Time and dark matter from the conformal symmetries of Euclidean space

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Abstract. The quotient of the conformal group of Euclidean 4-space by its Weyl subgroup results in a geometry possessing many of the properties of relativistic phase space, including both a natural symplectic form and non-degenerate Killing metric. We show that the general solution posesses orthogonal Lagrangian submanifolds, with the induced metric and the spin connection on the submanifolds necessarily Lorentzian, despite the Euclidean starting pont. By examining the structure equations of the biconformal space in an orthonormal frame adapted to its phase space properties, we also find that two new tensor fields exist in this geometry, not present in Riemannian geometry. The first is a combination of the Weyl vector with the scale factor on the metric, and determines the timelike directions on the submanifolds. The second comes from the components of the spin connection, symmetric with respect to the new metric. Though this field comes from the spin connection it transforms homogeneously. Finally, we show that in the absence of conformal curvature or sources, the configuration space has geometric terms equivalent to a perfect fluid and a cosmological constant.

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1. Introduction

We develop a gauge theory based on the conformal group of a Euclidean space, and show that its group properties necessarily lead to a Lorentzian phase spacetime with *vacuum* solutions carrying both a cosmological constant and a cosmological perfect fluid as part of the generalized Einstein tensor. In curved models, this geometric background may explain or contribute to dark matter and dark energy. To emphasize the purely geometric character of the construction, we give a description of our use of the quotient manifold method for building gauge theories. Our use of the conformal group, together with our choice of local symmetry, lead to several structures not present in other related gauge theories. Specifically, we show the generic presence of a symplectic form, that there exists an induced metric from the non-degenerate Killing form, demonstrate (but do not use) Kähler structure, and find natural orthogonal, Lagrangian submanifolds. All of these properties arise directly from group theory.

In the remainder of this introduction, we give a brief historical overview of techniques leading up to, related to, or motivating our own, then describe the layout of our presentation.

As mathematicians began studying the various incarnations of non-Euclidean geometry, Klein started his Erlangen Program in 1872 as a way to classify all forms of geometries that could be constructed using quotients of groups. These *homogeneous spaces* allowed for straightforward classification of the spaces dependent on their symmetry properties. Much of the machinery necessary to understand these spaces originated with Cartan, beginning with his doctoral dissertation [1]. The classification of these geometries according to symmetry foreshadowed gauge theory, the major tool that would be used by theoretical physicists as the twentieth century continued. We will go into extensive detail about how these methods are used in a modern context in section 2. Most of the development, in modern language, can be found in [2].

The use of symmetries to construct physical theories can be greatly credited to Weyl's attempts at constructing a unified theory of gravity and electromagnetism by adding dilatational symmetry to general relativity. These attempts failed until Weyl looked at a $U(1)$ symmetry of the action instead, thus constructing the first gauge theory of electromagnetism. These efforts were extended to non-Abelian groups by Yang and Mills [3], including all $SU(n)$ and described by the Yang-Mills action. The success of these theories as quantum pre-cursors inspired relativists to try and construct general relativity as a gauge theory. Utiyama [4] looked at GR based on the the Lorentz group, followed by Kibble [5] who first gauged the Poincaré group to form general relativity.

Standard approaches to gauge theory begin with a matter action globally invariant under some symmetry group $\mathcal{H}$. This action generally fails to be locally symmetric due to the derivatives of the fields, but can be made locally invariant by introducing an $\mathcal{H}$-covariant derivative. The connection fields used for this derivative are called gauge fields. The final step is to make the gauge fields dynamical by constructing their field strengths, which may be thought of as curvatures of the connection, and including them in a modified action.

In the 1970's the success of the standard model and the growth of supersymmetric gravity theories inspired physicists to extend the symmetry used to construct a gravitational theory. MacDowell and Mansouri [6] obtained general relativity by gauging the de Sitter or anti-de Sitter groups, and using a Wigner-Inönu contraction to recover Poincaré symmetry. As a pre-cursor to supersymmetrizing Weyl gravity, two groups [7–10] looked at a gravitational theory based on the conformal group, using the Weyl curvature-squared action. These approaches are top-down, in the sense that they often start with a physical matter action and generalize to a local symmetry that leads to interactions. However, as this work expanded, physicists started using the techniques of Cartan and Klein to organize and develop the structures systematically.

In [11,12] Ne'eman and Regge develop what they refer to as the quotient manifold technique to construct a gauge theory of gravity based on the Poincaré group. Theirs is the first construction of a gravitational gauge theory that uses Klein (homogeneous) spaces as generalized versions of tangent spaces, applying methods developed by Cartan [13] to characterize a more general geometry. In their 1982 papers [14,15], Ivanov and Niederle exhaustively considered quotients of the Poincaré, de Sitter, anti-de Sitter and Lorentzian conformal groups ($ISO\,(3,1)$, $SO\,(4,1)$, $SO\,(3,2)$ and $SO\,(4,2)$) by various subgroups containing the Lorentz group.

There are a number of more recent implementations of Cartan geometry in the modern literature. One good introduction is Wise's use of Cartan methods to look at the MacDowell-Mansouri action [16]. The "waywiser" approach of visualizing these geometries is advocated strongly, and gives a clear geometric way of undertsanding Cartan geometry. The use of Cartan techniques in [17] to look at the Chern-Simons action in $2 + 1$ dimensions provides a nice example of the versatility of the method. This action can be viewed as having either Minkowski, de Sitter or anti-de Sitter symmetry and Cartan methods allow a straightforward characterization of the theory given the various symmetries. The analysis is extended to look first at the conformal representation of these groups on the Euclidean surfaces of the theory (2-dimensional spatial slices). The authors then look specifically at shape dynamics, which is found equivalent to the case when the Chern-Simons action has de Sitter symmetry. Tractor calculus is yet another example using a quotient of the conformal group, in which the associated tensor bundles are based on a linear, $(n+2)$-dim representation of the group. This is a distinct gauging from the one we study here, but one studied in [18].

Our research focuses primarily on gaugings of the conformal group. Initially motivated by a desire to understand the physical role of local scale invariance, the growing prospects of twistor string formulations of gravity [19] elevate the importance of understanding its low-energy limit, which is expected to be a conformal gauge theory of gravity. Interestingly, there are two distinct ways to formulate gravitational theories based on the conformal group, first identified in [14, 15] and developed in [18, 20, 21]. Both of these lead directly to scale-invariant general relativity. This is surprising since the best known conformal gravity theory is the fourth-order theory developed by Weyl [22–26] and Bach [27]. When a Palatini style variation is applied to Weyl gravity, it becomes second-order, scale-invariant general relativity [18].

The second gauging of the conformal group identified in these works is the *biconformal* gauging. Leading to scale-invariant general relativity formulated on a $2n$-dimensional symplectic manifold, the approach took a novel twist for homogeneous spaces in [28]. There it is shown that, because the biconformal gauging leads to a zero-signature manifold of doubled dimension, we can start with the conformal symmetry of a non-Lorentzian space while still arriving at spacetime gravity. We describe the resulting signature theorem in detail below, and considerably strengthen its conclusions. In addition to necessarily developing a direction of time from a Euclidean-signature starting point, we show that these models give a group-theoretically driven candidate for dark matter.

In the next Section, we describe the quotient manifold method in detail, providing an example by applying it to the Poincaré group to produce Cartan and Riemannian geometries. Then, in Section 3, we apply the method to the conformal group in the two distinct ways outlined above. The first, called the auxiliary gauging, reproduces Weyl gravity. Focusing on the second, we identify a number of properties posessed by the homogeneous space of the biconformal gauging. In Section 4, we digress to complete both gaugings by modifying the quotient manifold and connections, then writing appropriate action functionals, thereby establishing physical theories of gravity. We return to study the homogeneous space of the biconformal gauging in Section 5, developing the Maurer-Cartan structure equations in an adapted basis. Then, in the next Section, we transform a known solution to the structure equations into the adapted basis and identify the properties of the resulting space. This reveals two previously unknown objects, one a tensor of rank three, and the other a vector. In Section 6 we find the form of the connection and basis forms when restricted to the configuration and momentum submanifolds. This reveals the possibility of Riemannian curvature of the submanifolds, even though the Cartan curvature of the full space vanishes. Imposing the form of the solution, we find the configuration space has a generalized Einstein tensor which contains both a cosmological constant and cosmological dust in addition to the usual Einstein tensor. Finally, we summarize our results.

2. Quotient Manifold Method

We are interested in geometries – ultimately spacetime geometries – which have continuous local symmetries. The structure of such systems is that of a principal fiber bundle with Lie group fibers. The quotient method starts with a Lie group, $\mathcal{G}$, with the desired local symmetry as a proper Lie subgroup. To develop the local properties any representation will give equivalent results, so without loss of generality we assume a linear representation, i.e. a vector space $\mathcal{V}^{n+2}$ on which $\mathcal{G}$ acts. Typically this will be either a signature (p, q) (pseudo-)Euclidean space or the corresponding spinor space. This vector space is useful for describing the larger symmetry group, but is only a starting point and will not appear in the theory.

The quotient method, laid out below, is identical in many respects to the approaches of [16, 17]. The nice geometric interpretation of using a Klein space in place of a tangent space to both characterize a curved manifold and take advantage of its metric structure are also among the motivations for using the quotient method. In what follows not all the manifolds we look at will be interpreted as spacetime, so we choose not to use the interpretation of a Klein space moving around on spacetime in a larger ambient space. Rather we directly generalize the homogeneous space to add curvatures. The homogeneous space becomes a local model for a more general curved space, similar to the way that $\mathbb{R}^n$ provides a local model for an n-dim Riemannian manifold.

We include a concise introduction here, but the reader can find a more detailed exposition in [2]. Our intention is to make it clear that our ultimate conclusions have rigorous roots in group theory, rather than to present a comprehensive mathematical description.

2.1. Construction of a principal $\mathcal{H}$-bundle $\mathfrak{B}(\mathcal{G}, \pi, \mathcal{H}, M_0)$ with connection

Consider a Lie group, $\mathcal{G}$, and a non-normal Lie subgroup, $\mathcal{H}$, on which $\mathcal{G}$ acts effectively and transitively. The quotient of these is a homogeneous manifold, M_0. The points of M_0 are the left cosets,

$$g\mathcal{H} = \{g' \mid g' = gh \, for\ some\ h \in \mathcal{H}\}$$

so there is a natural $1-1$ mapping $g\mathcal{H} \leftrightarrow \mathcal{H}$. The cosets are disjoint from one another and together cover $\mathcal{G}$. There is a projection, $\pi : \mathcal{G} \to M_0$, defined by $\pi(g) = g\mathcal{H} \in M_0$. There is also a right action of $\mathcal{G}$, $g\mathcal{H}\mathcal{G}$, given for all elements of $\mathcal{G}$ by right multiplication.

Therefore, $\mathcal{G}$ is a principal $\mathcal{H}$-bundle, $\mathfrak{B}(\mathcal{G}, \pi, \mathcal{H}, M_0)$, where the fibers are the left cosets. This is the mathematical object required to carry a gauge theory of the symmetry group $\mathcal{H}$. Let the dimension of $\mathcal{G}$ be m, the dimension of $\mathcal{H}$ be k. Then the dimension of the manifold is $n = m - k$ and we write $M_0^{(n)}$. Choosing a gauge amounts to picking a cross-section of this bundle, i.e., one point from each of these copies of $\mathcal{H}$. Local symmetry amounts to dynamical laws which are independent of the choice of cross-section.

Lie groups have a natural Cartan connection given by the one-forms, $\boldsymbol{\xi}^A$, dual to the group generators, G_A. Rewriting the Lie algebra in terms of these dual forms leads immediately to the Maurer-Cartan structure equations,

$$\mathbf{d}\boldsymbol{\xi}^A = -\frac{1}{2} c^A_{\ BC} \boldsymbol{\xi}^B \wedge \boldsymbol{\xi}^C \tag{1}$$

where $c^A_{\ BC}$ are the group structure constants, and $\wedge$ is the wedge product. The integrability condition for this equation follows from the Poincaré lemma, $\mathbf{d}^2 = 0$, and turns out to be precisely the Jacobi identity. Therefore, the Maurer-Cartan equations together with their integrability conditions are completely equivalent to the Lie algebra of $\mathcal{G}$.

Let $\boldsymbol{\xi}^a$ (where $a = 1, \ldots, k$) be the subset of one-forms dual to the generators of the subgroup, $\mathcal{H}$. Let the remaining independent forms be labeled $\boldsymbol{\chi}^\alpha$. Then the $\boldsymbol{\xi}^a$ give a connection on the fibers while the $\boldsymbol{\chi}^\alpha$ span the co-tangent spaces to $M_0^{(n)}$. We denote the manifold with connection by $\mathcal{M}_0^{(n)} = \left(M_0^{(n)}, \boldsymbol{\xi}^A\right)$.

2.2. Cartan generalization

For a gravity theory, we require in general a curved geometry, $\mathcal{M}^{(n)}$. To achieve this, the quotient method allows us to generalize both the connection and the manifold. Since the principal fiber bundle from the quotient is a local direct product, this is not changed if we allow a generalization of the manifold, $M_0^{(n)} \to M^{(n)}$. We will not consider such topological issues here. Generalizing the connection is more subtle. If we change $\boldsymbol{\xi}^A = (\boldsymbol{\xi}^a, \boldsymbol{\chi}^\alpha)$ to a new connection $\boldsymbol{\xi}^A \to \boldsymbol{\omega}^A, \boldsymbol{\xi}^a \to \boldsymbol{\omega}^a, \boldsymbol{\chi}^\alpha \to \boldsymbol{\omega}^\alpha$ arbitrarily, the Maurer-Cartan equation is altered to

$$\mathbf{d}\boldsymbol{\omega}^A = -\frac{1}{2} c^A_{\ BC} \boldsymbol{\omega}^B \wedge \boldsymbol{\omega}^C + \boldsymbol{\Omega}^A$$

where $\mathbf{\Omega}^A$ is a 2-form determined by the choice of the new connection. We need restrictions on $\mathbf{\Omega}^A$ so that it represents curvature of the geometry $\mathcal{M}^{(n)} = \left(M^{(n)}, \boldsymbol{\omega}^A\right)$ and not of the full bundle $\mathfrak{B}$. We restrict $\mathbf{\Omega}^A$ by requiring it to be independent of lifting, i.e., horizontality of the curvature.

To define horizontality, recall that the integral of the connection around a closed curve in the bundle is given by the integral of $\mathbf{\Omega}^A$ over any surface bounded by the curve. We require this integral to be independent of lifting, i.e., horizontal. It is easy to show that this means that the two-form basis for the curvatures $\mathbf{\Omega}^A$ cannot include any of the one-forms, $\boldsymbol{\omega}^a$, that span the fiber group, $\mathcal{H}$. With the horizontality condition, the curvatures take the simpler form

$$\mathbf{\Omega}^A = \frac{1}{2}\Omega^A_{\ \alpha\beta}\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}^\beta$$

More general curvatures than this will destroy the homogeneity of the fibers, so we would no longer have a *principal* $\mathcal{H}$-bundle.

In addition to horizontality, we require integrability. Again using the Poincaré lemma, $\mathbf{d}^2\boldsymbol{\omega}^A \equiv 0$, we always find a term $\frac{1}{2}c^A_{\ B[C}c^B_{\ DE]}\boldsymbol{\omega}^C \wedge \boldsymbol{\omega}^D \wedge \boldsymbol{\omega}^E$ which vanishes by the Jacobi identity, $c^A_{\ B[C}c^B_{\ DE]} \equiv 0$, while the remaining terms give the general form of the Bianchi identities,

$$\mathbf{d}\mathbf{\Omega}^A + c^A_{\ BC}\boldsymbol{\omega}^B \wedge \mathbf{\Omega}^C = 0$$

2.3. Example: Pseudo-Riemannian manifolds

To see how this works in a familiar example, consider the construction of the pseudo-Riemannian spacetimes used in general relativity, for which we take the quotient of the 10-dim Poincaré group by its 6-dim Lorentz subgroup. The result is a principal Lorentz bundle over $\mathbb{R}^4$. Writing the one-forms dual to the Lorentz $(M^a_{\ b})$ and translation (P_a) generators as $\boldsymbol{\xi}^a_{\ b}$ and $\boldsymbol{\omega}^a$, respectively, the ten Maurer-Cartan equations are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\xi}^a_{\ b} &= \boldsymbol{\xi}^c_{\ b} \wedge \boldsymbol{\xi}^a_{\ c} \\
\mathbf{d}\boldsymbol{\omega}^a &= \boldsymbol{\omega}^b \wedge \boldsymbol{\xi}^a_{\ b}
\end{aligned}$$

Notice that the first describes a pure gauge spin connection, $\mathbf{d}\boldsymbol{\xi}^a_{\ b} = -\bar{\Lambda}^c_{\ b}\mathbf{d}\Lambda^a_{\ c}$ where $\Lambda^a_{\ c}$ is a local Lorentz transformation. Therefore, there exists a local Lorentz gauge such that $\boldsymbol{\xi}^a_{\ b} = 0$. The second equation then shows the existence of an exact orthonormal frame, which tells us that the space is Minkowski.

