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Electron parallel closures for the 3 + 1 fluid model

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Linear closures are obtained for arbitrary collisionality for the 3 + 1 fluid model which includes the evolution of density, flow velocity, and pressure both parallel and perpendicular to a preferred direction, usually a magnetic field. A large set of 6400 moment equations is solved to provide closures that are accurate in the collisional regime and well into the collisionless regime. The closures in the collisionless limit are determined by solving the kinetic equation with a model collision operator. Simple fits for the kernel functions that define the closures are obtained for arbitrary collisionality in wave number space. The results are linearly accurate to within 3% across the entire range of collisionality. *Published by AIP Publishing.* <https://doi.org/10.1063/1.5014996>

I. INTRODUCTION

Capturing kinetic effects in the fluid description of plasmas has been the subject of great interest and has often been discussed using Landau-fluid and gyro-fluid models.^{1–6} Since the Braginskii closure relations are valid for high collisionality, the models focus on determining the effective collisionality in the collisionless limit in order to understand kinetic behavior such as Landau damping. In a recent work, Joseph and Dimits⁷ provided detailed matching between the collisional Braginskii and collisionless Hammett-Perkins regimes for the 3 + 1 Landau fluid model. Heat flux models for intermediate collisionality have also been developed in order to handle the heat transport in a steep temperature gradient.^{8–11} This intermediate collisionality regime is important for describing the edge and scrape-off layer plasmas of interest to magnetic confinement fusion devices. Fluid models for the entire collisionality regime have been developed in the conventional five moment model.^{12–14} Efforts to include the finite-Larmor-radius corrections for collisionless plasmas have been made in Refs. 15–17.

The 3 + 1 Landau model dynamically evolves three moments parallel to the magnetic field, density n , parallel velocity V_{\parallel} , and parallel pressure p^{\parallel} , and one moment perpendicular to the field, perpendicular pressure p^{\perp} . This model allows dynamic evolution of pressure anisotropy, present in many interesting phenomena such as micro-instabilities for the ion and electron temperature gradient modes,³ a relatively strong pressure anisotropy at the edge of a diverted tokamak,¹⁸ and micro-instabilities that relax an equilibrium temperature anisotropy (see, e.g., Ref. 5). Note that, in the conventional five moment model, the parallel viscosity closure (π_{\parallel}) which describes the pressure anisotropy is derived within the stationary and linearized approximations. Thus, by including the dynamic evolution of π_{\parallel} , the 3 + 1 model will provide a more accurate description of the high pressure anisotropy regime.

The goal of this work is to provide a 3 + 1 Landau model for electrons that is accurate for all collisionality regimes. This can be achieved by combining solutions of moment equations and a simplified kinetic equation in wave

number space. We evaluate the collisional effects accurately using exact moments of the Landau-Fokker-Planck operator and the collisionless limit using the method of Ref. 19. The closure scheme is straightforward in the moment hierarchy because the fluid and closure moment equations appear separately. When solving the kinetic equation, we carefully remove the fluid equations from the kinetic equation to obtain closures. While the moment approach has limitations that it requires continually increasing the number of moments in order to approach the collisionless limit, the kinetic approach has limitations in accurately evaluating collision operators for the collisional regime. Combining the moment approach with accurate collision operators in the high to nearly collisionless regime and the kinetic approach in the collisionless limit, we obtain closures for the 3 + 1 model for the entire range of collisionality.

When a system is inhomogeneous along one direction (say \hat{z}) with planar symmetry, the system can be fully described by moment components parallel to the direction. The parallel moment equations can be obtained either by taking parallel component of the general moment equations²⁰ or by taking moments of the parallel kinetic equation.²¹ Similarly, magnetized plasmas can be described by the parallel moments in the leading order of the usual Ω^{-1} expansion where $\hat{z} = \mathbf{B}/B$, \mathbf{B} is the magnetic field, and Ω is the Larmor frequency. In this case, the gyroaveraged distribution function satisfies the parallel moment equation called the drift kinetic equation (Sec. 4.2 of Ref. 22). Since the parallel moments alone describe the leading order only, it may be necessary for more accurate description to calculate higher Ω^{-1} order corrections that include the perpendicular moments. In this work, we focus only on parallel closures for electrons. The closures developed here can be used conveniently without solving the kinetic equation or higher order moment equations in closing electron fluid equations.

In Sec. II, we write the 3 + 1 fluid equations and general moment equations for the closures. The moment equations are then solved to produce closures in wave number space. In Sec. III, we solve a reduced kinetic equation for the 3 + 1 closure model with a Krook-type operator and obtain linearly

exact closures within the collision operator approximation. The solution is used to obtain closures in the collisionless limit. In Sec. IV, we combine the moment closures and the collisionless limit closures in order to obtain general closures for arbitrary collisionality. In Sec. V, we discuss possible extensions of this work for more accurate fluid models.

II. THE 3 + 1 FLUID MODEL AND GENERAL MOMENT EQUATIONS

In this section, we define the fluid and closure moments for the 3 + 1 fluid model. Then we introduce linearized parallel moment equations for *electrons* and solve a truncated set of equations to obtain parallel closures in wave number space. Throughout the paper, the electron species index “e” should be understood unless stated otherwise.

In the 3 + 1 fluid model,^{1,5,7} a fluid system is described by fluid moments $\{n, u, p_{\parallel}, p_{\perp}\}$. They are defined as

$$n = \int d\mathbf{v}f, \quad (1)$$

$$u = n^{-1} \int d\mathbf{v}v_{\parallel}f, \quad (2)$$

$$p^{\parallel} = \int d\mathbf{v}mw_{\parallel}^2f, \quad (3)$$

and

$$p^{\perp} = \int d\mathbf{v}\frac{1}{2}mw_{\perp}^2f, \quad (4)$$

where m is the mass, $u = V_{\parallel} = \hat{z} \cdot \mathbf{V}$, $\mathbf{w} = \mathbf{v} - \mathbf{V}$, and $\mathbf{V} = n^{-1} \int d\mathbf{v}\mathbf{v}f = \mathbf{V}_{\parallel} + \mathbf{V}_{\perp}$ is the flow velocity. Note that a superscript is used for the parallel (\parallel) and perpendicular (\perp) parts of $w^2 = w_{\parallel}^2 + w_{\perp}^2$, while a subscript is used for the \parallel and \perp component of a vector. Their evolution equations are

