d} \epsilon_{me \ldots f}
\]

\[
= (-1)^n (n-1)! (n-1)! \beta \delta_{mn}^{kc} T^{cm} \epsilon_{mnke \ldots f} + (-1)^n (n-1)! \beta \delta_{mn}^{ae} S_{e \ mn}
\]

so that

\[
\beta \left( T^{km} m - T^{nk} n - S_{m k} m \right) = 0
\]

### 3.2 Spin connection variation

Now vary the spin connection,

\[
0 = \delta S
\]

\[
= \alpha \int \left( \partial \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \right) + \alpha \int \left( d \omega^a_{\ b} - \omega^a_{\ g} \omega^a_{\ g} - \omega^a_{\ g} \delta \omega^a_{\ g} \right) \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} + \alpha \int \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \left( \epsilon_{e d f} \epsilon_{f e} \right)
\]

\[
= \alpha \int \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \left( \epsilon_{e d f} \epsilon_{f e} \right) + \alpha \int \left( \omega^a_{\ g} \delta \omega^a_{\ g} \right) \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} + \alpha \int \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \left( \epsilon_{e d f} \epsilon_{f e} \right)
\]

Now substitute from the structure equations and expand the spin connection and its variation as

\[
\omega^a_{\ g} = \omega^a_{\ g} \epsilon^i + \omega^a_{\ g} \epsilon^i
\]

\[
\delta \omega^a_{\ b} = A^a_{\ b} \epsilon^i + B^a_{\ b} \epsilon^i
\]

Then

\[
0 = \alpha \int \delta \omega^a_{\ b} \left( (n-1) \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \epsilon_{e d f} \epsilon_{f e} \right) + \alpha \int \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \epsilon_{e d f} \epsilon_{f e} + \alpha \int \epsilon_{ae \ldots f} \epsilon_{dc \ldots f} \epsilon_{e d f} \epsilon_{f e}
\]

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\[- \alpha \int \varepsilon_{ac...de} \left( (A^\alpha_{be} e^i + B^\alpha_{bf}) f_i + (\omega^a_{ij} e^j + \omega^a_{if} f_j) + (\omega^a_{bj} e^j + \omega^a_{bf}) f_j \right) e^c e^d f_{e...f} \]

\[= \alpha \int (n - 1) \varepsilon_{ach...de} (\omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \varepsilon_{ac...d} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f}) + \alpha \varepsilon_{ach...de} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \omega_{mj} e^j f_i + f_i e^j f_{e...f} \]

so we have two equations,

\[0 = \alpha \int (n - 1) \Delta_{bm}^{ac...de} (\varepsilon_{ach...de} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \omega_{mj} e^j f_i + f_i e^j f_{e...f}) \]

Look at the Weyl vector pieces,

\[0 = \alpha \int (n - 1) \Delta_{bn}^{ac...de} (\varepsilon_{ach...de} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \omega_{mj} e^j f_i + f_i e^j f_{e...f}) \]

and

\[0 = \alpha \int (n - 1) \Delta_{bn}^{ac...de} (\varepsilon_{ach...de} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \omega_{mj} e^j f_i + f_i e^j f_{e...f}) \]

Now the spin connection terms, dropping the full volume form,

\[\sigma = \alpha \int (n - 1) \Delta_{bn}^{ac...de} (\varepsilon_{ach...de} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f} + \omega_{mj} e^j f_i + f_i e^j f_{e...f}) \]

\[- \alpha \int \frac{1}{e^c e^d f_{e...f} + \varepsilon_{ac...d} \omega^a_{mj} e^j f_i + f_i e^j f_{e...f}} \]
and

\[
\sigma' = \alpha (n-1) (-1)^n \Delta_{b}^{\alpha} \omega_{e}^{g} \varepsilon_{ach...de} e^{h} e^{i} e^{j} e^{k} e_{l} e_{m} e_{n} e_{o} e_{p} e_{q} e_{r} e_{s} e_{t} e_{u} e_{v} e_{w} e_{x} e_{y} e_{z}
\]