Now generalize the geometry, $\left(M^4_0, \boldsymbol{\xi}^A\right) \to \left(M^4, \boldsymbol{\omega}^A\right)$ where $M^4_0 = \mathbb{R}^4$ and we denote the new connection forms by $\boldsymbol{\omega}^A = \left(\boldsymbol{\omega}^a_{\ b}, \mathbf{e}^b\right)$. In the structure equations, this leads to the presence of ten curvature 2-forms,

$$\begin{aligned}
\mathbf{d}\boldsymbol{\omega}^a_{\ b} &= \boldsymbol{\omega}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c} + \mathbf{R}^a_{\ b} \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{\ b} + \mathbf{T}^a
\end{aligned}$$

Since the $\boldsymbol{\omega}^a_{\ b}$ span the Lorentz subgroup, horizontality is accomplished by restricting the curvatures to

$$\begin{aligned}
\mathbf{R}^a_{\ b} &= \frac{1}{2}R^a_{\ bcd}\mathbf{e}^c \wedge \mathbf{e}^d \\
\mathbf{T}^a &= \frac{1}{2}T^a_{\ bc}\mathbf{e}^b \wedge \mathbf{e}^c
\end{aligned}$$

that is, there are no terms such as, for example, $\frac{1}{2}R^{a\ \ c}_{\ b\ \ de}\boldsymbol{\omega}^d_{\ c} \wedge \mathbf{e}^e$ or $\frac{1}{2}T^{a\ c\ e}_{\ b\ d}\boldsymbol{\omega}^b_{\ c} \wedge \boldsymbol{\omega}^d_{\ e}$. Finally, integrability is guaranteed by the pair of Bianchi identities,

$$\begin{aligned}
\mathbf{d}\mathbf{R}^a_{\ b} + \mathbf{R}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c} - \mathbf{R}^a_{\ c} \wedge \boldsymbol{\omega}^c_{\ b} &= 0 \\
\mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \boldsymbol{\omega}^a_{\ b} + \mathbf{e}^b \wedge \mathbf{R}^a_{\ b} &= 0
\end{aligned}$$

By looking at the transformation of $\mathbf{R}^a_{\ b}$ and $\mathbf{T}^a$ under local Lorentz transformations, we find that despite originating as components of a single Poincaré-valued curvature, they are independent Lorentz tensors. The translations of the Poincaré symmetry were broken when we curved the base manifold (see [5,11,12], but note that Kibble effectively uses a 14-dimensional bundle, whereas ours and related approaches require only 10-dim). We recognize $\mathbf{R}^a_{\ b}$ and $\mathbf{T}^a$ as the Riemann curvature and the torsion two-forms, respectively. Since the torsion is an independent tensor under the fiber group, it is consistent to consider the subclass of Riemannian geometries, $\mathbf{T}^a = 0$. Alternatively (see Sec. 4 below), vanishing torsion follows from the Einstein-Hilbert action.

With vanishing torsion, the quotient method has resulted in the usual solder form, $\mathbf{e}^a$, and related metric-compatible spin connection, $\boldsymbol{\omega}^a_{\ b}$,

$$\mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \boldsymbol{\omega}^a_{\ b} = 0,$$

the expression for the Riemannian curvature in terms of these,

$$\mathbf{R}^a_{\ b} = \mathbf{d}\boldsymbol{\omega}^a_{\ b} - \boldsymbol{\omega}^c_{\ b} \wedge \boldsymbol{\omega}^a_{\ c},$$

and the first and second Bianchi identities,

$$\begin{aligned} \mathbf{e}^b \wedge \mathbf{R}^a_{\ b} &= 0 \\ \mathbf{D}\mathbf{R}^a_{\ b} &= 0. \end{aligned}$$

This is a complete description of the class of Riemannian geometries.

Many further examples were explored by Ivanov and Niederle [14, 15].

3. Quotients of the conformal group

3.1. General properties of the conformal group

Physically, we are interested in measurements of relative magnitudes, so the relevant group is the conformal group, $\mathcal{C}$, of compactified $\mathbb{R}^n$. The one-point compactification at infinity allows a global definition of inversion, with translations of the point at infinity defining the special conformal transformation. Then $\mathcal{C}$ has a real linear representation in $n+2$ dimensions, $\mathcal{V}^{n+2}$ (alternatively we could choose the complex representation $\mathbb{C}^{2^{[(n+2)/2]}}$ for $Spin\,(p+1, q+1)$). The isotropy subgroup of $\mathbb{R}^n$ is the rotations, $SO\,(p,q)$, together with dilatations. We call this subgroup the homogeneous Weyl group, $\mathcal{W}$ and require our fibers to contain it. There are then only three allowed subgroups: $\mathcal{W}$ itself; the inhomogeneous Weyl group, $\mathcal{IW}$, found by appending the translations; and $\mathcal{W}$ together with special conformal transformations, isomorphic to $\mathcal{IW}$. The quotient of the conformal group by either inhomogeneous Weyl group, called the *auxiliary gauging*, leads most naturally to Weyl gravity [for a review, see [18]]. We concern ourselves with the only other meaningful conformal quotient, the *biconformal gauging*: the principal $\mathcal{W}$-bundle formed by the quotient of the conformal group by its Weyl subgroup. To help clarify the method and our model, it is useful to consider both these gaugings.

All parts of this construction work for any (p,q) with $n = p+q$. The conformal group is then $SO\,(p+1, q+1)$ (or $Spin\,(p+1, q+1)$ for the twistor representation). The Maurer-Cartan structure equations are immediate. In addition to the $\frac{n(n-1)}{2}$ generators $M^\alpha_{\ \beta}$ of $SO\,(p,q)$ and n translational generators P_α, there are n generators of translations of a point at infinity ("special conformal transformations") K^α, and a single dilatational generator D. Dual to these, we have the connections $\xi^\alpha_{\ \beta}, \boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha, \boldsymbol{\delta}$, respectively. Substituting the structure constants into the Maurer-Cartan dual form of the Lie algebra, eq.(1) gives

$$\mathbf{d}\boldsymbol{\xi}^\alpha_{\ \beta} = \boldsymbol{\xi}^\mu_{\ \beta} \wedge \boldsymbol{\xi}^\alpha_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\pi}_\mu \wedge \boldsymbol{\chi}^\nu \tag{2}$$

$$\mathbf{d}\boldsymbol{\chi}^\alpha = \boldsymbol{\chi}^\beta \wedge \boldsymbol{\xi}^\alpha_{\ \beta} + \boldsymbol{\delta} \wedge \boldsymbol{\chi}^\alpha \tag{3}$$

$$\mathbf{d}\boldsymbol{\pi}_\alpha = \boldsymbol{\xi}^\beta_{\ \alpha} \wedge \boldsymbol{\pi}_\beta - \boldsymbol{\delta} \wedge \boldsymbol{\pi}_\alpha \tag{4}$$

$$\mathbf{d}\boldsymbol{\delta} = \boldsymbol{\chi}^\alpha \wedge \boldsymbol{\pi}_\alpha \tag{5}$$

where $\Delta^{\alpha\mu}_{\nu\beta} \equiv \frac{1}{2}\left(\delta^\alpha_\nu \delta^\mu_\beta - \delta^{\alpha\mu}\delta_{\nu\beta}\right)$ antisymmetrizes *with respect to the original* (p,q) *metric*, $\delta_{\mu\nu} = diag\,(1,\ldots,1,-1,\ldots,-1)$. These equations, which are the same regardless of the gauging chosen, describe the Cartan connection on the conformal group manifold. Before proceeding to the quotients, we note that the conformal group has a nondegenerate Killing form,

$$K_{AB} \equiv tr\,(G_A G_B) = c^C_{\ AD} c^D_{\ BC} = \begin{pmatrix} \Delta^{ac}_{db} & & & \\ & 0 & \delta^a_b & \\ & \delta^a_b & 0 & \\ & & & 1 \end{pmatrix}$$

This provides a metric on the conformal Lie algebra. As we show below, when restricted to $\mathcal{M}_0$, it may or may not remain nondegenerate, depending on the quotient.

Finally, we note that the conformal group is invariant under inversion. Within the Lie algebra, this manifests itself as the interchange between the translations and special conformal transformations $P_\alpha \leftrightarrow \delta_{\alpha\beta}K^\beta$ along with the interchange of conformal weights, $D \to -D$. The corresponding transformation of the connection forms, is easily seen to leave eqs.(2)-(5) invariant. In the biconformal gauging, below, we show that this symmetry leads to a Kähler structure.

3.2. Curved generalizations

In this sub-Section and Section 4 we will complete the development of the curved auxiliary and biconformal geometries and show how one can easily construct actions with the curvatures. In this sub-Section, we construct the two possible fiber bundles, $\mathcal{C}/\mathcal{H}$ where $\mathcal{W} \subseteq \mathcal{H}$. For each, we carry out the generalization of the manifold and connection. The results in this sub-Section depend only on whether the local symmetry is $\mathcal{H} = \mathcal{IW}$ or $\mathcal{H} = \mathcal{W}$. In Section 5 and Section 6 we will return to the un-curved case to present a number of new calculations characterizing the homogenous space formed from the biconformal gauging.

The first sub-Section below describes the *auxiliary gauging*, given by the quotient of the conformal group by the inhomogeneous Weyl group, $\mathcal{IW}$.

Since $\mathcal{IW}$ is a parabolic subgroup of the conformal group, the resulting quotient can be considered a *tractor space*, for which there are numerous results [29]. *Tractor calculus* is a version of the auxiliary gauging where the original conformal group is tensored with $\mathbb{R}^{(p+1,q+1)}$. This allows for a linear representation of the conformal group with $(n+2)$-dimensional tensorial (physical) entities called *tractors*. This linear representation, first introduced by Dirac [30], makes a number of calculations much easier and also allows for straightforward building of tensors of any rank. The main physical differences stem from the use of Dirac's action, usually encoded as the scale tractor squared in the $n+2$-dimensional linear representation, instead of the Weyl action we introduce in Sec 4.

In sub-Section 3.2.2 below, we quotient by the homogeneous Weyl group, giving the *biconformal gauging*. This is *not* a parabolic quotient and therefore represents a less conventional option which turns out to have a number of rich structures not present in the auxiliary gauging. The biconformal gauging will occupy our attention for the bulk of our subsequent discussion.

3.2.1. The auxiliary gauging: $\mathcal{H} = \mathcal{IW}$ Given the quotient $\mathcal{C}/\mathcal{IW}$, the one-forms $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\pi}_\mu, \boldsymbol{\delta}\right)$ span the $\mathcal{IW}$-fibers, with $\boldsymbol{\beta}^\alpha$ spanning the co-tangent space of the remaining n independent directions. This means that $\mathcal{M}_0^{(n)}$ has the same dimension, n, as the original space. Generalizing the connection, we replace $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha, \boldsymbol{\delta}\right) \to \left(\boldsymbol{\omega}^\alpha_{\ \beta}, \mathbf{e}^\alpha, \boldsymbol{\omega}_\alpha, \boldsymbol{\omega}\right)$ and the Cartan equations now give the Cartan curvatures in terms of the new connection forms,

$$\mathbf{d}\boldsymbol{\omega}^\alpha_{\ \beta} = \boldsymbol{\omega}^\mu_{\ \beta} \wedge \boldsymbol{\omega}^\alpha_{\ \mu} + 2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}^\nu + \boldsymbol{\Omega}^\alpha_{\ \beta} \tag{6}$$

$$\mathbf{d}\mathbf{e}^\alpha = \mathbf{e}^\beta \wedge \boldsymbol{\omega}^\alpha_{\ \beta} + \boldsymbol{\omega} \wedge \mathbf{e}^\alpha + \mathbf{T}^\alpha \tag{7}$$

$$\mathbf{d}\boldsymbol{\omega}_\alpha = \boldsymbol{\omega}^\beta_{\ \alpha} \wedge \boldsymbol{\omega}_\beta - \boldsymbol{\omega} \wedge \boldsymbol{\omega}_\alpha + \mathbf{S}_\alpha \tag{8}$$

$$\mathbf{d}\boldsymbol{\omega} = \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha + \boldsymbol{\Omega} \tag{9}$$

Up to local gauge transformations, the curvatures depend only on the n non-vertical forms, $\mathbf{e}^\alpha$, so the curvatures are similar to what we find in an n-dim Riemannian geometry. For example, the $SO(p,q)$ piece of the curvature takes the form $\boldsymbol{\Omega}^\alpha_{\ \beta} = \frac{1}{2}\Omega^\alpha_{\ \beta\mu\nu}\mathbf{e}^\alpha \wedge \mathbf{e}^\beta$. The coefficients have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry.

Finally, each of the curvatures has a corresponding Bianchi identity, to guarantee integrability of the modified structure equations,

$$0 = \mathbf{D}\boldsymbol{\Omega}^\alpha_{\ \beta} + 2\Delta^{\alpha\mu}_{\nu\beta}\left(\boldsymbol{\Omega}_\mu \wedge \boldsymbol{\omega}^\nu - \boldsymbol{\omega}_\mu \wedge \boldsymbol{\Omega}^\nu\right) \tag{10}$$

$$0 = \mathbf{D}\mathbf{T}^\alpha - \mathbf{e}^\beta \wedge \boldsymbol{\Omega}^\alpha_{\ \beta} + \boldsymbol{\Omega} \wedge \mathbf{e}^\alpha \tag{11}$$

$$0 = \mathbf{D}\mathbf{S}_\alpha + \boldsymbol{\Omega}^\alpha_{\ \beta} \wedge \boldsymbol{\omega}_\beta - \boldsymbol{\omega}_\alpha \wedge \boldsymbol{\Omega} \tag{12}$$

$$0 = \mathbf{D\Omega} + \mathbf{T}^\alpha \wedge \boldsymbol{\omega}_\alpha - \boldsymbol{\omega}^\alpha \wedge \mathbf{S}_\alpha \tag{13}$$

where D is the Weyl covariant derivative,

$$\begin{aligned}
\mathbf{D\Omega}^\alpha_{\ \beta} &= \mathbf{d\Omega}^\alpha_{\ \beta} + \mathbf{\Omega}^\mu_{\ \beta} \wedge \boldsymbol{\omega}^\alpha_{\ \mu} - \mathbf{\Omega}^\alpha_{\ \mu} \wedge \boldsymbol{\omega}^\mu_{\ \beta} \\
\mathbf{DT}^\alpha &= \mathbf{dT}^\alpha + \mathbf{T}^\beta \wedge \boldsymbol{\omega}^\alpha_{\ \beta} - \boldsymbol{\omega} \wedge \mathbf{T}^\alpha \\
\mathbf{DS}_\alpha &= \mathbf{dS}_\alpha - \boldsymbol{\omega}^\beta_{\ \alpha} \wedge \mathbf{S}_\beta + \mathbf{S}_\alpha \wedge \boldsymbol{\omega} \\
\mathbf{D\Omega} &= \mathbf{d\Omega}
\end{aligned}$$

Equations eq.(6-9) give the curvature two-forms in terms of the connection forms. We have therefore constructed an n-dim geometry based on the conformal group with local $\mathcal{IW}$ symmetry.

