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Studies in torsion free biconformal spaces. Case 2: \gamma_{-} = 0

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Abstract

We show that the solutions for the symmetric part of the connection in homogeneous biconformal space also satisfy the more general field equations of curved biconformal spaces in the case when $\gamma_- = 0$.

1 Introduction

Starting with the conformal symmetries of Euclidean space, we have shown in [1] how construct a manifold where time manifests as a part of the geometry. The result is based on a theorem by Spencer and Wheeler [3] detailing when this is possible. In addition, though there is no matter present in the geometries studied in [1], geometric terms analogous to dark energy and dark matter appear in the Einstein tensor.

Specifically, the quotient of the conformal group of Euclidean four-space by its Weyl subgroup results in a *biconformal* geometry possessing many of the properties of relativistic phase space, including both a natural symplectic form and non-degenerate Killing metric. It is shown that the general solution for this homogeneous space posesses orthogonal Lagrangian submanifolds, with the induced metric and the spin connection on the submanifolds necessarily Lorentzian, despite the Euclidean starting point. A detailed presentation of our general methods is given in [1], and a synopsis of the model studied here in [2].

Though an explicit gravitational theory is presented in [1], the main results of that work apply to the homogeneous space. It is of great interest to extend this study to nontrivial gravitational solutions. This has been accomplished for certain restricted cases [4], but a more general approach is desirable. In [2], we showed that when $\gamma_+ = 0$, the torsion and co-torsion field equations for Euclidean biconformal space may be solved by starting with the solution for the symmetric part of the connection in flat biconformal space. Here we extend this same conclusion to the $\gamma_- = 0$ case.

2 The vanishing torsion field equations with the homogeneous solution

2.1 The torsion free field equations

As seen in [2], the torsion-free field equations reduce to

$$0 = \Delta^{ap}_{qb}\left(\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ e}_{\ e} - \delta^c_a\eta^{bf}v_f\right) - \left(\eta^{cd}\mu^{b}_{\ da} - \delta^c_a\eta^{bd}\mu^{f}_{\ df} - \eta^{cd}\delta^b_d v_a\right)\right) \tag{1}$$

while for the co-torsion we have

$$\begin{aligned}
0 &= S_b^{\ ab} \\
0 &= S_a^{\ b}{}_{\ b} - S_b^{\ b}{}_{\ a} \\
0 &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e} + \eta_{cd}\eta^{de}\mu^{b}_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{f}_{\ ef}\right) \\
&\quad +\Delta^{ap}_{qb}\left(\eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}u^b + \delta^b_c\eta_{ea}u^e\right)
\end{aligned} \tag{2}$$

The involution conditions become

$$
\begin{aligned}
0 &= \eta^{ad}\rho^{b\ c}_{\ d} - \eta^{ad}\rho^{c\ b}_{\ d} + \eta^{ac}u^b - \eta^{ab}u^c \\
k\eta^{ae}S_{ebc} &= \mu^a_{\ bc} - \mu^a_{\ cb} - \delta^a_b v_c + \delta^a_c v_b
\end{aligned} \tag{3}
$$

2.2 The homogeneous solution

From the zero curvature solution, we have

$$
\boldsymbol{\mu}^a_{\ b} = \left(\delta^a_b v_c - k\gamma_+\left(\delta^a_b v_c + \delta^a_c v_b + \eta^{ad}\eta_{bc}v_d + \frac{2}{\beta^2}\eta^{ad}v_b v_c v_d\right)\right)\mathbf{e}^c \tag{4}
$$

$$
\boldsymbol{\rho}^a_{\ b} = \left(\delta^a_b u^c + k\gamma_-\left(\delta^a_b u^c + \delta^c_b u^a + \eta^{ac}\eta_{bd}u^d + \frac{2}{\beta^2}\eta_{bd}u^a u^c u^d\right)\right)\mathbf{f}_c \tag{5}
$$