Expanding the torsion and co-torsion and simplifying the first,

\[
0 = \alpha \Delta_{b}^{\alpha} \left( \varepsilon_{ach...de} e^{h} e^{i} e^{j} e^{k} e_{l} e_{m} e_{n} e_{o} e_{p} e_{q} e_{r} e_{s} e_{t} e_{u} e_{v} e_{w} e_{x} e_{y} e_{z} \right)
\]

and therefore

\[
\frac{\alpha \Delta_{b}^{\alpha}(T_{eb} a - T_{eb} c \delta_{a})}{e_{b} e_{a}} = \alpha \Delta_{b}^{\alpha} \frac{\tilde{S}_{e}}{e_{b} e_{a}}
\]

There is only one independent trace; trace \( in \),

\[
\frac{\alpha \Delta_{b}^{\alpha}(T_{eb} a - T_{eb} c \delta_{a})}{e_{b} e_{a}} = \alpha \Delta_{b}^{\alpha} \frac{\tilde{S}_{e}}{e_{b} e_{a}}
\]

So we have two results,

\[
2 \Delta_{m_{b}}^{n_{b}} T_{e_{n} \epsilon_{m_{b}}} = \delta_{n_{b}} \delta_{e_{m_{b} e_{m_{b}}} e_{m_{b}}} \epsilon_{m_{b} e_{b} e_{c} e_{d}} + \delta_{n_{b}} \epsilon_{m_{b} e_{c} e_{d}} e_{m_{b}} e_{b} e_{c} e_{d} e_{m_{b}}
\]

\[
\tilde{S}_{e_{b} e_{c} e_{d}} e_{m_{b}} e_{b} e_{c} e_{d} e_{m_{b}} = - \frac{1}{n} \left( n_{b} \epsilon_{m_{b} e_{c} e_{d}} e_{m_{b}} e_{b} e_{c} e_{d} e_{m_{b}} \right)
\]
For the second, 

\[
0 = \alpha \left( \frac{1}{2} (-1)^n \varepsilon_{ach\ldots d} e^{be\ldots f} T^{\ell c} \epsilon_{km} e^k e^m e^h \ldots e^d e^f + \varepsilon_{ac\ldots d} e^{be\ldots f} e^c \ldots e^d S^e_k m f^e m f^h \ldots f_f \right)
\]

\[
= \alpha \left( \frac{1}{2} (-1)^n \varepsilon_{ach\ldots d} e^{be\ldots f} T^{\ell c} \epsilon_{km} \epsilon_{m \alpha \varepsilon_{d \varepsilon_{e f}}} + (-1)^n \epsilon_{e^c k m \alpha \varepsilon_{d \varepsilon_{e f}}} f \right) \Phi_n
\]

\[
= (-1)^n 2 (n-1)! (n-2)! \alpha \left( \frac{1}{2} \epsilon_{km} \epsilon_{m \alpha \varepsilon_{d \varepsilon_{e f}}} + \epsilon_{e^c k m \alpha \varepsilon_{d \varepsilon_{e f}}} f \right) \Phi_n
\]

\[
= (-1)^n (n-1)! (n-2)! \alpha \left( \epsilon_{km} \epsilon_{m \alpha \varepsilon_{d \varepsilon_{e f}}} + \epsilon_{e^c k m \alpha \varepsilon_{d \varepsilon_{e f}}} f \right) \Phi_n
\]

Therefore,

\[
\alpha \left( \delta^b_i T^{ac} \epsilon_{ac} + \tilde{S}^b_i a - \tilde{S}_e e^b \delta^b_i \right) = 0
\]

Trace \(b_i\),

\[
n \tilde{T}^{ac} \epsilon_{ac} = (n-1) \tilde{S}^b_a
\]

\[
\tilde{T}^{ac} \epsilon_{ac} = \frac{1}{n} (n-1) \tilde{S}^b_a
\]

Then

\[
\alpha \left( \frac{1}{n} \tilde{S}^c_e a \delta^b_i + \tilde{S}^b_i a \right) = 0
\]

\[
\tilde{S}^b_a = \frac{1}{n} \delta^b_n \tilde{S}^c_e a
\]

so we get two relations:

\[
\tilde{S}^b_a = \frac{1}{n} \delta^b_n \tilde{S}^c_e a
\]

\[
\tilde{T}^{ac} \epsilon_{ac} = \frac{1}{n} (n-1) \tilde{S}^b_a
\]

