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Studies in torsion free biconformal spaces

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Abstract

We study whether the solutions for the symmetric part of the connection in homogeneous biconformal space also satisfy the more general field equation of curved biconformal spaces. We show that the six field equations for the torsion and co-torsion are satisfied by vanishing torsion together with the Lorentzian form of the metric when $\gamma_+ = 0$.

1 Introduction

Starting with the conformal symmetries of Euclidean space, we have shown in [1] how construct a manifold where time manifests as a part of the geometry. The result is based on a theorem by Spencer and Wheeler [2] detailing when this is possible. In addition, though there is no matter present in the geometries studied in [1], geometric terms analogous to dark energy and dark matter appear in the Einstein tensor.

Specifically, the quotient of the conformal group of Euclidean four-space by its Weyl subgroup results in a *biconformal* geometry possessing many of the properties of relativistic phase space, including both a natural symplectic form and non-degenerate Killing metric. It is shown that the general solution for this homogeneous space posesses orthogonal Lagrangian submanifolds, with the induced metric and the spin connection on the submanifolds necessarily Lorentzian, despite the Euclidean starting point.

Though an explicit gravitational theory is presented in [1], the main results of that work apply to the homogeneous space. It is of great interest to extend this study to nontrivial gravitational solutions. This has been accomplished for certain restricted cases [3], but a more general approach is desirable. In this and subsequent studies, we accomplish various steps toward a general torsion-free solution.

2 Biconformal gravity

We review the construction of biconformal gravity theory.

2.1 General properties of the conformal group

We begin with the conformal group, $\mathcal{C}$, of compactified $\mathbb{R}^n$. The one-point compactification at infinity allows a global definition of inversion, with translations of the point at infinity defining the special conformal transformation. Then $\mathcal{C}$ has a real linear representation in $n+2$ dimensions, $\mathcal{V}^{n+2}$ (alternatively we could choose the complex representation $\mathbb{C}^{2^{[(n+2)/2]}}$ for $Spin\,(p+1, q+1)$). The isotropy subgroup of $\mathbb{R}^n$ is the rotations, $SO\,(p,q)$, together with dilatations. We call this subgroup the homogeneous Weyl group, $\mathcal{W}$ and require our fibers to contain it. There are then only three allowed subgroups for a quotient: $\mathcal{W}$ itself; the inhomogeneous Weyl group, $\mathcal{IW}$, found by appending the translations; and $\mathcal{W}$ together with special conformal transformations, isomorphic to $\mathcal{IW}$. The quotient of the conformal group by either inhomogeneous Weyl group, called the *auxiliary gauging*, leads most naturally to Weyl gravity (see [4]). We concern ourselves with the only other meaningful conformal quotient, the *biconformal gauging*, generalizing the principal $\mathcal{W}$-bundle formed by the quotient of the conformal group by its Weyl subgroup.

All parts of this construction work for any (p,q) with $n = p+q$. The conformal group is then $SO(p+1,q+1)$ (or $Spin(p+1,q+1)$ for the twistor representation). The Maurer-Cartan structure equations are immediate. In addition to the $\frac{n(n-1)}{2}$ generators $M^{\alpha}_{\ \beta}$ of $SO(p,q)$ and n translational generators P_α, there are n generators of translations of a point at infinity (special conformal transformations, or *co-translations*) K^α, and a single dilatational generator D. Dual to these, we have the connections $\xi^{\alpha}_{\ \beta}, \boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha, \boldsymbol{\delta}$, respectively. Substituting the structure constants into the Maurer-Cartan dual form of the Lie algebra [5] gives

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\xi}^{\alpha}_{\ \beta} &= \boldsymbol{\xi}^{\mu}_{\ \beta} \wedge \boldsymbol{\xi}^{\alpha}_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\pi}_\mu \wedge \boldsymbol{\chi}^\nu &(1)\\
\mathbf{d}\boldsymbol{\chi}^{\alpha} &= \boldsymbol{\chi}^{\beta} \wedge \boldsymbol{\xi}^{\alpha}_{\ \beta} + \boldsymbol{\delta} \wedge \boldsymbol{\chi}^\alpha &(2)\\
\mathbf{d}\boldsymbol{\pi}_{\alpha} &= \boldsymbol{\xi}^{\beta}_{\ \alpha} \wedge \boldsymbol{\pi}_\beta - \boldsymbol{\delta} \wedge \boldsymbol{\pi}_\alpha &(3)\\
\mathbf{d}\boldsymbol{\delta} &= \boldsymbol{\chi}^\alpha \wedge \boldsymbol{\pi}_\alpha &(4)
\end{aligned}
$$

where $\Delta^{\alpha\mu}_{\nu\beta} \equiv \frac{1}{2}\left(\delta^\alpha_\nu \delta^\mu_\beta - \delta^{\alpha\mu}\delta_{\nu\beta}\right)$ antisymmetrizes *with respect to the original* (p,q) *metric*,

$$\delta_{\mu\nu} = diag\,(1,\ldots,1,-1,\ldots,-1)$$

These equations, which are the same regardless of the gauging chosen, describe the Cartan connection on the conformal group manifold. Before proceeding to the quotient, we note that the conformal group has a nondegenerate Killing form,

$$K_{AB} \equiv tr\,(G_A G_B) = c^{C}_{\ AD} c^{D}_{\ BC} = \begin{pmatrix} \Delta^{ac}_{db} & & & \\ & 0 & \delta^a_b & \\ & \delta^a_b & 0 & \\ & & & 1 \end{pmatrix}$$

This provides a metric on the conformal Lie algebra. When restricted to $\mathcal{M}_0$, it remains nondegenerate.

Finally, we note that the conformal group is invariant under inversion. Within the Lie algebra, this manifests itself as the interchange between the translations and special conformal transformations $P_\alpha \leftrightarrow \delta_{\alpha\beta}K^\beta$ along with the interchange of conformal weights, $D \rightarrow -D$. The corresponding transformation of the connection forms, $\boldsymbol{\chi}^\alpha \leftrightarrow \delta^{\alpha\beta}\boldsymbol{\pi}_\beta, \boldsymbol{\delta} \rightarrow -\boldsymbol{\delta}$, is easily seen to leave eqs.(1)-(4) invariant. This symmetry leads to complex and Kähler structures.

2.2 The homogeneous quotient $\mathcal{C}/\mathcal{W}$

In the conformal group, translations and special conformal transformations are related by inversion. Indeed, a special conformal tranformation is a translation centered at the point at infinity instead of the origin. Because the biconformal gauging maintains the symmetry between translations and special conformal transformations, it is useful to name the corresponding connection forms and curvatures to reflect this. Therefore, the biconformal basis will be described as the solder form and the co-solder form, and the corresponding curvatures as the torsion and co-torsion. Thus, when we speak of "torsion-free biconformal space" we do not imply that the co-torsion (Cartan curvature of the co-solder form) vanishes. In phase space interpretations, the solder form is taken to span the cotangent spaces of the spacetime manifold, while the co-solder form is taken to span the cotangent spaces of the momentum space. The opposite convention is equally valid.

Unlike other quotient manifolds arising in conformal gaugings, the biconformal quotient manifold posesses natural invariant structures arising from the underlying groups. The first is the restriction of the Killing metric, which is non-degenerate,

$$\left.\begin{pmatrix} \Delta^{ac}_{db} & & \\ & \begin{bmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{bmatrix} & \\ & & 1 \end{pmatrix}\right|_{\mathcal{M}^{(2n)}} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{pmatrix}_{2n\times 2n},$$

This gives an inner product for the basis,

$$\left[\begin{array}{cc}\left\langle\boldsymbol{\omega}^{\alpha},\boldsymbol{\omega}^{\beta}\right\rangle & \left\langle\boldsymbol{\omega}^{\alpha},\boldsymbol{\omega}_{\beta}\right\rangle\\ \left\langle\boldsymbol{\omega}_{\alpha},\boldsymbol{\omega}^{\beta}\right\rangle & \left\langle\boldsymbol{\omega}_{\alpha},\boldsymbol{\omega}_{\beta}\right\rangle\end{array}\right]\equiv\left[\begin{array}{cc}0 & \delta_{\beta}^{\alpha}\\ \delta_{\alpha}^{\beta} & 0\end{array}\right] \tag{5}$$

This metric remains unchanged by the generalization to curved base manifolds.

The second natural invariant property is the generic presence of a symplectic form. The original fiber bundle always has this, because the structure equation, eq.(4), shows that $\boldsymbol{\chi}^{\alpha}\wedge\boldsymbol{\pi}_{\alpha}$ is exact hence closed, $\mathbf{d}^2\boldsymbol{\omega}=0$, while it is clear that the two-form product is non-degenerate because $(\boldsymbol{\chi}^{\alpha},\boldsymbol{\pi}_{\alpha})$ together span $\mathcal{M}_0^{(2n)}$. Moreover, the symplectic form is canonical,

$$[\Omega]_{AB} = \left[\begin{array}{cc}0 & \delta_{\alpha}^{\beta}\\ -\delta_{\beta}^{\alpha} & 0\end{array}\right]$$

so that $\boldsymbol{\chi}^{\alpha}$ and $\boldsymbol{\pi}_{\alpha}$ are canonically conjugate, in the sense that they form a canonical basis for the symplectic form $\boldsymbol{\Omega}$. The symplectic form persists for the 2-form, $\boldsymbol{\omega}^{\alpha}\wedge\boldsymbol{\omega}_{\alpha}+\boldsymbol{\Omega}$ (see below), as long as it is non-degenerate, so curved biconformal spaces are generically symplectic.

Next, we consider the effect of inversion symmetry. As a $\left(\begin{array}{c}1\\1\end{array}\right)$ tensor, the basis interchange takes the form

$$I_{\ B}^{A}\boldsymbol{\chi}^{B}=\left(\begin{array}{cc}0 & \delta^{\alpha\nu}\\ \delta_{\beta\mu} & 0\end{array}\right)\left(\begin{array}{c}\boldsymbol{\chi}^{\mu}\\ \boldsymbol{\pi}_{\nu}\end{array}\right)=\left(\begin{array}{c}\delta^{\alpha\nu}\boldsymbol{\pi}_{\nu}\\ \delta_{\beta\mu}\boldsymbol{\chi}^{\mu}\end{array}\right)$$

In order to interchange conformal weights, $I_{\ B}^{A}$ must anticommute with the conformal weight operator, which is given by

$$W_{\ B}^{A}\boldsymbol{\chi}^{B}=\left(\begin{array}{cc}\delta_{\mu}^{\alpha} & 0\\ 0 & -\delta_{\beta}^{\nu}\end{array}\right)\left(\begin{array}{c}\boldsymbol{\chi}^{\mu}\\ \boldsymbol{\pi}_{\nu}\end{array}\right)=\left(\begin{array}{c}+\boldsymbol{\chi}^{\alpha}\\ -\boldsymbol{\pi}_{\beta}\end{array}\right)$$

This is the case: we easily check that $\{I,W\}_{\ B}^{A}=I_{\ C}^{A}W_{\ B}^{C}+W_{\ C}^{A}I_{\ B}^{C}=0$. The commutator gives a new object,

$$J_{\ B}^{A}\equiv[I,W]_{\ B}^{A}=\left(\begin{array}{cc}0 & -\delta^{\alpha\beta}\\ \delta_{\alpha\beta} & 0\end{array}\right)$$

Squaring, $J_{\ C}^{A}J_{\ B}^{C}=-\delta_{B}^{A}$, we see that $J_{\ B}^{A}$ provides an almost complex structure. That the almost complex structure is integrable follows immediately in this (global) basis by the obvious vanishing of the Nijenhuis tensor,

$$N_{\ BC}^{A}=J_{\ C}^{D}\partial_{D}J_{\ B}^{A}-J_{\ C}^{D}\partial_{D}J_{\ B}^{A}-J_{\ D}^{A}\left(\partial_{C}J_{\ B}^{D}-\partial_{B}J_{\ C}^{D}\right)=0$$

Next, using the symplectic form to define the compatible metric

$$g(u,v)\equiv\Omega(u,Jv)$$

we find that in this basis $g=\left(\begin{array}{cc}\delta_{\alpha\beta} & 0\\ 0 & \delta^{\alpha\beta}\end{array}\right)$, and we check the remaining compatibility conditions of the triple (g,J,Ω),

$$\begin{aligned}\omega(u,v) &= g(Ju,v)\\ J(u) &= (\phi_g)^{-1}(\phi_{\omega}(u))\end{aligned}$$

where ϕ_{ω} and ϕ_g are defined by

$$\begin{aligned}\phi_{\omega}(u) &= \omega(u,\cdot)\\ \phi_g(u) &= g(u,\cdot)\end{aligned}$$

These are easily checked to be satisified, showing that that $\mathcal{M}_0^{(2n)}$ is a Kähler manifold. Notice, however, that the metric of the Kähler manifold is *not* the restricted Killing metric which we use in the following considerations. While Kähler manifolds play a role in string theory and geometric quantization, this relationship lies beyond the scope of this work.