Extracting the coefficients and finding the traces, we have

$$
\begin{aligned}
\rho^{c\ b}_{\ a} &= \delta^c_a u^b + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right) \\
\rho^{b\ f}_{\ f} &= \left(1 + k\gamma_-(n+2) + \frac{2k\gamma_-}{\beta^2}u^2\right)u^b \\
\mu^b_{\ da} &= \delta^b_d v_a - k\gamma_+\left(\delta^b_d v_a + \delta^b_a v_d + \eta^{be}\eta_{da}v_e + \frac{2}{\beta^2}\eta^{be}v_d v_a v_e\right) \\
\mu^f_{\ df} &= \left(1 - k\gamma_+(n+2) - \frac{2k\gamma_+}{\beta^2}v^2\right)v_d
\end{aligned}
$$

where

$$
\gamma_\pm \equiv \frac{1}{2\beta}\left(1 \pm k\beta^2\right)
$$

and we define

$$
\begin{aligned}
u^2 &\equiv \eta_{ab}u^a u^b \\
U^2 &\equiv \delta_{ab}u^a u^b \\
v^2 &\equiv \eta^{ab}v_a v_b \\
V^2 &\equiv \delta^{ab}v_a v_b
\end{aligned}
$$

2.3 Combining these

We would like to know whether the cubic forms of $\rho^{c\ b}_{\ a}$ and $\mu^b_{\ da}$ satisfy the field equation. Substituting,

$$
\begin{aligned}
0 &= \Delta^{ap}_{qb}\left(\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ f}_{\ f} - \delta^c_a\eta^{bf}v_f\right) - \left(\eta^{cd}\mu^b_{\ da} - \delta^c_a\eta^{bd}\mu^f_{\ df} - \eta^{cd}\delta^b_d v_a\right)\right) \\
&= \Delta^{ap}_{qb}\left(\delta^c_a u^b + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\right) \\
&\quad -\Delta^{ap}_{qb}\left(\left(1 + k\gamma_-(n+2) + \frac{2k\gamma_-}{\beta^2}u^2\right)u^b + \eta^{bf}v_f\right) \\
&\quad -\Delta^{ap}_{qb}\left(\eta^{bc}v_a - k\gamma_+\left(\eta^{bc}v_a + \delta^b_a\eta^{cd}v_d + \delta^c_a\eta^{be}v_e + \frac{2}{\beta^2}\eta^{cd}\eta^{be}v_d v_e v_a\right)\right) \\
&\quad +\Delta^{ap}_{qb}\left(\left(1 - k\gamma_+(n+2) - \frac{2k\gamma_+}{\beta^2}v^2\right)\delta^c_a\eta^{bd}v_d + \eta^{bc}v_a\right)
\end{aligned}
$$

$$
\begin{aligned}
&= k\gamma_- \Delta^{ap}_{qb} \left(\delta^b_a u^c + \eta^{cb}\eta_{ad} u^d + \frac{2}{\beta^2}\eta_{ad} u^c u^b u^d - \left((n+1) + \frac{2u^2}{\beta^2} \right) \delta^c_a u^b \right) \\
&\quad + k\gamma_+ \Delta^{ap}_{qb} \left(\eta^{bc} v_a + \delta^b_a \eta^{cd} v_d + \frac{2}{\beta^2}\eta^{cd}\eta^{be} v_d v_e v_a - \left((n+1) + \frac{2v^2}{\beta^2} \right) \delta^c_a \eta^{bd} v_d \right)
\end{aligned}
$$

Expanding $\Delta^{ap}_{qb} = \frac{1}{2}\left(\delta^a_q \delta^p_b - \delta^{ab}\delta_{bq}\right)$ This becomes