3.2.1 Check

Vary,

\[
0 = \delta S
\]

\[
= \alpha \int (\delta \Omega^b) \varepsilon_{ac\ldots d} e^{be\ldots f} e^c \ldots e^d f_c \ldots f_f
\]

\[
= \alpha \int (d \omega^a \delta \omega^b - \delta \omega^a \omega^a \omega^a - \omega^a \delta \omega^a) \varepsilon_{ac\ldots d} e^{be\ldots f} e^c \ldots e^d f_c \ldots f_f
\]

\[
= \alpha \int \varepsilon_{ac\ldots d} e^{be\ldots f} \omega^a D (e^c \ldots e^d f_c \ldots f_f)
\]

\[
= (n-1) \alpha \int \delta \omega_m \Delta^m_n (\varepsilon_{ac\ldots d} e^{be\ldots f} \tilde{T}^g e^c \ldots e^d f_c \ldots f_f + (-1)^{n-1} \varepsilon_{ac\ldots d} e^{be\ldots f} e^c \ldots e^d \tilde{S}^g e^c \ldots e^d f_c \ldots f_f)
\]

\[
= (n-1) \alpha \int (A_m k e^k + B_m k f_k) \Delta^m_n (\varepsilon_{ac\ldots d} e^{be\ldots f} \tilde{T}^g e^c \ldots e^d f_c \ldots f_f + (-1)^{n-1} \varepsilon_{ac\ldots d} e^{be\ldots f} e^c \ldots e^d \tilde{S}^g e^c \ldots e^d f_c \ldots f_f)
\]

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Then

\[
0 = (n-1) \alpha \int A^q_{abq} \left( \varepsilon_{age...d} \delta T^e \delta f \delta e... \delta f \right) + \varepsilon_{age...d} \delta b \delta e... \delta f \delta e... \delta f
\]

so the field equations are

\[
0 = (n-1) \alpha \Delta^p_{aq} \left( \varepsilon_{age...d} \delta T^e \delta f \delta e... \delta f \right) + \varepsilon_{age...d} \delta b \delta e... \delta f \delta e... \delta f
\]

\[
0 = (n-1) \alpha \Delta^p_{aq} \left( \varepsilon_{age...d} \delta T^e \delta f \delta e... \delta f \right) + \varepsilon_{age...d} \delta b \delta e... \delta f \delta e... \delta f
\]

and therefore,

\[
\alpha \Delta^p_{aq} \left( \varepsilon_{age...d} \delta T^e \delta f \delta e... \delta f \right) + \varepsilon_{age...d} \delta b \delta e... \delta f \delta e... \delta f = 0
\]

These agree with the previous results.

3.3 Solder form variation

The solder form variation is:

\[
0 = \delta S = \int (\alpha \Omega^a_b + \beta T^a_b + \gamma \delta e^a f_b) \varepsilon_{ac...d} \delta e... \delta f \delta e... \delta f
\]

\[
+ \int (n-1) (\alpha \Omega^a_b + \beta T^a_b + \gamma \delta e^a f_b) \varepsilon_{ac...d} \delta e... \delta f \delta e... \delta f
\]

\[
= \int (\alpha (\Delta^a_b \delta T^e \delta e^k + h_{ck} \delta e^k) \varepsilon_{ac...d} \delta e... \delta f \delta e... \delta f
\]

\[
+ \int (n-1) (\alpha \Omega^a_b + \beta T^a_b + \gamma \delta e^a f_b) \varepsilon_{ac...d} \delta e... \delta f \delta e... \delta f
\]

Expand the variations as

\[
\delta e^k = A^k_m e^m + B^{km} f_m
\]

\[
\delta T^e = \varepsilon^e \varepsilon \delta e^k + \delta e^k
\]

\[
= h_{me} A^k_m + h_{km} A^e_m
\]

\[
h_{ck} \delta e^k + \delta h_{ck} h_{cf}
\]

\[
- \delta h_{ck} h_{cf} f_m + h_{ck} h_{m} A^k_m + h_{ck} h_{km} h_{cf} A^e_m
\]

\[
\delta h_{cf} = -h_{ck} A^k f - h_{fk} A^k c
\]
this becomes

\[ 0 = \int A^k_m \left( -\Delta^{ac}_{ab} h^c_k \epsilon^m \delta \epsilon f_c - \Delta^a_{ab} h^c_k \epsilon^d \delta \epsilon f_c + \Delta^b_{ab} h^c_k \epsilon^e \delta \epsilon f_c - \Delta^c_{ab} \epsilon^{ef} \delta \epsilon f_c + \Delta^d_{ab} \epsilon^{ef} \delta \epsilon f_c + \Delta^e_{ab} \epsilon^{ef} \delta \epsilon f_c \right) \varepsilon_{ac} \]