$$d_t n + n \nabla \cdot \mathbf{V} = 0, \quad (5)$$

$$mnd_t u + \hat{z} \nabla : \mathbf{p} - nqE_{\parallel} = R_{\parallel}, \quad (6)$$

$$d_t p^{\parallel} + p^{\parallel} \nabla \cdot \mathbf{V} + 2\mathbf{p} : \nabla \mathbf{V}_{\parallel} + \nabla \cdot \mathbf{q}^{\parallel} = C_{p^{\parallel}}, \quad (7)$$

$$d_t p^{\perp} + p^{\perp} \nabla \cdot \mathbf{V} + \mathbf{p} : \nabla \mathbf{V}_{\perp} + \nabla \cdot \mathbf{q}^{\perp} = C_{p^{\perp}}, \quad (8)$$

where $d_t = \partial/\partial t + \mathbf{V} \cdot \nabla$,

$$\mathbf{p} = \int d\mathbf{v}m\mathbf{w}\mathbf{w}f, \quad (9)$$

$$\mathbf{q}^{\parallel} = \int d\mathbf{v}mw_{\parallel}^2\mathbf{w}f, \quad (10)$$

$$\mathbf{q}^{\perp} = \int d\mathbf{v}\frac{1}{2}mw_{\perp}^2\mathbf{w}f, \quad (11)$$

$$R_{\parallel} = \int d\mathbf{v}mw_{\parallel}C(f), \quad (12)$$

$$C_{p^{\parallel}} = \int d\mathbf{v}mw_{\parallel}^2C(f), \quad (13)$$

and

$$C_{p^{\perp}} = \int d\mathbf{v}\frac{1}{2}mw_{\perp}^2C(f). \quad (14)$$

For plasmas with planar symmetry, inhomogeneous along \hat{z} direction only, the moments can be written as

$$\mathbf{V} = u\hat{z}, \quad (15)$$

$$\mathbf{p} = p^{\perp}\hat{x}\hat{x} + p^{\perp}\hat{y}\hat{y} + p^{\parallel}\hat{z}\hat{z}, \quad (16)$$

$$\mathbf{q}^{\perp} = q^{\perp}\hat{z}, \quad (17)$$

and

$$\mathbf{q}^{\parallel} = q^{\parallel}\hat{z}. \quad (18)$$

For magnetized plasmas, the leading-order moments in the Ω^{-1} expansion can be written in the same way. Then Eqs. (5)–(8) can be simplified as

$$d_t n + u\partial_{\parallel}n = 0, \quad (19)$$

$$mnd_t u + \partial_{\parallel}p^{\parallel} - nqE_{\parallel} = R_{\parallel}, \quad (20)$$

$$d_t p^{\parallel} + p^{\parallel}\partial_{\parallel}u + 2p^{\parallel}\partial_{\parallel}u + \partial_{\parallel}q^{\parallel} = C_{p^{\parallel}}, \quad (21)$$

$$d_t p^{\perp} + p^{\perp}\partial_{\parallel}u + \partial_{\parallel}q^{\perp} = C_{p^{\perp}}, \quad (22)$$

where $d_t = \partial/\partial t + u\partial_{\parallel}$. This fluid system should be closed by expressing closure variables $\{q^{\parallel}, q^{\perp}, R_{\parallel}, C_{p^{\parallel}}, C_{p^{\perp}}\}$ in terms of fluid variables $\{n, u, p^{\parallel}, p^{\perp}\}$.

For obtaining quantitative closures for arbitrary collisionality, an accurate evaluation of the collision operator is essential. The collisional moments are analytically computed in Refs. 23 and 24. The 3 + 1 fluid moments are related to the conventional moments by

$$p^{\parallel} = p + \pi_{\parallel}, \quad (23)$$

$$p^{\perp} = p - \frac{1}{2}\pi_{\parallel}, \quad (24)$$

$$q^{\parallel} = \frac{6}{5}h_{\parallel} + \sigma_{\parallel}, \quad (25)$$

$$q^{\perp} = \frac{2}{5}h_{\parallel} - \frac{1}{2}\sigma_{\parallel}, \quad (26)$$

where

$$p = \int d\mathbf{v}\frac{1}{3}mw^2f, \quad (27)$$

$$\pi_{\parallel} = \int d\mathbf{v}m\left(w_{\parallel}^2 - \frac{1}{3}w^2\right)f, \quad (28)$$

$$h_{\parallel} = \int d\mathbf{v}\frac{1}{2}mw^2w_{\parallel}f, \quad (29)$$

$$\sigma_{\parallel} = \int d\mathbf{v}\left(w_{\parallel}^3 - \frac{3}{5}w^2w_{\parallel}\right)f. \quad (30)$$

Therefore, the 3 + 1 fluid model $\{n, u, p^{\parallel}, p^{\perp}\}$ is equivalent to the fluid model F = $\{n, u, p, \pi_{\parallel}\}$. Similarly, the 3 + 1 collisional moment closures are related to the conventional collisional moments by

$$C_{p\parallel} = \frac{2}{3}Q + S_{\parallel}, \quad (31)$$

$$C_{p\perp} = \frac{2}{3}Q - \frac{1}{2}S_{\parallel}, \quad (32)$$

where

$$Q = \int d\mathbf{v} \frac{1}{2} m v^2 C(f), \quad (33)$$

and

$$S_{\parallel} = \int d\mathbf{v} m \left(w_{\parallel}^2 - \frac{1}{3} w^2 \right) C(f). \quad (34)$$

Therefore, the closure set $C_{3+1} = \{q^{\parallel}, q^{\perp}, R_{\parallel}, C_{p\parallel}, C_{p\perp}\}$ for the 3+1 model is equivalent to the closure set $C_F = \{h_{\parallel}, \sigma_{\parallel}, R_{\parallel}, Q, S_{\parallel}\}$ for the conventional fluid model F. The fluid equations corresponding to (21) and (22) are

$$\frac{3}{2} d_t p + \frac{5}{2} p \nabla \cdot \mathbf{V} + \partial_{\parallel} h_{\parallel} + \frac{3}{2} \partial_{\parallel} V_{\parallel} \pi_{\parallel} - \frac{1}{2} \pi_{\parallel} \nabla \cdot \mathbf{V} = Q \quad (35)$$

and

$$d_t \pi_{\parallel} + \frac{4}{3} \pi_{\parallel} \nabla \cdot \mathbf{V} + \pi_{\parallel} \partial_{\parallel} V_{\parallel} + \frac{8}{15} \partial_{\parallel} h_{\parallel} + \partial_{\parallel} \sigma_{\parallel} + 2p \partial_{\parallel} V_{\parallel} - \frac{2}{3} p \nabla \cdot \mathbf{V} = S_{\parallel}. \quad (36)$$

