

Weyl geometry

James T. Wheeler*

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Abstract

We develop the properties of Weyl geometry, beginning with a review of the conformal properties of Riemannian spacetimes. Decomposition of the Riemann curvature into trace and traceless parts allows an easy proof that the Weyl curvature tensor is the conformally invariant part of the Riemann curvature, and shows the explicit change in the Ricci and Schouten tensors required to insure conformal invariance. We include a proof of the well-known condition for the existence of a conformal transformation to a Ricci-flat spacetime. We generalize this to a derivation of the condition for the existence of a conformal transformation to a spacetime satisfying the Einstein equation with matter sources. Then, enlarging the symmetry from Poincaré to Weyl, we develop the Cartan structure equations of Weyl geometry, the form of the curvature tensor and its relationship to the Riemann curvature of the corresponding Riemannian geometry. We present a simple theory of Weyl-covariant gravity based on a curvature-linear action, and show that it is conformally equivalent to general relativity. This theory is invariant under local dilatations, but not the full conformal group.

1 Introduction

In 1918, H. Weyl introduced an additional symmetry into Riemannian geometry in an attempt to unify electromagnetism with gravity as a fully geometric model [1, 2]. The idea was to allow both the orientation and the length of vectors to vary under parallel transport, instead of just the orientation as in Riemannian geometry. The resulting *geometries* are called Weyl geometries, and they form a completely consistent generalization of Riemannian geometries. However, Weyl’s attempt to identify the vector part of the connection associated with stretching and contraction with the vector potential of electromagnetism failed. As Einstein pointed out immediately following Weyl’s first paper on the subject [3], the identification implies that identical atoms which move in such a way as to enclose some electromagnetic flux would be different sizes after the motion. Different sized atoms would have different spectra, and it is easy to show that change in frequency resulting from the size change would be vastly inconsistent with the known precision of spectral lines.

Many attempts were made to patch up the theory. In the end, following some notable work by London [4], Weyl showed that a satisfactory theory of electromagnetism is achieved if the scale factor is replaced by a complex phase. This is the origin of $U(1)$ gauge theory. Many interesting details are discussed in O’Raifeartaigh [5], and a basic introduction to Weyl geometry is given in [6].

In modern language, the new vector part of the connection introduced by Weyl is the dilatational gauge vector, often called the Weyl vector. When this vector is given by the gradient of a function, then there exists a scale transformation (understandable as a change of units, or a dilatation) that sets the vector to zero. When this is possible, the Weyl geometry is called *integrable*: vectors parallel transported about closed paths return with their lengths unaltered. Integrable Weyl geometries are trivial in the sense that there exists a subclass of global gauges in which the geometry is Riemannian.

While Weyl’s theory of electromagnetism fails, Weyl geometry does not. Indeed, although no new physical predictions have emerged directly from its use, there are at least the following three considerations for seeking a deeper understanding of general relativity formulated within an integrable Weyl geometry:

*Utah State University Dept of Physics email: jim.wheeler@usu.edu

1. General relativity is naturally invariant under global changes of units. By formulating general relativity in an integrable Weyl geometry, this scale invariance becomes local. We refer to this generalization as *scale invariant general relativity*. As soon as we make a definition of a fundamental standard of length – for example, as the distance light travels in one second¹ – scale invariant general relativity reduces to general relativity.
2. In [7], Ehlers, Pirani and Schild make the following argument. First, the paths of light pulses may be used to determine a conformal connection on spacetime. Then, a projective connection is found by tracing trajectories of massive test particles (“dust”). Finally, requiring the two connections to approach one another in the limit of near-light velocities reduces the possible connection to that of a Weyl geometry. When this program is carried out with mathematical precision [8], the resulting geometry is an *integrable Weyl geometry*.
3. Deeper physical interest in Weyl geometry also arises in higher symmetry approaches to gravity. Gravitational theories based on the full conformal group ([1],[9]-[19]) often yield general relativity formulated on an integrable Weyl geometry rather than on a Riemannian one and are therefore equivalent to general relativity while providing additional natural structures.

For these reasons, it is useful to recognize the typical forms and meaning of the connection and curvature of Weyl geometry.

Here we use the techniques of modern gauge theory [20, 21] to develop the properties of Weyl geometry, beginning in the next section with a review of the conformal properties of Riemannian spacetimes. Decomposition of the Riemann curvature into trace and traceless parts allows an easy proof that the Weyl curvature tensor is the conformally invariant part of the Riemann curvature, and shows the explicit change in the Ricci and Schouten tensors required to insure conformal invariance. We include a proof of the well-known condition for the existence of a conformal transformation to a Ricci-flat spacetime, and generalize this to a derivation of the condition for the existence of a conformal transformation to a spacetime satisfying the Einstein equation with matter sources. Then, in the final section, we enlarge the symmetry from Poincaré to Weyl to develop the Cartan structure equations of Weyl geometry, the form of the curvature tensor, and its relationship to the Riemann curvature of the corresponding Riemannian geometry. We conclude with a simple theory of Weyl-covariant gravity based on a curvature-linear action, and show that its vacuum solutions are conformal equivalence classes of Ricci-flat metrics in an integrable Weyl geometry. This theory is invariant under dilatations, but not the full conformal group.