\[ + \int A^k_m \left( -\beta \delta^a_{b} \delta^b_{c} + \gamma \delta^a_{c} \delta^b_{b} \right) \varepsilon_{ac} \cdots \varepsilon_{bf} \cdots \varepsilon_{ef} \cdots f_f \]

\[ + \int (n - 1) A^k_m \left( \alpha \Omega^i_{b} j + \beta \delta^a_{b} \Omega^i_{j} \right) \delta \varepsilon_{ak} \cdots \varepsilon_{bf} \cdots \varepsilon_{ef} \cdots f_f \]

\[ + \int (n - 1) B^k_m \left( \alpha \Omega^i_{b} j + \beta \delta^a_{b} \Omega^i_{j} \right) \left( -1 \right)^n \varepsilon_{ak} \cdots \varepsilon_{bf} \cdots \varepsilon_{ef} \cdots f_f \]

resulting in two field equations,

\[ 0 = \left( -1 \right)^{n-1} \alpha \left( -\Delta^a_{ab} h^c_k \epsilon^m \delta \epsilon f_c - \Delta^a_{ab} h^c_k \epsilon^d \delta \epsilon f_c + \Delta^a_{ab} h^c_k \epsilon^e \delta \epsilon f_c - \Delta^a_{ab} \epsilon^{ef} \delta \epsilon f_c + \Delta^a_{ab} \epsilon^{ef} \delta \epsilon f_c + \Delta^a_{ab} \epsilon^{ef} \delta \epsilon f_c \right) \varepsilon_{ac} \]

\[ + \left( -1 \right)^{n-1} \beta \left( \delta^a_{b} \delta^b_{c} + \gamma \delta^a_{c} \delta^b_{b} \right) \varepsilon_{ac} \cdots \varepsilon_{bf} \cdots \varepsilon_{ef} \cdots f_f \]

\[ + \left( -1 \right)^{n-1} \left( \alpha \Omega^i_{b} j + \beta \delta^a_{b} \Omega^i_{j} \right) \delta \varepsilon_{ak} \cdots \varepsilon_{bf} \cdots \varepsilon_{ef} \cdots f_f \]

Expand the first,

\[ 0 = -\frac{1}{2} \alpha \delta^m_{k} \epsilon^m \delta \epsilon f_c - \frac{1}{2} \alpha \delta^m_{k} \epsilon^d \delta \epsilon f_c + \frac{1}{2} \alpha \delta^m_{k} \epsilon^e \delta \epsilon f_c - \frac{1}{2} \alpha \epsilon^{ef} \delta \epsilon f_c + \frac{1}{2} \alpha \epsilon^{ef} \delta \epsilon f_c + \frac{1}{2} \alpha \epsilon^{ef} \delta \epsilon f_c \]

\[ + \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) - \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) + \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) + \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) + \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) + \frac{1}{2} \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) \]

The field equation is therefore

\[ 0 = \alpha \Omega^m_{b k} - \alpha \Omega^m_{b k} \delta^m_{k} + \beta \Omega^m_{k} - \beta \Omega^m_{k} \delta^m_{k} + \frac{1}{2} \left( \alpha \Omega^m_{b} \eta^m_{c} \delta \epsilon f_c \right) \]

Check the trace,

\[ 0 = -\left( n - 1 \right) \left( \alpha \Omega^m_{b} \eta^m_{c} + \beta \Omega^m_{k} \right) \]

Therefore, we may write

\[ 0 = \alpha \Omega^m_{b k} + \beta \Omega^m_{k} - \left( \alpha \Omega^m_{b} \eta^m_{c} + \beta \Omega^m_{k} \right) \delta^m_{k} \]

\[ + \frac{1}{2} \left( \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) - \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) \right) \delta^m_{k} \]