We obtain these closures for arbitrary collisionality following the steps of the method of Refs. 13 and 21: (i) solve a system of general moment equations for the collisional to nearly collisionless regime, (ii) solve a reduced kinetic equation for the collisionless limit, and (iii) combine the results in two regimes for closures in arbitrary collisionality. This will be done conveniently in wave number space.

A truncated system of L Legendre polynomials and K associated Laguerre polynomials can be obtained by taking the parallel component of the general moment equations.²⁴ The linearized equations for $jp \notin F$ are

$$\sum_{lk \notin F} \psi^{jp, lk} \frac{\partial n^{lk}}{\partial \eta} = \sum_{lk \notin F} c^{jp, lk} n^{lk} + g^{jp}, \quad (37)$$

where the summation excludes the fluid moments as

$$p, k = \begin{cases} 2, 3, \dots, K+1, & \text{for } l = 0, \\ 1, 2, \dots, K, & \text{for } l = 1, 2, \\ 0, 1, \dots, K-1, & \text{for } l = 3, 4, \dots, L-1, \end{cases}$$

and, as a consequence, the fluid moment terms appear as thermodynamic drives g^{jp} . The same jp moment equation (37) can be obtained by taking \hat{P}^{jp} moment of the parallel kinetic equation

$$\frac{\partial f}{\partial t} + v_{\parallel} \frac{\partial f}{\partial z} + \frac{q}{m} E_{\parallel} \frac{\partial f}{\partial v_{\parallel}} = C(f), \quad (38)$$

with the expansion

$$f = \sum_{lk} \hat{f}_0^M(\mathbf{c}) \hat{P}^{lk}(\mathbf{c}) n^{lk}, \quad (39)$$

$$n^{lk} = \int d\mathbf{v} \hat{P}^{lk}(\mathbf{c}) f, \quad (40)$$

where

$$\hat{f}_0^M = n \hat{f}_0^M = \frac{n}{\pi^{3/2} v_0^3} e^{-c^2}. \quad (41)$$

Although we can define \mathbf{c} with the local flow velocity $\mathbf{V}(t, \mathbf{r})$ and temperature $T(t, \mathbf{r})$, for the linearized equations, it is convenient to define $\mathbf{c} = (\mathbf{v} - \mathbf{V}_0)/v_0$ and $v_0 = \sqrt{2T_0}/v_0$ with constant \mathbf{V}_0 and T_0 . The orthonormal polynomials are

$$\hat{P}^{lk} = \frac{1}{\sqrt{\bar{\sigma}_{lk}}} P^{lk}(\mathbf{c}), \quad P^{lk}(\mathbf{c}) = c^l P_l(c_{\parallel}/c) L^{(l+1/2)}(c^2), \quad (42)$$

with the normalization constant

$$\bar{\sigma}_{lk} = \bar{\sigma}_l \lambda_{lk}, \quad \bar{\sigma}_l = \frac{1}{2l+1}, \quad \lambda_{lk} = \frac{(l+k+1/2)!}{k!(1/2)!}, \quad (43)$$

where P_l is the Legendre polynomial and $L_k^{(l+1/2)}$ is the associated Laguerre polynomial.

The matrix elements for the free-streaming operator are

$$\psi^{jp, lk} = \frac{1}{\bar{\sigma}_j} \int d\mathbf{v} v_{\parallel} \hat{P}^{jp} \partial_{\parallel} \hat{f}_0^M \hat{P}^{lk}, \quad (44)$$

and the matrix elements for the linearized collision operator are

$$c^{jp, lk} = \delta_{jl} c^{jp, lk}, \quad c^{jp, lk} = a_{ee}^{jp, lk} + b_{ee}^{jp, lk} + Z a_{ei}^{jp, lk}, \quad (45)$$

with

$$a_{ab}^{jp, lk} = \frac{\tau_{ab}}{n_a} \int d\mathbf{v} \hat{P}^{jp} C(f_{a0}^M \hat{P}_a^{jk}, f_{b0}^M) = \tau_{ab} \hat{A}_{ab}^{jp, lk}, \quad (46)$$

$$b_{ab}^{jp, lk} = \frac{\tau_{ab}}{n_b} \int d\mathbf{v} \hat{P}^{jp} C(f_{a0}^M, f_{b0}^M \hat{P}_b^{jk}) = \tau_{ab} \hat{B}_{ab}^{jp, lk}. \quad (47)$$

The formulas for matrix elements are presented in Refs. 23 and 24. The electron-electron collision time τ_{ee} has been absorbed into η as $d\eta = dz/v_0 \tau_{ee}$ in Eq. (37). For electrons, the non-vanishing drives are

$$g^{1k} = \delta_{1k} \frac{\sqrt{5} n dT_*}{2 T d\eta} + c^{1k0} \sqrt{2n} \hat{V}_{ei\parallel}, \quad (48)$$

$$g^{2k} = c^{2k0} \frac{\sqrt{3} \pi_{\parallel}}{2T}, \quad (49)$$

$$g^{30} = -\frac{3}{2} \sqrt{\frac{3}{10T}} \frac{1}{d\eta} \frac{d\pi_{\parallel}}{d\eta}, \quad (50)$$

where $\hat{V}_{ei\parallel} = V_{ei\parallel}/v_0$, $V_{ei\parallel} = V_{e\parallel} - V_{i\parallel}$, and

$$\frac{dT_*}{d\eta} = \frac{dT}{d\eta} + \frac{2}{5n} \frac{d\pi_{\parallel}}{d\eta}. \quad (51)$$