$$
\begin{aligned}
0 &= k\gamma_- \left(\delta^p_q u^c + \eta^{cp}\eta_{qd} u^d + \frac{2}{\beta^2}\eta_{qd} u^c u^p u^d - \left((n+1) + \frac{2u^2}{\beta^2} \right) \delta^c_q u^p \right) \\
&\quad - k\gamma_- \left(\delta^{ap}\delta_{bq}\delta^b_a u^c + \delta^{ap}\delta_{bq}\eta^{cb}\eta_{ad} u^d + \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad} u^c u^b u^d - \left((n+1) + \frac{2u^2}{\beta^2} \right) \delta^{ap}\delta_{bq}\delta^c_a u^b \right) \\
&\quad + k\gamma_+ \left(\eta^{pc} v_q + \delta^p_q \eta^{cd} v_d + \frac{2}{\beta^2}\eta^{cd}\eta^{pe} v_d v_e v_q - \left((n+1) + \frac{2v^2}{\beta^2} \right) \delta^c_q \eta^{pd} v_d \right) \\
&\quad - k\gamma_+ \left(\delta^{ap}\delta_{bq}\eta^{bc} v_a + \delta^{ap}\delta_{bq}\delta^b_a \eta^{cd} v_d + \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta^{cd}\eta^{be} v_d v_e v_a - \left((n+1) + \frac{2v^2}{\beta^2} \right) \delta^{ap}\delta_{bq}\delta^c_a \eta^{bd} v_d \right) \\
&= k\gamma_- \left(\eta^{cp}\eta_{qb} u^b + (n+1)\,\delta^{pc}\delta_{bq} u^b - \left((n+1) + \frac{2u^2}{\beta^2} \right) \delta^c_q u^p - \eta^{cb}\delta_{bq}\delta^{ap}\eta_{ad} u^d \right) \\
&\quad + k\gamma_- \left(\frac{2}{\beta^2}\eta_{qd} u^c u^p u^d - \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad} u^c u^b u^d + \frac{2u^2}{\beta^2}\delta^{pc}\delta_{bq} u^b \right) \\
&\quad + k\gamma_+ \left(\eta^{pc} v_q + (n+1)\,\delta^{cp}\delta_{bq}\eta^{bd} v_d - \left((n+1) + \frac{2v^2}{\beta^2} \right) \delta^c_q \eta^{pd} v_d - \delta^{ap}\delta_{bq}\eta^{bc} v_a \right) \\
&\quad + k\gamma_+ \left(\frac{2}{\beta^2}\eta^{cd}\eta^{pe} v_d v_e v_q - \frac{2}{\beta^2}\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be} v_d v_e v_a + \frac{2v^2}{\beta^2}\delta^{cp}\delta_{bq}\eta^{bd} v_d \right)
\end{aligned}
$$

Therefore, the torsion field equation is:

$$
\begin{aligned}
0 &= k\gamma_- \left(\eta^{cp}\eta_{qb} u^b + (n+1)\,\delta^{pc}\delta_{bq} u^b - \left((n+1) + \frac{2u^2}{\beta^2} \right) \delta^c_q u^p - \eta^{cb}\delta_{bq}\delta^{ap}\eta_{ad} u^d \right) \\
&\quad + k\gamma_- \left(\frac{2}{\beta^2}\eta_{qd} u^c u^p u^d - \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad} u^c u^b u^d + \frac{2u^2}{\beta^2}\delta^{pc}\delta_{bq} u^b \right) \\
&\quad + k\gamma_+ \left(\eta^{pc} v_q + (n+1)\,\delta^{cp}\delta_{bq}\eta^{bd} v_d - \left((n+1) + \frac{2v^2}{\beta^2} \right) \delta^c_q \eta^{pd} v_d - \delta^{ap}\delta_{bq}\eta^{bc} v_a \right) \\
&\quad + k\gamma_+ \left(\frac{2}{\beta^2}\eta^{cd}\eta^{pe} v_d v_e v_q - \frac{2}{\beta^2}\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be} v_d v_e v_a + \frac{2v^2}{\beta^2}\delta^{cp}\delta_{bq}\eta^{bd} v_d \right) \qquad (6)
\end{aligned}
$$

3 The $\gamma_- = 0$ case: necessary and sufficient conditions

In this Section we find the necessary and sufficient conditions for the solution of eq.(6) when $\gamma_- = 0$.

3.1 The necessary form of the inverse metric

Consider the $\gamma_- = \frac{1}{2\beta}\left(1 - k\beta^2\right) = 0$ case. Then we have $k = 1$ and $\beta^2 = 1$, so letting $\beta = \gamma_+ = \pm 1$,

$$
\begin{aligned}
0 &= \eta^{pc} v_q + (n+1)\,\delta^{cp}\delta_{bq}\eta^{bd} v_d - \left((n+1) + 2v^2 \right) \delta^c_q \eta^{pd} v_d - \delta^{ap}\delta_{bq}\eta^{bc} v_a \\
&\quad + 2\eta^{cd}\eta^{pe} v_d v_e v_q - 2\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be} v_d v_e v_a + 2v^2\delta^{cp}\delta_{bq}\eta^{bd} v_d \qquad (7)
\end{aligned}
$$