Therefore, we may write

\[ 0 = \alpha \Omega^m_{b k} + \beta \Omega^m_{k} - \left( \alpha \Omega^m_{b} \eta^m_{c} + \beta \Omega^m_{k} \right) \delta^m_{k} \]

\[ + \frac{1}{2} \left( \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) - \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) \right) \delta^m_{k} \]

\[ + \frac{1}{2} \left( \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) - \alpha \eta^m_{c} \delta \epsilon f_c \left( \eta_{db} \delta^d_{h} \right) \right) \delta^m_{k} \]
The second equation is

$$0 = \frac{1}{2} \alpha ((n-2) h_{mk} + \eta_{km} (\eta^{an} h_{na}) - \alpha \Omega^m_{nak} - \beta \Omega_{mk}$$

### 3.4 Variation of the co-solder form

Finally, we vary $f_a$.

$$0 = \delta S = \int (\alpha \delta \Omega^a_b + \beta \delta_b^a \delta \Omega + \gamma e^a \delta f_a) \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_e \ldots f_f$$

$$+ \int (\alpha \Omega^a_b + \beta \delta_b^a \Omega + \gamma e^a f_b) (n-1) \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_g \ldots f_f$$

$$= \int \left( \alpha \left( -\Delta^a_{jb} (\delta f_g + h_{gy} e^i) (e^i - h_{kjb} f_k) - \Delta^a_{jb} (f_g + h_{gy} e^i) (-\delta h_{kjb} f_k - h_{jkb} \delta f_k) \right) - \beta \delta_b^a \delta f_g + \gamma e^a f_b \right) \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_g \ldots f_f$$

$$+ \int (\alpha \Omega^a_b + \beta \delta_b^a \Omega + \gamma e^a f_b) (n-1) \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_g \ldots f_f$$

To find how $h_{ab}$ varies we have

$$\langle \delta f_a, f_b \rangle + \langle f_a, \delta f_b \rangle = -\delta h_{ab}$$

so with $\delta f_a = C_{ab} e^b + D_a^b f_b$,

$$D_a^c h_{cb} + D_b^c h_{ac} = \delta h_{ab}$$

and therefore,

$$h^n_a D_a^c h_{cb} + D_b^n h_{ac} = h^n_a \delta h_{ab}$$

$$-h^n_a D_a^m - h^n_{mb} D_b^n = \delta h^{nm}$$

Now substitute for the variations and collect terms.

$$0 = \int C_{gr} \left( \alpha \Delta^a_{jb} h^{j_s} - \alpha h^{j_s} \Delta^a_{jb} \right) (-1)^{n-1} \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_e \ldots f_f$$

$$+ \int C_{gm} \left( \frac{1}{2} \alpha \Omega^a_{grs} + \frac{1}{2} \beta \delta_b^a \Omega^s \right) (-1)^{n-1} (n-1)^{n-1}$$

$$+ \int \left( \alpha \Delta^a_{rb} D_g^r - \alpha \Omega^a_{rb} \right) m h_{mj} h^{j_s} + \alpha \Delta^a_{jb} D_r^m h_{gm} h^{j_s} - \alpha \Delta^a_{jb} h_{gr} h^{jm} D_m^j - \alpha \Delta^a_{jb} h_{gr} h^{jm} D_m^j - \alpha \Delta^a_{jb} h_{gr} h^{jm} D_m^j$$

$$+ \int \left( \alpha \Omega^a_{rb} \right) m (\alpha \Omega^a_{rb} + \beta \delta_b^a \Omega^s - \gamma \delta_b^a \delta^s) (n-1) (-1)^{n} \varepsilon_{ac...d} e^{be...f} e^c \ldots e^d f_e \ldots f_f$$

The two field equations are:

$$0 = \frac{1}{2} \alpha ((n-2) h_{mk} + \eta_{km} (\eta^{an} h_{na}) - \alpha \Omega^m_{nak} - \beta \Omega_{mk}$$

$$0 = \alpha \Omega^m_{ak} - \beta \Omega_{ak} - \frac{1}{2} \eta_{ak} (\eta_{bj} h_{jb})$$

or (from the check, and with better indices),

$$0 = \alpha \Omega^b_{ki} + \beta \Omega^k_i - \frac{1}{2} (n-2) \alpha h_{ki}^i - \frac{1}{2} \alpha \eta_{ki} (\eta_{bj} h_{jb})$$

$$0 = \alpha \Omega^m_{ak} - \beta \Omega_{ak} - \frac{1}{2} \eta_{ak} (\eta_{bj} h_{jb})$$

$$+ \frac{1}{2} \left( n \alpha - 2 \beta + 2n^2 \gamma \right) \delta^r_s - \alpha \eta_{rc} h_{ac} (\eta_{ab} h_{cb}) + \frac{1}{2} \alpha \eta_{ac} (\eta_{ab} h_{ac})$$