We note no additional special properties of these geometries from the group structure. In particular, the restriction (in square brackets, [ ], below) of the Killing metric, K_{AB}, to $\mathcal{M}^{(n)}$ vanishes identically,

$$\left.\begin{pmatrix} \Delta^{ac}_{db} & & & \\ & [0] & \delta^a_b & \\ & \delta^a_b & 0 & \\ & & & 1 \end{pmatrix}\right|_{\mathcal{M}^{(n)}} = \begin{pmatrix} 0 \end{pmatrix}_{n\times n},$$

so there is no induced metric on the spacetime manifold. We may add the usual metric by hand, of course, but our goal here is to find those properties which are intrinsic to the underlying group structures.

3.2.2. The biconformal gauging: $\mathcal{H} = \mathcal{W}$ We next consider the biconformal gauging, first considered by Ivanov and Niederle [15], given by the quotient of the conformal group by its Weyl subgroup. The resulting geometry has been shown to contain the structures of general relativity [20, 21].

Given the quotient $\mathcal{C}/\mathcal{W}$, the one-forms $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\delta}\right)$ span the $\mathcal{W}$-fibers, with $(\boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha)$ spanning the remaining $2n$ independent directions. This means that $\mathcal{M}^{(2n)}_0$ has twice the dimension of the original compactified $\mathbb{R}^{(n)}$. Generalizing, we replace $\left(\boldsymbol{\xi}^\alpha_{\ \beta}, \boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha, \boldsymbol{\delta}\right) \to \left(\boldsymbol{\omega}^\alpha_{\ \beta}, \boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha, \boldsymbol{D}\right)$ and the modified structure equations *appear* identical to eqs.(6-9). However, the curvatures now depend on the $2n$ non-vertical forms, $(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha)$, so there are far more components than for an n-dim Riemannian geometry. For example,

$$\mathbf{\Omega}^\alpha_{\ \beta} = \frac{1}{2}\Omega^\alpha_{\ \beta\mu\nu}\boldsymbol{\omega}^\mu \wedge \boldsymbol{\omega}^\nu + \Omega^{\alpha\ \mu}_{\ \beta\ \nu}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}^\nu + \frac{1}{2}\Omega^{\alpha\ \mu\nu}_{\ \beta}\boldsymbol{\omega}_\mu \wedge \boldsymbol{\omega}_\nu$$

The coefficients of the pure terms, $\Omega^\alpha_{\ \beta\mu\nu}$ and $\Omega^{\alpha\ \mu\nu}_{\ \beta}$ each have the same number of degrees of freedom as the Riemannian curvature of an n-dim Weyl geometry, while the cross-term coefficients $\Omega^{\alpha\ \mu}_{\ \beta\ \nu}$ have more, being asymmetric on the final two indices.

For our purpose, it is important to notice that the spin connection, $\boldsymbol{\xi}^\alpha_{\ \beta}$, is antisymmetric with respect to the original (p, q) metric, $\delta_{\alpha\beta}$, in the sense that

$$\boldsymbol{\xi}^\alpha_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\xi}^\nu_{\ \mu}$$

It is crucial to note that $\boldsymbol{\omega}^\alpha_{\ \beta}$ retains this property, $\boldsymbol{\omega}^\alpha_{\ \beta} = -\delta^{\alpha\mu}\delta_{\beta\nu}\boldsymbol{\omega}^\nu_{\ \mu}$. This expresses metric compatibility with the $SO(p, q)$-covariant derivative, since it implies

$$\mathbf{D}\delta_{\alpha\beta} \equiv \mathbf{d}\delta_{\alpha\beta} - \delta_{\mu\beta}\boldsymbol{\omega}^\mu_{\ \alpha} - \delta_{\alpha\mu}\boldsymbol{\omega}^\mu_{\ \beta} = 0$$

Therefore, the curved generalization has a connection which is compatible with a locally (p, q)-metric. This relationship is general. If $\kappa_{\alpha\beta}$ is any metric, its compatible spin connection will satisfy $\boldsymbol{\omega}^\alpha_{\ \beta} = -\kappa^{\alpha\mu}\kappa_{\beta\nu}\boldsymbol{\omega}^\nu_{\ \mu}$. Since we also have local scale symmetry, the full covariant derivative we use will also include a Weyl vector term.

The Bianchi identities, written as 2-forms, also *appear* the same as eqs.(10-13), but expand into more components.

In the conformal group, translations and special conformal transformations are related by inversion. Indeed, a special conformal tranformation is a translation centered at the point at infinity instead of the origin. Because the biconformal gauging maintains the symmetry between translations and special conformal transformations, it is useful to name the corresponding connection forms and curvatures to reflect this. Therefore, the biconformal basis will be described as the solder form and the co-solder form, and the corresponding curvatures as the torsion and co-torsion. Thus, when we speak of "torsion-free biconformal space" we do not imply that the co-torsion (Cartan curvature of the co-solder form) vanishes. In phase space interpretations, the solder form is taken to span the cotangent spaces of the spacetime manifold, while the co-solder form is taken to span the cotangent spaces of the momentum space. The opposite convention is equally valid.

Unlike other quotient manifolds arising in conformal gaugings, the biconformal quotient manifold posesses natural invariant structures. The first is the restriction of the Killing metric, which is now non-degenerate,

$$\left.\begin{pmatrix} \Delta^{ac}_{db} & & \\ & \begin{bmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{bmatrix} & \\ & & 1 \end{pmatrix}\right|_{\mathcal{M}^{(2n)}} = \begin{pmatrix} 0 & \delta^a_b \\ \delta^a_b & 0 \end{pmatrix}_{2n\times 2n},$$

and this gives an inner product for the basis,

$$\begin{bmatrix} \langle \boldsymbol{\omega}^\alpha, \boldsymbol{\omega}^\beta \rangle & \langle \boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\beta \rangle \\ \langle \boldsymbol{\omega}_\alpha, \boldsymbol{\omega}^\beta \rangle & \langle \boldsymbol{\omega}_\alpha, \boldsymbol{\omega}_\beta \rangle \end{bmatrix} \equiv \begin{bmatrix} 0 & \delta^\alpha_\beta \\ \delta^\beta_\alpha & 0 \end{bmatrix} \tag{14}$$

This metric remains unchanged by the generalization to curved base manifolds.

The second natural invariant property is the generic presence of a symplectic form. The original fiber bundle always has this, because the structure equation, eq.(5), shows that $\boldsymbol{\chi}^\alpha \wedge \boldsymbol{\pi}_\alpha$ is exact hence closed, $\mathbf{d}^2\boldsymbol{\omega} = 0$, while it is clear that the two-form product is non-degenerate because $(\boldsymbol{\chi}^\alpha, \boldsymbol{\pi}_\alpha)$ together span $\mathcal{M}_0^{(2n)}$. Moreover, the symplectic form is canonical,

$$[\Omega]_{AB} = \begin{bmatrix} 0 & \delta^\beta_\alpha \\ -\delta^\alpha_\beta & 0 \end{bmatrix}$$

so that $\boldsymbol{\chi}^\alpha$ and $\boldsymbol{\pi}_\alpha$ are canonically conjugate. The symplectic form persists for the 2-form, $\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha + \boldsymbol{\Omega}$, as long as it is non-degenerate, so curved biconformal spaces are generically symplectic.

Next, we consider the effect of inversion symmetry. As a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor, the basis interchange takes the form

$$I^A_{\ B}\boldsymbol{\chi}^B = \begin{pmatrix} 0 & \delta^{\alpha\nu} \\ \delta_{\beta\mu} & 0 \end{pmatrix}\begin{pmatrix} \boldsymbol{\chi}^\mu \\ \boldsymbol{\pi}_\nu \end{pmatrix} = \begin{pmatrix} \delta^{\alpha\nu}\boldsymbol{\pi}_\nu \\ \delta_{\beta\mu}\boldsymbol{\chi}^\mu \end{pmatrix}$$

In order to interchange conformal weights, $I^A_{\ B}$ must anticommute with the conformal weight operator, which is given by

$$W^A_{\ B}\boldsymbol{\chi}^B = \begin{pmatrix} \delta^\alpha_\mu & 0 \\ 0 & -\delta^\nu_\beta \end{pmatrix}\begin{pmatrix} \boldsymbol{\chi}^\mu \\ \boldsymbol{\pi}_\nu \end{pmatrix} = \begin{pmatrix} +\boldsymbol{\chi}^\alpha \\ -\boldsymbol{\pi}_\beta \end{pmatrix}$$

This is the case: we easily check that $\{I, W\}^A_{\ B} = I^A_{\ C}W^C_{\ B} + W^A_{\ C}I^C_{\ B} = 0$. The commutator gives a new object,

$$J^A_{\ B} \equiv [I, W]^A_{\ B} = \begin{pmatrix} 0 & -\delta^{\alpha\beta} \\ \delta_{\alpha\beta} & 0 \end{pmatrix}$$

Squaring, $J^A_{\ C}J^C_{\ B} = -\delta^A_B$, we see that $J^A_{\ B}$ provides an almost complex structure. That the almost complex structure is integrable follows immediately in this (global) basis by the obvious vanishing of the Nijenhuis tensor,

$$N^A_{\ BC} = J^D_{\ C}\partial_D J^A_{\ B} - J^D_{\ C}\partial_D J^A_{\ B} - J^A_{\ D}\left(\partial_C J^D_{\ B} - \partial_B J^D_{\ C}\right) = 0$$

Next, using the symplectic form to define the compatible metric

$$g(u, v) \equiv \Omega(u, Jv)$$

we find that in this basis $g = \begin{pmatrix} \delta_{\alpha\beta} & 0 \\ 0 & \delta^{\alpha\beta} \end{pmatrix}$, and we check the remaining compatibility conditions of the triple (g, J, Ω),

$$\begin{aligned} \omega(u,v) &= g(Ju,v) \\ J(u) &= (\phi_g)^{-1}(\phi_\omega(u)) \end{aligned}$$

where ϕ_ω and ϕ_g are defined by

$$\begin{aligned} \phi_\omega(u) &= \omega(u,\cdot) \\ \phi_g(u) &= g(u,\cdot) \end{aligned}$$

These are easily checked to be satisified, showing that that $\mathcal{M}_0^{(2n)}$ is a Kähler manifold. Notice, however, that the metric of the Kähler manifold is *not* the restricted Killing metric which we use in the following considerations.

Finally, a surprising result emerges if we require $\mathcal{M}_0^{(2n)}$ to match our usual expectations for a relativistic phase space. To make the connection to phase space clear, the precise requirements were studied in [28], where it was shown that the flat biconformal gauging of $SO(p,q)$ in any dimension $n = p + q$ will have *Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have non-degenerate n-dim restrictions of the metric* only if the original space is Euclidean or signature zero $\left(p \in \left\{0, \frac{n}{2}, n\right\}\right)$, and then the signature of the submanifolds is severely limited $(p \to p \pm 1)$, leading in the two Euclidean cases to Lorentzian configuration space, and hence the origin of time. For the case of flat, 8-dim biconformal space [28] proves the following theorem:

Flat 8-dim biconformal space is a metric phase space with Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have non-degenerate n-dim metric restrictions of the biconformal metric if and only if the initial 4-dim space we gauge is Euclidean or signature zero. In either of these cases the resulting configuration sub-manifold is necessarily Lorentzian [28].

Thus, it is possible to impose the conditions necessary to make biconformal space a metric phase space only in a restricted subclass of cases, and the configuration space metric *must* be Lorentzian. In [28], it was found that with a suitable choice of gauge, the metric may be written in coordinates y_α as

$$h_{\alpha\beta} = \frac{1}{(y^2)^2}\left(2y_\alpha y_\beta - y^2\delta_{\alpha\beta}\right) \tag{15}$$

where the signature changing character of the metric is easily seen.

In the metric above, eq.(15), $y_\alpha = W_\alpha$ is the Weyl vector of the space. This points to another unique characteristic of flat biconformal space. The structures of the conformal group, treated as described above, give rise to a natural *direction* of time, given by the gauge field of dilatations. The situation is reminiscent of previous studies. In 1979, Stelle and West introduced a special vector field to choose the local symmetry of the MacDowell-Mansouri theory. The vector breaks the de Sitter symmetry, eliminating the need for the Wigner-Inönu contraction. Recently, Westman and Zlosnik [31] have looked in depth at both the de Sitter and anti-de Sitter cases using a class of actions which extend that of Stelle and West by including derivative terms for the vector field and therefore lead to dynamical symmetry breaking. In [32, 33] and Einstein-Aether theory [34], there is also a special vector field introduced into the action by hand that can make the Lorentzian metric Euclidean. These approaches are distinct from that of the biconformal approach, where the vector necessary for specifying the timelike direction occurs *naturally* from the underlying group structure. We will have more to say about this below, where we show explicitly that the Euclidean gauge theory necessarily posesses a special *vector*, $\mathbf{v} = \boldsymbol{\omega} - \frac{1}{2}\eta_{ab}\mathbf{d}\eta^{bc}$. This vector gives the time direction on two Lagrangian submanifolds, making them necessarily Lorentzian. The full manifold retains its original symmetry.

4. A brief note on gravitation

Notice that our development to this point was based solely on group quotients and generalization of the resulting principal fiber bundle. We have arrived at the form of the curvatures in terms of the Cartan connection, and Bianchi identities required for integrability, thereby describing certain classes of geometry. Within the biconformal quotient, the demand for *orthogonal Lagrangian submanifolds with non-degenerate n-dim restrictions of the Killing metric* leads to the selection of certain Lorentzian submanifolds. Though our present concern has to do with the geometric background rather than with gravitational theories on those backgrounds, for completeness we briefly digress to specify the action functionals for gravity. The main results of the current study, taken up again in the final three Sections, concern only the homogeneous space, $\mathcal{M}_0^{(2n)}$.

We are guided in the choice of action functionals by the example of general relativity. Given the Riemannin geometries of Section 2.3, we may write the Einstein-Hilbert action and proceed. More systematically, however, we may write the most general, even-parity action linear in the curvature and torsion. This still turns out to be the Einstein-Hilbert action, and, as noted above, one of the classical field equations under a full variation of the connection $\left(\delta\mathbf{e}^b, \delta\boldsymbol{\omega}^a_{\ b}\right)$, implies vanishing torsion. The latter, more robust approach is what we follow for conformal gravity theories.

It is generally of interest to build the simplest class of actions possible, and we use the following criteria:

(i) The pure-gravity action should be built from the available curvature tensor(s) and other tensors which occur in the geometric construction.

(ii) The action should be of lowest possible order ≥ 1 in the curvatures.

(iii) The action should be of even parity.

These are of sufficient generality not to bias our choice. It may also be a reasonable assumption to set certain tensor fields, for example, the spacetime torsion to zero. This can significantly change the available tensors, allowing a wider range of action functionals.

Notice that if we perform an infinitesmal *conformal* transformation to the curvatures, $\left(\boldsymbol{\Omega}^\alpha_{\ \beta}, \boldsymbol{\Omega}^\alpha, \boldsymbol{\Omega}_\beta, \boldsymbol{\Omega}\right)$, they all mix with one another, since the conformal curvature is really a single Lie-algebra-valued two form. However, the generalization to a curved manifold breaks the non-vertical symmetries, allowing these different components to become independent tensors under the remaining Weyl group. Thus, to find the available tensors, we apply an infinitesmal transformation of the *fiber symmetry.* Tensors are those objects which transform linearly and homogeneously under these transformations.

4.1. The auxiliary gauging and Weyl gravity

The generalization of the auxiliary quotient, $\mathcal{C}/\mathcal{IW}$, breaks translational symmetry, and a local transformation of the connection components immediately shows that the solder form, $\mathbf{e}^\alpha$, becomes a tensor. Correspondingly, the torsion, $\mathbf{T}^\alpha$, no longer mixes with the other curvature components. This suggests the possibility of a teleparallel theory based on the torsion, but this would involve little of the conformal structure. Instead we choose to set $\mathbf{T}^\alpha = 0$ as an additional condition on our model. This gives us Riemannian or Weyl geometries instead of Cartan geometries and is therefore more in line with the requirements of general relativity.