Contracing with $\delta^{qe} v_e$ we find

$$
0 = V^2\eta^{pc} + (n+1)\, v^2\delta^{cp} - \left((n+1) + 2v^2 \right) \delta^{ce} v_e \eta^{pd} v_d - \eta^{bc} v_b \delta^{ap} v_a
$$

$$
\begin{aligned}
& && +2V^2\eta^{cd}\eta^{pf}v_dv_f - 2v^2\eta^{cd}\delta^{ap}v_dv_a + 2v^2v^2\delta^{cp} \\
\left(2v^2+(n+1)\right)v^2\delta^{cp} &= && -V^2\eta^{pc} + \left((n+1)+2v^2\right)\delta^{ce}v_e\eta^{pd}v_d + \eta^{bc}v_b\delta^{ap}v_a \\
& && -2V^2\eta^{cd}\eta^{pf}v_dv_f + 2v^2\eta^{cd}\delta^{ap}v_dv_a
\end{aligned}
$$

Contract again. With v_p we get an identity,

$$
\begin{aligned}
\left(2v^2+(n+1)\right)v^2\delta^{cp}v_p &= -V^2\eta^{pc}v_p + \left((n+1)+2v^2\right)v^2\delta^{ce}v_e + V^2\eta^{bc}v_b - 2v^2V^2\eta^{cd}v_d + 2v^2V^2\eta^{cd}v_d \\
\left(2v^2+(n+1)\right)v^2\delta^{cp}v_p &= \left((n+1)+2v^2\right)v^2\delta^{ce}v_e
\end{aligned}
$$

while contracting with v_c we get

$$
\begin{aligned}
\left(2v^2+(n+1)\right)v^2\delta^{cp}v_c &= -V^2\eta^{pc}v_c + \left((n+1)+2v^2\right)V^2\eta^{pd}v_d + v^2\delta^{ap}v_a - 2V^2v^2\eta^{pf}v_f + 2v^2v^2\delta^{ap}v_a \\
nv^2\delta^{ap}v_a &= nV^2\eta^{pf}v_f \\
\delta^{ap}v_a &= \frac{V^2}{v^2}\eta^{pf}v_f
\end{aligned}
$$

Notice that $V^2 > 0$ implies v^2 is nonzero.

Making this replacement in the expression for δ^{cp},

$$
\left(n+1+2v^2\right)v^2\delta^{cp} = -V^2\eta^{pc} + \left(n+2+2v^2\right)\frac{V^2}{v^2}\eta^{cd}v_d\eta^{pf}v_f
$$

so that the inverse metric is given by

$$
\delta^{cp} = -\frac{V^2}{\left(n+1+2v^2\right)v^2}\left(\eta^{pc} - \frac{1}{v^2}\left(n+2+2v^2\right)\eta^{cd}v_d\eta^{pf}v_f\right) \tag{8}
$$

3.2 The metric

Now invert to find the metric. Since the answer must be unique, we may begin with the ansatz

$$
\delta_{pd} = -\frac{\left(n+1+2v^2\right)v^2}{V^2}\left(\eta_{pd} + \alpha v_pv_d\right)
$$

Then α is determined by

$$
\begin{aligned}
\delta^c_d &= \delta^{cp}\delta_{pd} \\
&= \frac{V^2}{n+1+2v^2}\left(\eta^{pc} - \frac{1}{v^2}\left((n+2)+2v^2\right)\eta^{cb}v_b\eta^{pa}v_a\right)\frac{n+1+2v^2}{V^2}\left(\eta_{pd}+\alpha v_pv_d\right) \\
&= \delta^c_d + \alpha\eta^{pc}v_pv_d - \frac{1}{v^2}\left((n+2)+2v^2\right)\left(\eta^{cb}v_bv_d + \alpha v^2\eta^{cb}v_bv_d\right) \\
0 &= \left(\alpha - \frac{1}{v^2}\left((n+2)+2v^2\right)\left(1+\alpha v^2\right)\right)\eta^{cb}v_bv_d \\
\alpha &= -\frac{1}{v^2}\left(\frac{n+2+2v^2}{n+1+2v^2}\right)
\end{aligned}
$$

so the metric is

$$
\delta_{pd} = -\frac{\left(n+1+2v^2\right)v^2}{V^2}\left(\eta_{pd} - \frac{1}{v^2}\left(\frac{n+2+2v^2}{n+1+2v^2}\right)v_pv_d\right) \tag{9}
$$