Finally, a surprising result emerges if we require $\mathcal{M}_0^{(2n)}$ to match our usual expectations for a relativistic phase space. To make the connection to phase space clear, the precise requirements were studied in [2]. There it was required that the flat biconformal gauging of $SO(p,q)$ in any dimension $n = p + q$ have *Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have non-degenerate n-dim restrictions of the metric.* This is found to be possible only if the original space is Euclidean or signature zero $\left(p \in \left\{0, \frac{n}{2}, n\right\}\right)$. The signature of the Lagrangian submanifolds is severely limited $(p \to p \pm 1)$, leading in the two Euclidean cases to Lorentzian configuration space, and hence the origin of time. For the case of flat, 8-dim biconformal space we paraphrase the the following theorem from [2]:

Flat 8-dim biconformal space is a metric phase space with the following properties:

1. *There exist Lagrangian submanifolds orthogonal with respect to the 2n-dim biconformal (Killing) metric*
2. *The restriction of the Killing metric to each Lagrangian submanifold is non-degenerate*

if and only if the initial 4-dim space we gauge is Euclidean or signature zero. In either of these cases the resulting configuration sub-manifold is necessarily Lorentzian [2].

Thus, it is possible to impose the conditions necessary to make biconformal space a metric phase space only in a restricted subclass of cases, and the configuration space metric *must* be Lorentzian. In [1], it was shown that coordinates (w^α, s_β) exist such that the metric takes the form

$$k_{\alpha\beta} = s^2 \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right) \tag{6}$$

where $\delta_{\alpha\beta}$ is the Euclidean metric and $s^2 = \delta^{\alpha\beta} s_\alpha s_\beta$. The signature changing character is easily seen.

The Euclidean gauge theory necessarily posesses a special *vector*, $\mathbf{v} = \boldsymbol{\omega} - \frac{1}{2}\eta_{ab}\mathbf{d}\eta^{bc}$, built from the Weyl vector and the conformal gauge of the metric [1]. It is preferable to write $k_{\alpha\beta}$ in terms of the components of this vector, u^α and $v_\alpha$defined above. In the (w^α, s_β) coordinates, $v_\alpha = -k\beta s_\alpha$ and $u^\alpha = \beta k^{\alpha\beta} s_\beta$, so we find

$$k_{\alpha\beta} = \frac{1}{\beta^2} v^2 \left(\delta_{\alpha\beta} - \frac{2}{v^2} v_\alpha v_\beta\right)$$

with inverse $k^{\alpha\beta} = \frac{\beta^2}{v^2}\left(\delta^{\alpha\beta} - \frac{2}{v^2}\delta^{\alpha\mu}\delta^{\beta\nu} v_\mu v_\nu\right)$. This shows that the vector v_α determines the timelike directions. . This vector gives the time direction on two Lagrangian submanifolds, making them necessarily Lorentzian. The full manifold retains its original symmetry.

Thus, the structures of the conformal group give rise to a natural *direction* of time. The situation is reminiscent of previous studies. In 1979, Stelle and West introduced a special vector field to choose the local symmetry of the MacDowell-Mansouri theory. The vector breaks the de Sitter symmetry, eliminating the need for the Wigner-Inönu contraction. Recently, Westman and Zlosnik[9] have looked in depth at both the de Sitter and anti-de Sitter cases using a class of actions which extend that of Stelle and West by including derivative terms for the vector field and therefore lead to dynamical symmetry breaking. In [10, 11] and Einstein-Aether theory [12], there is also a special vector field introduced into the action by hand that can make the Lorentzian metric Euclidean. These approaches are distinct from that of the biconformal approach, where the vector necessary for specifying the timelike direction occurs *naturally* from the underlying group structure.

We may also express $u^\alpha$and v_α in coordinates (x^α, y_β) adapted to the Lagrangian submanifolds, one set of which is given by

$$\begin{aligned} x^\alpha &= k\gamma_+ w^\alpha + \beta\delta_{\alpha\beta}\frac{s_\alpha}{s^2} \\ y_\mu &= k\beta\left(\frac{s_\mu}{s^2}\right) - \gamma_- \delta_{\mu\nu} w^\nu \end{aligned}$$

where $\gamma_\pm = \frac{1}{2\beta}\left(1 \pm k\beta^2\right)$. Inverting to solve for w^α, s_β, we readily find

$$s_\mu = \frac{\gamma_- \delta_{\mu\nu} x^\nu + k y_\mu}{\gamma_-^2 x^2 + 2k\gamma_- x^\alpha y_\alpha + y^2}$$

and therefore

$$\begin{aligned} v_\mu &= -k\beta\left(\frac{\gamma_- \delta_{\mu\alpha} x^\alpha + y_\mu}{\gamma_-^2 x^2 + 2\gamma_- y_\mu x^\mu + y^2}\right) \\ u^\mu &= -\beta\left(\gamma_- x^\mu + \delta^{\mu\alpha} y_\alpha\right) \end{aligned}$$

2.3 Gauging

By gauge theory, we typically understand a theory (i.e., the specification of an action functional) which is invariant under a local symmetry group – the gauge symmetry. Thus, there may be many gauge theories having the same gauge group. However, gauge theories having the same gauge group share a common structure: the underlying principal fiber bundle in which the base manifold is spacetime or some other world manifold and the fibers are copies of the gauge group. Such a principal fiber bundle is most simply constructed as the quotient of a larger group by the symmetry group. Constructed in this way, we have immediate access to relevant tensor fields: any group invariant tensors, the curvatures of the bundle, and the vectors of the group representation. Then any functional built invariantly from these tensors is a gauge theory. For example, the quotient of the Poincaré group by its Lorentz subgroup may be generalized to a principal fiber bundle with Lorentz group fibers and a general base manifold having arbitrary Riemannian curvature. Identifying the curvature, solder form, Lorentz metric, and Levi-Civita tensor as tensors with respect to this local Lorentz symmetry, it is clear that any functional built invariantly from them is a gauge theory. In addition, if we use a linear representation, $SO(3,1)$ or $SL(2,C)$, of the Lorentz group then the action functional may include vectors or spinors from that representation and their covariant derivatives. For these reasons, we will define a *gauging* to be the fiber bundle of a specific quotient, along with the identification of its associated tensors. A *gauge theory* remains the specification of an action functional invariant on this bundle.

The biconformal gauging was first considered by Ivanov and Niederle [6]. They considered a curvature-quadratic action, with the extra four dimensions restricted to the minimum necessary for consistency with conformal symmetry. The full geometry was first studied in [7, 8], where tensorial constraints on the full higher-dimensional curvatures were shown to reduce the geomety to general relativity [7]. It was soon realized that the dimensionless volume form permits an action linear in the curvatures [8], and the resulting field equations were found to reduce to the system described in [7].

In the fiber bundle produced by the quotient $\mathcal{C}/\mathcal{W}$, the one-forms $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\delta}\right)$ span the $\mathcal{W}$-fibers, with $(\boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha)$ spanning the remaining $2n$ independent directions of the homogeneous base manifold. We now generalize the connection (and the manifold, if desired) by replacing $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha, \boldsymbol{\delta}\right) \rightarrow \left(\boldsymbol{\omega}^\alpha_{\ \beta}, \mathbf{e}^\alpha, \boldsymbol{\omega}_\alpha, \boldsymbol{\omega}\right)$ in the Maurer-Cartan equations, eqs.(1-4) to give the Cartan curvatures in terms of the new connection forms,

$$\begin{aligned} \mathbf{d}\boldsymbol{\omega}^\alpha_{\ \beta} &= \boldsymbol{\omega}^\mu_{\ \beta} \wedge \boldsymbol{\omega}^\alpha_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}^\nu + \boldsymbol{\Omega}^\alpha_{\ \beta} && (7) \\ \mathbf{d}\mathbf{e}^\alpha &= \mathbf{e}^\beta \wedge \boldsymbol{\omega}^\alpha_{\ \beta} + \boldsymbol{\omega} \wedge \mathbf{e}^\alpha + \mathbf{T}^\alpha && (8) \\ \mathbf{d}\boldsymbol{\omega}_\alpha &= \boldsymbol{\omega}^\beta_{\ \alpha} \wedge \boldsymbol{\omega}_\beta - \boldsymbol{\omega} \wedge \boldsymbol{\omega}_\alpha + \mathbf{S}_\alpha && (9) \\ \mathbf{d}\boldsymbol{\omega} &= \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha + \boldsymbol{\Omega} && (10) \end{aligned}$$

Equations eq.(7-10) give the curvature two-forms in terms of the connection forms. We have therefore constructed a $2n$-dim manifold based on the conformal group with local $\mathcal{W}$ symmetry.

Since the curvatures now depend on the $2n$ non-vertical forms, $(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha)$, so there are far more components than for an n-dim Riemannian geometry. For example,

$$\boldsymbol{\Omega}^\alpha_{\ \beta} = \frac{1}{2}\Omega^\alpha_{\ \beta\mu\nu}\boldsymbol{\omega}^\mu \wedge \boldsymbol{\omega}^\nu + \Omega^{\alpha\ \mu}_{\ \beta\ \nu}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}^\nu + \frac{1}{2}\Omega^{\alpha\ \mu\nu}_{\ \beta}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}_\nu$$

The coefficients of the pure terms, $\Omega^{\alpha}_{\beta\mu\nu}$ and $\Omega^{\alpha\ \mu\nu}_{\beta}$ each have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry, while the cross-term coefficients $\Omega^{\alpha\ \mu}_{\beta\ \nu}$ have more, being asymmetric on the final two indices.

For our purpose, it is important to notice that the spin connection, $\boldsymbol{\xi}^{\alpha}_{\ \beta}$, is antisymmetric with respect to the original (p,q) metric, $\delta_{\alpha\beta}$, in the sense that

$$\boldsymbol{\xi}^{\alpha}_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\xi}^{\nu}_{\ \mu}$$

It is crucial to note that $\boldsymbol{\omega}^{\alpha}_{\ \beta}$ retains this property, $\boldsymbol{\omega}^{\alpha}_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\omega}^{\nu}_{\ \mu}$. This expresses metric compatibility with the $SO(p,q)$-covariant derivative, since it implies

$$\mathbf{D}\delta_{\alpha\beta} \equiv \mathbf{d}\delta_{\alpha\beta} - \delta_{\mu\beta}\boldsymbol{\omega}^{\mu}_{\ \alpha} - \delta_{\alpha\mu}\boldsymbol{\omega}^{\mu}_{\ \beta} = 0$$

Therefore, the curved generalization has a connection which is compatible with a locally (p,q)-metric. This relationship is general. If $\kappa_{\alpha\beta}$ is any metric, its compatible spin connection will satisfy $\boldsymbol{\omega}^{\alpha}_{\ \beta} = -\kappa^{\alpha\mu}\kappa_{\beta\nu}\boldsymbol{\omega}^{\nu}_{\ \mu}$. Since we also have local scale symmetry, the full covariant derivative we use will also include a Weyl vector term.

Each of the curvatures has a corresponding Bianchi identity, to guarantee integrability of the modified structure equations,

$$0 = \mathbf{D}\boldsymbol{\Omega}^{\alpha}_{\ \beta} + 2\Delta^{\alpha\mu}_{\nu\beta}\left(\boldsymbol{\Omega}_{\mu}\wedge\boldsymbol{\omega}^{\nu} - \boldsymbol{\omega}_{\mu}\wedge\boldsymbol{\Omega}^{\nu}\right) \tag{11}$$

$$0 = \mathbf{D}\mathbf{T}^{\alpha} - \mathbf{e}^{\beta}\wedge\boldsymbol{\Omega}^{\alpha}_{\ \beta} + \boldsymbol{\Omega}\wedge\mathbf{e}^{\alpha} \tag{12}$$

$$0 = \mathbf{D}\mathbf{S}_{\alpha} + \boldsymbol{\Omega}^{\alpha}_{\ \beta}\wedge\boldsymbol{\omega}_{\beta} - \boldsymbol{\omega}_{\alpha}\wedge\boldsymbol{\Omega} \tag{13}$$

$$0 = \mathbf{D}\boldsymbol{\Omega} + \mathbf{T}^{\alpha}\wedge\boldsymbol{\omega}_{\alpha} - \boldsymbol{\omega}^{\alpha}\wedge\mathbf{S}_{\alpha} \tag{14}$$

where D is the Weyl covariant derivative,

$$\begin{aligned}
\mathbf{D}\boldsymbol{\Omega}^{\alpha}_{\ \beta} &= \mathbf{d}\boldsymbol{\Omega}^{\alpha}_{\ \beta} + \boldsymbol{\Omega}^{\mu}_{\ \beta}\wedge\boldsymbol{\omega}^{\alpha}_{\ \mu} - \boldsymbol{\Omega}^{\alpha}_{\ \mu}\wedge\boldsymbol{\omega}^{\mu}_{\ \beta} \\
\mathbf{D}\mathbf{T}^{\alpha} &= \mathbf{d}\mathbf{T}^{\alpha} + \mathbf{T}^{\beta}\wedge\boldsymbol{\omega}^{\alpha}_{\ \beta} - \boldsymbol{\omega}\wedge\mathbf{T}^{\alpha} \\
\mathbf{D}\mathbf{S}_{\alpha} &= \mathbf{d}\mathbf{S}_{\alpha} - \boldsymbol{\omega}^{\beta}_{\ \alpha}\wedge\mathbf{S}_{\beta} + \mathbf{S}_{\alpha}\wedge\boldsymbol{\omega} \\
\mathbf{D}\boldsymbol{\Omega} &= \mathbf{d}\boldsymbol{\Omega}
\end{aligned}$$

Notice that our development to this point was based solely on group quotients and generalization of the resulting principal fiber bundle. We have arrived at the form of the curvatures in terms of the Cartan connection, and Bianchi identities required for integrability, thereby describing certain classes of geometry with local symmetry. Within the biconformal quotient, the demand for *orthogonal Lagrangian submanifolds with non-degenerate n-dim restrictions of the Killing metric* leads to the selection of certain Lorentzian submanifolds.