When the torsion is maintained at zero, both the rotational curvature, $\boldsymbol{\Omega}^\alpha_{\ \beta}$, and the dilatational curvature, $\boldsymbol{\Omega}$, become tensorial. Because the n-dim volume form has conformal weight n there is no curvature-linear action. Together with the orthonormal metric and the Levi-Civita tensor, we build the most general even parity curvature-quadratic action,

$$S = \int \left(\alpha\boldsymbol{\Omega}^\alpha_{\ \beta} \wedge {}^*\boldsymbol{\Omega}^\beta_{\ \alpha} + \beta\boldsymbol{\Omega} \wedge {}^*\boldsymbol{\Omega}\right)$$

This was partially studied in the 1970s with an eye to supersymmetry [7–10, 35], where the $\beta = 0$ case is shown to lead to Weyl gravity. Indeed, assuming a suitable metric dependence of the remaining connection components, $\left(\boldsymbol{\omega}^\alpha_{\ \beta}, \mathbf{f}_\alpha, \boldsymbol{\omega}\right)$, metric variation leads to the fourth-order Bach equation [27]. However, it has recently been shown that varying all connection forms independently leads to scale-invariant general relativity [18].

In dimensions higher than four, our criteria lead to still higher order actions. Alternatively, curvature-linear actions can be written in any dimension by introducing a suitable power of a scalar field [30, 36]. This latter reference, [36], gives the $\phi^2 R$ action often used in tractor studies.

4.2. Gravity in the biconformal gauging

The biconformal gauging, based on $\mathcal{C}/\mathcal{W}$, also has tensorial basis forms $(\boldsymbol{\omega}^{\alpha}, \boldsymbol{\omega}_{\alpha})$. Moreover, each of the component curvatures, $\left(\boldsymbol{\Omega}^{\alpha}_{\ \beta}, \boldsymbol{\Omega}^{\alpha}, \boldsymbol{\Omega}_{\beta}, \boldsymbol{\Omega}\right)$, becomes an independent tensor under the Weyl group.

In the biconformal case, the volume form $e^{\rho\sigma\ldots\lambda}{}_{\alpha\beta\ldots\nu}\boldsymbol{\omega}^{\alpha}\wedge\boldsymbol{\omega}^{\beta}\wedge\ldots\wedge\boldsymbol{\omega}^{\nu}\wedge\boldsymbol{\omega}_{\rho}\wedge\boldsymbol{\omega}_{\sigma}\wedge\ldots\wedge\boldsymbol{\omega}_{\lambda}$ has zero conformal weight. Since both $\boldsymbol{\Omega}^{\alpha}_{\ \beta}$ and $\boldsymbol{\Omega}$ also have zero conformal weight, there exists a curvature-linear action in any dimension [20]. The most general linear case is

$$S=\int\left(\alpha\boldsymbol{\Omega}^{\alpha}_{\ \beta}+\beta\boldsymbol{\Omega}\delta^{\alpha}_{\beta}+\gamma\boldsymbol{\omega}^{\alpha}\wedge\boldsymbol{\omega}_{\beta}\right)\wedge e^{\beta\rho\ldots\sigma}{}_{\alpha\mu\ldots\nu}\boldsymbol{\omega}^{\mu}\wedge\ldots\wedge\boldsymbol{\omega}^{\nu}\wedge\boldsymbol{\omega}_{\rho}\wedge\ldots\wedge\boldsymbol{\omega}_{\sigma}$$

Notice that we now have three important properties of biconformal gravity that arise because of the doubled dimension: (1) the non-degenerate conformal Killing metric induces a non-degenerate metric on the manifold, (2) the dilatational structure equation generically gives a symplectic form, and (3) there exists a Weyl symmetric action functional linear in the curvature, valid in any dimension.

There are a number of known results following from the linear action. In [20] torsion-constrained solutions are found which are consistent with scale-invariant general relativity. Subsequent work along the same lines shows that the torsion-free solutions are determined by the spacetime solder form, and reduce to describe spaces conformal to Ricci-flat spacetimes on the corresponding spacetime submanifold [37]. A supersymmetric version is presented in [38], and studies of Hamiltonian dynamics [39, 40] and quantum dynamics [41] support the idea that the models describe some type of relativistic phase space determined by the configuration space solution.

5. Homogeneous biconformal space in a conformally orthonormal, symplectic basis

The central goal of the remainder of this manuscript is to examine properties of the homogeneous manifold, $\mathcal{M}_0^{(2n)}$, which become evident in a conformally orthonormal basis, that is, a basis which is orthonormal up to an overall conformal factor. Generically, the properties we discuss will be inherited by the related gravity theories as well.

As noted above, biconformal space is immediately seen to possess several structures not seen in other gravitational gauge theories: a non-degenerate restriction of the Killing metric‡, a symplectic form, and Kähler structure. In addition, the signature theorem in [28] shows that if the original space has signature $\pm n$ or zero, the *imposition of involution conditions* leads to orthogonal Lagrangian submanifolds that have non-degenerate n-dim restrictions of the Killing metric. Further, *constraining the momentum space* to be as flat as permitted requires the restricted metrics to be Lorentzian. We strengthen these results in this Section and the next. Concerning ourselves only with elements of the *geometry* of the Euclidean ($s=\pm n$) cases (as opposed to the additional restrictions of the field equations, involution conditions or other constraints), we show the presence of exactly such Lorentzian signature Lagrangian submanifolds *without further assumptions.*

We go on to study the transformation of the spin connection when we transform the basis of an 8-dim biconformal space to one adapted to the Lagrangian submanifolds. We show that in addition to the Lorentzian metric, a Lorentzian connection emerges on the configuration and momentum spaces and there are two new tensor fields. Finally, we examine the curvature of these Lorentzian connections and find both a cosmological constant and cosmological "dust". While it is premature to make quantitative predictions, these new geometric features provide novel candidates for dark energy and dark matter.

5.1. The biconformal quotient

We start with the biconformal gauging of Section 3, specialized to the case of compactified, *Euclidean* $\mathbb{R}^4$ in a conformally orthonormal, symplectic basis. The Maurer-Cartan structure equations are

$$\mathbf{d}\boldsymbol{\omega}^{\alpha}_{\ \beta}=\boldsymbol{\omega}^{\mu}_{\ \beta}\wedge\boldsymbol{\omega}^{\alpha}_{\ \mu}+2\Delta^{\alpha\mu}_{\nu\beta}\boldsymbol{\omega}_{\mu}\wedge\boldsymbol{\omega}^{\nu} \tag{16}$$

$$\mathbf{d}\boldsymbol{\omega}^{\alpha}=\boldsymbol{\omega}^{\beta}\wedge\boldsymbol{\omega}^{\alpha}_{\ \beta}+\boldsymbol{\omega}\wedge\boldsymbol{\omega}^{\alpha} \tag{17}$$

‡ There are non-degenerate restrictions in anti-de Sitter and de Sitter gravitational gauge theories.

$$\mathbf{d}\boldsymbol{\omega}_\alpha = \boldsymbol{\omega}^\beta_{\ \alpha} \wedge \boldsymbol{\omega}_\beta + \boldsymbol{\omega}_\alpha \wedge \boldsymbol{\omega} \tag{18}$$

$$\mathbf{d}\boldsymbol{\omega} = \boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha \tag{19}$$

where the connection one-forms represent $SO\,(4)$ rotations, translations, special conformal transformations and dilatations respectively. The projection operator $\Delta^{\alpha\mu}_{\gamma\beta} \equiv \frac{1}{2}\left(\delta^\alpha_\gamma \delta^\mu_\beta - \delta^{\alpha\mu}\delta_{\gamma\beta}\right)$ in eq.(16) gives that part of any $\binom{1}{1}$-tensor antisymmetric with respect to the original Euclidean metric, $\delta_{\alpha\beta}$. As discussed in Section 3.2.2, this group has a non-degenerate, 15-dim Killing metric. We stress that the structure equations and Killing metric – and hence their restrictions to the quotient manifold – are intrinsic to the conformal symmetry.

The gauging begins with the quotient of this conformal group, $SO\,(5,1)$, by its Weyl subgroup, spanned by the connection forms $\boldsymbol{\omega}^\alpha_\beta$ (here dual to $SO(4)$ generators) and $\boldsymbol{\omega}$. The co-tangent space of the quotient manifold is then spanned by the solder form, $\boldsymbol{\omega}^\alpha$, and the co-solder form, $\boldsymbol{\omega}_\alpha$, and the full conformal group becomes a principal fiber bundle with local Weyl symmetry over this 8-dim quotient manifold. The independence of $\boldsymbol{\omega}^\alpha$ and $\boldsymbol{\omega}_\alpha$ in the biconformal gauging makes the 2-form $\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha$ non-degenerate, and eq.(19) immediately shows that $\boldsymbol{\omega}^\alpha \wedge \boldsymbol{\omega}_\alpha$ is a symplectic form. The basis $(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha)$ is canonical.

The involution evident in eq.(17) shows that the solder forms, $\boldsymbol{\omega}^\alpha$, span a submanifold, and from the simultaneous vanishing of the symplectic form this submanifold is Lagrangian. Similarly, eq.(18) shows that the $\boldsymbol{\omega}_\beta$ span a Lagrangian submanifold. However, notice that neither of these submanifolds, spanned by either $\boldsymbol{\omega}^\alpha$ or $\boldsymbol{\omega}_\alpha$ alone, has an induced metric, since by eq.14, $\langle \boldsymbol{\omega}^\alpha, \boldsymbol{\omega}^\beta \rangle = \langle \boldsymbol{\omega}_\alpha, \boldsymbol{\omega}_\beta \rangle = 0$. The *orthonormal* basis will make the Killing metric block diagonal, guaranteeing that its restriction to the configuration and momentum submanifolds have well-defined, non-degenerate metrics.

It was shown in [28] this it is consistent (for signatures $\pm n$, 0 *only*) to impose involution conditions and momentum flatness in this rotated basis in such a way that the new basis still gives Lagrangian submanifolds. Moreover, the restriction of the Killing metric to these new submanifolds is necessarily Lorentzian. In what follows, we do not need the assumptions of momentum flatness or involution, and work only with intrinsic properties of $\mathcal{M}^{(2n)}_0$. This Section describes the new basis and resulting connection, while the next establishes that for initial Euclidean signature, the principal results of [28] follow necessarily. Our results show that the timelike directions in these models arise from intrinsically conformal structures.

We now change to a new canonical basis, adapted to the Lagrangian submanifolds.

5.2. The conformally-orthonormal Lagrangian basis

In [28] the $(\boldsymbol{\omega}^\alpha, \boldsymbol{\omega}_\alpha)$ basis is rotated so that the metric, h_{AB} becomes block diagonal

$$\left[\begin{array}{cc} 0 & \delta^\alpha_\beta \\ \delta^\alpha_\beta & 0 \end{array}\right] \Rightarrow [h_{AB}] = \left[\begin{array}{cc} h_{ab} & 0 \\ 0 & -h^{ab} \end{array}\right]$$

while the symplectic form remains canonical. This makes the Lagrangian submanifolds orthogonal with a non-degenerate restriction to the metric. Here we use the same basis change, but in addition define coefficients, $h_a^{\ \alpha}$ to relate the orthogonal metric to one conformally orthonormal on the submanifolds, $\eta_{ab} = h_a^{\ \alpha} h_{\alpha\beta} h_b^{\ \beta}$, where η_{ab} is conformal to $diag\,(\pm 1, \pm 1, \pm 1, \pm 1)$. From [28] we know that h_{ab} is necessarily Lorentzian, $h_{ab} = \eta_{ab} = e^{2\phi} diag\,(-1,1,1,1) = e^{2\phi}\eta^0_{ab}$ and we give a more general proof below. Notice that the definition of η_{ab} includes an unknown conformal factor. The required change of basis is then

$$\mathbf{e}^a = h_\alpha^{\ a}\left(\boldsymbol{\omega}^\alpha + \frac{1}{2}h^{\alpha\beta}\boldsymbol{\omega}_\beta\right) \tag{20}$$

$$\mathbf{f}_a = h_a^{\ \alpha}\left(\frac{1}{2}\boldsymbol{\omega}_\alpha - h_{\alpha\beta}\boldsymbol{\omega}^\beta\right) \tag{21}$$

with inverse basis change

$$\boldsymbol{\omega}^\alpha = \frac{1}{2}h_a^{\ \alpha}\left(\mathbf{e}^a - \eta^{ab}\mathbf{f}_b\right) \tag{22}$$

$$\boldsymbol{\omega}_\alpha = h_\alpha^{\ a}\left(\mathbf{f}_a + \eta_{ab}\mathbf{e}^b\right) \tag{23}$$

Using (14), the Killing metric is easily checked to be

$$\begin{bmatrix} \langle \mathbf{e}^a, \mathbf{e}^b \rangle & \langle \mathbf{e}^a, \mathbf{f}_b \rangle \\ \langle \mathbf{f}_a, \mathbf{e}^b \rangle & \langle \mathbf{f}_a, \mathbf{f}_b \rangle \end{bmatrix} = \begin{bmatrix} h_\alpha^{\ a} h_\beta^{\ b} h^{(\alpha\beta)} & 0 \\ 0 & -h_a^{\ \alpha} h_b^{\ \beta} h_{(\alpha\beta)} \end{bmatrix}$$
$$= \begin{bmatrix} e^{-2\phi}\eta_0^{ab} & 0 \\ 0 & -e^{2\phi}\eta_{ab}^0 \end{bmatrix}$$

where $h_{\alpha\beta} = h_{(\alpha\beta)}$, and $h^{\alpha\beta} h_{\beta\gamma} = \delta^\alpha_\gamma$.

The new basis is also canonical, as we see by transforming the dilatation equation, eq.(19), to find $\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^a \mathbf{f}_a$. We refer to the $\mathbf{f}_a = 0$ and $\mathbf{e}^a = 0$ submanifolds as the configuration and momentum submanifolds respectively.

5.3. Properties of the structure equations in the new basis

We now explore the properties of the biconformal system in this adapted basis. Rewriting the remaining structure equations, eqs.(16, 17, 18), in terms of $\mathbf{e}^a$ and $\mathbf{f}_a$, we show some striking cancelations that lead to the emergence of a connection compatible with the Lorentzian metric, and two new tensors.

We begin with the exterior derivative of eq.(20), using structure equations eq.(17) and eq.(18), and then using the basis change equations eqs.(22, 23). Because eqs.(22, 23) involve the sum and difference of $\mathbf{e}^a$ and $\mathbf{f}_b$, separating by these new basis forms leads to a separation of symmetries. This leads to a cumbersome expansion, which reduces considerably and in significant ways, to

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \Theta^{ad}_{cb} \boldsymbol{\tau}^c_{\ d} - \eta^{bc} \mathbf{f}_c \wedge \Xi^{ae}_{db} \boldsymbol{\tau}^d_{\ e} + \frac{1}{2}\eta_{bc}\mathbf{d}\eta^{ab} \wedge \mathbf{e}^c + \frac{1}{2}\mathbf{d}\eta^{ab} \wedge \mathbf{f}_b + 2\eta^{ab}\mathbf{f}_b \wedge \boldsymbol{\omega} \tag{24}$$

where we define projections $\Theta^{ac}_{db} \equiv \frac{1}{2}\left(\delta^a_d \delta^c_b - \eta^{ac}\eta_{bd}\right)$ and $\Xi^{ad}_{cb} \equiv \frac{1}{2}\left(\delta^a_c \delta^d_b + \eta^{ad}\eta_{cb}\right)$ that separate symmetries with respect to the new metric η_{ab} rather than $\delta_{\alpha\beta}$. These give the antisymmetric and symmetric parts, respectively, of a $\binom{1}{1}$-tensor with respect to the *new* orthonormal metric, η_{ab}. Notice that these projections are independent of the conformal factor on η_{ab}.