3.3 Sufficiency

We now return to the original equation, eq.(7), to show that the specification of the metric is also sufficient to solve the torsion equation,

$$\begin{aligned} 0 &= \eta^{pc}v_q + (n+1)\,\delta^{cp}\delta_{bq}\eta^{bd}v_d - \left((n+1)+2v^2\right)\delta^c_q\eta^{pd}v_d - \delta^{ap}\delta_{bq}\eta^{bc}v_a \\ &\quad +2\eta^{cd}\eta^{pe}v_dv_ev_q - 2\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be}v_dv_ev_a + 2v^2\delta^{cp}\delta_{bq}\eta^{bd}v_d \end{aligned}$$

We need the combinations

$$\begin{aligned} \delta_{bq}\eta^{bd}v_d &= \frac{v^2}{V^2}v_q \\ \delta^{ap}v_a &= \frac{V^2}{v^2}\eta^{ap}v_a \end{aligned}$$

Then we find,

$$\begin{aligned} 0 &= \eta^{pc}v_q + (n+1)\,\delta^{cp}\delta_{bq}\eta^{bd}v_d - \left((n+1)+2v^2\right)\delta^c_q\eta^{pd}v_d - \delta^{ap}\delta_{bq}\eta^{bc}v_a \\ &\quad +2\eta^{cd}\eta^{pe}v_dv_ev_q - 2\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be}v_dv_ev_a + 2v^2\delta^{cp}\delta_{bq}\eta^{bd}v_d \\ &= \eta^{pc}v_q + (n+1)\left(-\frac{V^2}{(n+1+2v^2)\,v^2}\left(\eta^{pc} - \frac{1}{v^2}\left(n+2+2v^2\right)\eta^{cd}v_d\eta^{pf}v_f\right)\right)\frac{v^2}{V^2}v_q - \left((n+1)+2v^2\right)\delta^c_q\eta^{pd}v_d \\ &\quad -\left(-\frac{\left(n+1+2v^2\right)v^2}{V^2}\left(\eta_{bq} - \frac{1}{v^2}\left(\frac{n+2+2v^2}{n+1+2v^2}\right)v_bv_q\right)\right)\eta^{bc}\frac{V^2}{v^2}\eta^{ap}v_a + 2\eta^{cd}\eta^{pe}v_dv_ev_q \\ &\quad -2\eta^{cd}\frac{V^2}{v^2}\eta^{ap}v_a\frac{v^2}{V^2}v_qv_d + 2v^2\left(-\frac{V^2}{(n+1+2v^2)\,v^2}\left(\eta^{pc} - \frac{1}{v^2}\left(n+2+2v^2\right)\eta^{cd}v_d\eta^{pf}v_f\right)\right)\frac{v^2}{V^2}v_q \\ &= \eta^{pc}v_q - \frac{(n+1)}{(n+1+2v^2)}\eta^{pc}v_q + \frac{1}{v^2}\frac{(n+1)}{(n+1+2v^2)}\left(n+2+2v^2\right)\eta^{cd}v_d\eta^{pf}v_fv_q - \left((n+1)+2v^2\right)\delta^c_q\eta^{pd}v_d \\ &\quad +\left(n+1+2v^2\right)\delta^c_q\eta^{ap}v_a - \frac{1}{v^2}\frac{\left(n+1+2v^2\right)^2}{n+1+2v^2}\eta^{ap}\eta^{cb}v_av_bv_q + 2\eta^{cd}\eta^{pe}v_dv_ev_q \\ &\quad -2\eta^{cd}\eta^{ap}v_av_qv_d - \frac{2v^2}{(n+1+2v^2)}\eta^{pc}v_q + \frac{2}{(n+1+2v^2)}\left(n+2+2v^2\right)\eta^{cd}\eta^{pf}v_dv_fv_q \\ &= \left(1 - \frac{(n+1)}{(n+1+2v^2)} - \frac{2v^2}{(n+1+2v^2)}\right)\eta^{pc}v_q \\ &\quad +\left(\left(n+1+2v^2\right) - \left(n+1+2v^2\right)\right)\delta^c_q\eta^{pd}v_d \\ &\quad +\frac{\left(n+2+2v^2\right)}{(n+1+2v^2)}\left(\frac{1}{v^2}(n+1) - \frac{1}{v^2}\left(n+1+2v^2\right)+2\right)\eta^{cd}\eta^{pe}v_dv_ev_q \\ &\equiv 0 \end{aligned}$$

which shows that the form of the metric given by eq.(9) is necessary and sufficient for the $\gamma_+ = 0$ flat solution to solve the field equation.