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3.4.1 Check

From the action,

\[
0 = \delta S = \int (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a_b f_b) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
= \int (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a_b f_b) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (n-1) (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a_b f_b) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
= \int (-\alpha \Delta^a_{jb} (\delta f_j + \delta h_{ik} e^k) (e^j - h^{jm} f_m) - \alpha \Delta^a_{jb} (f_j + h_{ik} e^k) (-\delta h^{jm} f_m - h^{jm} \delta f_m)) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

Now substitute variations,

\[
\delta f_a = C_{ab} b^b + D_a b^b
\]

\[
\delta h_{ab} = \{ f_a, C_{ab} e^c + D_a e^c f_b \} = D_b \varepsilon_h c_{ac} + D_a \varepsilon_h c_{cb}
\]

\[
\delta h^{de} = -h^{ac} D_a d - h^{ad} D_a e
\]

to get

\[
0 = \int (-\alpha \Delta^a_{jb} (C_{ik} e^k + D_i k^k f_k + D_k m h_{mi} e^i + D_i m h_{mk} e^k) (e^j - h^{jm} f_m)) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (-\alpha \Delta^a_{jb} (f_j + h_{ik} e^k) (h^{nj} D_n m f_m + h^{nm} D_n j f_m - h^{jm} C_{mk} e^k - h^{jm} D_{mk} k f_k)) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (-\beta \delta^b_d e^a D_g e^b e^c e^d f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (n-1) (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a_b f_b) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

Then

\[
0 = \int (\alpha \Delta^a_{jb} C_{ik} h^{jm} e^k f_m + \alpha \Delta^a_{jb} D_i k^k f_k + \alpha \Delta^a_{jb} D_k m h_{mi} h^{jn} e^j f_n + \alpha \Delta^a_{jb} D_i m h_{mk} h^{jn} e^k f_n) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (-\alpha \Delta^a_{jb} h^{nj} D_n m h_{ik} e^k f_m - \alpha \Delta^a_{jb} h^{jm} D_m j h_{ik} e^k f_m - \alpha \Delta^a_{jb} h^{jm} C_{mk} e^k f_i + \alpha \Delta^a_{jb} h^{jm} D_{mk} k h_{in} e^n f_k) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int (-\beta \delta^b_d D_g k^k + \gamma D_b k^k \delta^a_b \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int C_{ek} (n-1) \left( \frac{1}{2} \alpha \Omega^m_m + \frac{1}{2} \beta \delta^a_b \Omega^m_m \right) (-1)^{n-1} \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]

\[
+ \int D_e k (n-1) (\alpha \Omega^a_b + \beta \delta^a_b \Omega + \gamma e^a_b f_b) \varepsilon_{ac...de} f_c ... f_d e^e ... f_e \delta f_e \delta f_f ...
\]
so that

\[
0 = \int \left( \alpha \Delta_{jk}^p C_{pk} h^{jn} + \alpha \Delta_{bk}^p D_k \cdot n + \alpha \Delta_{ib}^p D_i \cdot m h_{mk} h^{jn} + \alpha \Delta_{jk}^p D_k \cdot m h_{mj} h^{jn} \right) \left( -1 \right)^{n-1} \varepsilon_{ac...d} \varepsilon_{b...f} e^k e^c \cdots e^f f_n f_e \cdots f_f
\]

\[
+ \int \left( -\alpha \Delta_{jk}^p h^{nj} D_n \cdot m h_{ik} - \alpha \Delta_{bj}^p h^{mj} D_j \cdot m h_{ik} - \alpha \Delta_{jm}^p h^{ji} C_{ik} + \alpha \Delta_{bj}^p h^{jm} D_n \cdot m h_{ik} \right) \left( -1 \right)^{n-1} \varepsilon_{ac...d} \varepsilon_{b...f} e^k e^c \cdots e^f f_m f_n f_e \cdots f_f
\]

\[
+ \int \left( -\beta \delta_{b}^p D_k \cdot g + \gamma D_k \cdot b \delta_{g}^p \right) \left( -1 \right)^{n-1} \varepsilon_{ac...d} \varepsilon_{b...f} e^k e^c \cdots e^f f_k f_e \cdots f_f
\]