Once the general moment equations have been solved, the closures are obtained from the general moments as

$$h_{\parallel} = -\frac{\sqrt{5}}{2} v_T T n^{11}, \quad (52)$$

$$\sigma_{\parallel} = \sqrt{\frac{6}{5}} v_T T n^{30}, \quad (53)$$

$$R_{\parallel} = \frac{m v_T}{\sqrt{2} \tau_{ee}} \sum_{k=0} c^{10k} n^{1k}, \quad (54)$$

$$S_{\parallel} = \frac{2T}{\sqrt{3} \tau_{ee}} \sum_{k=0} c^{20k} n^{2k}. \quad (55)$$

Now we solve a system of differential equation (37) in wave number (k) space

$$\sum_{B=1}^N ik \psi_{AB} \tilde{n}_B = \sum_{B=1}^N c_{AB} \tilde{n}_B + \tilde{g}_B, \quad (56)$$

where the moment indices (j, p) are denoted by a single index $A, B, \dots, N=LK$, and a tilde is used for a Fourier transformed quantity. First, we diagonalize Eq. (56) using the eigensystem of $[\psi^{-1}c]$

$$\sum_B [\psi^{-1}c]_{AB} W_{BC} = k_C W_{AC}. \quad (57)$$

By transforming

$$\tilde{n}_B = \sum_C W_{BC} \tilde{e}_C. \quad (58)$$

Equation (56) can be diagonalized for \tilde{e}_A

$$ik \tilde{e}_B = k_B \tilde{e}_B + \sum_C W_{BC}^{-1} [\psi^{-1} \tilde{g}]_C, \quad (59)$$

from which the solution can be easily obtained

$$\tilde{e}_B = \frac{1}{ik - k_B} \sum_C W_{BC}^{-1} [\psi^{-1} \tilde{g}]_C. \quad (60)$$

The solution of Eq. (56) is

$$\tilde{n}_A = \sum_D \tilde{K}_{AD} \tilde{g}_D, \quad (61)$$

where

$$\tilde{K}_{AD} = \sum_B \gamma_{AD}^B \frac{1}{ik - k_B}, \quad (62)$$

with

$$\gamma_{AD}^B = \sum_C W_{AB} W_{BC}^{-1} \psi_{CD}^{-1}. \quad (63)$$

Note that the inverse Fourier transform of Eq. (61) yields the integral closure in the configuration space

$$n_A(\eta) = \sum_D \int d\eta' K_{AD}(\eta - \eta') g_D(\eta'), \quad (64)$$

by the convolution theorem.

We define \tilde{K}_{AD} and γ_{AD}^B using the closure indices $A, D = h, \sigma, R, S$ instead of the moment indices (l, k) by absorbing some dimensionless factors in Eqs. (48)–(55) into γ_{AD}^B as

$$\begin{aligned} \gamma_{hh}^B &= \frac{5}{2} \gamma_{11,11}^B, \\ \gamma_{h\sigma}^B &= \frac{3}{4} \sqrt{\frac{3}{2}} \gamma_{11,30}^B = \frac{3}{8} \gamma_{\sigma h}^B, \\ \gamma_{hR}^B &= -\sqrt{\frac{5}{2}} \sum_{k=1} \gamma_{11,1k}^B c^{1k0} = \gamma_{Rh}^B, \\ \gamma_{hS}^B &= -\frac{\sqrt{15}}{4} \sum_{k=1} \gamma_{11,2k}^B c^{2k0} = \frac{3}{8} \gamma_{Sh}^B, \\ \gamma_{\sigma\sigma}^B &= \frac{9}{10} \gamma_{30,30}^B, \\ \gamma_{\sigma R}^B &= -2\sqrt{\frac{3}{5}} \sum_{k=1} \gamma_{30,1k}^B c^{1k0} = \frac{8}{3} \gamma_{R\sigma}^B, \\ \gamma_{\sigma S}^B &= \frac{3}{\sqrt{10}} \sum_{k=1} \gamma_{30,2k}^B c^{2k0} = \gamma_{S\sigma}^B, \\ \gamma_{RR}^B &= \sum_{k,q=1} c^{10k} \gamma_{1k,1q}^B c^{1q0}, \\ \gamma_{RS}^B &= \frac{1}{2} \sqrt{\frac{3}{2}} \sum_{k,q=1} c^{10k} \gamma_{1k,2q}^B c^{2q0} = \frac{3}{8} \gamma_{SR}^B, \\ \gamma_{SS}^B &= \sum_{k,q=1} c^{20k} \gamma_{2k,2q}^B c^{2q0}. \end{aligned} \quad (65)$$

The eigenvalues appear in positive and negative pairs, and the corresponding coefficients satisfy $\gamma_{AD}^{-B} = \gamma_{AD}^B$ for $AD = hS, \sigma S, RS \equiv \text{odd}$, and $\gamma_{AD}^{-B} = -\gamma_{AD}^B$ for $AD = hh, h\sigma, hR, \sigma\sigma, \sigma R, RR, RS, SS \equiv \text{even}$, where $-B$ stands for $-k_B$. Then the kernel (62) can be written as

$$\tilde{K}_{AD} = \begin{cases} \sum_B \frac{-\gamma_{AD}^B k_B}{k^2 + k_B^2} = \hat{K}_{AD}, & \text{for even } AD, \\ -i \sum_B \frac{\gamma_{AD}^B k}{k^2 + k_B^2} = -i \hat{K}_{AD}, & \text{for odd } AD, \end{cases} \quad (66)$$

where \hat{K}_{AD} is real. Finally, we write the closure relations

$$\tilde{h}_{\parallel} = -\frac{1}{2} \hat{h}_h n_0 v_0 i \tilde{T}_* + \hat{h}_\sigma v_T i \tilde{\pi}_{\parallel} + \hat{h}_R p_0 V_{ei\parallel} + \hat{h}_S \tilde{\pi}_{\parallel}, \quad (67)$$

$$\tilde{\sigma}_{\parallel} = \frac{4}{3} \hat{\sigma}_\sigma n_0 v_0 i \tilde{T}_* - \hat{\sigma}_\sigma v_0 i \tilde{\pi}_{\parallel} + \hat{\sigma}_R p_0 V_{ei\parallel} + \hat{\sigma}_S v_0 \tilde{\pi}_{\parallel}, \quad (68)$$