4 The remaining field equations

We have solved eq.(7). For the co-torsion we already have $S_b^{\ ab} = 0$ and $S_a^{\ b}{}_b = S_b^{\ b}{}_a$, and the involution condition gives $S_{abc} = 0$. We now examine the final field equation, eq.(2):

$$\begin{aligned} 0 &= \Delta^{ap}_{qb}\left(S_c^{\ b}{}_a - \delta^b_cS_e^{\ e}{}_a + \eta_{cd}\eta^{de}\mu^{\ b}_{ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{\ f}_{ef}\right) \\ &\quad +\Delta^{ap}_{qb}\left(\eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}u^b + \delta^b_c\eta_{ea}u^e\right) \\ 0 &= \Delta^{ap}_{qb}\left(\left(S_c^{\ b}{}_a - \delta^b_cS_e^{\ e}{}_a\right) + \left(\mu^{\ b}_{ca} + \eta_{ec}\rho^{e\ b}_{\ a} - \eta_{ac}u^b\right) - \delta^b_c\left(\mu^{\ f}_{af} + \eta_{da}\rho^{d\ e}_{\ e} - \eta_{ea}u^e\right)\right) \end{aligned}$$

This task is simplified by comparing the present solution for $\rho^{c\ \ b}_{\ a}$ and $\mu^{b}_{\ da}$ to the solution in the $\gamma_+ = 0$ case found in [2],

$$
\begin{aligned}
\rho^{c\ \ b}_{\ a} &= \delta^c_a u^b - \gamma_- \left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + 2\eta_{ad}u^c u^b u^d\right) \\
\mu^{b}_{\ da} &= \delta^b_d v_a
\end{aligned}
$$

to their form in the present ($\gamma_- = 0$) case,

$$
\begin{aligned}
\rho^{c\ \ b}_{\ a} &= \delta^c_a u^b \\
\mu^{b}_{\ da} &= \delta^b_d v_a - \gamma_+ \left(\delta^b_d v_a + \delta^b_a v_d + \eta^{be}\eta_{da}v_e + \frac{2}{\beta^2}\eta^{be}v_d v_a v_e\right)
\end{aligned}
$$

These are equal with the interchange of $\rho^{c\ \ b}_{\ a}$ and $\mu^{b}_{\ da}$, γ_+ and γ_-, and $u^a \leftrightarrow v_a$, but the correspondence we need is even simpler since eq.(2) depends only on the combination

$$
\left(\mu^{b}_{\ ca} + \eta_{ec}\rho^{e\ \ b}_{\ a} - \eta_{ac}u^b\right) - \delta^b_c\left(\mu^{f}_{\ af} + \eta_{da}\rho^{d\ \ e}_{\ e} - \eta_{ea}u^e\right)
$$

where the second piece is a trace of the first. For the $\gamma_+ = 0$ solution (see [2]), this combination reduced to

$$
-\gamma_- \left(\eta_{ac}u^b + \delta^b_a\eta_{ec}u^e - \left(n+1+2u^2\right)\delta^b_c\eta_{ae}u^e + 2\eta_{ec}\eta_{ad}u^e u^b u^d\right)
$$

In the $\gamma_- = 0$ case, we have

$$
\begin{aligned}
\mu^{b}_{\ ca} + \eta_{ec}\rho^{e\ \ b}_{\ a} - \eta_{ac}u^b &= \delta^b_c v_a - \gamma_+\left(\delta^b_c v_a + \delta^b_a v_c + \eta^{be}\eta_{ca}v_e + 2\eta^{be}v_c v_a v_e\right) + \eta_{ec}\left(\delta^e_a u^b\right) - \eta_{ac}u^b \\
&= \delta^b_c v_a - \gamma_+\left(\delta^b_c v_a + \delta^b_a v_c + \eta^{be}\eta_{ca}v_e + 2\eta^{be}v_c v_a v_e\right) \\
\mu^{f}_{\ cf} + \eta_{ec}\rho^{e\ \ f}_{\ f} - \eta_{fc}u^f &= v_c - \gamma_+\left(n+2+2v^2\right)v_c
\end{aligned}
$$