The significance of the reduction lies in how the symmetries separate between the different subspaces. Just as the curvatures split into three parts, eq.(24) and each of the remaining structure equations splits into three parts. Expanding these independent parts separately allows us to see the Riemannian structure of the configuration and momentum spaces. It is useful to first define

$$\boldsymbol{\tau}^a_{\ b} \equiv \boldsymbol{\alpha}^a_{\ b} + \boldsymbol{\beta}^a_{\ b} \tag{25}$$

where $\boldsymbol{\alpha}^a_{\ b} \equiv \Theta^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$ and $\boldsymbol{\beta}^a_{\ b} \equiv \Xi^{ad}_{cb}\boldsymbol{\tau}^c_{\ d}$. Then, to facilitate the split into $\mathbf{e}^a \wedge \mathbf{e}^b$, $\mathbf{e}^a \wedge \mathbf{f}_b$ and $\mathbf{f}_a \wedge \mathbf{f}_b$ parts, we partition the spin connection and Weyl vector by submanifold, defining

$$\boldsymbol{\alpha}^a_{\ b} \equiv \boldsymbol{\sigma}^a_{\ b} + \boldsymbol{\gamma}^a_{\ b} = \sigma^a_{\ bc}\mathbf{e}^c + \gamma^{a\ c}_{\ b}\mathbf{f}_c \tag{26}$$
$$\boldsymbol{\beta}^a_{\ b} \equiv \boldsymbol{\mu}^a_{\ b} + \boldsymbol{\rho}^a_{\ b} = \mu^a_{\ bc}\mathbf{e}^c + \rho^{a\ c}_{\ b}\mathbf{f}_c \tag{27}$$
$$\boldsymbol{\omega} \equiv W_a \mathbf{e}^a + W^a \mathbf{f}_a \tag{28}$$

We also split the exterior derivative, $\mathbf{d} = \mathbf{d}_{(x)} + \mathbf{d}_{(y)}$, where coordinates x^α and y_α are used on the $\mathbf{e}^a = e_\alpha^{\ \ a}\mathbf{d}x^\alpha$ and $\mathbf{f}_a = f_a^{\ \ \alpha}\mathbf{d}y_\alpha$ submanifolds, respectively. Using these, we expand each of the structure equations into three $\mathcal{W}$-invariant parts. The complete set (with curvatures included for completeness) is given in Appendix 1.

The simplifying features and notable properties include:

(i) *The new connection:* The first thing that is evident is that all occurences of the spin connection $\boldsymbol{\omega}^\alpha_{\ \beta}$ may be written in terms of the combination

$$\boldsymbol{\tau}^a_{\ b} \equiv h_\alpha^{\ a}\boldsymbol{\omega}^\alpha_{\ \beta} h_b^{\ \beta} - h_b^{\ \alpha}\mathbf{d}h_\alpha^{\ a} \tag{29}$$

which, as we show below, transforms as a Lorentz spin connection. Although the basis change is not a gauge transformation, the change in the connection has a similar inhomogeneous form. Because $h_\alpha^{\ a}$ is a change of basis rather than local $SO(n)$ or local Lorentz, the inhomogeneous term has no particular symmetry property, so $\boldsymbol{\tau}^a_{\ b}$ will have both symmetric and antisymmetric parts.

(ii) *Separation of symmetric and antisymmetric parts:* Notice in eq.(24) how the antisymmetric part of the new connection, $\boldsymbol{\alpha}^a_{\ b}$, is associated with $\mathbf{e}^b$, while the symmetric part, $\boldsymbol{\beta}^a_{\ b}$ pairs with $\mathbf{f}_c$. This surprising correspondence puts the symmetric part into the cross-terms while leaving the connection of the configuration submanifold metric compatible, up to the conformal factor.

(iii) *Cancellation of the submanifold Weyl vector:* The Weyl vector terms cancel on the configuration submanifold, while the $\mathbf{f}_a$ terms add. The expansion of the $\mathbf{df}_a$ structure equation shows that the Weyl vector also drops out of the momentum submanifold equations. Nonetheless, these submanifold equations are scale invariant because of the residual metric derivative. Recognizing the combination of $\mathbf{d}h$ terms that arises as $\mathbf{d}\eta^{ab}$, and recalling that $\eta_{ab} = e^{2\phi}\eta^0_{ab}$, we have $-\frac{1}{2}\mathbf{d}\eta^{ac}\eta_{cb} = \delta^a_b\mathbf{d}\phi$. When the metric is rescaled, this term changes with the same inhomogeneous term as the Weyl vector.

(iv) *Covariant derivative and a second Weyl-type connection:* It is natural to define the $\boldsymbol{\tau}^b_{\ c}$-covariant derivative of the metric. Since $\eta^{cb}\boldsymbol{\alpha}^a_{\ c} + \eta^{ac}\boldsymbol{\alpha}^b_{\ c} = 0$, it depends only on $\boldsymbol{\beta}^a_{\ c}$ and the Weyl vector,

$$\mathbf{D}\eta^{ab} \equiv \mathbf{d}\eta^{ab} + \eta^{cb}\boldsymbol{\tau}^a_{\ c} + \eta^{ac}\boldsymbol{\tau}^b_{\ c} - 2\boldsymbol{\omega}\eta^{ae} \tag{30}$$

$$= \mathbf{d}\eta^{ab} + 2\eta^{cb}\boldsymbol{\beta}^a_{\ c} - 2\boldsymbol{\omega}\eta^{ab} \tag{31}$$

This derivative allows us to express the structure of the biconformal space in terms of the Lorentzian properties.

When all of the identifications and definitions are included, and carrying out similar calculations for the remaining structure equations, the full set becomes

$$\mathbf{d}\boldsymbol{\tau}^a_{\ b} = \boldsymbol{\tau}^c_{\ b} \wedge \boldsymbol{\tau}^a_{\ c} + \Delta^{ae}_{db}\eta_{ec}\mathbf{e}^c \wedge \mathbf{e}^d - \Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c \wedge \mathbf{f}_d + 2\Delta^{ae}_{fb}\Xi^{fc}_{de}\mathbf{f}_c \wedge \mathbf{e}^d \tag{32}$$

$$\mathbf{de}^a = \mathbf{e}^c \wedge \boldsymbol{\alpha}^a_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac} \wedge \mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ab} \wedge \mathbf{f}_b \tag{33}$$

$$\mathbf{df}_a = \boldsymbol{\alpha}^b_{\ a} \wedge \mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab} \wedge \mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ab} \wedge \mathbf{e}^b \tag{34}$$

$$\mathbf{d}\boldsymbol{\omega} = \mathbf{e}^a \wedge \mathbf{f}_a \tag{35}$$

with the complete $\mathcal{W}$-invariant separation in Appendix 1.

5.4. Gauge transformations and new tensors

The biconformal bundle now allows local Lorentz transformations and local dilatations on $\mathcal{M}^{(2n)}_0$. Under local Lorentz transformations, $\Lambda^a_{\ c}$, the connection $\boldsymbol{\tau}^a_{\ b}$ changes with an inhomogeneous term of the form $\bar{\Lambda}^c_{\ b}\mathbf{d}\Lambda^a_{\ c}$. Since this term lies in the Lie algebra of the Lorentz group, it is antisymmetric with respect to η_{ab}, $\Theta^{ac}_{db}\left(\bar{\Lambda}^e_{\ c}\mathbf{d}\Lambda^d_{\ e}\right) = \bar{\Lambda}^e_{\ b}\mathbf{d}\Lambda^a_{\ e}$and therefore only changes the corresponding Θ^{ac}_{db}-antisymmetric part of the connection, with the symmetric part transforming homogeneously:

$$\tilde{\boldsymbol{\alpha}}^a_{\ b} = \Lambda^a_{\ c}\boldsymbol{\alpha}^c_{\ d}\bar{\Lambda}^d_{\ b} - \bar{\Lambda}^c_{\ b}\mathbf{d}\Lambda^a_{\ c}$$
$$\tilde{\boldsymbol{\beta}}^a_{\ b} = \Lambda^a_{\ c}\boldsymbol{\beta}^c_{\ d}\bar{\Lambda}^d_{\ b}$$

Having no inhomogeneous term, $\boldsymbol{\beta}^a_{\ b}$ is a Lorentz tensor. This new tensor field $\boldsymbol{\beta}^a_{\ b}$ necessarily includes degrees of freedom from the original connection that cannot be present in $\boldsymbol{\alpha}^a_{\ b}$, the total equaling the degrees of freedom present in $\boldsymbol{\tau}^a_{\ b}$. As there is no obvious constraint on the connection $\boldsymbol{\alpha}^a_{\ b}$, we expect $\boldsymbol{\beta}^a_{\ b}$ to be highly constrained. Clearly, $\boldsymbol{\alpha}^c_{\ d}$ transforms as a Lorentzian spin connection, and the addition of the tensor $\boldsymbol{\beta}^a_{\ b}$ preserves this property, so $\boldsymbol{\tau}^a_{\ b}$ is a local Lorentz connection.

Transformation of the connection under dilatations reveals another new tensor. The Weyl vector transforms inhomogeneously in the usual way, $\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \mathbf{d}f$, but, as noted above, the expression $\frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}$ also transforms,

$$\frac{1}{2}\tilde{\eta}_{cb}\mathbf{d}\tilde{\eta}^{ac} = \delta^a_b\mathbf{d}\tilde{\phi} = \delta^a_b\left(\mathbf{d}\phi - \mathbf{d}f\right)$$

so that the combination

$$\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi$$

is scale invariant. Notice the presence of two distinct scalars here. Obviously, given $\frac{1}{2}\eta^{ac}\mathbf{d}\eta_{cb} = \delta^a_b\mathbf{d}\phi$ we can choose a gauge function, $f_1 = -\phi$, such that $\frac{1}{2}\eta^{ac}\mathbf{d}\eta_{cb} = 0$. We also have, $\mathbf{d}\boldsymbol{\omega} = 0$, on the configuration submanifold, so that $\boldsymbol{\omega} = \mathbf{d}f_2$, for some scalar f_2 and this might be gauged to zero instead. But while one or the other of $\boldsymbol{\omega}$ or $\mathbf{d}\phi$ can be gauged to zero, their *sum* is gauge invariant. As we show below, it is the resulting vector $\mathbf{v}$ which determines the timelike directions.

Recall that certain involution relationships must be satisfied to ensure that spacetime and momentum space are each submanifolds. The involution conditions in homogeneous biconformal space are

$$0 = \boldsymbol{\mu}^a_{\ b} \wedge \mathbf{e}^b - \mathbf{v}_{(x)} \wedge \mathbf{e}^a \tag{36}$$

$$0 = \boldsymbol{\rho}^b_{\ a} \wedge \mathbf{f}_b - \mathbf{u}_{(y)} \wedge \mathbf{f}_a \tag{37}$$

where $\mathbf{v} \equiv \mathbf{v}_{(x)} + \mathbf{u}_{(y)} \equiv v_a\mathbf{e}^a + u^a\mathbf{f}_a$. These were imposed as constraints in [28], but are shown below to hold automatically in Euclidean cases.

6. Riemannian spacetime in Euclidean biconformal space

The principal result of [28] was to show that the flat biconformal space $\mathcal{M}_0^{(2n)}$ arising from any $SO(p,q)$ symmetric biconformal gauging can be identified with a metric phase space only when the initial n-space is of signature $\pm n$ or zero. To make the identification, involution of the Lagrangian submanifolds was imposed, and it was assumed that the momentum space is conformally flat. With these assumptions the Lagrangian configuration and momentum submanifolds of the signature $\pm n$ cases are necessarily Lorentzian.

Here we substantially strengthen this result, by considering only the Euclidean case. We are able to show that further assumptions are unnecessary. The gauging *always* leads to Lorentzian configuration and momentum submanifolds, the involution conditions are automatically satisfied by the structure equations, and both the configuration and momentum spaces are conformally flat. We make no assumptions beyond the choice of the quotient $\mathcal{C}/\mathcal{W}$ and the structures that follow from these groups.

Because this result shows the development of the Lorentzian metric on the Lagrangian submanifolds, we give details of the calculation.

6.1. Solution of the structure equations

A complete solution of the structure equations in the original basis, eqs.(16-19) is given in [21] and derived in [20], with a concise derivation presented in [39]. By choosing the gauge and coordinates (w^α, s_β) appropriately, where Greek indices now refer to *coordinates* and will do so for the remainder of this manuscript$^+$, the solution may be given the form,

$$\boldsymbol{\omega}^\alpha_{\ \beta} = 2\Delta^{\alpha\mu}_{\nu\beta}s_\mu\mathbf{d}w^\nu \tag{38}$$

$$\boldsymbol{\omega}^\alpha \ = \mathbf{d}w^\alpha \tag{39}$$

$$\boldsymbol{\omega}_\alpha \ = \mathbf{d}s_\alpha - \left(s_\alpha s_\beta - \frac{1}{2}s^2\delta_{\alpha\beta}\right)\mathbf{d}w^\beta \tag{40}$$

$$\boldsymbol{\omega} \ \ = -\,s_\alpha\mathbf{d}w^\alpha \tag{41}$$

as is easily checked by direct substitution. Our first goal is to express this solution in the adapted basis and find the resulting metric.

From the original form of the Killing metric, eq.(14), we find

$$\begin{bmatrix} \langle\mathbf{d}w^\alpha, \mathbf{d}w^\beta\rangle & \langle\mathbf{d}w^\alpha, \mathbf{d}s_\beta\rangle \\ \langle\mathbf{d}s_\alpha, \mathbf{d}w^\beta\rangle & \langle\mathbf{d}s_\alpha, \mathbf{d}s_\beta\rangle \end{bmatrix} = \begin{bmatrix} 0 & \delta^\alpha_\beta \\ \delta^\beta_\alpha & -k_{\alpha\beta} \end{bmatrix}$$

where we define $k_{\alpha\beta} \equiv s^2\delta_{\alpha\beta} - 2s_\alpha s_\beta$. This shows that $\mathbf{d}w^\alpha$ and $\mathbf{d}s_\alpha$ do not span orthogonal subspaces. We want to find the most general set of orthogonal Lagrangian submanifolds, and the restriction of the Killing metric to them.

$^+$ The connection forms could be written with distinct indices, for example as $\boldsymbol{\omega}^a = \delta^a_\alpha\mathbf{d}w^\alpha$, but this is unnecessarily cumbersome.

Suppose we find linear combinations of the orginal basis $\boldsymbol{\kappa}^\beta, \boldsymbol{\lambda}_\alpha$ that make the metric block diagonal, with $\boldsymbol{\lambda}_\alpha = 0$ and $\boldsymbol{\kappa}^\beta = 0$ giving Lagrangian submanifolds. Then any further transformation,

$$\begin{aligned}\tilde{\boldsymbol{\kappa}}^\alpha &= A^\alpha_{\ \beta}\boldsymbol{\kappa}^\beta \\ \tilde{\boldsymbol{\lambda}}_\alpha &= B^\beta_{\ \alpha}\boldsymbol{\lambda}_\beta\end{aligned}$$

leaves these submanifolds unchanged and is therefore equivalent. Now suppose one of the linear combinations is

$$\begin{aligned}\tilde{\boldsymbol{\lambda}}_\alpha &= \alpha A^\beta_{\ \alpha}\mathbf{d}s_\beta + \beta\tilde{C}_{\alpha\mu}\mathbf{d}w^\mu \\ &= A^\beta_{\ \alpha}\left(\alpha\mathbf{d}s_\beta + \beta C_{\beta\mu}\mathbf{d}w^\mu\right)\end{aligned}$$

where $C = A^{-1}\tilde{C}$ and the constants are required to keep the transformation nondegenerate. Then $\boldsymbol{\lambda}_\alpha = \alpha\mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta$ spans the same subspace. A similar argument holds for $\tilde{\boldsymbol{\kappa}}^\beta$, so if we can find a basis at all, there is also one of the form

$$\begin{aligned}\boldsymbol{\lambda}_\alpha &= \alpha\mathbf{d}s_\alpha + \beta C_{\alpha\beta}\mathbf{d}w^\beta \\ \boldsymbol{\kappa}^\alpha &= \mu\mathbf{d}w^\alpha + \nu B^{\alpha\beta}\mathbf{d}s_\beta\end{aligned}$$

Now check the symplectic condition,

$$\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha = \left(\mu\beta C_{\alpha\mu}\right)\mathbf{d}w^\alpha\mathbf{d}w^\mu + \alpha\mu\left(\delta^\beta_\mu - \nu\beta C_{\alpha\mu}B^{\alpha\beta}\right)\mathbf{d}w^\mu\mathbf{d}s_\beta + \left(\nu\alpha B^{\alpha\beta}\right)\mathbf{d}s_\beta\mathbf{d}s_\alpha$$

To have $\boldsymbol{\kappa}^\alpha\boldsymbol{\lambda}_\alpha = \mathbf{d}w^\alpha\mathbf{d}s_\alpha$, $B^{\alpha\beta}$ and $C_{\alpha\beta}$ must be symmetric and

$$B = B^t = \frac{\alpha\mu - 1}{\nu\beta}C^{-1} \equiv \alpha\beta\bar{C}$$

Replacing $B_{\alpha\beta}$ in the basis, we look at orthogonality of the inner product, requiring

$$\begin{aligned}0 &= \left\langle\boldsymbol{\kappa}^\alpha, \boldsymbol{\lambda}_\beta\right\rangle \\ &= \left\langle\mu\mathbf{d}w^\alpha + \frac{\alpha\mu - 1}{\beta}\bar{C}^{\alpha\mu}\mathbf{d}s_\mu, \alpha\mathbf{d}s_\beta + \beta C_{\beta\nu}\mathbf{d}w^\nu\right\rangle \\ &= \left(2\alpha\mu - 1\right)\delta^\alpha_\beta - \frac{1}{\beta}\alpha\left(\alpha\mu - 1\right)\bar{C}^{\alpha\mu}k_{\mu\beta}\end{aligned}$$

with solution $C_{\alpha\beta} = \frac{\alpha(\alpha\mu-1)}{\beta(2\alpha\mu-1)}k_{\alpha\beta}$. Therefore, the basis

$$\begin{aligned}\boldsymbol{\lambda}_\alpha &= \alpha\mathbf{d}s_\alpha + \frac{\alpha\left(\alpha\mu - 1\right)}{\left(2\alpha\mu - 1\right)}k_{\alpha\beta}\mathbf{d}w^\beta \\ \boldsymbol{\kappa}^\alpha &= \mu\mathbf{d}w^\alpha + \frac{2\alpha\mu - 1}{\alpha}k^{\alpha\beta}\mathbf{d}s_\beta\end{aligned}$$

satisfies the required properties and is equivalent to any other which does.