We are guided in the choice of action functional by the example of general relativity. Given the Riemannin geometries of Section 2.3, we may write the Einstein-Hilbert action and proceed. More systematically, however, we may write the most general, even-parity action linear in the curvature and torsion. This turns out to be the tetradic Palatini action, $S_P = \int \mathbf{R}^{ab}\wedge\mathbf{e}^{c}\wedge\mathbf{e}^{d}\varepsilon_{abcd}$, and, as noted above, a full variation of the connection, $\left(\delta\mathbf{e}^{b}, \delta\boldsymbol{\omega}^{a}_{\ b}\right)$, implies vanishing torsion in addition to the Einstein equation. The latter, more robust approach is what we follow for conformal gravity theories.

It is generally of interest to build the simplest class of actions possible, and we use the following criteria:

1. The pure-gravity action should be built from the available curvature tensor(s) and other tensors which occur in the geometric construction.

2. The action should be of lowest possible order ≥ 1 in the curvatures.

3. The action should be of even parity.

These are of sufficient generality not to bias our choice. It may also be a reasonable assumption to set certain tensor fields, for example, the spacetime torsion to zero. This can significantly change the available tensors, allowing a wider range of action functionals.

Notice that if we perform an infinitesmal *conformal* transformation to the curvatures, $\left(\mathbf{\Omega}^{\alpha}_{\ \beta}, \mathbf{\Omega}^{\alpha}, \mathbf{\Omega}_{\beta}, \mathbf{\Omega}\right)$, they all mix with one another, since the conformal curvature is really a single Lie-algebra-valued two form. However, the generalization to a curved manifold breaks the non-vertical symmetries, allowing these different components to become independent tensors under the remaining Weyl group. Thus, to find the available tensors, we apply an infinitesmal transformation of the *fiber symmetry.* Tensors are those objects which transform linearly and homogeneously under these transformations. Note that this is not symmetry breaking, but rather simply part of the construction of the manifold and local symmetry.

The biconformal gauging, based on $\mathcal{C}/\mathcal{W}$, also has tensorial basis forms $(\boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\alpha})$. Moreover, each of the component curvatures, $\left(\mathbf{\Omega}^{\alpha}_{\ \beta}, \mathbf{\Omega}^{\alpha}, \mathbf{\Omega}_{\beta}, \mathbf{\Omega}\right)$, becomes an independent tensor under the Weyl group. In the biconformal case, the volume form $e^{\rho\sigma\ldots\lambda}{}_{\alpha\beta\ldots\nu}\boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}^{\beta} \wedge \ldots \wedge \boldsymbol{\omega}^{\nu} \wedge \boldsymbol{\omega}_{\rho} \wedge \boldsymbol{\omega}_{\sigma} \wedge \ldots \wedge \boldsymbol{\omega}_{\lambda}$ has zero conformal weight. Since both $\mathbf{\Omega}^{\alpha}_{\ \beta}$ and $\mathbf{\Omega}$ also have zero conformal weight, there exists a curvature-linear action in any dimension [8]. The most general linear case is

$$S = \int \left(\alpha \mathbf{\Omega}^{\alpha}_{\ \beta} + \beta \mathbf{\Omega} \delta^{\alpha}_{\beta} + \gamma \boldsymbol{\omega}^{\alpha} \wedge \boldsymbol{\omega}_{\beta}\right) \wedge e^{\beta\rho\ldots\sigma}{}_{\alpha\mu\ldots\nu} \boldsymbol{\omega}^{\mu} \wedge \ldots \wedge \boldsymbol{\omega}^{\nu} \wedge \boldsymbol{\omega}_{\rho} \wedge \ldots \wedge \boldsymbol{\omega}_{\sigma} \tag{15}$$

Notice that we now have three important properties of biconformal gravity that arise because of the doubled dimension: (1) the non-degenerate conformal Killing metric induces a non-degenerate metric on the manifold, (2) the dilatational structure equation generically gives a symplectic form, and (3) there exists a Weyl symmetric action functional linear in the curvature, valid in any dimension.

There are a number of known results following from the linear action. In [8] torsion-constrained solutions are found which are consistent with scale-invariant general relativity. Subsequent work along the same lines shows that the torsion-free solutions are determined by the spacetime solder form, and reduce to describe spaces conformal to Ricci-flat spacetimes on the corresponding spacetime submanifold. A supersymmetric version is presented in [13], and studies of Hamiltonian dynamics [16, 15] and quantum dynamics [16] support the idea that the models describe some type of relativistic phase space determined by the configuration space solution.

In an orthonormal frame field, $(\mathbf{e}^{a}, \mathbf{f}_{a})$, adapted to the Lagrangian submanifolds, two new tensor fields emerge. The first, $\mathbf{v} = v_{a}\mathbf{e}^{a} + u^{a}\mathbf{f}_{a}$ is a combination of the Weyl vector with the scale factor on the metric, and determines the timelike directions on the submanifolds. The second, $\boldsymbol{\beta}^{a}_{\ b} = \boldsymbol{\mu}^{a}_{\ b} + \boldsymbol{\rho}^{a}_{\ b} = \mu^{a}_{\ bc}\mathbf{e}^{c} + \rho^{a\ c}_{\ b}\mathbf{f}_{c}$, comes from the components of the spin connection, and is *symmetric* with respect to the new metric, $\boldsymbol{\beta}^{a}_{\ b} = \eta^{ac}\eta_{bd}\boldsymbol{\beta}^{d}_{\ c}$. Though this field is part of the original spin connection, it transforms homogeneously under local Lorentz transformations and local dilatations. A complete discussion is given in [1].

In the remaining discussion below, we show that in torsion-free, biconformal spaces, the $\gamma_{+} = 0$ solution for $\boldsymbol{\beta}^{a}_{\ b}$ from the homogeneous case also solves the field equation for the torsion provided the metric is Lorentzian.

3 The vanishing torsion field equations

The action given in eq.(15) lead to the following field equations for the torsion and co-torsion:

$$\begin{aligned}
0 &= \beta\left(T^{ba}{}_{b} - T^{ab}{}_{b} + S_{b}^{\ ab}\right) \\
0 &= \beta\left(T^{b}_{\ ab} + S_{a}^{\ b}{}_{b} - S_{b}^{\ b}{}_{a}\right) \\
0 &= \Delta^{ap}_{qb}\left(T^{cb}{}_{a} - \delta^{c}_{a}T^{eb}{}_{e} - \delta^{c}_{a}S_{e}^{\ be}\right) \\
&\quad + \Delta^{ap}_{qb}\left(-\frac{1}{2}\partial_{a}\eta^{cb} + \frac{1}{2}\delta^{c}_{a}\partial_{f}\eta^{bf} - \eta^{cd}\mu^{b}_{\ da} + \delta^{c}_{a}\eta^{bd}\mu^{f}_{\ df} + W_{a}\eta^{cb} - \delta^{c}_{a}\eta^{bf}W_{f}\right) \\
&\quad + \Delta^{ap}_{qb}\left(-\frac{1}{2}\eta^{cd}\partial^{b}\eta_{ad} + \frac{1}{2}\delta^{c}_{a}\eta^{bd}\partial^{e}\eta_{ed} + \rho^{c\ b}_{\ a} - \delta^{c}_{a}\rho^{b\ e}_{\ e}\right)
\end{aligned}$$

$$
\begin{aligned}
0 &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e} + \delta^b_c T^e_{ae}\right) \\
&+\Delta^{ap}_{qb}\left(-\frac{1}{2}\partial^b\eta_{ac} + \delta^b_c\frac{1}{2}\partial^e\eta_{ea} + \eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}W^b + \delta^b_c\eta_{ea}W^e\right) \\
&+\Delta^{ap}_{qb}\left(\frac{1}{2}\eta_{cd}\partial_a\eta^{bd} - \frac{1}{2}\delta^b_c\eta_{ad}\partial_f\eta^{df} + \eta_{cd}\eta^{de}\mu^b_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^f_{\ ef}\right)
\end{aligned}
$$

An additional four equations relate components of the rotational and dilatational curvatures, and do not concern us here. In addition, we have two involution conditions

$$
\begin{aligned}
T^{abc} &= \eta^{ad}\rho^{b\ c}_{\ d} - \eta^{ad}\rho^{c\ b}_{\ d} + \eta^{ac}u^b - \eta^{ab}u^c \\
&= \rho^{bac} - \rho^{cab} + \eta^{ac}u^b - \eta^{ab}u^c \\
S_{abc} &= k\eta_{ad}\mu^d_{\ bc} - k\eta_{ad}\mu^d_{\ cb} - k\eta_{ab}v_c + k\eta_{ac}v_b \\
&= k\left((\mu_{abc} - \eta_{ab}v_c) - (\mu_{acb} - \eta_{ac}v_b)\right)
\end{aligned}
$$

In general we wish to determine the six independent parts of the torsion and co-torsion,

$$
T^{abc},\ T^{ab}_{\ \ c},\ T^a_{bc},\ S_a^{\ bc},\ S_a^{\ b}{}_c,\ S_{abc}
$$

and the underlying connection.

From the symmetries $\mu_{abc} = \mu_{(ab)c}$ and $\rho^{abc} = \rho^{(ab)c}$, the involution conditions immediately give

$$
\begin{aligned}
T^{[abc]} &= 0 \\
S_{[abc]} &= 0
\end{aligned}
$$

Further progress is difficult without some assumptions. In keeping with ideas from Riemannian geometry, it is prudent (though not necessary) to assume vanishing torsion,

$$
\begin{aligned}
T^{abc} &= 0 \\
T^{ab}_{\ \ c} &= 0 \\
T^a_{bc} &= 0
\end{aligned}
$$

It is known to be overly constraining to also set the co-torsion to zero.

The field equation for the torsion reduces to

$$
0 = \Delta^{ap}_{qb}\left(\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ e}_{\ e} - \delta^c_a\eta^{bf}v_f\right) - \left(\eta^{cd}\mu^b_{\ da} - \delta^c_a\eta^{bd}\mu^f_{\ df} - \eta^{cd}\delta^b_d v_a\right)\right) \tag{16}
$$

while for the co-torsion we have

$$
\begin{aligned}
0 &= S_b^{\ ab} && (17)\\
0 &= S_a^{\ b}{}_b - S_b^{\ b}{}_a && (18)\\
0 &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e} + \eta_{cd}\eta^{de}\mu^b_{\ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^f_{\ ef}\right) && (19)\\
&\quad +\Delta^{ap}_{qb}\left(\eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}u^b + \delta^b_c\eta_{ea}u^e\right)
\end{aligned}
$$

The involution conditions become

$$
\begin{aligned}
0 &= \eta^{ad}\rho^{b\ c}_{\ d} - \eta^{ad}\rho^{c\ b}_{\ d} + \eta^{ac}u^b - \eta^{ab}u^c && (20)\\
k\eta^{ae}S_{ebc} &= \mu^a_{\ bc} - \mu^a_{\ cb} - \delta^a_b v_c + \delta^a_c v_b && (21)
\end{aligned}
$$

Eqs.(16-21) are the torsion-free field equations we wish to solve.

4 The homogeneous solution and the field equations

From the zero curvature solution [1], we have

$$\boldsymbol{\mu}^a_{\ b} = \left(\delta^a_b v_c - k\gamma_+\left(\delta^a_b v_c + \delta^a_c v_b + \eta^{ad}\eta_{bc}v_d + \frac{2}{\beta^2}\eta^{ad}v_b v_c v_d\right)\right)\mathbf{e}^c \qquad (22)$$

$$\boldsymbol{\rho}^a_{\ b} = \left(\delta^a_b u^c + k\gamma_-\left(\delta^a_b u^c + \delta^c_b u^a + \eta^{ac}\eta_{bd}u^d + \frac{2}{\beta^2}\eta_{bd}u^a u^c u^d\right)\right)\mathbf{f}_c \qquad (23)$$

This form holds in a particular coordinate system and gauge. Extracting the coefficients:

$$\begin{aligned}
\rho^{c\ \,b}_{\ a} &= \delta^c_a u^b + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\\
\rho^{b\ \,f}_{\ f} &= \left(1 + k\gamma_-(n+2) + \frac{2k\gamma_-}{\beta^2}u^2\right)u^b\\
\mu^b_{\ da} &= \delta^b_d v_a - k\gamma_+\left(\delta^b_d v_a + \delta^b_a v_d + \eta^{be}\eta_{da}v_e + \frac{2}{\beta^2}\eta^{be}v_d v_a v_e\right)\\
\mu^f_{\ df} &= \left(1 - k\gamma_+(n+2) - \frac{2k\gamma_+}{\beta^2}v^2\right)v_d
\end{aligned}$$

where

$$\begin{aligned}
\gamma_\pm &\equiv \frac{1}{2\beta}\left(1 \pm k\beta^2\right)\\
k &= \pm 1
\end{aligned}$$

Our goal is to examine whether these flat solutions (which did not need to satisfy any field equation at all), give a solution to the curved space field equations. Our principal result in this report is to show that the $\gamma_+ = 0$ case *does* solve the torsion and co-torsion field equations.