$$\tilde{R}_{\parallel} = -\hat{h}_R \frac{2\pi n_0}{\lambda} i \tilde{T}_* - \frac{3}{4} \hat{\sigma}_R \frac{2\pi}{\lambda} i \tilde{\pi}_{\parallel} - \hat{R}_R \frac{nm}{\tau_{ee}} \tilde{V}_{ei\parallel} + \frac{2\hat{R}_S}{\tau_{ee} v_0} \tilde{\pi}_{\parallel}, \quad (69)$$

$$\begin{aligned} \tilde{S}_{\parallel} &= -\frac{4}{3} \hat{h}_S \frac{2\pi n_0 v_0}{\lambda} i \tilde{T}_* - \hat{\sigma}_S \frac{2\pi v_0}{\lambda} i \tilde{\pi}_{\parallel} + \frac{8}{3} \hat{R}_S \frac{p_0}{\tau_{ee}} \frac{\tilde{V}_{ei\parallel}}{v_0} \\ &\quad - \frac{\hat{S}_S}{\tau_{ee}} \tilde{\pi}_{\parallel}, \end{aligned} \quad (70)$$

where the dimensionless closures ($\hat{A}_B, A, B = h, \sigma, R, S$) are defined to be positive as

$$\begin{aligned}
\hat{h}_h &= k\tilde{K}_{hh}, & \hat{h}_\sigma &= k\tilde{K}_{h\sigma}, & \hat{h}_R &= \tilde{K}_{hR}, & \hat{h}_S &= i\tilde{K}_{hS}, \\
\hat{\sigma}_\sigma &= k\tilde{K}_{\sigma\sigma}, & \hat{\sigma}_R &= \tilde{K}_{\sigma R}, & \hat{\sigma}_S &= i\tilde{K}_{\sigma S}, \\
\hat{R}_R &= 1 - \tilde{K}_{RR}, & \hat{R}_S &= i\tilde{K}_{RS}, \\
\hat{S}_S &= 2.05 - \tilde{K}_{SS}.
\end{aligned} \tag{71}$$

In the collisional limit, the closure relations become for the vector moments

$$\tilde{h}_\parallel = -3.20 \frac{p_0}{mv_0} ik\tilde{T}_* + 0.703p_0\tilde{V}_{ei\parallel}, \tag{72}$$

$$\tilde{R}_\parallel = -0.703 \frac{n_0}{\tau_{ee}v_0} ik\tilde{T}_* - 0.506 \frac{m_0n_0}{\tau_{ee}} \tilde{V}_{ei\parallel}, \tag{73}$$

the (2,0) friction, from the 6×6 calculation

$$\tilde{S}_\parallel = -1.36 \frac{1}{\tau_{ee}} \tilde{\pi}_\parallel, \tag{74}$$

and the (3,0) moment

$$\tilde{\sigma}_\parallel = -0.491v_0 ik\tilde{\pi}_\parallel. \tag{75}$$

The coefficients agree with the collisional theory.

The closures obtained from a truncated system of N moment equations are valid up to finite k . Increasing $N = 100, 400, 1600, 6400$, the convergence is checked (see Fig. 1). For $N=6400$, the closures are accurate for $k \lesssim 100$ but they involve 6400 terms as seen in Eq. (66). Instead of increasing N for lower collisionality, we solve the kinetic equation with a Krook-type collision operator to find closures in the collisionless limit.

III. LINEARLY EXACT CLOSURES WITH A KROOK-TYPE OPERATOR

In this section, we use the method of Ref. 19 to derive linearly exact closures for the $3+1$ model. We adopt a Krook type operator which conserves particle, momentum, and energy. Although the Krook-type operator is a crude approximation for the collisional regime, the final results in the collisionless limit are the same as those obtained from the exact Landau-Fokker-Planck operator, reproducing the exact linear response function.

To obtain closures, we decompose the distribution function into the fluid and the closure parts, $f = f^F + f^C$ and solve the kinetic equation for f^C

$$\frac{\partial f}{\partial t} + v_\parallel \partial_\parallel f + \frac{q}{m} E_\parallel \frac{\partial f}{\partial v_\parallel} = -\nu(f - f^M), \tag{76}$$

where f^M is the Maxwellian distribution. Since the solution f^C will be expressed in terms of f^F , taking closure moments of f^C will connect the closures to the fluid moments of f^F , yielding closure relations. For the fluid model $F = \{n, u, p, \pi_\parallel\}$, the fluid part of the distribution is, up to first order in perturbed fluid moments (denoted by the superscript 1)

$$f^F = f_0 + f_1^M + f_1^\pi, \tag{77}$$

where

$$f_0 = n_0 \hat{f}_0, \quad \hat{f}_0 = \frac{1}{\pi^{3/2} v_0^3} e^{-s^2}, \tag{78}$$

$$f_1^M = \left(\frac{n_1}{n_0} + 2 \frac{u_1}{v_0} P^{10} - \frac{T_1}{T_0} P^{01} \right) \hat{f}_0, \tag{79}$$

and

$$f_1^\pi = \frac{\pi_\parallel}{p_0} P^{20} \hat{f}_0. \tag{80}$$

The several lowest orders of orthogonal polynomials are

$$\begin{aligned}
P^{01} &= \frac{3}{2} - s^2, & P^{02} &= \frac{1}{8}(s^4 - 20s^2 + 15), \\
P^{10} &= s_\parallel, & P^{11} &= s_\parallel \left(\frac{5}{2} - s^2 \right), \\
P^{20} &= s_\parallel^2 - \frac{1}{2} s_\perp^2, & P^{30} &= s_\parallel^3 - \frac{3}{2} s_\parallel s_\perp^2,
\end{aligned} \tag{81}$$

where $s = v/v_0$, $v_0 = \sqrt{2T_0/m}$, P_l are Legendre polynomials, and $L_k^{(l+1/2)}$ are associated Laguerre polynomials.