The metric restrictions to the submanifolds are now immediate from the inner products:

$$\begin{aligned}\left\langle\boldsymbol{\kappa}^\alpha, \boldsymbol{\kappa}^\beta\right\rangle &= \frac{2\alpha\mu - 1}{\alpha^2}k^{\alpha\beta} \\ \left\langle\boldsymbol{\lambda}_\alpha, \boldsymbol{\lambda}_\beta\right\rangle &= -\frac{\alpha^2}{2\alpha\mu - 1}k_{\alpha\beta}\end{aligned}$$

This shows that the metric on the Lagrangian submanifolds is proportional to $k_{\alpha\beta}$, and we normalize the proportionality to 1 by choosing $\mu = \frac{1+k\alpha^2}{2\alpha}$ and $\beta \equiv k\alpha$, where $k = \pm 1$. This puts the basis in the form

$$\begin{aligned}\boldsymbol{\kappa}^\alpha &= \frac{k}{2\beta}\left(\left(k\beta^2 + 1\right)\mathbf{d}w^\alpha + 2k\beta^2 k^{\alpha\beta}\mathbf{d}s_\beta\right) \\ \boldsymbol{\lambda}_\alpha &= \frac{1}{2\beta}\left(2k\beta^2\mathbf{d}s_\alpha + \left(k\beta^2 - 1\right)k_{\alpha\beta}\mathbf{d}w^\beta\right)\end{aligned}$$

Now that we have established the metric

$$k_{\alpha\beta} = s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2}s_\alpha s_\beta\right)$$

where $\delta_{\alpha\beta}$ is the Euclidean metric and $s^2 = \delta^{\alpha\beta} s_\alpha s_\beta > 0$, and have found one basis for the submanifolds, we may form an orthonormal basis for each, setting $\eta_{ab} = h_a^{\ \alpha} h_b^{\ \beta} k_{\alpha\beta}$.

$$\mathbf{e}^a = \frac{k}{2\beta} h_\alpha^{\ a} \left(\left(1 + k\beta^2\right) \mathbf{d}w^\alpha + 2k\beta^2 k^{\alpha\beta} \mathbf{d}s_\beta \right) \tag{42}$$

$$\mathbf{f}_a = \frac{1}{2\beta} h_a^{\ \alpha} \left(2k\beta^2 \mathbf{d}s_\alpha - \left(1 - k\beta^2\right) k_{\alpha\beta} \mathbf{d}w^\beta \right) \tag{43}$$

We see from the form $k_{\alpha\beta} = s^2 \left(\delta_{\alpha\beta} - \frac{2}{s^2} s_\alpha s_\beta\right)$ that at any point s_α^0, a rotation that takes $\frac{1}{\sqrt{s^w}} s_\alpha^0$ to a fixed direction $\hat{\mathbf{n}}$ will take $k_{\alpha\beta}$ to $s^2 diag\left(-1, 1, \ldots, 1\right)$ so the orthonormal metric η_{ab} is Lorentzian. This is one of our central results. Since eqs.(38-41) provide an exact, general solution to the structure equations, the induced configuration and momentum spaces of Euclidean biconformal spaces are *always* Lorentzian, without restrictions.

We now find the connections forms in the orthogonal basis and check the involution conditions required to guarantee that the configuration and momentum subspaces are Lagrangian submanifolds.

6.2. The connection in the adapted solution basis

We have defined $\boldsymbol{\tau}^a_{\ b}$ in eq. (29) with antisymmetric and symmetric parts $\boldsymbol{\alpha}^a_{\ b}$ and $\boldsymbol{\beta}^a_{\ b}$, subdivided between the $\mathbf{e}^a$ and $\mathbf{f}_a$ subspaces, eq(26, 27). All quantities may be written in terms of the new basis. We will make use of $s_a \equiv h_a^{\ \alpha} s_\alpha$ and $\delta_{ab} \equiv h_a^{\ \alpha} h_b^{\ \beta} \delta_{\alpha\beta}$. In terms of these, the orthonormal metric is $\eta_{ab} = s^2 \left(\delta_{ab} - \frac{2}{s^2} s_a s_b\right)$, where $s^2 \equiv \delta^{ab} s_a s_b > 0$. Solving for δ_{ab}, we find $\delta_{ab} = \frac{1}{s^2}\eta_{ab} + \frac{2}{s^2} s_a s_b$. Similar relations hold for the inverses, η^{ab}, δ^{ab}, see Appendix 2. In addition, we may invert the basis change to write the coordinate differentials,

$$\mathbf{d}w^\beta = k\beta h_a^{\ \beta} \left(\mathbf{e}^a - k\eta^{ab} \mathbf{f}_b\right)$$

$$\mathbf{d}s_\alpha = \frac{1}{2\beta} h_\alpha^{\ a} \left(\left(1 - k\beta^2\right) \eta_{ab} \mathbf{e}^b + k \left(1 + k\beta^2\right) \mathbf{f}_a \right)$$

The known solution for the spin connection and Weyl form, eqs.(38,41) immediately become

$$\boldsymbol{\omega}^a_{\ b} = 2\Delta^{ac}_{db} s_c k\beta \left(\mathbf{e}^d - k\eta^{de} \mathbf{f}_e\right) \tag{44}$$

$$\boldsymbol{\omega} = -k\beta s_a \mathbf{e}^a + \beta \eta^{ab} s_a \mathbf{f}_b \tag{45}$$

where we easily expand the projection Δ^{ac}_{db} in terms of the new metric. Substituting this expansion to find $\boldsymbol{\tau}^a_{\ b}$, results in

$$\boldsymbol{\tau}^a_{\ b} = \beta \left(2\Theta^{ac}_{db} s_c + 2\eta^{ae} \eta_{bd} s_e + 2\eta^{ae} s_e s_b s_d\right) \left(k\mathbf{e}^d - \eta^{dg} \mathbf{f}_g\right) - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a}$$

The antisymmetric part is then $\boldsymbol{\alpha}^a_{\ b} \equiv \Theta^{ad}_{cb} \boldsymbol{\tau}^c_{\ d} = -\Theta^{ad}_{cb} h_d^{\ \alpha} \mathbf{d} h_\alpha^{\ c}$ with the remaining terms cancelling identically. Furthermore, as described above, $h_\alpha^{\ c}$ is a purely s_α-dependent rotation at each point. Therefore the remaining $h_d^{\ \alpha} \mathbf{d} h_\alpha^{\ c}$ term will lie totally in the subspace spanned by $\mathbf{d}s_\alpha$, giving the parts of $\boldsymbol{\alpha}^a_{\ b}$ as

$$\boldsymbol{\sigma}^a_{\ b} = -\frac{1 - k\beta^2}{2\beta} \Theta^{ad}_{cb} \left(h_b^{\ \alpha} \frac{\partial}{\partial s_\beta} h_\alpha^{\ a} \right) h_\beta^{\ c} \eta_{cd} \mathbf{e}^d \tag{46}$$

$$\boldsymbol{\gamma}^a_{\ b} = -\frac{k + \beta^2}{2\beta} \Theta^{ad}_{cb} \left(h_b^{\ \alpha} \frac{\partial}{\partial s_\beta} h_\alpha^{\ a} \right) h_\beta^{\ c} \mathbf{f}_c \tag{47}$$

Recall that the value of k or β in these expressions is essentially a gauge choice and should be physically irrelavant. If we choose $\beta^2 = 1$, we get either $\boldsymbol{\sigma}^a_{\ b} = 0$ or $\boldsymbol{\gamma}^a_{\ b} = 0$, depending on the sign of k.

Continuing, we are particularly interested in the symmetric pieces of the connection since they constitute a new feature of the theory. Applying the symmetric projection to $\boldsymbol{\tau}^a_{\ b}$, we expand

$$\boldsymbol{\beta}^a_{\ b} \equiv \Xi^{ad}_{cb} \boldsymbol{\tau}^c_{\ d}$$

Using $\Xi^{cd}_{ab} \left(h_d^{\ \mu} \mathbf{d} h_\mu^{\ a}\right) = \frac{1}{2} h_\alpha^{\ c} h_b^{\ \beta} k^{\alpha\mu} \mathbf{d} k_{\mu\beta}$ (see Appendix 3) to express the derivative term in terms of v_a, we find the independent parts

$$\boldsymbol{\mu}^a_{\ b} = \left(-k\beta \delta^a_b s_c + \beta\gamma_+ \left(\delta^a_b s_c + \delta^a_c s_b + \eta^{ad} \eta_{bc} s_d + 2\eta^{ad} s_b s_c s_d\right)\right) \mathbf{e}^c$$

$$\boldsymbol{\rho}^a_{\ b} = \left(\beta \delta^a_b \eta^{cd} s_d + k\beta\gamma_- \left(\delta^a_b \eta^{cd} s_d + \delta^c_b \eta^{ad} s_d + \eta^{ac} s_b + 2\eta^{ad} \eta^{ce} s_b s_d s_e\right)\right) \mathbf{f}_c$$

where $\gamma_{\pm} \equiv \frac{1}{2\beta}\left(1 \pm k\beta^2\right)$. Written in this form, the tensor character of $\boldsymbol{\mu}^a_{\ b}$ and $\boldsymbol{\rho}^a_{\ b}$ is not evident, but since we have chosen η_{ab} orthonormal (referred to later as the orthonormal gauge), $\phi = 0$, and $\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi = \boldsymbol{\omega}$ we have $\mathbf{v}_{(e)} + \mathbf{u}_{(f)} = -k\beta s_a \mathbf{e}^a + \beta\eta^{ab} s_a \mathbf{f}_b$ so that we may equally well write

$$\boldsymbol{\mu}^a_{\ b} = \left(\delta^a_b v_c - k\gamma_+\left(\delta^a_b v_c + \delta^a_c v_b + \eta^{ad}\eta_{bc}v_d + \frac{2}{\beta^2}\eta^{ad}v_b v_c v_d\right)\right)\mathbf{e}^c \tag{48}$$

$$\boldsymbol{\rho}^a_{\ b} = \left(\delta^a_b u^c + k\gamma_-\left(\delta^a_b u^c + \delta^c_b u^a + \eta^{ac}\eta_{bd}u^d + \frac{2}{\beta^2}\eta_{bd}u^a u^c u^d\right)\right)\mathbf{f}_c \tag{49}$$

which are manifestly tensorial.

The involution conditions, eqs.(36-37), are easily seen to be satisfied identically by eqs.(48, 49). Therefore, the $\mathbf{f}_a = 0$ and $\mathbf{e}^a = 0$ subspaces are Lagrangian submanifolds spanned respectively by $\mathbf{e}^a$ and $\mathbf{f}_a$. There exist coordinates such that these basis forms may be written

$$\mathbf{e}^a = e_\mu^{\ \ a}\mathbf{d}x^\mu \tag{50}$$

$$\mathbf{f}_a = f_a^{\ \ \mu}\mathbf{d}y_\mu \tag{51}$$

To find such submanifold coordinates, the useful thing to note is that $\mathbf{d}\left(\frac{s_\alpha}{s^2}\right) = \delta_{\alpha\nu}k^{\mu\nu}\mathbf{d}s_\mu$ so that the basis may be written as

$$\mathbf{e}^a = h_\alpha^{\ a}\mathbf{d}\left(k\gamma_+ w^\alpha + \beta\delta^{\alpha\beta}\left(\frac{s_\alpha}{s^2}\right)\right) \equiv h_\alpha^{\ a}\mathbf{d}x^\alpha$$

$$\mathbf{f}_a = \left(h_a^{\ \alpha}k_{\alpha\beta}\delta^{\beta\mu}\right)\mathbf{d}\left(k\beta\left(\frac{s_\mu}{s^2}\right) - \gamma_-\delta_{\mu\nu}w^\nu\right) \equiv f_a^{\ \mu}\mathbf{d}y_\mu$$

with $x^\alpha = k\gamma_+ w^\alpha + \beta\delta^{\alpha\beta}\left(\frac{s_\alpha}{s^2}\right)$ and $y_\mu = k\beta\left(\frac{s_\mu}{s^2}\right) - \gamma_-\delta_{\mu\nu}w^\nu$. This confirms the involution.

7. Curvature of the submanifolds

The nature of the configuration or momentum submanifold may be determined by restricting the structure equations by $\mathbf{f}_a = 0$ or $\mathbf{e}^a = 0$, respectively. To aid in the interpretation of the resulting submanifold structure equations, we define the curvature of the antisymmetric connection $\boldsymbol{\alpha}^a_{\ b}$

$$\mathbf{R}^a_{\ b} \equiv \mathbf{d}\boldsymbol{\alpha}^a_{\ b} - \boldsymbol{\alpha}^c_{\ b}\wedge\boldsymbol{\alpha}^a_{\ c} \tag{52}$$

$$= \frac{1}{2}R^a_{\ bcd}\mathbf{e}^c\wedge\mathbf{e}^d + R^{a\ c}_{\ b\ d}\mathbf{f}_c\wedge\mathbf{e}^d + \frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\wedge\mathbf{f}_d \tag{53}$$

While all components of the overall Cartan curvature, $\boldsymbol{\Omega}^A = (\boldsymbol{\Omega}^a_b, \mathbf{T}^a, \mathbf{S}_a, \boldsymbol{\Omega})$ are zero on $\mathcal{M}_0^{(2n)}$, the curvature, $\mathbf{R}^a_{\ b}$, and in particular the curvatures $\frac{1}{2}R^a_{\ bcd}\mathbf{e}^c\wedge\mathbf{e}^d$ and $\frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\wedge\mathbf{f}_d$ on the submanifolds, may or may not be. Here we examine this question, using the structure equations to find the *Riemannian* curvature of the connections, $\boldsymbol{\sigma}^a_{\ b}$ and $\boldsymbol{\gamma}^a_b$, of the Lorentzian submanifolds.