2 Conformal transformations in Riemannian geometry

2.1 Structure equations for Riemannian geometry

The Cartan structure equations of a Riemannian geometry are

$$\mathbf{R}_b^a = \mathbf{d}\alpha_b^a - \alpha_b^c \wedge \alpha_c^a \quad (1)$$

$$0 = \mathbf{d}e^a - e^b \wedge \alpha_b^a \quad (2)$$

where the solder form, $e^a = e_\mu^a \mathbf{d}x^\mu$, provides an orthonormal basis, α_b^a is the spin connection 1-form, and the curvature 2-form is $\mathbf{R}_b^a = \frac{1}{2}R_{bcd}^a e^c \wedge e^d$. Differential forms are written in boldface.

The structure equations satisfy integrability conditions, the Bianchi identities, found by applying the

¹The second is currently defined as the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom [physics.nist.gov]. The metre is defined as the length of the path travelled by light in a vacuum in $\frac{1}{299792458}$ second [17th General Conference on Weights and Measures (1983), Resolution 1].

Poincaré lemma, $\mathbf{d}^2 \equiv 0$:

$$\begin{aligned}
0 \equiv \mathbf{d}^2 \alpha_b^a &= \mathbf{d}(\alpha_b^c \wedge \alpha_c^a + \mathbf{R}_b^a) \\
&= \mathbf{d}\alpha_b^c \wedge \alpha_c^a - \alpha_b^c \wedge \mathbf{d}\alpha_c^a + \mathbf{d}\mathbf{R}_b^a \\
&= (\alpha_b^e \wedge \alpha_e^c + \mathbf{R}_b^c) \wedge \alpha_c^a - \alpha_b^c \wedge (\alpha_c^e \wedge \alpha_e^a + \mathbf{R}_c^a) + \mathbf{d}\mathbf{R}_b^a \\
&= \mathbf{d}\mathbf{R}_b^a + \mathbf{R}_b^c \wedge \alpha_c^a - \alpha_b^c \wedge \mathbf{R}_c^a \\
&\equiv \mathbf{D}\mathbf{R}_b^a \\
0 \equiv \mathbf{d}^2 \mathbf{e}^a &= \mathbf{d}(\mathbf{e}^b \wedge \alpha_b^a) \\
&= \mathbf{d}\mathbf{e}^b \wedge \alpha_b^a - \mathbf{e}^b \wedge \mathbf{d}\alpha_b^a \\
&= (\mathbf{e}^c \wedge \alpha_c^b) \wedge \alpha_b^a - \mathbf{e}^b \wedge (\alpha_b^c \wedge \alpha_c^a + \mathbf{R}_b^a) \\
&= -\mathbf{e}^b \wedge \mathbf{R}_b^a
\end{aligned}$$

In components, these take the familiar forms

$$\begin{aligned}
R^a_{b[cd;e]} &= 0 \\
R^a_{[bcd]} &= 0
\end{aligned}$$

Here we use Greek and Latin indices to distinguish different vector bases. Use of the covariantly constant coefficient matrix of the solder form, e_μ^a , allows us to convert freely between orthonormal components (Latin indices) and coordinate components (Greek indices), $R^\alpha_{\beta\mu\nu} = e_a^\alpha e_\beta^b e_\mu^c e_\nu^d R^a_{bcd}$.

2.2 Conformal transformation of the metric, solder form, and connection

A conformal transformation of the metric is the transformation

$$g_{\mu\nu} \rightarrow \tilde{g}_{\mu\nu} = e^{2\phi} g_{\mu\nu} \quad (3)$$

This is not an invariance of Riemannian geometry, but it is an invariance of Weyl geometry. If we make a change of this type in a Riemannian geometry, the solder form changes by

$$\mathbf{e}^a \rightarrow \tilde{\mathbf{e}}^a = e^\phi \mathbf{e}^a \quad (4)$$

since the solder form and metric are related via the orthonormal metric, $\eta_{ab} = \text{diag}(-1, 1, 1, 1)$ by

$$g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$$

The corresponding structure equation then gives the altered form of the metric compatible connection,

$$\begin{aligned}
\mathbf{d}\tilde{\mathbf{e}}^a &= \tilde{\mathbf{e}}^b \wedge \tilde{\alpha}_b^a \\
\mathbf{d}(e^\phi \mathbf{e}^a) &= e^\phi \mathbf{e}^b \wedge \tilde{\alpha}_b^a \\
e^\phi (\mathbf{d}\mathbf{e}^a + \mathbf{d}\phi \wedge \mathbf{e}^a) &= e^\phi \mathbf{e}^b \wedge \tilde{\alpha}_b^a
\end{aligned}$$

so to find the new spin connection we must solve

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \tilde{\alpha}_b^a - \mathbf{d}\phi \wedge \mathbf{e}^a \quad (5)$$