so the desired combination is

$$
\begin{aligned}
\left(\mu^{b}_{\ ca} + \eta_{ec}\rho^{e\ \ b}_{\ a} - \eta_{ac}u^b\right) - \delta^b_c\left(\mu^{f}_{\ af} + \eta_{da}\rho^{d\ \ e}_{\ e} - \eta_{ea}u^e\right) &= \delta^b_c v_a - \gamma_+\left(\delta^b_c v_a + \delta^b_a v_c + \eta^{be}\eta_{ca}v_e + 2\eta^{be}v_c v_a v_e\right) \\
&\quad -\delta^b_c\left(v_c - \gamma_+\left(n+2+2v^2\right)v_c\right) \\
&= -\gamma_+\left(\delta^b_a v_c + \eta^{be}\eta_{ca}v_e - \left(n+1+2v^2\right)\delta^b_c v_c + 2\eta^{be}v_c v_a v_e\right)
\end{aligned}
$$

Comparing these,

$$
\begin{aligned}
\left(\eta_{ac}u^b + \delta^b_a\eta_{ec}u^e - \left(n+1+2u^2\right)\delta^b_c\eta_{ae}u^e + 2\eta_{ec}\eta_{ad}u^e u^b u^d\right) &= \left(\delta^b_a\left(\eta_{ce}u^e\right) + \eta^{be}\eta_{ac}\left(\eta_{ef}u^f\right) - \left(n+1+2u^2\right)\delta^b_c\left(\eta_{ae}u^e\right) + \right. \\
&\Longrightarrow \left(\delta^b_a v_c + \eta^{be}\eta_{ca}v_e - \left(n+1+2v^2\right)\delta^b_c v_c + 2\eta^{be}v_c v_a v_e\right)
\end{aligned}
$$

where the last follows with the simple replacement $\eta_{ab}u^b \Longrightarrow v_a$ and $\gamma_- \Longrightarrow \gamma_+$.

Using this correspondence, we compare the metrics. When $\gamma_+ = 0$ we have

$$
\delta_{ab} = -\frac{\left(n+1+2u^2\right)U^2}{u^2}\left(\eta_{ab} - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{ac}\eta_{bd}u^c u^d\right)
$$

where

$$
\begin{aligned}
U^2 &= \delta_{ab}u^a u^b \\
u^2 &= \eta_{ab}u^a u^b
\end{aligned}
$$

Now replace $u^a = \eta^{ab}v_b$ everywhere, and we find

$$
U^2 = \delta_{ab}u^a u^b
$$

$$
\begin{aligned}
&= \delta_{ab}\eta^{ac}v_c\eta^{bd}v_d \\
&= \eta^{ac}v_c\left(\delta_{ab}\eta^{bd}v_d\right) \\
&= \eta^{ac}v_c\frac{v^2}{V^2}v_a \\
&= \frac{v^2v^2}{V^2} \\
u^2 &= \eta_{ab}u^au^b \\
&= \eta_{ab}\eta^{ae}v_e\eta^{bc}v_c \\
&= v^2
\end{aligned}
$$

so that

$$
\begin{aligned}
\delta_{ab} &= -\frac{\left(n+1+2u^2\right)U^2}{u^2}\left(\eta_{ab}-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{ac}\eta_{bd}u^cu^d\right) \\
&= -\frac{\left(n+1+2v^2\right)v^2}{V^2}\left(\eta_{ab}-\frac{n+2+2v^2}{v^2\left(n+1+2v^2\right)}v_av_b\right)
\end{aligned}
$$

and this is precisely the $\gamma_- = 0$ form of the metric. So all results for the $\gamma_+ = 0$ carry over to this case with a single simple substitution of $\eta^{ab}v_b$ in place of u^a and the interchange of γ_- and γ_+ (each of which is simply ± 1). In particular, it is shown in [2] that the symmetric part, $P_{abc} = \frac{1}{2}\left(\eta_{be}S_{a\ c}^{\ e} + \eta_{ce}S_{a\ b}^{\ e}\right)$ determines the full co-torsion.

References

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