7.1. Momentum space curvature

To see that the Lagrangian submanifold equations describe a Riemannian geometry, we set $\mathbf{e}^a = 0$ in the structure equations, eqs.(32-35) and choose the $\phi = 0$ (orthonormal) gauge (or see Appendix 1, eqs.(1.3*a*-1.3*d*, with the Cartan curvatures set to zero). Then, taking the Θ^{ac}_{db} projection, we have

$$0 = \frac{1}{2}R^{a\ cd}_{\ b}\mathbf{f}_c\wedge\mathbf{f}_d - \boldsymbol{\rho}^c_{\ b}\wedge\boldsymbol{\rho}^a_{\ c} + \Theta^{ac}_{db}\eta^{ac}\Delta^{eb}_{cf}\mathbf{f}_b\wedge\mathbf{f}_a \tag{54}$$

$$0 = \mathbf{d}_{(y)}\mathbf{f}_b - \boldsymbol{\gamma}^a_{\ b}\wedge\mathbf{f}_a$$

These are the structure equations of a Riemannian geometry with additional *geometric* terms, $-\boldsymbol{\rho}^c_{\ b}\wedge\boldsymbol{\rho}^a_{\ c} + \Theta^{ac}_{db}\eta^{ac}\Delta^{eb}_{cf}\mathbf{f}_b\wedge\mathbf{f}_a$, reflecting the difference between Lorentz curvature and conformal curvature. The symmetric projection is

$$\mathbf{D}^{(y)}\boldsymbol{\rho}^a_{\ b} = -k\Xi^{ac}_{db}\Delta^{df}_{ec}\eta^{eg}\mathbf{f}_f\wedge\mathbf{f}_g$$

$$\mathbf{d}_{(y)}\mathbf{u}_{(f)} = 0$$

where $\mathbf{u}_{(f)}, \boldsymbol{\gamma}^a_{\ b}$ and $\boldsymbol{\rho}^a_{\ b}$ are given by eqs.(45,47,49), respectively. Rather than computing $R^{a\ \ cd}_{\ b}$ directly from $\boldsymbol{\gamma}^a_{\ b}$, which requires a complicated expression for the local rotation, $h_\alpha^{\ a}$, we find it using the rest of eq. (54).

Letting $\beta = e^\lambda$ so that

$$k + \gamma_-^2 = \begin{cases} \cosh^2 \lambda & k = 1 \\ \sinh^2 \lambda & k = -1 \end{cases}$$

the curvature is

$$\frac{1}{2} R^{a\ \ cd}_{\ b} \mathbf{f}_c \mathbf{f}_d = \begin{cases} \cosh^2 \lambda \Theta^{ag}_{cb} \left(\eta^{cf} + 2\eta^{cd}\eta^{fe} s_d s_e\right) \mathbf{f}_f \wedge \mathbf{f}_g & k = 1 \\ \sinh^2 \lambda \Theta^{ag}_{cb} \left(\eta^{cf} + 2\eta^{cd}\eta^{fe} s_d s_e\right) \mathbf{f}_f \wedge \mathbf{f}_g & k = -1 \end{cases}$$

Now consider the symmetric equations. Notice that the Weyl vector has totally decoupled, with its equation showing that $\mathbf{u}_{(f)}$ is closed, a result which also follows from its definition. For the symmetric projection, we find $\Xi^{ac}_{db} \eta^{ac} \Delta^{eb}_{cf} \mathbf{f}_b \mathbf{f}_a \equiv 0$. Then, contraction of $D^a \rho^{a\ \ c}_{\ b}$ with $\eta_{ad}\eta_{ce} u^a u^e$, together with $\mathbf{d}_{(y)} \mathbf{u}_{(f)} = 0$ shows that u^a is covariantly constant, $D^a_{(y)} u^b = 0$.

If we choose $k = -1$ and $\lambda = 0$, the Riemann curvature of the momentum space vanishes. This is a stronger result than in [28], since there only the Weyl curvature could be set to zero consistently. In this case, the Lagrangian submanifold becomes a vector space and there is a natural interpretation as the co-tangent space of the configuration space. However, the orthonormal metric in this case, $\langle \mathbf{f}_a, \mathbf{f}_b \rangle = \eta_{ab}$, has the opposite sign from the metric of the configuration space, $\left\langle \mathbf{e}^a, \mathbf{e}^b \right\rangle = -\eta^{ab}$. This reversal of sign of the metric together with the the units, suggests that the physical (momentum) tangent space coordinates are related to the geometrical ones by $p_\alpha \sim i\hbar y_\alpha$. This has been suggested previously [42] and explored in the context of quantization [41].

Leaving β and k unspecified, we see that in general momentum space has non-vanishing Riemannian curvature of the connection $\boldsymbol{\gamma}^a_{\ b}$, a situation suggested long ago for quantum gravity [43, 44]. We consider this further in Section 7.3. Whatever the values of β and k, the momentum space is conformally flat. We see this from the decomposition of Riemannian curvature into the Weyl curvature, $\mathbf{C}^a_{\ b}$, and Schouten tensor, $\boldsymbol{\mathcal{R}}_a$, given by

$$\mathbf{R}^a_{\ b} = \mathbf{C}^a_{\ b} - 2\Theta^{ae}_{db} \boldsymbol{\mathcal{R}}_e \mathbf{e}^d$$

The Schouten tensor, $\boldsymbol{\mathcal{R}}_a \equiv \frac{1}{n-2}\left(R_{ab} - \frac{1}{2(n-1)} R\eta_{ab}\right)\mathbf{e}^b$ is algebraically equivalent to the Ricci tensor, R_{ab}. It is easy to prove that when the curvature 2-form can be expressed as a projection in the form $\mathbf{R}^a_{\ b} = -2\Theta^{ae}_{db} \mathbf{X}_e \mathbf{e}^d$, then $\mathbf{X}_a$ is the Schouten tensor, and the Weyl curvature vanishes. Vanishing Weyl curvature implies conformal flatness.

7.2. Spacetime curvature and geometric curvature

The curvature on the configuration space takes the same basic form. Still in the orthonormal gauge, and separating the symmetric and antisymmetric parts as before, we again find a Riemannian geometry with additional geometric terms,

$$0 = \mathbf{R}^a_{\ b}(\boldsymbol{\sigma}) - \boldsymbol{\mu}^c_{\ b} \boldsymbol{\mu}^a_{\ c} - \Theta^{ac}_{db} \Delta^{de}_{fc} \eta_{eg} \mathbf{e}^g \mathbf{e}^f \tag{55}$$

$$0 = \mathbf{d}_{(x)} \mathbf{e}^a - \mathbf{e}^b \boldsymbol{\sigma}^a_{\ b} \tag{56}$$

together with

$$0 = \mathbf{D}_{(x)} \boldsymbol{\mu}^a_{\ b} - \Xi^{ac}_{db} \Delta^{de}_{fc} \eta_{eg} \mathbf{e}^g \mathbf{e}^f$$

$$0 = \mathbf{d}_{(x)} \mathbf{v}$$

Looking first at all the Θ^{ad}_{cb}-antisymmetric terms and substituting in (48) for $\boldsymbol{\mu}^a_{\ b}$, we find that the Riemannian curvature is

$$\mathbf{R}^a_{\ b} = \left(\gamma_+^2 - k\right) \Theta^{ac}_{db} \left(\eta_{ce} + 2 s_c s_e\right) \mathbf{e}^d \mathbf{e}^e$$

so the Weyl curvature vanishes and the Schouten tensor is

$$\boldsymbol{\mathcal{R}}_a = \frac{1}{2}\left(\gamma_+^2 - k\right)\left(\eta_{ab} + 2 s_a s_b\right) \mathbf{e}^b \tag{57}$$

The vanishing Weyl curvature tensor shows that the spacetime is conformally flat. This result is discussed in detail below.

The equation, $\mathbf{d}_{(x)}\mathbf{v} = 0$ shows that $\mathbf{v}$ is hypersurface orthogonal. Expanding the remaining equation with $\mathbf{d}_{(x)}\mathbf{v} = 0$, $\mathbf{D}_{(x)}\eta_{ab} = 0$ and $\mathbf{D}_{(x)}\mathbf{e}^a = 0$, contractions involving η_{ab} and v_a quickly show that

$$D_a^{(x)} v_b = 0$$

This, combined with $\mathbf{D}^{(y)} u^a = 0$ and $u^a = -k\eta^{ab} v_b$ shows that the full covariant derivative vanishes, $D_a v_b = 0$. The scale vector is therefore a covariantly constant, hypersurface orthogonal, unit timelike Killing vector of the spacetime submanifold.

7.3. Curvature invariant

Substituting $\beta = e^\lambda$ as before, the components of the momentum and configuration curvatures become

$$\eta_{df}\eta_{eg}R^{a\ fg}_{\ b} = \begin{cases} \cosh^2\lambda \left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k = 1 \\ \sinh^2\lambda \left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k = -1 \end{cases}$$

and

$$R^a_{\ bde} = \begin{cases} \sinh^2\lambda \left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k = 1 \\ \cosh^2\lambda \left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c) & k = -1 \end{cases}$$

Subtracting these

$$\eta_{df}\eta_{eg}R^{a\ fg}_{\ b} - R^a_{\ bde} = k\left(\Theta^{ac}_{db}\delta^f_e - \Theta^{ac}_{eb}\delta^f_d\right)(\eta_{fc} + 2s_f s_c)$$

so that the difference of the configuration and momentum curvatures is independent of the linear combination of basis forms used. This coupling between the momentum and configuration space curvatures adds a sort of complementarity that goes beyond the suggestion by Born [43, 44] that momentum space might also be curved. As we continuously vary β^2, the curvature moves between momentum and configuration space but this difference remains unchanged. We may even make one or the other Lagrangian submanifold flat.

For the Einstein tensors,

$$\eta_{ac}\eta_{bd}G^{cd}_{(y)} - G^{(x)}_{ab} = \frac{1}{2}k\left((n-3)\eta_{ab} + (n-2)s_a s_b\right)$$

7.4. Candidate dark matter

There is a surprising consequence of the tensor $\boldsymbol{\mu}^a_{\ b}$ in the Lorentz structure equation. The structure equations for the configuration Lagrangian submanifold above describe an ordinary curved Lorentzian spacetime with certain extra terms from the conformal geometry that exist even in the *absence of matter*. We gain some insight into the nature of these additional terms from the metric and Einstein tensor. In coordinates, the metric takes the form

$$h_{\alpha\beta} = s^2\left(\delta_{\alpha\beta} - \frac{2}{s^2}s_\alpha s_\beta\right)$$

which is straightforwardly boosted to $s^2\eta^0_{\alpha\beta}$ at a point. Since the spacetime is conformally flat, gradients of the conformal factor must be in the time direction, s_α, so we may rescale the time, $dt' = \sqrt{s^2}dt$ to put the line element in the form

$$ds^2 = -dt'^2 + s^2(t')\left(dx^2 + dy^2 + dz^2\right)$$

That is, the *vacuum solution* is a spatially flat FRW cosmology. Putting the results in terms of the Einstein tensor and a coordinate basis, we expect an equation of the form $\tilde{G}_{\alpha\beta} = \kappa T^{matter}_{\alpha\beta}$ where the Cartan Einstein tensor is modified to

$$\tilde{G}_{\alpha\beta} \equiv G_{\alpha\beta} - 3(n-2)s^2 s_\alpha s_\beta + \frac{3}{2}(n-2)(n-3)s^2 h_{\alpha\beta} \tag{58}$$

where $G_{\alpha\beta}$ is the familiar Einstein tensor. The new geometric terms may be thought of as a combination of a *cosmological constant* and a *cosmological perfect fluid*. With this interpretation, we may write the new cosmological terms as

$$\kappa T^{cosm}_{\alpha\beta} = (\rho_0 + p_0) v_\alpha v_\beta + p_0 h_{\alpha\beta} - \Lambda h_{\alpha\beta}$$

where $\kappa T^{cosm}_{ab} \equiv 3(n-2) s^2 v_\alpha v_\beta - \frac{3}{2}(n-2)(n-3) s^2 h_{\alpha\beta}$. In $n = 4$-dimensions, $\frac{1}{2}(\rho_0 + p_0) = \Lambda - p_0$, with the equation of state and the overall scale undetermined. If we assume an equation of state $p_0 = w\rho_0$, this becomes

$$\frac{1}{2}(1+3w)\rho_0 = \Lambda$$

This relation alone does not account for the values suggested by the current Planck data: about 0.68 for the cosmological constant, 0.268 for the density of dark matter, and vanishing pressure, $w = 0$. However, these values are based on standard cosmology, while we have not yet included matter terms in eq.(58). Moreover, the proportions of the three geometric terms in eq.(58) may change when curvature is included. Such a change is suggested by the form of known solutions in the original basis, where $h_{\alpha\beta}$ is augmented by a Schouten term. If this modification also occurs in the adapted basis, the ratios above will be modified. We are currently examining such solutions.

8. Discussion

Using the quotient method of gauging, we constructed the class of biconformal geometries . The construction starts with the conformal group of an $SO(p,q)$-symmetric pseudo-metric space. The quotient by $\mathcal{W}(p,q) \equiv SO(p,q) \times$ *dilatations* gives the homogeneous manifold, $\mathcal{M}^{2n}_0$. We show that this manifold is metric and symplectic (as well as Kähler with a different metric). Generalizing the manifold and connection while maintaining the local $\mathcal{W}$ invariance, we display the resulting biconformal spaces, $\mathcal{M}^{2n}$ [14,15,21].

This class of locally symmetric manifolds becomes a model for gravity when we recall the most general curvature-linear action [20].

It is shown in [28] that $\mathcal{M}^{(2n)}_0(p,q)$ in any dimension $n = p+q$ will have *Lagrangian submanifolds that are orthogonal with respect to the 2n-dim biconformal (Killing) metric and have non-degenerate n-dim metric restrictions on those submanifolds* only if the original space is Euclidean or signature zero $\left(p \in \left\{0, \frac{n}{2}, n\right\}\right)$, and then the signature of the submanifolds is severely limited $(p \to p \pm 1)$. This leads in the two Euclidean cases to Lorenztian configuration space, and hence the origin of time [28]. For the case of flat, 8-dim biconformal space the Lagrangian submanifolds are necessarily Lorentzian.

Our investigation explores properties of the homogeneous manifold, $\mathcal{M}^{2n}_0(n,0)$. Starting with Euclidean symmetry, $SO(n)$, we clarify the emergence of Lorentzian signature Lagrangian submanifolds. We extend the results of [28], eliminating all but the group-theoretic assumptions. By writing the structure equations in an adapted basis, we reveal new features of these geometries. We summarize our new findings below.

A new connection

There is a natural $SO(n)$ Cartan connection on $\mathcal{M}^{2n}_0$. Rewriting the biconformal structure equations in an orthogonal, canonically conjugate, conformally orthonormal basis automatically introduces a Lorentzian connection and decouples the Weyl vector from the submanifolds. This structure emerges directly from the transformation of the structure equations, as detailed in points 1 through 4 in §5.3.

Specifically, we showed that all occurences of the $SO(4)$ spin connection $\boldsymbol{\omega}^\alpha_{\ \beta}$ may be written in terms of the new connection, $\boldsymbol{\tau}^a_{\ b} \equiv h_\alpha^{\ a} \boldsymbol{\omega}^\alpha_{\ \beta} h_b^{\ \beta} - h_b^{\ \alpha} \mathbf{d} h_\alpha^{\ a}$, which has both symmetric and antisymmetric parts. These symmetric and antisymmetric parts separate *automatically* in the structure equations, with only the Lorentz part of the connection, $\boldsymbol{\alpha}^a_{\ b} = \Theta^{ac}_{db} \boldsymbol{\tau}^d_{\ c}$ describing the evolution of the configuration submanifold solder form. The spacetime and momentum space connections are metric compatible, up to a conformal factor.

The Weyl vector terms drop out of the submanifold basis equations. The submanifold equations remain scale invariant because of the residual metric derivative, $\frac{1}{2}\mathbf{d}\eta^{ac}\eta_{cb} = \delta^a_b \mathbf{d}\phi$. When the metric is rescaled, this term changes with the negative of the inhomogeneous term acquired by the Weyl vector.

Two new tensors

It is especially striking how the Weyl vector and the symmetric piece of the connection are pushed from the basis submanifolds into the mixed basis equations. These extra degrees of freedom are embodied in two new *Lorentz tensors*.

The factor $\delta^a_b \mathbf{d}\phi$ which replaces the Weyl vector in the basis equations allows us to form a *scale-invariant* 1-form,

$$\mathbf{v} = \boldsymbol{\omega} + \mathbf{d}\phi$$

It is ultimately this vector which determines the time direction.