4.1 Involution conditions

We first check that these satisfy the involution conditions,

$$\begin{aligned}
0 &= \rho^{b\ \,c}_{\ a} - \rho^{c\ \,b}_{\ a} + \delta^c_a u^b - \delta^b_a u^c\\
&= \delta^b_a u^c + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\\
&\quad - \left(\delta^c_a u^b + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\right) + \delta^c_a u^b - \delta^b_a u^c\\
&= k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d - \left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\right)\\
&\equiv 0\\
k\eta^{ae}S_{ebc} &= \left(\delta^a_b v_c - k\gamma_+\left(\delta^a_b v_c + \delta^a_c v_b + \eta^{ae}\eta_{bc}v_e + \frac{2}{\beta^2}\eta^{ae}v_b v_c v_e\right)\right)\\
&\quad - \left(\delta^a_c v_b - k\gamma_+\left(\delta^a_c v_b + \delta^a_b v_c + \eta^{ae}\eta_{bc}v_e + \frac{2}{\beta^2}\eta^{ae}v_b v_c v_e\right)\right) - \delta^a_b v_c + \delta^a_c v_b\\
&= 0
\end{aligned}$$

so the involution conditions are satisfied with $S_{abc} = 0$.

4.2 Homogeneous solution as a solution to the field equations

We would like to know whether the cubic forms of $\rho^{c\ b}_{\ a}$ and $\mu^{b}_{\ da}$ satisfy the field equation. Substituting,

$$\begin{aligned}
0 &= \Delta^{ap}_{qb}\left(\left(\rho^{c\ b}_{\ a} - \delta^c_a\rho^{b\ f}_{\ f} - \delta^c_a\eta^{bf}v_f\right) - \left(\eta^{cd}\mu^{b}_{\ da} - \delta^c_a\eta^{bd}\mu^{f}_{\ df} - \eta^{cd}\delta^b_d v_a\right)\right) \\
&= \Delta^{ap}_{qb}\left(\delta^c_a u^b + k\gamma_-\left(\delta^c_a u^b + \delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d\right)\right) \\
&\quad -\Delta^{ap}_{qb}\left(\delta^c_a\left(1 + k\gamma_-\left(n+2\right) + \frac{2k\gamma_-}{\beta^2}u^2\right)u^b - \delta^c_a\eta^{bf}v_f\right) \\
&\quad -\Delta^{ap}_{qb}\left(\eta^{bc}v_a - k\gamma_+\left(\eta^{bc}v_a + \delta^b_a\eta^{cd}v_d + \delta^c_a\eta^{be}v_e + \frac{2}{\beta^2}\eta^{cd}\eta^{be}v_d v_e v_a\right)\right) \\
&\quad +\Delta^{ap}_{qb}\left(\left(1 - k\gamma_+\left(n+2\right) - \frac{2k\gamma_+}{\beta^2}v^2\right)\delta^c_a\eta^{bd}v_d - \eta^{bc}v_a\right) \\
&= k\gamma_-\Delta^{ap}_{qb}\left(\delta^b_a u^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^c u^b u^d - \left(\left(n+1\right) + \frac{2u^2}{\beta^2}\right)\delta^c_a u^b\right) \\
&\quad +k\gamma_+\Delta^{ap}_{qb}\left(\eta^{bc}v_a + \delta^b_a\eta^{cd}v_d + \frac{2}{\beta^2}\eta^{cd}\eta^{be}v_d v_e v_a - \left(\left(n+1\right) + \frac{2v^2}{\beta^2}\right)\delta^c_a\eta^{bd}v_d\right)
\end{aligned}$$

Expanding $\Delta^{ap}_{qb} = \frac{1}{2}\left(\delta^a_q\delta^p_b - \delta^{ab}\delta_{bq}\right)$ This becomes

$$\begin{aligned}
0 &= k\gamma_-\left(\delta^p_q u^c + \eta^{cp}\eta_{qd}u^d + \frac{2}{\beta^2}\eta_{qd}u^c u^p u^d - \left(\left(n+1\right) + \frac{2u^2}{\beta^2}\right)\delta^c_q u^p\right) \\
&\quad -k\gamma_-\left(\delta^{ap}\delta_{bq}\delta^b_a u^c + \delta^{ap}\delta_{bq}\eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad}u^c u^b u^d - \left(\left(n+1\right) + \frac{2u^2}{\beta^2}\right)\delta^{ap}\delta_{bq}\delta^c_a u^b\right) \\
&\quad +k\gamma_+\left(\eta^{pc}v_q + \delta^p_q\eta^{cd}v_d + \frac{2}{\beta^2}\eta^{cd}\eta^{pe}v_d v_e v_q - \left(\left(n+1\right) + \frac{2v^2}{\beta^2}\right)\delta^c_q\eta^{pd}v_d\right) \\
&\quad -k\gamma_+\left(\delta^{ap}\delta_{bq}\eta^{bc}v_a + \delta^{ap}\delta_{bq}\delta^b_a\eta^{cd}v_d + \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta^{cd}\eta^{be}v_d v_e v_a - \left(\left(n+1\right) + \frac{2v^2}{\beta^2}\right)\delta^{ap}\delta_{bq}\delta^c_a\eta^{bd}v_d\right) \\
&= k\gamma_-\left(\left(n+1\right)\delta^{pc}\delta_{bq}u^b + \eta^{cp}\eta_{qb}u^b - \left(\left(n+1\right) + \frac{2u^2}{\beta^2}\right)\delta^c_q u^p - \eta^{cb}\delta_{bq}\delta^{ap}\eta_{ad}u^d\right) \\
&\quad +k\gamma_-\left(\frac{2}{\beta^2}\eta_{qd}u^c u^p u^d - \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad}u^c u^b u^d + \frac{2u^2}{\beta^2}\delta^{pc}\delta_{bq}u^b\right) \\
&\quad +k\gamma_+\left(\left(n+1\right)\delta^{pc}\delta_{bq}\eta^{bd}v_d + \eta^{pc}v_q - \left(\left(n+1\right) + \frac{2v^2}{\beta^2}\right)\delta^c_q\eta^{pd}v_d - \delta^{ap}\delta_{bq}\eta^{bc}v_a\right) \\
&\quad +k\gamma_+\left(\frac{2}{\beta^2}\eta^{cd}\eta^{pe}v_d v_e v_q - \frac{2}{\beta^2}\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be}v_d v_e v_a + \frac{2v^2}{\beta^2}\delta^{cp}\delta_{bq}\eta^{bd}v_d\right)
\end{aligned}$$

The final form of the field equation is therefore,

$$\begin{aligned}
0 &= k\gamma_-\left(\left(n+1\right)\delta^{pc}\delta_{bq}u^b + \eta^{cp}\eta_{qb}u^b - \left(\left(n+1\right) + \frac{2u^2}{\beta^2}\right)\delta^c_q u^p - \eta^{cb}\delta_{bq}\delta^{ap}\eta_{ad}u^d\right) \\
&\quad +k\gamma_-\left(\frac{2}{\beta^2}\eta_{qd}u^c u^p u^d - \frac{2}{\beta^2}\delta^{ap}\delta_{bq}\eta_{ad}u^c u^b u^d + \frac{2u^2}{\beta^2}\delta^{pc}\delta_{bq}u^b\right) \\
&\quad +k\gamma_+\left(\left(n+1\right)\delta^{pc}\delta_{bq}\eta^{bd}v_d + \eta^{pc}v_q - \left(\left(n+1\right) + \frac{2v^2}{\beta^2}\right)\delta^c_q\eta^{pd}v_d - \delta^{ap}\delta_{bq}\eta^{bc}v_a\right) \\
&\quad +k\gamma_+\left(\frac{2}{\beta^2}\eta^{cd}\eta^{pe}v_d v_e v_q - \frac{2}{\beta^2}\eta^{cd}\delta^{ap}\delta_{bq}\eta^{be}v_d v_e v_a + \frac{2v^2}{\beta^2}\delta^{cp}\delta_{bq}\eta^{bd}v_d\right)
\end{aligned}$$

This equation implies conditions which must hold between u^a, v_a, δ_{ab} and η_{ab}. Solving from this general form for any one of these in terms of the others is algebraically difficult. Here we solve completely when $\gamma_+ = 0$.

5 The $\gamma_+ = 0$ case

Consider the $\gamma_+ = 0$ case. Then we have $k = -1$ and $\beta^2 = 1$, so the field equation reduces to

$$\begin{aligned} 0 &= (n+1)\,\delta^{pc}\delta_{bq}u^b + \eta^{cp}\eta_{qb}u^b - \left((n+1) + 2u^2\right)\delta^c_q u^p - \eta^{cb}\delta_{bq}\delta^{ap}\eta_{ad}u^d \\ &\quad +2\eta_{qd}u^c u^p u^d - 2\delta^{ap}\delta_{bq}\eta_{ad}u^c u^b u^d + 2u^2\delta^{pc}\delta_{bq}u^b \end{aligned}$$

so that contraction with u^q lets us solve for the inverse Euclidean metric, δ^{pc}, in terms of simpler quantities,

$$\begin{aligned} (n+1)\,\delta^{pc}U^2 &= -\eta^{cp}u^2 + \left((n+1) + 2u^2\right)u^c u^p + \eta^{cb}\delta_{bq}u^q\delta^{ap}\eta_{ad}u^d \\ &\quad -2u^2 u^c u^p + 2U^2\delta^{ap}\eta_{ad}u^c u^d - 2u^2U^2\delta^{pc} \end{aligned}$$

where we define $u^2 \equiv \eta_{ab}u^a u^b$ and $U^2 \equiv \delta_{ab}u^a u^b > 0$. Now contracting further with $\eta_{ce}u^e$ and collecting like terms,

$$\delta^{pc}\eta_{ce}u^e = \frac{u^2}{U^2}u^p$$

which, transferring the metric factors to the other side of the equation also gives $\eta^{ec}\delta_{cp}u^p = \frac{U^2}{u^2}u^e$. Substituting these back into the expression for the inverse metric now gives

$$\delta^{pc} = -\frac{u^2}{(n+1+2u^2)\,U^2}\left(\eta^{pc} - \frac{(n+2+2u^2)}{u^2}u^p u^c\right) \tag{24}$$

This gives the relationship between the Euclidean metric and the induced metric η_{ab} on the Lagrangian submanifold.