The linearized kinetic equation for f_1^C is

$$\begin{aligned}
\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z} + \nu \right) f_1^C &= -\nu f_1^\pi - \left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z} \right) (f_1^M + f_1^\pi) \\
&+ \frac{q}{m} E_\parallel \frac{\partial}{\partial v_\parallel} f_0.
\end{aligned} \tag{82}$$

Taking $\{P^{00}, mv_0 P^{10}, -T_0 P^{01}, \frac{4}{3} T_0 P^{20}\}$ moments of Eq. (82) produces linearized moment equations for $F = \{n, u, T, \pi_\parallel\}$

$$0 = -\frac{\partial n_1}{\partial t} - n_0 \frac{\partial u}{\partial z}, \tag{83}$$

$$0 = -mn_0 \frac{\partial u}{\partial t} - n_0 \frac{\partial T_1}{\partial z} - T_0 \frac{\partial n_1}{\partial z} - \frac{\partial \pi_\parallel}{\partial z}, \tag{84}$$

$$\frac{\partial h_\parallel}{\partial z} = -\frac{3}{2} n_0 \frac{\partial T_1}{\partial t} - p_0 \frac{\partial u}{\partial z}, \tag{85}$$

$$\frac{\partial \sigma_\parallel}{\partial z} + \frac{8}{15} \frac{\partial h_\parallel}{\partial z} = -\frac{\partial \pi_\parallel}{\partial t} - \frac{4}{3} p_0 \partial_\parallel u - \nu \pi_\parallel, \tag{86}$$

where we have used $\int d\mathbf{v} \{P^{00}, P^{10}, P^{01}, P^{20}\} f^C = \{0, 0, 0, 0\}$. Now we eliminate these fluid equations from the kinetic equation (82) by replacing the time derivative terms with Eqs. (83)–(86). Then the kinetic equation to be solved becomes

$$\begin{aligned}
&\left(\frac{\partial}{\partial t} + v_\parallel \frac{\partial}{\partial z} + \nu \right) f_1^C \\
&= \left[-\frac{2}{3T_0} \frac{\partial h_\parallel}{\partial z} P^{01} + \frac{1}{T_0} \left(\frac{\partial \sigma_\parallel}{\partial z} + \frac{8}{15} \frac{\partial h_\parallel}{\partial z} \right) P^{20} \right. \\
&\quad \left. + \frac{n_0 v_0}{T_0} \frac{\partial T_1}{\partial z} P^{11} - \frac{3v_0}{5T_0} \frac{\partial \pi_\parallel}{\partial z} P^{30} \right] \hat{f}_0,
\end{aligned} \tag{87}$$

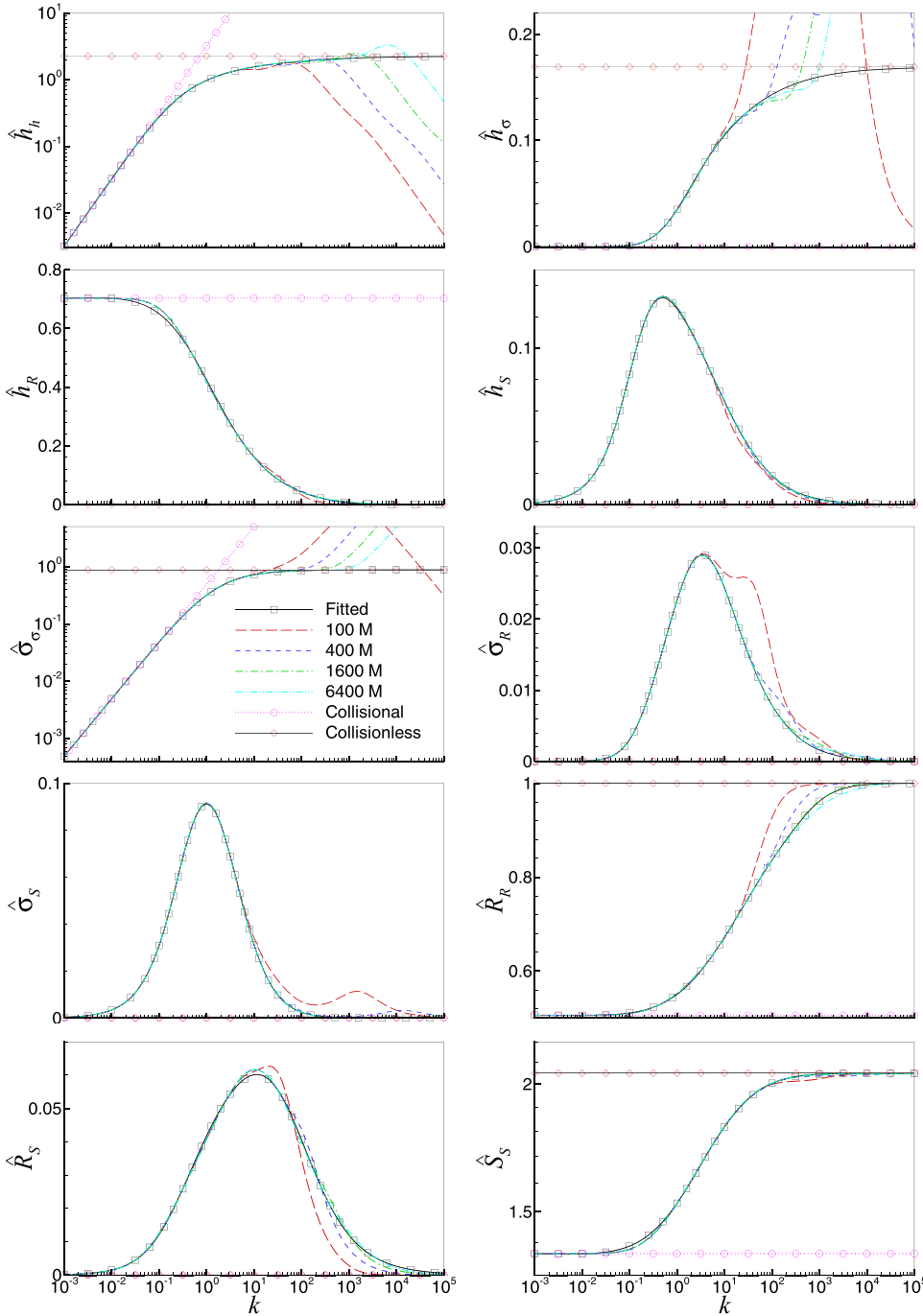


FIG. 1. Closures for $Z=1$ computed from $N=100$ (red, long-dashed), 400 (blue, short-dashed), 1600 (green, dashed-dotted), and 6400 (cyan, dashed-dotted-dotted). The fitted closures (black, solid with squares) are also shown. The collisional [magenta, dotted (with circles)] and collisionless [gray, thin solid (with diamonds)] closures are also shown.

where

$$T_* = T_1 + \frac{2}{5n_0} \pi_{\parallel}. \quad (88)$$

Note that the $\{P^{00}, P^{10}, P^{01}, P^{20}\}$ moment equations of Eq. (87) are trivially satisfied, which verifies that the fluid equations have been eliminated from the kinetic equation.