We showed that the symmetric part of the spin connection, $\boldsymbol{\beta}^a_{\ b}$, despite being a piece of the connection, transforms as a tensor. The solution of the structure equations shows that the two tensors, $\mathbf{v}$ and $\boldsymbol{\beta}^a_{\ b}$ are related, with $\boldsymbol{\beta}^a_{\ b}$ constructed cubically, purely from $\mathbf{v}$ and the metric. Although the presence of $\boldsymbol{\beta}^a_{\ b}$ changes the form of the momentum space curvature, we find the same signature changing metric as found in [28]. Rather than imposing vanishing momentum space curvature as in [28], we make use of a complete solution of the Maurer-Cartan equations to derive the metric. The integrability of the Lagrangian submanifolds, the Lorentzian metric and connection, and the possibility of a flat momentum space are all now seen as direct consequences of the structure equations, without assumptions.

Riemannian spacetime and momentum space

The configuration and momentum submanifolds have vanishing dilatational curvature, making them gauge equivalent to Riemannian geometries. Together with the signature change from the original Euclidean space to these Lorentzian manifolds, we arrive at a suitable arena for general relativity in which time is constructed covariantly from a scale-invariant Killing field. This field is provided automatically from the group structure.

Effective cosmological fluid and cosmological constant

Though we work in the homogeneous space, $\mathcal{M}^{2n}_0$, so that there are no Cartan curvatures, there is a net Riemannian curvature remaining on the spacetime submanifold. We show this to describe a conformally flat spacetime with the deviation from flatness provided by additional geometric terms of the form

$$\tilde{G}_{\alpha\beta} \equiv G_{\alpha\beta} - \rho_0 v_\alpha v_\beta + \Lambda h_{\alpha\beta} = 0$$

that is, a background dust and a cosmological constant. The values $\rho_0 = 3(n-2)s^2$ and $\Lambda = \frac{3}{2}(n-2)(n-3)s^2$ give, in the absence of physical sources, the relation $(2+3w)\rho_0 = \Lambda$ for an equation of state $p_0 = w\rho_0$. An examination of more realistic cosmological models involving matter fields and curved biconformal spaces, $\mathcal{M}^{2n}$, is underway.

Appendix 1: Subparts of the structure equations

Here we write the structure equations, including Cartan curvature. We expand the configuration, mixed and momentum terms separately. Note that the $\mathbf{f}_a\mathbf{f}_b$ part of the $\mathbf{d}\mathbf{e}^a$ equation and the $\mathbf{e}^a\mathbf{e}^b$ part of the $\mathbf{d}\mathbf{f}_a$ equation are set to zero. These are the involution conditions, which guarantee that the configuration and momentum subspaces are integrable submanifolds by the Frobenius theorem.

In the conformal-orthonormal basis, we have $g^{ab}\mathbf{d}g_{bc} = e^{-2\phi}\eta^{ab}\mathbf{d}\left(e^{2\phi}\eta_{bc}\right) = 2\delta^a_c\mathbf{d}\phi$. The structure equations in the conformal-orthonormal basis are

$$\begin{aligned}
\mathbf{d}\boldsymbol{\tau}^a_{\ b} &= \boldsymbol{\tau}^c_{\ b}\boldsymbol{\tau}^a_{\ c} + \Delta^{ae}_{db}\eta_{ec}\mathbf{e}^c\mathbf{e}^d - \Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d + 2\Delta^{ae}_{fb}\Xi^{fc}_{de}\mathbf{f}_c\mathbf{e}^d + \boldsymbol{\Omega}^a_{\ b} \\
\mathbf{d}\mathbf{e}^a &= \mathbf{e}^c\boldsymbol{\alpha}^a_{\ c} + \frac{1}{2}\eta_{cb}\mathbf{d}\eta^{ac}\mathbf{e}^b + \frac{1}{2}\mathbf{D}\eta^{ab}\mathbf{f}_b + \mathbf{T}^a \\
\mathbf{d}\mathbf{f}_a &= \boldsymbol{\alpha}^b_{\ a}\mathbf{f}_b + \frac{1}{2}\eta^{bc}\mathbf{d}\eta_{ab}\mathbf{f}_c - \frac{1}{2}\mathbf{D}\eta_{ab}\mathbf{e}^b + \mathbf{S}_a \\
\mathbf{d}\boldsymbol{\omega} &= \mathbf{e}^a\mathbf{f}_a + \boldsymbol{\Omega}
\end{aligned}$$

Then defining

$$\begin{aligned}
\mathbf{D}^{(x)}\boldsymbol{\mu}^a_{\ b} &\equiv \mathbf{d}^{(x)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\mu}^a_{\ c} \\
\mathbf{D}^{(x)}\boldsymbol{\rho}^a_{\ b} &\equiv \mathbf{d}^{(x)}\boldsymbol{\rho}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\sigma}^a_{\ c} - \boldsymbol{\sigma}^c_{\ b}\boldsymbol{\rho}^a_{\ c} \\
\mathbf{D}^{(y)}\boldsymbol{\mu}^a_{\ b} &\equiv \mathbf{d}^{(y)}\boldsymbol{\mu}^a_{\ b} - \boldsymbol{\mu}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\mu}^a_{\ c} \\
\mathbf{D}^{(y)}\boldsymbol{\rho}^a_{\ b} &\equiv \mathbf{d}^{(y)}\boldsymbol{\rho}^a_{\ b} - \boldsymbol{\rho}^c_{\ b}\boldsymbol{\gamma}^a_{\ c} - \boldsymbol{\gamma}^c_{\ b}\boldsymbol{\rho}^a_{\ c}
\end{aligned}$$

the separation of the structure equations into independent parts gives:

Configuration space:

$$\frac{1}{2}\Omega^{a}_{\ bcd}\mathbf{e}^c\mathbf{e}^d = \mathbf{d}^{(x)}\boldsymbol{\sigma}^{a}_{\ b} - \boldsymbol{\sigma}^{c}_{\ b}\boldsymbol{\sigma}^{a}_{\ c} + \mathbf{D}^{(x)}\boldsymbol{\mu}^{a}_{\ b} - \boldsymbol{\mu}^{c}_{\ b}\boldsymbol{\mu}^{a}_{\ c} - k\Delta^{ac}_{eb}\eta_{cd}\mathbf{e}^d\mathbf{e}^e \tag{1.1a}$$

$$\frac{1}{2}T^{a}_{\ bc}\mathbf{e}^b\mathbf{e}^c = \mathbf{d}_{(x)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\sigma}^{a}_{\ b} + \frac{1}{2}\eta^{ac}\mathbf{d}^{(x)}\eta_{cb}\mathbf{e}^b \tag{1.1b}$$

$$\frac{1}{2}S_{abc}\mathbf{e}^b\mathbf{e}^c = k\eta_{ab}\mathbf{e}^c\left(\boldsymbol{\mu}^{b}_{\ c} - \delta^b_c W_d\mathbf{e}^d + \frac{1}{2}\eta_{ce}\mathbf{d}^{(x)}\eta^{be}\right) \tag{1.1c}$$

$$\frac{1}{2}\Omega_{ab}\mathbf{e}^a\mathbf{e}^b = \mathbf{d}_{(x)}\left(W_a\mathbf{e}^a\right) \tag{1.1d}$$

Cross-term:

$$\begin{aligned}\Omega^{a\ \ c}_{\ b\ d}\mathbf{f}_c\mathbf{e}^d &= \mathbf{d}^{(y)}\boldsymbol{\sigma}^{a}_{\ b} + \mathbf{d}^{(x)}\boldsymbol{\gamma}^{a}_{\ b} - \boldsymbol{\gamma}^{c}_{\ b}\boldsymbol{\sigma}^{a}_{\ c} - \boldsymbol{\sigma}^{c}_{\ b}\boldsymbol{\gamma}^{a}_{\ c} \\ &\quad + \mathbf{D}^{(x)}\boldsymbol{\rho}^{a}_{\ b} + \mathbf{D}^{(y)}\boldsymbol{\mu}^{a}_{\ b} - \boldsymbol{\rho}^{c}_{\ b}\boldsymbol{\mu}^{a}_{\ c} - \boldsymbol{\mu}^{c}_{\ b}\boldsymbol{\rho}^{a}_{\ c} \\ &\quad - 2\Delta^{ac}_{db}\Xi^{fd}_{ce}\mathbf{f}_f\mathbf{e}^e\end{aligned} \tag{1.2a}$$

$$\begin{aligned}T^{ab}_{\ \ c}\mathbf{f}_b\mathbf{e}^c &= \mathbf{d}^{(y)}\mathbf{e}^a - \mathbf{e}^b\boldsymbol{\gamma}^{a}_{\ b} + \frac{1}{2}\eta^{ac}\mathbf{d}^{(y)}\eta_{cb}\mathbf{e}^b \\ &\quad - k\eta^{ac}\left(\boldsymbol{\mu}^{b}_{\ c}\mathbf{f}_b + W_d\mathbf{f}_c\mathbf{e}^d - \frac{1}{2}\eta^{bd}\mathbf{d}^{(x)}\eta_{cd}\mathbf{f}_b\right)\end{aligned} \tag{1.2b}$$

$$\begin{aligned}S_{a\ c}^{\ b}\mathbf{f}_b\mathbf{e}^c &= \mathbf{d}^{(x)}\mathbf{f}_a - \boldsymbol{\sigma}^{b}_{\ a}\mathbf{f}_b - \frac{1}{2}\eta^{cb}\mathbf{d}^{(x)}\eta_{ac}\mathbf{f}_b \\ &\quad + k\eta_{ab}\left(\mathbf{e}^c\boldsymbol{\rho}^{b}_{\ c} + W^c\mathbf{f}_c\mathbf{e}^b + \frac{1}{2}\eta^{bc}\mathbf{d}^{(y)}\eta_{cd}\mathbf{e}^d\right)\end{aligned} \tag{1.2c}$$

$$\Omega^{a}_{\ b}\mathbf{f}_a\mathbf{e}^b = \mathbf{d}_{(y)}\left(W_a\mathbf{e}^a\right) + \mathbf{d}_{(x)}\left(W^a\mathbf{f}_a\right) - \mathbf{e}^a\mathbf{f}_a \tag{1.2d}$$

Momentum space:

$$\frac{1}{2}\Omega^{a\ \ cd}_{\ b}\mathbf{f}_c\mathbf{f}_d = \mathbf{d}^{(y)}\boldsymbol{\gamma}^{a}_{\ b} - \boldsymbol{\gamma}^{c}_{\ b}\boldsymbol{\gamma}^{a}_{\ c} + \mathbf{D}\boldsymbol{\rho}^{a}_{\ b} - \boldsymbol{\rho}^{c}_{\ b}\boldsymbol{\rho}^{a}_{\ c} + k\Delta^{ac}_{eb}\eta^{ed}\mathbf{f}_c\mathbf{f}_d \tag{1.3a}$$

$$\frac{1}{2}S_{a}^{\ bc}\mathbf{f}_b\mathbf{f}_c = \mathbf{d}^{(y)}\mathbf{f}_a - \boldsymbol{\gamma}^{b}_{\ a}\mathbf{f}_b - \frac{1}{2}\eta^{cb}\mathbf{d}^{(y)}\eta_{ac}\mathbf{f}_b \tag{1.3b}$$

$$\frac{1}{2}T^{abc}\mathbf{f}_b\mathbf{f}_c = -k\eta^{ac}\left(\boldsymbol{\rho}^{b}_{\ c}\mathbf{f}_b - W^b\mathbf{f}_b\mathbf{f}_c - \frac{1}{2}\eta^{bd}\mathbf{d}^{(y)}\eta_{cd}\mathbf{f}_b\right) \tag{1.3c}$$

$$\frac{1}{2}\Omega^{bc}\mathbf{f}_b\mathbf{f}_c = \mathbf{d}^{(y)}\left(W^a\mathbf{f}_a\right) \tag{1.3d}$$

Appendix 2: Coordinate to orthonormal basis

The Euclidean and Lorentzian metric components are related in the orthonormal basis by:

$$\begin{aligned}\eta_{ab} &= s^2\left(\delta_{ab} - \frac{2}{s^2}s_a s_b\right) \\ \eta^{ab} &= \frac{1}{s^2}\left(\delta^{ab} - \frac{2}{s^2}\delta^{ac}\delta^{bd}s_c s_d\right) \\ \delta_{ab} &= \frac{1}{s^2}\left(\eta_{ab} + 2s_a s_b\right) \\ \delta^{ab} &= s^2\left(\eta^{ab} + 2\eta^{ac}s_c\eta^{ad}s_d\right)\end{aligned}$$

where $s^2 = \delta^{\alpha\beta}s_\alpha s_\beta > 0$.

Appendix 3: Symmetric projection of the derivative of the solder form

For the calculation of the symmetric pieces of the connection, we need to express the symmetric part, $\Xi^{ad}_{cb} h_d^{\ \alpha} \mathbf{d} h_\alpha^{\ c}$, in terms of the metric. Expanding the metric derivatives,

$$\begin{aligned}
k^{\alpha\mu} \mathbf{d} k_{\mu\beta} &= k^{\alpha\mu} \mathbf{d} \left(h_\mu^{\ a} h_\beta^{\ b} \eta_{ab} \right) \\
&= h_c^{\ \alpha} h_d^{\ \mu} \eta^{cd} \left(\mathbf{d} h_\mu^{\ a} h_\beta^{\ b} \eta_{ab} + h_\mu^{\ a} \mathbf{d} h_\beta^{\ b} \eta_{ab} \right) \\
&= h_c^{\ \alpha} h_d^{\ \mu} h_\beta^{\ b} \eta^{cd} \eta_{ab} \mathbf{d} h_\mu^{\ a} + h_b^{\ \alpha} \mathbf{d} h_\beta^{\ b} \\
&= h_c^{\ \alpha} h_\beta^{\ b} \eta^{cd} \eta_{ab} \left(h_d^{\ \mu} \mathbf{d} h_\mu^{\ a} \right) + h_b^{\ \alpha} h_\beta^{\ c} \left(h_c^{\ \mu} \mathbf{d} h_\mu^{\ b} \right) \\
&= 2 h_c^{\ \alpha} h_\beta^{\ b} \Xi^{cd}_{ab} \left(h_d^{\ \mu} \mathbf{d} h_\mu^{\ a} \right)
\end{aligned}$$

so that we can write $\Xi^{ad}_{cb} \left(h_d^{\ \mu} \mathbf{d} h_\mu^{\ c} \right)$ explicitly,

$$\begin{aligned}
\Xi^{ad}_{cb} \left(h_d^{\ \mu} \mathbf{d} h_\mu^{\ c} \right) &= \frac{1}{2} h_\alpha^{\ a} h_b^{\ \beta} k^{\alpha\mu} \mathbf{d} k_{\mu\beta} \\
&= \frac{1}{2} h_\alpha^{\ a} h_b^{\ \beta} k^{\alpha\mu} \mathbf{d} \left(s^2 \delta_{\mu\beta} - 2 s_\mu s_\beta \right) \\
&= h_\alpha^{\ a} h_b^{\ \beta} \frac{1}{s^2} \left(\delta^\alpha_\beta \delta^{\nu\rho} s_\rho - \delta^{\alpha\nu} s_\beta + \delta^\nu_\beta \delta^{\alpha\mu} s_\mu \right) \mathbf{d} s_\nu \\
&= \frac{1}{s^2} \left(\delta^a_b \delta^{cd} s_d - \delta^{ac} s_b + \delta^c_b \delta^{ad} s_d \right) h_c^{\ \nu} \mathbf{d} s_\nu \\
&= -\frac{\left(1 - k\beta^2\right)}{2\beta} \left(\delta^a_b \eta^{cd} s_d + \delta^c_b \eta^{ad} s_d + \eta^{ac} s_b + 2\eta^{af} \eta^{ce} s_b s_e s_f \right) \eta_{cf} \mathbf{e}^f \\
&\quad - \frac{k\left(1 + k\beta^2\right)}{2\beta} \left(\delta^a_b \eta^{cd} s_d + \delta^c_b \eta^{ad} s_d + \eta^{ac} s_b + 2\eta^{af} \eta^{ce} s_b s_e s_f \right) \mathbf{f}_c
\end{aligned}$$

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