Next, we invert the metric. Starting from the ansatz $\delta_{pa} = -\frac{(n+1+2u^2)U^2}{u^2}\left(\eta_{pa} - \alpha\eta_{pc}\eta_{ad}u^c u^d\right)$ we demand $\delta^{cp}\delta_{pa} = \delta^c_a$ and solve for α,

$$\alpha = \frac{n+2+2u^2}{u^2\,(n+1+2u^2)}$$

and uniqueness of the inverse gives us the metric,

$$\delta_{ab} = -\frac{\left(n+1+2u^2\right)U^2}{u^2}\left(\eta_{ab} - \frac{n+2+2u^2}{u^2\,(n+1+2u^2)}\eta_{ac}\eta_{bd}u^c u^d\right) \tag{25}$$

Using this, we immediately find $\delta_{ab}u^b = \frac{U^2}{u^2}\eta_{ab}u^b$. Then substituting into the full field equation,

$$\begin{aligned} 0 &= (n+1)\,\delta^{pc}\delta_{bq}u^b - \eta^{cb}\delta_{bq}\left(\delta^{ap}\eta_{ad}u^d\right) + \frac{2u^2}{\beta^2}\delta^{pc}\left(\delta_{bq}u^b\right) - \frac{2}{\beta^2}u^c\left(\delta_{bq}u^b\right)\left(\delta^{ap}\eta_{ad}u^d\right) \\ &\quad +\eta^{cp}\eta_{qb}u^b - \left((n+1) + \frac{2u^2}{\beta^2}\right)\delta^c_q u^p + \frac{2}{\beta^2}\eta_{qd}u^c u^p u^d \\ &= (n+1)\left(-\frac{u^2}{(n+1+2u^2)\,U^2}\left(\eta^{cp} - \frac{(n+2+2u^2)}{u^2}u^c u^p\right)\right)\left(\frac{U^2}{u^2}\eta_{qb}u^b\right) \\ &\quad -\eta^{cb}\delta_{bq}\left(\frac{u^2}{U^2}u^p\right) - \frac{2u^2}{(n+1+2u^2)}\left(\eta^{cp} - \frac{(n+2+2u^2)}{u^2}u^c u^p\right)\eta_{qb}u^b - \frac{2}{\beta^2}u^c\left(\frac{U^2}{u^2}\eta_{qb}u^b\right)\left(\frac{u^2}{U^2}u^p\right) \\ &\quad +\eta^{cp}\eta_{qb}u^b - \left((n+1) + \frac{2u^2}{\beta^2}\right)\delta^c_q u^p + \frac{2}{\beta^2}\eta_{qd}u^c u^p u^d \end{aligned}$$

Then

$$\begin{aligned}
0 &= -\frac{(n+1)}{(n+1+2u^2)}\eta^{cp}\eta_{qb}u^b + \frac{(n+1)\left(n+2+2u^2\right)}{u^2\left(n+1+2u^2\right)}\eta_{qb}u^bu^cu^p \\
&\quad -\frac{u^2}{U^2}\eta^{cb}\delta_{bq}u^p - \frac{2u^2}{(n+1+2u^2)}\eta^{cp}\eta_{qb}u^b + \frac{2\left(n+2+2u^2\right)}{(n+1+2u^2)}\eta_{qb}u^bu^cu^p - 2\eta_{qb}u^bu^pu^c \\
&\quad +\eta^{cp}\eta_{qb}u^b - \left(n+1+2u^2\right)\delta^c_qu^p + 2\eta_{qd}u^cu^pu^d \\
&= -\left(n+1+2u^2\right)\delta^c_qu^p + \left(n+1+2u^2\right)\delta^c_qu^p \\
&\quad + \left(1 - \frac{(n+1)}{(n+1+2u^2)} - \frac{2u^2}{(n+1+2u^2)}\right)\eta^{cp}\eta_{qb}u^b \\
&\quad + \left(-\frac{n+2+2u^2}{u^2} + \frac{(n+1)\left(n+2+2u^2\right)}{u^2\left(n+1+2u^2\right)} + \frac{2\left(n+2+2u^2\right)}{(n+1+2u^2)}\right)\eta_{qb}u^bu^pu^c \\
&= \frac{1}{n+1+2u^2}\left(\left(n+1+2u^2\right) - (n+1) - 2u^2\right)\eta^{cp}\eta_{qb}u^b \\
&\quad + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(-\left(n+1+2u^2\right) + n+1+2u^2\right)\eta_{qb}u^bu^pu^c \\
&= 0
\end{aligned}$$

So this form of the metric is necessary and sufficient for the $\gamma_+ = 0$ flat solution to solve the field equation.

6 The remaining field equations

We have solved eq.(16). For the co-torsion we have $S_b^{\ ab} = 0$ and $S_a^{\ b}{}_b = S_b^{\ b}{}_a$, and the involution condition gives $S_{abc} = 0$. We can now substitute fully into the final field equation, eq.(19):

$$\begin{aligned}
0 &= \Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_cS_{e\ a}^{\ e} + \eta_{cd}\eta^{de}\mu^{\ b}_{\ \ ea} - \delta^b_c\eta_{ad}\eta^{de}\mu^{\ f}_{\ \ ef}\right) \\
&\quad +\Delta^{ap}_{qb}\left(\eta_{ec}\rho^{e\ b}_{\ a} - \delta^b_c\eta_{da}\rho^{d\ e}_{\ e} - \eta_{ac}u^b + \delta^b_c\eta_{ea}u^e\right) \\
0 &= \Delta^{ap}_{qb}\left(\left(S_{c\ a}^{\ b} - \delta^b_cS_{e\ a}^{\ e}\right) + \left(\mu^{\ b}_{\ \ ca} + \eta_{ec}\rho^{e\ b}_{\ a} - \eta_{ac}u^b\right) - \delta^b_c\left(\mu^{\ f}_{\ \ af} + \eta_{da}\rho^{d\ e}_{\ e} - \eta_{ea}u^e\right)\right)
\end{aligned}$$

6.1 Field equation for the co-torsion cross term

Substituting from

$$\begin{aligned}
\rho^{c\ b}_{\ a} &= \delta^c_au^b + k\gamma_-\left(\delta^c_au^b + \delta^b_au^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^cu^bu^d\right) \\
\rho^{b\ f}_{\ f} &= \left(1 + k\gamma_-(n+2) + \frac{2k\gamma_-}{\beta^2}u^2\right)u^b \\
\mu^{\ b}_{\ \ da} &= \delta^b_dv_a - k\gamma_+\left(\delta^b_dv_a + \delta^b_av_d + \eta^{be}\eta_{da}v_e + \frac{2}{\beta^2}\eta^{be}v_dv_av_e\right) \\
\mu^{\ f}_{\ \ df} &= \left(1 - k\gamma_+(n+2) - \frac{2k\gamma_+}{\beta^2}v^2\right)v_d
\end{aligned}$$

with $\gamma_+ = 0$, $\gamma_- = \beta = \pm1$, $k = -1$ and $\beta^2 = 1$,

$$\rho^{c\ b}_{\ a} = \delta^c_au^b - \gamma_-\left(\delta^c_au^b + \delta^b_au^c + \eta^{cb}\eta_{ad}u^d + \frac{2}{\beta^2}\eta_{ad}u^cu^bu^d\right)$$

$$
\begin{aligned}
\rho^{b\ \ f}_{\ f} &= \left(1-\gamma_-\left(n+2\right)-\frac{2\gamma_-}{\beta^2}u^2\right)u^b \\
\mu^b_{\ da} &= \delta^b_d v_a \\
\mu^f_{\ df} &= v_d
\end{aligned}
$$

we find

$$
\begin{aligned}
\left(\mu^b_{\ ca}+\eta_{ec}\rho^{e\ b}_{\ a}-\eta_{ac}u^b\right) &= \delta^b_c v_a+\eta_{ec}\left(\delta^e_a u^b-\gamma_-\left(\delta^e_a u^b+\delta^b_a u^e+\eta^{eb}\eta_{ad}u^d+2\eta_{ad}u^e u^b u^d\right)\right)-\eta_{ac}u^b \\
&= \delta^b_c v_a-\gamma_-\left(\eta_{ac}u^b+\delta^b_a\eta_{ec}u^e+\delta^b_c\eta_{ae}u^e+2\eta_{ec}\eta_{ad}u^e u^b u^d\right) \\
\left(\mu^f_{\ af}+\eta_{da}\rho^{d\ e}_{\ e}-\eta_{ea}u^e\right) &= v_a-\gamma_-\left(n+2+2u^2\right)\eta_{ae}u^e
\end{aligned}
$$

and the remaining co-torsion field equation becomes

$$
\begin{aligned}
0 &= \Delta^{ap}_{qb}\left(\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)+\left(\mu^b_{\ ca}+\eta_{ec}\rho^{e\ b}_{\ a}-\eta_{ac}u^b\right)-\delta^b_c\left(\mu^f_{\ af}+\eta_{da}\rho^{d\ e}_{\ e}-\eta_{ea}u^e\right)\right) \\
&= \Delta^{ap}_{qb}\left(\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)+\delta^b_c v_a-\gamma_-\left(\eta_{ac}u^b+\delta^b_a\eta_{ec}u^e+\delta^b_c\eta_{ae}u^e+2\eta_{ec}\eta_{ad}u^e u^b u^d\right)\right) \\
&\quad +\Delta^{ap}_{qb}\left(-\delta^b_c v_a+\gamma_-\left(n+2+2u^2\right)\delta^b_c\eta_{ae}u^e\right) \\
&= \Delta^{ap}_{qb}\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right) \\
&\quad -\gamma_-\Delta^{ap}_{qb}\left(\eta_{ac}u^b+\delta^b_a\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^b_c\eta_{ae}u^e+2\eta_{ec}\eta_{ad}u^e u^b u^d\right)
\end{aligned}
$$

Next, we substitute the expression, eq.(25), for the metric,

$$
\begin{aligned}
0 &= 2\Delta^{ap}_{qb}\left(\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)-\gamma_-\left(\eta_{ac}u^b+\delta^b_a\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^b_c\eta_{ae}u^e+2\eta_{ec}\eta_{ad}u^e u^b u^d\right)\right) \\
&= 2\Delta^{ap}_{qb}\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)-\gamma_-\left(\eta_{qc}u^p+\delta^p_q\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^p_c\eta_{qe}u^e+2\eta_{ec}\eta_{qd}u^e u^p u^d\right) \\
&\quad +\gamma_-\left(\eta^{pa}-\frac{\left(n+2+2u^2\right)}{u^2}u^p u^a\right)\left(\eta_{qb}-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{qf}\eta_{bg}u^f u^g\right) \\
&\quad \times\left(\eta_{ac}u^b+\delta^b_a\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^b_c\eta_{ae}u^e+2\eta_{ec}\eta_{ad}u^e u^b u^d\right) \\
&= 2\Delta^{ap}_{qb}\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)-\gamma_-\left(\eta_{qc}u^p+\delta^p_q\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^p_c\eta_{qe}u^e+2\eta_{ec}\eta_{qd}u^e u^p u^d\right) \\
&\quad +\gamma_-\left(\eta^{pa}-\frac{\left(n+2+2u^2\right)}{u^2}u^p u^a\right) \\
&\quad \times\left[\eta_{ac}\eta_{qb}u^b-\frac{n+2+2u^2}{n+1+2u^2}\eta_{ac}\eta_{qf}u^f+\eta_{qa}\eta_{ec}u^e-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{ec}\eta_{qf}\eta_{ag}u^f u^g u^e\right. \\
&\quad -\left(n+1+2u^2\right)\eta_{qc}\eta_{ae}u^e+\frac{n+2+2u^2}{u^2}\eta_{qf}\eta_{bg}\delta^b_c\eta_{ae}u^e u^f u^g \\
&\quad \left.+2\eta_{qb}\eta_{ec}\eta_{ad}u^e u^b u^d-2\frac{n+2+2u^2}{\left(n+1+2u^2\right)}\eta_{qf}\eta_{ec}\eta_{ad}u^e u^d u^f\right] \\
&= 2\Delta^{ap}_{qb}\left(S^{\ b}_{c\ a}-\delta^b_c S^{\ e}_{e\ a}\right)-\gamma_-\left(\eta_{qc}u^p+\delta^p_q\eta_{ec}u^e-\left(n+1+2u^2\right)\delta^p_c\eta_{qe}u^e+2\eta_{ec}\eta_{qd}u^e u^p u^d\right) \\
&\quad +\gamma_-\left(\delta^p_c\eta_{qb}u^b-\frac{\left(n+2+2u^2\right)}{u^2}\eta_{ac}\eta_{qb}u^b u^p u^a-\frac{n+2+2u^2}{n+1+2u^2}\delta^p_c\eta_{qf}u^f\right) \\
&\quad +\gamma_-\left(\frac{n+2+2u^2}{n+1+2u^2}\frac{\left(n+2+2u^2\right)}{u^2}\eta_{ac}\eta_{qf}u^f u^p u^a\right) \\
&\quad +\gamma_-\left(\delta^p_q\eta_{ec}u^e-\frac{\left(n+2+2u^2\right)}{u^2}\eta_{qa}\eta_{ec}u^e u^p u^a-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{ec}\eta_{qf}u^f u^p u^e\right)
\end{aligned}
$$

$$+\gamma_- \left(\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)} \left(n+2+2u^2\right) \eta_{ec}\eta_{qf} u^f u^e u^p \right)$$
$$+\gamma_- \left(\left(n+1+2u^2\right)\left(n+1+2u^2\right) \eta_{qc} u^p + \frac{n+2+2u^2}{u^2} \eta_{qf}\eta_{cg} u^p u^f u^g \right)$$
$$+\gamma_- \left(-\frac{n+2+2u^2}{u^2} \left(n+2+2u^2\right) \eta_{qf}\eta_{cg} u^f u^g u^p \right)$$
$$+\gamma_- \left(2\eta_{qb}\eta_{ec} u^e u^b u^p - 2\left(n+2+2u^2\right) \eta_{qb}\eta_{ec} u^e u^b u^p \right)$$
$$+\gamma_- \left(-2\frac{n+2+2u^2}{\left(n+1+2u^2\right)} \eta_{qf}\eta_{ec} u^e u^p u^f + 2\frac{n+2+2u^2}{\left(n+1+2u^2\right)} \left(n+2+2u^2\right) \eta_{qf}\eta_{ec} u^e u^f u^p \right)$$