\[
+ \int C_{ek} \left( n-1 \right) \left( \frac{1}{2} \Omega_{b}^{mn} + \frac{1}{2} \beta \delta_{b}^p \Omega_{mn} \right) \left( -1 \right)^{n-1} \varepsilon_{ac...d} \varepsilon_{b...f} e^k e^c \cdots e^f f_m f_n f_e \cdots f_f
\]

\[
+ \int D_e \left( n-1 \right) \left( -\alpha \Omega_{b}^{mn} - \beta \delta_{b}^p \Omega_{nm} + \gamma \delta_{b}^p \delta_{n}^a \right) \left( -1 \right)^{n-1} \varepsilon_{ac...d} \varepsilon_{b...f} e^k e^c \cdots e^f f_m f_n f_e \cdots f_f
\]

Next,

\[
0 = \alpha \Delta_{jk}^p h^{jn} \delta_a^b \delta_{c}^a - \alpha \Delta_{bk}^p h^{ji} \delta_{c}^b \delta_{m}^k + 2 \left( \frac{1}{2} \alpha \Omega_{b}^{mn} + \frac{1}{2} \beta \delta_{b}^p \Omega_{mn} \right) \delta_{a}^b \delta_{m}^n
\]

so that

\[
0 = \alpha \Omega_{b}^{bi} + \beta \Omega_{ki} - \frac{1}{2} \left( n - 2 \right) \alpha h^{ki} - \frac{1}{2} \eta_{j}^{b} \eta_{j}^{b} \eta_{j}^{b} \eta_{j}^{b}
\]

\[
0 = \alpha \Omega_{s}^{ar} a - \alpha \Omega_{s}^{am} a \delta_s^a + \beta \Omega_{r} - \beta \Omega_{a} a \delta_s^a
\]

\[
+ \frac{1}{2} \left( n \alpha - 2 \beta + 2 n^2 \gamma \right) \delta_s^a - \alpha \eta_{rc} h_{sc} \left( \eta_{ab} h_{ab} \right) + \frac{1}{2} \alpha \eta_{ac} h_{rc} \left( \eta_{ab} h_{ab} \right)
\]

4 Summary of field equations

Collecting the field equations:

\[
\beta \tilde{S}_{jk} = \beta \left( \tilde{T}_{jk} - \tilde{T}_{kj} \right)
\]

\[
\beta \tilde{T}_{jc} = \beta \left( \tilde{S}_{c} - \tilde{S}_{j} \right)
\]

\[
0 = \alpha \Delta_{biq} \delta_{k}^{b} \delta_{c}^{a} + \tilde{S}_{a} - \delta_{k}^{b} \tilde{S}_{m} \delta_{a}^{m}
\]

\[
0 = \alpha \Delta_{biq} \left( \tilde{T}_{a} - \delta_{a}^{b} \tilde{T}_{n} \delta_{b}^{n} \right)
\]

\[
0 = \alpha \Omega_{b}^{-k} - \alpha \Omega_{b}^{am} a \delta_{m}^a + \beta \Omega_{m} - \beta \Omega_{a} a \delta_{m}^a + \frac{1}{2} \left( n \alpha - 2 \beta + 2 n^2 \gamma \right) \delta_{m}^a - \alpha \epsilon_{bk} h_{ck} \left( \eta_{ab} h_{ab} \right) + \frac{1}{2} \alpha \epsilon_{mc} h_{ck} \left( \eta_{ab} h_{ab} \right)
\]

\[
0 = \alpha \Omega_{m}^{-a} a - \alpha \Omega_{b}^{-a} a \delta_{m}^a + \beta \Omega_{a} - \beta \Omega_{m} a \delta_{m}^a + \frac{1}{2} \left( n \alpha - 2 \beta + 2 n^2 \gamma \right) \delta_{m}^a - \alpha \epsilon_{ag} h_{ma} \left( \eta_{ab} h_{ab} \right) + \frac{1}{2} \alpha \epsilon_{mg} h_{ag} \left( \eta_{ak} h_{ak} \right)
\]
Notice that the first four equations involve only the torsion and co-torsion, while the remaining four depend only on the Lorentz and dilatational curvatures.

References