We perform a Fourier transform of Eq. (87): $A(t, z) \rightarrow \tilde{A}(\omega, k)$ for a function A , $\partial/\partial t \rightarrow -i\omega$, and $\partial/\partial z \rightarrow ik$, and write the solution in k -space as

$$\tilde{f}_1^C = \frac{1}{ikv_0 s_{\parallel} - \zeta} \tilde{g}^C, \quad (89)$$

where \tilde{g}^C is the Fourier transform of the right hand side of Eq. (87) and

$$\zeta = \frac{\omega + i\nu}{kv_0}. \quad (90)$$

Next, we take moments using $\tilde{h}_{\parallel} = -v_0 T_0 \int d\nu P^{11} \tilde{f}_1^C$ and $\tilde{\sigma}_{\parallel} = \frac{4}{5} v_0 T_0 \int d\nu P^{30} \tilde{f}_1^C$ to obtain

$$\begin{pmatrix} \tilde{h}_{\parallel} \\ \tilde{\sigma}_{\parallel} \end{pmatrix} = \frac{1}{\Delta} \begin{pmatrix} S_{hh} & S_{h\sigma} \\ S_{\sigma h} & S_{\sigma\sigma} \end{pmatrix} \begin{pmatrix} \frac{1}{2} n_0 v_0 \tilde{T}_* \\ v_0 \tilde{\pi}_{\parallel} \end{pmatrix}, \quad (91)$$

where

$$\begin{aligned}\Delta &= (1 - 2\zeta^2)Z^2 - 2\zeta Z, \\ S_{hh} &= (5\zeta - 6\zeta^3)Z^2 + (4 - 10\zeta^2)Z - 4\zeta, \\ S_{h\sigma} &= \frac{6}{5}\zeta^3 Z^2 + \left(3\zeta^2 - \frac{3}{10}\right)Z + \frac{9}{5}\zeta, \\ S_{\sigma h} &= \frac{8}{3}S_{h\sigma}, \\ S_{\sigma\sigma} &= \zeta\left(\frac{9}{5} - \frac{66}{25}\zeta^2\right)Z^2 - \left(\frac{24}{5}\zeta^2 + \frac{39}{25}\right)Z - \frac{54}{25}\zeta,\end{aligned}\quad (92)$$

and Z is the plasma dispersion function.

The results have been obtained by setting $V_{0\parallel} = u_0 = 0$ for simplicity. Since the u_0 term appears in the form of $u_0\partial/\partial z$, it can be combined with $\partial/\partial t$. Therefore, the closure relations for $u_0 \neq 0$ can be obtained by the replacement $\omega \rightarrow \omega - u_0k$, that is, by defining $\zeta = (\omega - ku_0 + i\nu)/kv_0$ in Eq. (92). It also should be emphasized that $k = 2\pi/\lambda$ is the wave number in this section while $k = 2\pi\lambda_C/\lambda$ is the dimensionless wave number (normalized by the collision length) in other sections.

The closure relations are linearly exact and should reproduce the linear kinetic response

$$\tilde{n}_1 = -\frac{n_0q}{T_0}(1 + \zeta Z)\tilde{\phi}_1. \quad (93)$$

Using Eq. (91) for \tilde{h}_\parallel and $\tilde{\sigma}_\parallel$ in Fourier transform of Eqs. (83)–(86)

$$0 = \zeta\tilde{n}_1 - n_0\frac{\tilde{u}}{v_0}, \quad (94)$$

$$0 = 2p_0\zeta\frac{\tilde{u}}{v_0} - n_0\tilde{T}_1 - T_0\tilde{n}_1 - \tilde{\pi}_\parallel - n_0q\tilde{\phi}_1, \quad (95)$$

$$\tilde{h}_\parallel = \frac{3}{2}n_0v_0\zeta\tilde{T}_1 - p_0\tilde{u}, \quad (96)$$

$$\tilde{\sigma}_\parallel + \frac{8}{15}\tilde{h}_\parallel = \zeta v_0\tilde{\pi}_\parallel - \frac{4}{3}p_0\tilde{u}, \quad (97)$$

one can derive Eq. (93). This confirms that the closures (91) are linearly exact.

In the stationary (closure ordering) and collisionless limits ($\zeta \rightarrow 0$)

$$\begin{pmatrix} \tilde{h}_\parallel \\ \tilde{\sigma}_\parallel \end{pmatrix} = \frac{-i}{\sqrt{\pi}}\frac{k}{|k|} \begin{pmatrix} 4 & -\frac{3}{10} \\ -\frac{4}{5} & \frac{39}{25} \end{pmatrix} \begin{pmatrix} \frac{1}{2}n_0v_0\tilde{T}_* \\ v_0\tilde{\pi}_\parallel \end{pmatrix}, \quad (98)$$

or

$$\tilde{h}_\parallel = \frac{-i}{\sqrt{\pi}}\frac{k}{|k|} \left(2n_0v_0\tilde{T}_1 + \frac{v_0}{2}\tilde{\pi}_\parallel\right), \quad (99)$$

$$\tilde{\sigma}_\parallel = \frac{-i}{\sqrt{\pi}}\frac{k}{|k|} \left(-\frac{2}{5}n_0v_0\tilde{T}_1 + \frac{7v_0}{5}\tilde{\pi}_\parallel\right), \quad (100)$$

where we have used Eq. (88) to replace \tilde{T}_* with \tilde{T}_1 and $\tilde{\pi}_\parallel$. Finally, we compare the results with those in Ref. 7. Using

the relations (25) and (26), $\tilde{T}_1 = \frac{1}{3}(\tilde{T}^\parallel + 2\tilde{T}^\perp)$, and $\tilde{\pi} = n_0(\tilde{T}^\parallel - \tilde{T}^\perp)$, we can reproduce the same results of Ref. 7

$$\tilde{q}^\parallel = \frac{-i}{\sqrt{\pi}}\frac{k}{|k|}2n_0v_0\tilde{T}^\parallel, \quad (101)$$

$$\tilde{q}^\perp = \frac{-i}{\sqrt{\pi}}\frac{k}{|k|}n_0v_0\tilde{T}^\perp. \quad (102)$$