Simplify,

$$\begin{aligned}
0 &= 2\Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) - \gamma_-\left(\eta_{qc}u^p + \delta^p_q \eta_{ec}u^e - \left(n+1+2u^2\right)\delta^p_c \eta_{qe}u^e + 2\eta_{ec}\eta_{qd}u^e u^p u^d\right) \\
&\quad + \gamma_- \left(-\frac{1}{n+1+2u^2}\delta^p_c \eta_{qb}u^b + \delta^p_q \eta_{ec}u^e + \left(n+1+2u^2\right)\left(n+1+2u^2\right)\eta_{qc}u^p \right) \\
&\quad + \gamma_- \left(2\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(n+2+2u^2\right) - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)} - \frac{\left(n+2+2u^2\right)}{u^2} \right) \eta_{qa}\eta_{ec}u^e u^p u^a \\
&\quad + \gamma_- \left(2 - \frac{n+2+2u^2}{u^2}\left(n+2+2u^2\right) \right) \eta_{qa}\eta_{ec}u^e u^p u^a \\
&\quad + \gamma_- \left(-2\left(n+2+2u^2\right) - 2\frac{n+2+2u^2}{\left(n+1+2u^2\right)} + 2\frac{n+2+2u^2}{\left(n+1+2u^2\right)}\left(n+2+2u^2\right) \right) \eta_{qb}\eta_{ec}u^e u^b u^p \\
&= 2\Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) \\
&\quad + \gamma_- \left(\left(\left(n+1+2u^2\right) - \frac{1}{n+1+2u^2} \right) \delta^p_c \eta_{qb}u^b + \left(\left(n+1+2u^2\right)\left(n+1+2u^2\right) - 1 \right) \eta_{qc}u^p \right) \\
&\quad - \gamma_- \frac{1}{u^2}\frac{n+2+2u^2}{\left(n+1+2u^2\right)} \left(4u^2u^2 + 4\left(n+1\right)u^2 + n\left(n+2\right) \right) \eta_{qa}\eta_{ec}u^e u^p u^a \\
&= 2\Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) \\
&\quad + \gamma_- \chi \left(\delta^p_c \eta_{qb}u^b + \left(n+1+2u^2\right)\eta_{qc}u^p - \frac{1}{u^2}\left(n+2+2u^2\right)\eta_{qa}\eta_{ec}u^e u^p u^a \right)
\end{aligned}$$

where we define

$$\chi \equiv n+1+2u^2 - \frac{1}{n+1+2u^2}$$

and have used

$$\begin{aligned}
\left(4u^2u^2 + 4\left(n+1\right)u^2 + n\left(n+2\right)\right) &= 2u^2 2u^2 + 2\left(n+1\right)2u^2 + \left(n+1\right)^2 + n\left(n+2\right) - \left(n+1\right)^2 \\
&= \left(2u^2 + \left(n+1\right)\right)^2 - 1
\end{aligned}$$

Check the symmetry on pq,

$$\eta_{pc}u_q + \left(n+1+2u^2\right)\eta_{qc}u_p - \frac{1}{u^2}\left(n+2+2u^2\right)u_c u_p u_q$$

This is *symmetric* on pq! This is not a contradiction — symmetry with respect to η_{ab} is not inconsistent with antisymmetry with respect to δ_{ab}.

Now we ask if this equation determines the co-torsion, $S_{c\ a}^{\ b}$. The general form of the equation is

$$2\Delta^{ap}_{qb}\left(S_{c\ a}^{\ b} - \delta^b_c S_{e\ a}^{\ e}\right) = K_{c\ a}^{\ b}$$

where $K_{c\ a}^{\ b}$ is given by

$$K_{c\ a}^{\ b} = -\gamma_{-}\chi\left(\delta_c^b\eta_{ae}u^e + \left(n+1+2u^2\right)\eta_{ac}u^b - \frac{1}{u^2}\left(n+2+2u^2\right)\eta_{ad}\eta_{ce}u^bu^du^e\right)$$

Notice that the combination $u^2 = \eta_{ab}u^au^b$ has zero conformal weight. If $u^2 = -\frac{n}{2}$, then χ vanishes and $K_{c\ a}^{\ b} = 0$. This is the same value that puts the metric into standard form:

$$\begin{aligned}\eta_{ab}\big|_{u^2=-\frac{n}{2}} &= -\frac{1}{(n+1+2u^2)}\frac{u^2}{U^2}\left(\delta_{ab} - \frac{n+2+2u^2}{U^2}\delta_{ac}u^c\delta_{bc}u^d\right)\Bigg|_{u^2=-\frac{n}{2}} \\ &= -\frac{u^2}{U^2}\left(\delta_{ab} - \frac{2}{U^2}\delta_{ac}u^c\delta_{bc}u^d\right)\end{aligned}$$

and it would not be surprising to find that this value is required. Since $U^2 > 0$, this form of η_{ab} makes the signature clear.

To study the symmetries of $K_{c\ a}^{\ b}$ it is convenient to briefly allow lowering of indices. For the remainder of this section we allow lowering of Latin indices using η_{ab}.

$$\begin{aligned}K_{cba} &= -\gamma_{-}\chi\left(\eta_{bc}u_a + \left(n+1+2u^2\right)\eta_{ac}u_b - \frac{1}{u^2}\left(n+2+2u^2\right)u_au_bu_c\right) \\ &= -\gamma_{-}\chi\left(\left(\eta_{bc} - \frac{1}{u^2}u_bu_c\right)u_a + \left(n+1+2u^2\right)\left(\eta_{ac} - \frac{1}{u^2}u_au_c\right)u_b\right) \\ K_{cb}^{\ \ b} &= 0 \\ K_{cab}u^au^b &= 0 \\ K_{cba}u^a &= -\gamma_{-}\chi u^2\left(\eta_{bc} - \frac{1}{u^2}u_bu_c\right)\end{aligned}$$

Using these general properties of K_{abc}, we proceed to study the limitations on the co-torsion.

6.2 Solving for the co-torsion cross-term

We now explore how much of the co-torsion cross-term the field equation determines.

Expanding Δ_{qb}^{ap},

$$\begin{aligned}K_{c\ b}^{\ a} &= \left(S_{c\ b}^{\ a} - \delta_c^aS_{e\ b}^{\ e}\right) - \delta^{ap}\delta_{bq}\left(S_{c\ p}^{\ q} - \delta_c^qS_{e\ p}^{\ e}\right) \\ &= S_{c\ b}^{\ a} - \delta_c^aS_{e\ b}^{\ e} \\ &\quad -\left(\eta^{pa} - \frac{\left(n+2+2u^2\right)}{u^2}u^pu^a\right)\left(\eta_{qb} - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\eta_{qg}\eta_{bh}u^gu^h\right)\left(S_{c\ p}^{\ q} - \delta_c^qS_{e\ p}^{\ e}\right) \\ &= S_{c\ b}^{\ a} - \delta_c^aS_{e\ b}^{\ e} - \eta^{pa}\eta_{be}S_{c\ p}^{\ e} + \frac{\left(n+2+2u^2\right)}{u^2}u^pu^a\eta_{be}S_{c\ p}^{\ e} + \eta^{ap}\eta_{bc}S_{e\ p}^{\ e} - \frac{\left(n+2+2u^2\right)}{u^2}u^pu^a\eta_{bc}S_{e\ p}^{\ e} \\ &\quad + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\eta^{pa}\eta_{qg}\eta_{bh}S_{c\ p}^{\ q}u^gu^h - \frac{\left(n+2+2u^2\right)}{u^2}\eta_{qg}S_{c\ p}^{\ q}u^g\eta_{bh}u^hu^pu^a\right) \\ &\quad - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\eta^{pa}S_{e\ p}^{\ e}\eta_{cg}\eta_{bh}u^gu^h - \frac{\left(n+2+2u^2\right)}{u^2}S_{e\ p}^{\ e}\eta_{cg}\eta_{bh}u^gu^hu^pu^a\right) \\ &= S_{c\ b}^{\ a} - \eta^{pa}\eta_{be}S_{c\ p}^{\ e} + \frac{\left(n+2+2u^2\right)}{u^2}u^pu^a\eta_{be}S_{c\ p}^{\ e} \\ &\quad + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\eta^{pa}\eta_{qg}\eta_{bh}S_{c\ p}^{\ q}u^gu^h - \frac{\left(n+2+2u^2\right)}{u^2}\eta_{qg}S_{c\ p}^{\ q}u^g\eta_{bh}u^hu^pu^a\right)\end{aligned}$$

$$-\delta^a_c S_{e\ b}^{\ e} + \eta^{ap}\eta_{bc} S_{e\ p}^{\ e} - \frac{\left(n+2+2u^2\right)}{u^2} u^p u^a \eta_{bc} S_{e\ p}^{\ e}$$
$$-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\eta^{pa} S_{e\ p}^{\ e}\eta_{cg}\eta_{bh}u^g u^h - \frac{\left(n+2+2u^2\right)}{u^2} S_{e\ p}^{\ e}\eta_{cg}\eta_{bh}u^g u^h u^p u^a\right)$$

Contract ac,

$$\begin{aligned}
K_{a\ b}^{\ a} &= -\left(n-2\right) S_{e\ b}^{\ e} - \eta_{be}\eta^{pa} S_{a\ p}^{\ e} + \frac{\left(n+2+2u^2\right)}{u^2}\eta_{be} S_{a\ p}^{\ e} u^p u^a \\
&\quad + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\eta^{pa} S_{a\ p}^{\ q}\eta_{qg}u^g\right)\eta_{bh}u^h \\
&\quad - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}\left(\eta_{qg} S_{a\ p}^{\ q} u^g u^p u^a\right)\eta_{bh}u^h \\
&= -\left(n-2\right) S_{e\ b}^{\ e} - \eta_{be}\eta^{pa} S_{a\ p}^{\ e} + \frac{\left(n+2+2u^2\right)}{u^2}\eta_{be} S_{a\ p}^{\ e} u^p u^a \\
&\quad + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(\left(\eta^{pa} S_{a\ p}^{\ q}\eta_{qg}u^g\right) - \frac{\left(n+2+2u^2\right)}{u^2}\left(\eta_{qg} S_{a\ p}^{\ q} u^g u^p u^a\right)\right)\eta_{bh}u^h
\end{aligned}$$

We may express the antisymmetric part of the co-torsion in terms of the symmetric part.

Again, we allow raising and lowering of indices, but note that we are computing $S_{a\ c}^{\ b}$ here while S_{abc} is a distinct tensor; dropping all indices we have

$$\begin{aligned}
S_{cba} - S_{cab} &= -K_{cab} + \frac{\left(n+2+2u^2\right)}{u^2} u_a S_{cbe} u^e + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)} u_b S_{cea} u^e \\
&\quad - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2} u_a u_b S_{cef} u^e u^f \\
&\quad - \eta_{ac} S_{e\ b}^{\ e} + \eta_{bc} S_{e\ a}^{\ e} - \frac{\left(n+2+2u^2\right)}{u^2} u_a \eta_{bc} S_{e\ f}^{\ e} u^f \\
&\quad - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)} S_{e\ a}^{\ e} u_c u_b + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2} S_{e\ f}^{\ e} u^f u_a u_b u_c
\end{aligned}$$

Write S_{abc} in terms of symmetric and antisymmetric parts,

$$\begin{aligned}
S_{abc} &= S_{a(bc)} + S_{a[bc]} \\
&\equiv P_{abc} + R_{abc}
\end{aligned}$$

Using

$$\begin{aligned}
S_{e\ a}^{\ e} &= S_{a\ e}^{\ e} \\
R_{a\ e}^{\ e} &= 0 \\
P_{e\ a}^{\ e} + R_{e\ a}^{\ e} &= P_{a\ e}^{\ e} \\
R_{e\ a}^{\ e} &= P_{a\ e}^{\ e} - P_{e\ a}^{\ e}
\end{aligned}$$

we show that the trace of the antisymmetric part may be expressed in terms of the symmetric part. To begin, expand

$$\begin{aligned}
2R_{cba} &= -K_{cab} + \frac{\left(n+2+2u^2\right)}{u^2} u_a\left(P_{cbe} + R_{cbe}\right)u^e + \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)} u_b\left(P_{cea} + R_{cea}\right)u^e \\
&\quad - \frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2} u_a u_b\left(P_{cef} + R_{cef}\right)u^e u^f
\end{aligned}$$

$$
\begin{aligned}
& -\eta_{ac}\left(P_{e\ b}^{\ e}+R_{e\ b}^{\ e}\right)+\eta_{bc}\left(P_{e\ a}^{\ e}+R_{e\ a}^{\ e}\right)-\frac{\left(n+2+2u^2\right)}{u^2}u_a\eta_{bc}\left(P_{e\ f}^{\ e}+R_{e\ f}^{\ e}\right)u^f \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(P_{e\ a}^{\ e}+R_{e\ a}^{\ e}\right)u_c u_b+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}\left(P_{e\ f}^{\ e}+R_{e\ f}^{\ e}\right)u^f u_a u_b u_c \\
= & -K_{cab}+\frac{\left(n+2+2u^2\right)}{u^2}u_a\left(P_{cbe}+R_{cbe}\right)u^e+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b\left(P_{cea}+R_{cea}\right)u^e \\
& -\eta_{ac}P_{b\ e}^{\ e} \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}u_a u_b P_{cef}u^e u^f+\eta_{bc}P_{a\ e}^{\ e}-\frac{\left(n+2+2u^2\right)}{u^2}u_a\eta_{bc}P_{f\ e}^{\ e}u^f \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}P_{a\ e}^{\ e}u_c u_b+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}P_{f\ e}^{\ e}u^f u_a u_b u_c
\end{aligned}
$$

Now we need $R_{cbe}u^e$:

$$
\begin{aligned}
2R_{cba}u^a = & -K_{cab}u^a+\left(n+2+2u^2\right)\left(P_{cbe}+R_{cbe}\right)u^e-u_c P_{b\ e}^{\ e} \\
& +\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b P_{cea}u^a u^e-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(n+2+2u^2\right)u_b P_{cef}u^e u^f \\
& +\eta_{bc}\left(P_{a\ e}^{\ e}u^a\right)-\left(n+2+2u^2\right)\eta_{bc}P_{f\ e}^{\ e}u^f \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(P_{a\ e}^{\ e}u^a\right)u_c u_b+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\left(n+2+2u^2\right)P_{f\ e}^{\ e}u^f u_b u_c \\
R_{cbe}u^e = & \frac{1}{n+2u^2}\left(K_{cab}u^a-\left(n+2+2u^2\right)P_{cbe}u^e+u_c P_{b\ e}^{\ e}+\frac{n+2+2u^2}{u^2}u_b P_{cef}u^e u^f\right) \\
& +\frac{1}{n+2u^2}\left(\left(n+1+2u^2\right)P_{a\ e}^{\ e}u^a\eta_{bc}-\frac{n+2+2u^2}{u^2}P_{a\ e}^{\ e}u^a u_b u_c\right)
\end{aligned}
$$

Substitute,

$$
\begin{aligned}
2R_{cba} = & -K_{cab}+\frac{\left(n+2+2u^2\right)}{u^2}u_a P_{cbe}u^e+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b P_{cea}u^e \\
& +\frac{\left(n+2+2u^2\right)}{u^2}\left(u_a R_{cbe}u^e-\frac{1}{n+1+2u^2}u_b R_{cae}u^e\right) \\
& -\eta_{ac}P_{b\ e}^{\ e}-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}u_a u_b P_{cef}u^e u^f+\eta_{bc}P_{a\ e}^{\ e}-\frac{\left(n+2+2u^2\right)}{u^2}u_a\eta_{bc}P_{f\ e}^{\ e}u^f \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}P_{a\ e}^{\ e}u_c u_b+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}P_{f\ e}^{\ e}u^f u_a u_b u_c
\end{aligned}
$$

$$
\begin{aligned}
2R_{cba} = & -K_{cab}+\frac{\left(n+2+2u^2\right)}{u^2}u_a P_{cbe}u^e+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b P_{cea}u^e \\
& +\frac{\left(n+2+2u^2\right)}{u^2\left(n+2u^2\right)}u_a\left(K_{ceb}u^e-\left(n+2+2u^2\right)P_{cbe}u^e+u_c P_{b\ e}^{\ e}+\frac{n+2+2u^2}{u^2}u_b P_{cef}u^e u^f\right) \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b\left(K_{cea}u^e-\left(n+2+2u^2\right)P_{cae}u^e+u_c P_{a\ e}^{\ e}+\frac{n+2+2u^2}{u^2}u_a P_{cef}u^e u^f\right) \\
& +\frac{\left(n+2+2u^2\right)}{u^2\left(n+2u^2\right)}u_a\left(\left(n+1+2u^2\right)P_{d\ e}^{\ e}u^d\eta_{bc}-\frac{n+2+2u^2}{u^2}P_{d\ e}^{\ e}u^d u_b u_c\right) \\
& -\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}u_b\left(\left(n+1+2u^2\right)P_{d\ e}^{\ e}u^d\eta_{ac}-\frac{n+2+2u^2}{u^2}P_{d\ e}^{\ e}u^d u_a u_c\right)
\end{aligned}
$$

$$-\eta_{ac}P_{b\ e}^{\ e}-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}u_au_bP_{cef}u^eu^f+\eta_{bc}P_{a\ e}^{\ e}-\frac{\left(n+2+2u^2\right)}{u^2}u_a\eta_{bc}P_{f\ e}^{\ e}u^f$$
$$-\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}P_{a\ e}^{\ e}u_cu_b+\frac{n+2+2u^2}{u^2\left(n+1+2u^2\right)}\frac{\left(n+2+2u^2\right)}{u^2}P_{f\ e}^{\ e}u^fu_au_bu_c$$

Therefore, R_{abc} is constructed completely from K_{abc} and the symmetric part, P_{abc}.

Appendix 1: Conformal weights and the basis change

The original solution to the field equations is

$$\begin{aligned}
\boldsymbol{\omega}^{\alpha}_{\ \ \beta} &= 2\Delta^{\alpha\mu}_{\nu\beta}s_\mu\mathbf{d}w^\nu \\
\boldsymbol{\omega}^{\alpha} &= \mathbf{d}w^\alpha \\
\boldsymbol{\omega}_{\alpha} &= \mathbf{d}s_\alpha-\left(s_\alpha s_\beta-\frac{1}{2}s^2\delta_{\alpha\beta}\right)\mathbf{d}w^\beta \\
\boldsymbol{\omega} &= s_\alpha\mathbf{d}w^\alpha
\end{aligned}$$

and we know the conformal weights. Scaling of the gauge fields is determined by the group properties.

$$\begin{aligned}
\boldsymbol{\omega}^{\alpha}_{\ \ \beta} &\rightarrow \boldsymbol{\omega}^{\alpha}_{\ \ \beta} \\
\boldsymbol{\omega}^{\alpha} &\rightarrow e^{\phi}\boldsymbol{\omega}^{\alpha} \\
\boldsymbol{\omega}_{\alpha} &\rightarrow e^{-\phi}\boldsymbol{\omega}_{\alpha} \\
\boldsymbol{\omega} &\rightarrow \boldsymbol{\omega}+\mathbf{d}\phi
\end{aligned}$$

Keep the frame field and coordinate indices distinct. Coordinates do not participate in the dilatation, but the coefficients do. For example, if we rewrite

$$\begin{aligned}
\boldsymbol{\omega}^{a}_{\ \ b} &= 2e_\alpha^{\ \ a}e_b^{\ \ \beta}\Delta^{\alpha\mu}_{\nu\beta}s_\mu\mathbf{d}w^\nu \\
\boldsymbol{\omega}^{a} &= e_\alpha^{\ \ a}\mathbf{d}w^\alpha \\
\boldsymbol{\omega}_{a} &= e_a^{\ \ \alpha}\left(\mathbf{d}s_\alpha-\left(s_\alpha s_\beta-\frac{1}{2}s^2\delta_{\alpha\beta}\right)\mathbf{d}w^\beta\right) \\
\boldsymbol{\omega} &= s_\alpha\mathbf{d}w^\alpha
\end{aligned}$$

where

$$e_\alpha^{\ \ a}=\delta^a_\alpha$$

in this coordinate system and gauge. Under dilatations and local rotations $e_\alpha^{\ \ a}\rightarrow e^\phi e_\alpha^{\ \ a}\Lambda^a_b$. Repeating the basis change with these matrices explicit should keep the weights straight.

The inner product shows the form and weight of the metric. Writing

$$\begin{aligned}
\boldsymbol{\omega}^{a} &= e_\alpha^{\ \ a}\mathbf{d}w^\alpha \\
\boldsymbol{\omega}_{a} &= e_a^{\ \ \alpha}\mathbf{d}s_\alpha-e_a^{\ \ \alpha}e_b^{\ \ \beta}\left(s_\alpha s_\beta-\frac{1}{2}s^2\delta_{\alpha\beta}\right)\boldsymbol{\omega}^b \\
&= e_a^{\ \ \alpha}\mathbf{d}s_\alpha+k_{ab}\boldsymbol{\omega}^b
\end{aligned}$$

we see that $k_{ab}\rightarrow e^{-2\phi}k_{ab}$ and find

$$\begin{aligned}
\left\langle\boldsymbol{\omega}^a,\boldsymbol{\omega}^b\right\rangle &= 0 \\
\left\langle\boldsymbol{\omega}^a,\boldsymbol{\omega}_b\right\rangle &= \delta^a_b=\left\langle\boldsymbol{\omega}^a,e_a^{\ \ \alpha}\mathbf{d}s_\alpha\right\rangle \\
0 &= \left\langle\boldsymbol{\omega}_a,\boldsymbol{\omega}_b\right\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \left\langle e_a{}^{\alpha}\mathbf{d}s_\alpha + k_{ac}\boldsymbol{\omega}^c, e_b{}^{\alpha}\mathbf{d}s_\alpha + k_{bd}\boldsymbol{\omega}^d\right\rangle \\
&= \langle e_a{}^{\alpha}\mathbf{d}s_\alpha, e_b{}^{\alpha}\mathbf{d}s_\alpha\rangle + \langle k_{ac}\boldsymbol{\omega}^c, e_b{}^{\alpha}\mathbf{d}s_\alpha\rangle + \left\langle e_a{}^{\alpha}\mathbf{d}s_\alpha, k_{bd}\boldsymbol{\omega}^d\right\rangle \\
&= \langle e_a{}^{\alpha}\mathbf{d}s_\alpha, e_b{}^{\alpha}\mathbf{d}s_\alpha\rangle + k_{ab} + k_{ba} \\
&= e_a{}^{\alpha}e_b{}^{\alpha}\langle \mathbf{d}s_\alpha, \mathbf{d}s_\alpha\rangle + 2k_{ab} \\
&= e_a{}^{\alpha}e_b{}^{\beta}\left(\langle \mathbf{d}s_\alpha, \mathbf{d}s_\alpha\rangle + \left(s^2\delta_{\alpha\beta} - 2s_\alpha s_\beta\right)\right) \\
e_a{}^{\alpha}e_b{}^{\beta}\left(2s_\alpha s_\beta - s^2\delta_{\alpha\beta}\right) &\equiv 2s_a s_b - s^2\delta_{ab} \\
&\to e^{-2\phi}\left(2s_a s_b - s^2\delta_{ab}\right)
\end{aligned}
$$

noting along the way that

$$
\begin{aligned}
\left\langle \mathbf{d}w^\alpha, \mathbf{d}w^\beta\right\rangle &= 0 \\
\langle \mathbf{d}w^\alpha, \mathbf{d}s_\beta\rangle &= \delta^\alpha_\beta \\
\langle \mathbf{d}s_\alpha, \mathbf{d}s_\beta\rangle &= \left(2s_\alpha s_\beta - s^2\delta_{\alpha\beta}\right) = -2k_{\alpha\beta}
\end{aligned}
$$

$$
\frac{\alpha\mu - 1}{\beta\nu}\bar{C}^{\mu\nu}
$$

Weights match the Latin indices, $s^a \to e^{\phi}s^a$ and $s_a \to e^{-\phi}s_a$.

Now let a change of basis be given by

$$
\begin{aligned}
\boldsymbol{\lambda}_\alpha &= \alpha\mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta \\
\boldsymbol{\kappa}^\alpha &= \mu\mathbf{d}w^\alpha + \nu B^{\alpha\beta}\mathbf{d}s_\beta
\end{aligned}
$$

The parameters and coordinates are all dimensionless. Then imposing the symplectic condition $\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha = \mathbf{d}w^\alpha\mathbf{d}s_\alpha$ we have

$$
\begin{aligned}
\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha &= \left(\mu\mathbf{d}w^\alpha + \nu B^{\alpha\mu}\mathbf{d}s_\mu\right)\left(\alpha\mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta\right) \\
&= \alpha\mu\mathbf{d}w^\alpha\mathbf{d}s_\alpha + \beta\mu C_{\alpha\beta}\mathbf{d}w^\alpha\mathbf{d}w^\beta + \alpha\nu B^{\alpha\mu}\mathbf{d}s_\mu\mathbf{d}s_\alpha + \beta\nu B^{\alpha\mu}\mathbf{d}s_\mu C_{\alpha\beta}\mathbf{d}w^\beta \\
&= \beta\mu C_{\alpha\beta}\mathbf{d}w^\alpha\mathbf{d}w^\beta + \left(\alpha\mu\delta^\mu_\beta - \beta\nu B^{\alpha\mu}C_{\alpha\beta}\right)\mathbf{d}w^\beta\mathbf{d}s_\mu + \alpha\nu B^{\alpha\mu}\mathbf{d}s_\mu\mathbf{d}s_\alpha \\
&\equiv \mathbf{d}w^\alpha\mathbf{d}s_\alpha
\end{aligned}
$$

The equality requires $B = B^t$ and $C = C^t$, and

$$
\alpha\mu\delta^\mu_\beta - \beta\nu B^{\alpha\mu}C_{\alpha\beta} = \delta^\mu_\beta
$$

It is here we see the need for $\alpha\mu \neq 1$. Solving for $B^{\alpha\mu}$,

$$
\begin{aligned}
B^{\alpha\mu}C_{\alpha\beta} &= \frac{\alpha\mu - 1}{\beta\nu}\delta^\mu_\beta \\
B^{\alpha\mu}C_{\alpha\beta}\bar{C}^{\beta\nu} &= \frac{\alpha\mu - 1}{\beta\nu}\delta^\mu_\beta\bar{C}^{\beta\nu} \\
B^{\mu\nu} &= \frac{\alpha\mu - 1}{\beta\nu}\bar{C}^{\mu\nu}
\end{aligned}
$$