IV. FITTED KERNELS FOR ARBITRARY COLLISIONALITY

The closures obtained from the 6400 moment solution are accurate for $k \leq 100$. All kernel functions can be fitted to the following kernel functions with high accuracy, within $\sim 2.6\%$ error for $k \leq 100$:

$$\tilde{K}_{AB} = \frac{ak^\alpha}{1 + d_1k^\delta + d_2k^{2\delta} + d_3k^{3\delta} + d_4k^{4\delta} + d_5k^{5\delta} + d_6k^{6\delta}}, \quad (103)$$

where fitted parameters are given in Table I. The parameters a and α can be determined by the collisional limit. The integer power α is the difference of tensor ranks (Legendre orders) between the closure A and thermodynamic drive B , $l(A) - l(B)$, where $l(h) = h(R) = 1$, $l(S) = 2$, and $l(\sigma) = 3$. This is the consequence of the Chapman-Enskog theory for a small Knudsen number (high collisionality). For $\alpha = 0$, the coefficient a can be determined precisely from the collisional coefficients. For $\alpha > 0$, a is determined by extrapolation for $k \ll 1$. The parameter δ can be determined from the collisionless response, $6\delta = \alpha + 1$. The parameter d_6 can be exactly determined from the collisionless theory for no friction indices R and S . For friction related kernels, d_6 is determined by extrapolation for $k \geq 100$. Although the parameters determined by extrapolation may be inaccurate, the kernel values are very small in the extrapolation regimes and the errors can be ignored. Other parameters are fitted to produce convergent closure values for $0.01 \leq k \leq 100$ from the 6400 moment calculations. Therefore, the kernels are quite accurate in practice for the entire collisionality regime.

V. DISCUSSION

For the 3 + 1 fluid model, we have derived parallel electron closures for arbitrary collisionality. The closure relations are highly accurate in the stationary and linear response limits. They are represented as functions of the normalized wave number and can be implemented using a variety of numerical techniques. For example, these closures can be implemented in the BOUT++ code using the fast non-Fourier method.²⁵

The 3 + 1 model is accurate in dealing with $d_t\pi_\parallel$ and nonlinear interactions between π_\parallel and $\partial_\parallel u$ while they are ignored in obtaining π_\parallel closure for the standard fluid model $\{n, u, T\}$. Therefore, the 3 + 1 fluid model can describe the regime of high pressure anisotropy $p^\parallel - p^\perp$ more accurately. The formalism introduced in this work can be extended to develop a hierarchy of fluid models with increasing numbers of moments with corresponding closures. The next level of

TABLE I. Fitted parameters in Eq. (103). The maximum error occurs for a very narrow range of k for each kernel, mostly where the closure values are very small. The maximum errors for $\tilde{R}_R = 1 - \tilde{K}_{RR}$ and $\tilde{S}_S = 2.05 - \tilde{K}_{SS}$ are in the parentheses. Note that the odd kernel functions in configuration space are pure imaginary and odd functions in wave number space.

	a	α	δ	d_1	d_2	d_3	d_4	d_5	d_6	err. (%)
\tilde{K}_{hh}	3.20	0	1/6	0.675	-2.89	2.41	0.677	-0.0249	1.42	2.5
$\tilde{K}_{h\sigma}$	2.03	2	1/2	0.124	-3.18	31.2	-5.18	21.9	12.0	2.0
\tilde{K}_{hR}	0.703	0	1/6	0.00	0.733	-4.18	7.01	-3.62	0.713	1.5
$i\tilde{K}_{hS}$	1.18	1	1/3	1.40	-7.90	15.8	-3.53	2.55	0.111	0.9
$\tilde{K}_{\sigma\sigma}$	0.491	0	1/6	0.164	-0.721	0.653	-0.115	0.00	0.558	1.6
$\tilde{K}_{\sigma R}$	0.450	2	1/2	-0.735	3.09	12.32	-0.00147	4.017	0.156	1.1
$i\tilde{K}_{\sigma S}$	0.336	1	1/2	-0.536	2.45	0.155	0.534	0.0798	0.00	1.6
\tilde{K}_{RR}	0.494	0	1/6	0.00	0.114	-0.606	0.956	-0.4386	0.0730	0.5(0.4)
$i\tilde{K}_{RS}$	0.130	1	1/3	-0.979	1.88	1.29	-0.193	0.113	0.00	2.6
\tilde{K}_{SS}	0.685	0	1/6	0.00	0.258	-1.55	2.76	-1.57	0.417	1.7(0.8)

Landau fluid model will be the $4+2$ model^{5,7} which includes q^{\parallel} and q^{\perp} , equivalently h_{\parallel} and σ_{\parallel} in the conventional fluid model, and the next task would be to develop $4+2$ closures that are accurate in the linear and stationary approximations. In the $4+2$ model, the time-dependence of q^{\parallel} and q^{\perp} (h_{\parallel} and σ_{\parallel}) and the nonlinear effects of couplings between the additional fluid variables and closures will be treated accurately. Thus, extended fluid models such as $3+1$ and $4+2$ may prove to be sufficiently accurate to be used as a substitute for a fully kinetic model.

For magnetized plasmas, the parallel dynamics of the $3+1$ fluid system can be described by Eqs. (19)–(22) in the leading order of the Ω^{-1} expansion. For higher order corrections, the perpendicular closures are required in Eqs. (5)–(8) in addition to the parallel closures obtained in this work. The perpendicular flow velocity \mathbf{V}_{\perp} can be either evolved by the fluid equation or closed by the Ω^{-1} expansion method (see, e.g., Sec. 6.3 of Ref. 22). For general moments, a method of obtaining perpendicular closures has been developed in Ref. 20. Finally, although the parallel direction \hat{z} may refer to a full magnetic field \mathbf{B} , it is often convenient to define \hat{z} as the direction of the background magnetic field \mathbf{B}_0 in practical applications such as in turbulence simulations.^{26,27}

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