Now, replacing B, we demand orthogonality of the subspaces:

$$
\begin{aligned}
\left\langle \boldsymbol{\kappa}^\alpha, \boldsymbol{\kappa}^\beta\right\rangle &= \left\langle \mu\mathbf{d}w^\alpha + \nu\frac{\alpha\mu - 1}{\beta\nu}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \mu\mathbf{d}w^\beta + \nu\frac{\alpha\mu - 1}{\beta\nu}\bar{C}^{\beta\nu}\mathbf{d}s_\nu\right\rangle \\
&= \left\langle \mu\mathbf{d}w^\alpha, \frac{\alpha\mu - 1}{\beta}\bar{C}^{\beta\nu}\mathbf{d}s_\nu\right\rangle + \left\langle \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \mu\mathbf{d}w^\beta\right\rangle + \left\langle \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \frac{\alpha\mu - 1}{\beta}\bar{C}^{\beta\nu}\mathbf{d}s_\nu\right\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= \mu\frac{\alpha\mu-1}{\beta}\bar{C}^{\beta\alpha} + \mu\frac{\alpha\mu-1}{\beta}\bar{C}^{\alpha\beta} + \left(\frac{\alpha\mu-1}{\beta}\right)^2 \bar{C}^{\alpha\mu}\bar{C}^{\beta\nu}\left(-\left(2s_\alpha s_\beta - s^2\delta_{\alpha\beta}\right)\right) \\
&= \frac{2\mu}{\beta}\left(\alpha\mu-1\right)\bar{C}^{\alpha\beta} - 2\left(\frac{\alpha\mu-1}{\beta}\right)^2\bar{C}^{\mu\alpha}k_{\alpha\beta}\bar{C}^{\beta\nu} \\
0 &= \langle\boldsymbol{\kappa}^\alpha, \boldsymbol{\lambda}_\beta\rangle \\
&= \left\langle \mu\mathbf{d}w^\alpha + \nu\frac{\alpha\mu-1}{\beta\nu}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \alpha\mathbf{d}s_\beta + \beta C_{\beta\nu}\mathbf{d}w^\nu\right\rangle \\
&= \mu\alpha\left\langle\mathbf{d}w^\alpha, \mathbf{d}s_\beta\right\rangle + \mu\beta C_{\beta\nu}\left\langle\mathbf{d}w^\alpha, \mathbf{d}w^\nu\right\rangle + \alpha\frac{\alpha\mu-1}{\beta}\bar{C}^{\alpha\mu}\left\langle\mathbf{d}s_\mu, \mathbf{d}s_\beta\right\rangle + \beta\frac{\alpha\mu-1}{\beta}\bar{C}^{\alpha\mu}C_{\beta\nu}\left\langle\mathbf{d}s_\mu, \mathbf{d}w^\nu\right\rangle \\
&= \mu\alpha\delta^\alpha_\beta - 2\alpha\frac{\alpha\mu-1}{\beta}\bar{C}^{\alpha\mu}k_{\mu\beta} + \beta\frac{\alpha\mu-1}{\beta}\bar{C}^{\alpha\mu}C_{\beta\mu} \\
&= \left(2\alpha\mu-1\right)\delta^\alpha_\beta - \frac{2\alpha}{\beta}\left(\alpha\mu-1\right)\bar{C}^{\alpha\mu}k_{\mu\beta} \\
C_{\nu\beta} &= \frac{2\alpha}{\beta}\frac{\alpha\mu-1}{2\alpha\mu-1}k_{\nu\beta} \\
\langle\boldsymbol{\lambda}_\alpha, \boldsymbol{\lambda}_\beta\rangle &= \langle\alpha\mathbf{d}s_\alpha + \beta C_{\alpha\mu}\mathbf{d}w^\mu, \alpha\mathbf{d}s_\beta + \beta C_{\beta\nu}\mathbf{d}w^\nu\rangle \\
&= \alpha^2\left\langle\mathbf{d}s_\alpha, \mathbf{d}s_\beta\right\rangle + \alpha\beta C_{\beta\nu}\left\langle\mathbf{d}s_\alpha, \mathbf{d}w^\nu\right\rangle + \alpha\beta C_{\alpha\mu}\left\langle\mathbf{d}w^\mu, \mathbf{d}s_\beta\right\rangle + \beta^2 C_{\alpha\mu}C_{\beta\nu}\left\langle\mathbf{d}w^\mu, \mathbf{d}w^\nu\right\rangle \\
&= -2\alpha^2 k_{\alpha\beta} + 2\alpha\beta C_{\alpha\beta} \\
&= 2\alpha^2\left(2\frac{\alpha\mu-1}{2\alpha\mu-1} - 1\right)k_{\alpha\beta} \\
&= -\frac{2\alpha^2}{2\alpha\mu-1}k_{\alpha\beta}
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left\langle\boldsymbol{\kappa}^\alpha, \boldsymbol{\kappa}^\beta\right\rangle &= \frac{2\alpha\mu-1}{2\alpha^2}k^{\alpha\beta} \\
\langle\boldsymbol{\kappa}^\alpha, \boldsymbol{\lambda}_\beta\rangle &= 0 \\
\langle\boldsymbol{\lambda}_\alpha, \boldsymbol{\lambda}_\beta\rangle &= -\frac{2\alpha^2}{2\alpha\mu-1}k_{\alpha\beta}
\end{aligned}
$$

We may choose the normalizing constant to give $\langle\boldsymbol{\lambda}_\alpha, \boldsymbol{\lambda}_\beta\rangle = s^2\delta_{\alpha\beta} - 2s_\alpha s_\beta$ by setting $-\frac{\alpha^2}{2\alpha\mu-1} = \pm 1 = k$ so that

$$
\begin{aligned}
\frac{k-\alpha^2}{2\alpha k} &= \mu \\
\mu &= \frac{1-k\alpha^2}{2\alpha}
\end{aligned}
$$

The final change of basis is then

$$
\begin{aligned}
\boldsymbol{\kappa}^\alpha &= \frac{1-k\alpha^2}{2\alpha}\mathbf{d}w^\alpha - \frac{k\alpha}{2}k^{\alpha\beta}\mathbf{d}s_\beta \\
\boldsymbol{\lambda}_\alpha &= \alpha\mathbf{d}s_\alpha + \frac{1+k\alpha^2}{k\alpha}k_{\alpha\beta}\mathbf{d}w^\beta
\end{aligned}
$$

None of the fields above with Greek indices has conformal weight. In the weighted basis, $\mathbf{e}^a = e_\alpha^{\ \ a}\boldsymbol{\kappa}^\alpha, \mathbf{f}_a = e_a^{\ \ \alpha}\boldsymbol{\lambda}_\alpha$,

$$
\begin{aligned}
\mathbf{e}^a &= \frac{1-k\alpha^2}{2\alpha}\mathbf{d}w^a - \frac{k\alpha}{2}k^{ab}\mathbf{d}s_b \\
\mathbf{f}_a &= \alpha\mathbf{d}s_a + \frac{1+k\alpha^2}{k\alpha}k_{ab}\mathbf{d}w^b
\end{aligned}
$$

with the constants non-scaling. The conformal weights are as expected, with each raised Latin index contributing $+1$ and each lowered Latin index contributing -1 to the conformal weight.

Appendix 2: Conformal weight of the connection

The structure equations in the new basis are

$$
\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b}\boldsymbol{\omega}^a_{\ c} + \Delta^{ah}_{gb}\eta_{hj}\mathbf{e}^j\mathbf{e}^g - \Delta^{ah}_{gb}\eta^{gi}\mathbf{f}_h\mathbf{f}_i + 2\Delta^{ah}_{gb}\Xi^{gi}_{jh}\mathbf{f}_i\mathbf{e}^j + \boldsymbol{\Omega}^a_{\ b} \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}\mathbf{f}_a &= \Theta^{bc}_{da}\boldsymbol{\tau}^d_{\ c}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ac}\mathbf{e}^c + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}
$$

where the $\mathbf{e}^c$ components of $\mathbf{D}\eta^{ab}$ become

$$
\begin{aligned}
\mathbf{D}\eta^{ab} &= \mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\alpha}^a_{\ c} + \eta^{ac}\boldsymbol{\alpha}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \eta^{cb}\left(\boldsymbol{\sigma}^a_{\ c} + \boldsymbol{\mu}^a_{\ c}\right) + \eta^{ac}\left(\boldsymbol{\sigma}^b_{\ c} + \boldsymbol{\mu}^b_{\ c}\right) - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + \left(\eta^{cb}\boldsymbol{\sigma}^a_{\ c} + \eta^{ac}\boldsymbol{\sigma}^b_{\ c}\right) + \eta^{cb}\boldsymbol{\mu}^a_{\ c} + \eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} \\
&= \mathbf{d}\eta^{ab} + 2\eta^{ac}\boldsymbol{\mu}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab}
\end{aligned}
$$

Under conformal transformation, this last gives

$$
\begin{aligned}
\mathbf{D}\tilde{\eta}^{ab} &= \mathbf{d}\left(e^{2\phi}\eta^{ab}\right) + 2\left(e^{2\phi}\eta^{ac}\right)\tilde{\boldsymbol{\mu}}^b_{\ c} - 2e^{-\phi}\left(W_c + \partial_c\phi\right)e^{\phi}\mathbf{e}^c\left(e^{2\phi}\eta^{ab}\right) \\
&= e^{2\phi}\mathbf{d}\eta^{ab} + 2e^{2\phi}\eta^{ab}\mathbf{d}\phi + 2\left(e^{2\phi}\eta^{ac}\right)\tilde{\boldsymbol{\mu}}^b_{\ c} - 2\left(W_c + \partial_c\phi\right)\mathbf{e}^c\left(e^{2\phi}\eta^{ab}\right) \\
&= e^{2\phi}\left(\mathbf{d}\eta^{ab} + 2\eta^{ab}\mathbf{d}\phi + 2\eta^{ac}\tilde{\boldsymbol{\mu}}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab} - 2\partial_c\phi\mathbf{e}^c\eta^{ab}\right) \\
&= e^{2\phi}\left(\mathbf{d}\eta^{ab} + 2\eta^{ac}\tilde{\boldsymbol{\mu}}^b_{\ c} - 2W_c\mathbf{e}^c\eta^{ab}\right) \\
&= e^{2\phi}\mathbf{D}\eta^{ab}
\end{aligned}
$$

provided $\tilde{\boldsymbol{\mu}}^b_{\ c} = \boldsymbol{\mu}^b_{\ c}$,

Now look at the basis equation before and after a dilatation,

$$
\begin{aligned}
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ae}\mathbf{f}_e + \mathbf{T}^a \\
\mathbf{d}\left(e^{\phi}\mathbf{e}^a\right) &= \left(e^{\phi}\mathbf{e}^b\right)\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + \frac{1}{2}e^{-2\phi}\eta_{cb}\mathbf{d}\left(e^{2\phi}\eta^{ac}\right)e^{\phi}\mathbf{e}^b + \frac{1}{2}e^{2\phi}\left(\mathbf{D}\eta^{ae}\right)e^{-\phi}\mathbf{f}_e + e^{\phi}\mathbf{T}^a
\end{aligned}
$$

Subtracting e^{ϕ} times the first from the second,

$$
\begin{aligned}
\mathbf{d}\left(e^{\phi}\mathbf{e}^a\right) - e^{\phi}\mathbf{d}\mathbf{e}^a &= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + \frac{1}{2}e^{-2\phi}\eta_{cb}\mathbf{d}\left(e^{2\phi}\eta^{ac}\right)e^{\phi}\mathbf{e}^b + \frac{1}{2}e^{2\phi}\left(\mathbf{D}\eta^{ae}\right)e^{-\phi}\mathbf{f}_e + e^{\phi}\mathbf{T}^a \\
&\quad - e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^{\phi}\mathbf{D}\eta^{ae}\mathbf{f}_e - e^{\phi}\mathbf{T}^a \\
0 &= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} + e^{\phi}\mathbf{e}^a\mathbf{d}\phi - e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c} + \frac{1}{2}\eta_{cb}\left(2\eta^{ac}\mathbf{d}\phi + \mathbf{d}\eta^{ac}\right)e^{\phi}\mathbf{e}^b - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b \\
&= e^{\phi}\mathbf{e}^b\left(\Theta^{ac}_{db}\tilde{\boldsymbol{\tau}}^d_{\ c} - \Theta^{ac}_{db}\boldsymbol{\tau}^d_{\ c}\right) + e^{\phi}\mathbf{e}^a\mathbf{d}\phi + \mathbf{d}\phi e^{\phi}\mathbf{e}^a + \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b - \frac{1}{2}e^{\phi}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b \\
&= e^{\phi}\mathbf{e}^b\Theta^{ac}_{db}\left(\tilde{\boldsymbol{\tau}}^d_{\ c} - \boldsymbol{\tau}^d_{\ c}\right)
\end{aligned}
$$

so the equation is satisfied with the connection dilatationally invariant.

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