Since the spin connection is antisymmetric, $\tilde{\alpha}_b^a = -\eta^{ad}\eta_{bc}\tilde{\alpha}_d^c$, this is solved by setting

$$\tilde{\alpha}_b^a = \alpha_b^a + 2\Delta_{db}^{ac} e_c^\mu \partial_\mu \phi \mathbf{e}^d \quad (6)$$

where e_c^μ is inverse to e_μ^a . The convenient symbol Δ_{db}^{ac} is defined as

$$\Delta_{db}^{ac} \equiv \frac{1}{2} (\delta_d^a \delta_b^c - \eta^{ac} \eta_{bd}) \quad (7)$$

This is meant to act on any $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor according to

$$\begin{aligned}\Delta_{db}^{ac} T_c^d &\equiv \frac{1}{2} (\delta_d^a \delta_b^c - \eta^{ac} \eta_{bd}) T_c^d \\ &= \frac{1}{2} (T_b^a - \eta^{ac} \eta_{bd} T_c^d) \\ &= \frac{1}{2} \eta^{ac} (T_{cb} - T_{bc})\end{aligned}$$

which is just an antisymmetric $\begin{pmatrix} 0 \\ 2 \end{pmatrix}$ tensor with the first index raised to give a $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ tensor. It is a projection since it is idempotent,

$$\begin{aligned}\Delta_{db}^{ac} \Delta_{cf}^{ed} &= \frac{1}{4} (\delta_d^a \delta_b^c - \eta^{ac} \eta_{bd}) (\delta_c^e \delta_f^d - \eta^{ed} \eta_{cf}) \\ &= \frac{1}{4} (\delta_b^e \delta_f^a - \eta^{ae} \eta_{bf} - \eta^{ea} \eta_{bf} + \delta_f^a \delta_b^e) \\ &= \Delta_{fb}^{ae}\end{aligned}$$

Returning to check eq.(6) in eq.(5), we have,

$$\begin{aligned}\mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \tilde{\alpha}_b^a - \mathbf{d}\phi \wedge \mathbf{e}^a \\ &= \mathbf{e}^b \wedge (\alpha_b^a + 2\Delta_{db}^{ac} e_c^\mu \partial_\mu \phi \mathbf{e}^d) - \mathbf{d}\phi \wedge \mathbf{e}^a \\ &= \mathbf{e}^b \wedge \alpha_b^a + (\delta_d^a \delta_b^c - \eta^{ac} \eta_{bd}) e_c^\mu \partial_\mu \phi \mathbf{e}^b \wedge \mathbf{e}^d - \mathbf{d}\phi \wedge \mathbf{e}^a \\ &= \mathbf{e}^b \wedge \alpha_b^a + e_b^\mu \partial_\mu \phi \mathbf{e}^b \wedge \mathbf{e}^a - \mathbf{d}\phi \wedge \mathbf{e}^a \\ &= \mathbf{e}^b \wedge \alpha_b^a + \mathbf{d}\phi \wedge \mathbf{e}^a - \mathbf{d}\phi \wedge \mathbf{e}^a \\ &= \mathbf{e}^b \wedge \alpha_b^a\end{aligned}$$

as required. Since the spin connection is uniquely determined (up to local Lorentz transformations) by the structure equation, this is the unique solution.

2.3 Transformation of the curvature

Now compute the new curvature tensor. For this longer calculation it is convenient to define the orthonormal component of $\mathbf{d}\phi = \phi_b \mathbf{e}^b$,

$$\phi_a \equiv e_a^\mu \partial_\mu \phi.$$

Then $\tilde{\alpha}_b^a = \alpha_b^a + 2\Delta_{db}^{ac} \phi_c \mathbf{e}^d$ and the conformal transformation changes the curvature to:

$$\begin{aligned}\frac{1}{2} \tilde{R}_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d &= \mathbf{d}\tilde{\alpha}_b^a - \tilde{\alpha}_b^c \wedge \tilde{\alpha}_c^a \\ &= \mathbf{d}\alpha_b^a - \alpha_b^c \wedge \alpha_c^a + \mathbf{d}(2\Delta_{db}^{ac} \phi_c \mathbf{e}^d) - \alpha_b^c \wedge (2\Delta_{dc}^{ae} \phi_e \mathbf{e}^d) \\ &\quad - (2\Delta_{db}^{ce} \phi_e \mathbf{e}^d) \wedge \alpha_c^a - (2\Delta_{db}^{ce} \phi_e \mathbf{e}^d) \wedge (2\Delta_{fc}^{ag} \phi_g \mathbf{e}^f) \\ &= \frac{1}{2} R_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d + \mathbf{D}(2\Delta_{db}^{ac} \phi_c \mathbf{e}^d) - (2\Delta_{db}^{ce} \phi_e \mathbf{e}^d) \wedge (2\Delta_{fc}^{ag} \phi_g \mathbf{e}^f)\end{aligned}$$

Since

$$\mathbf{D}\mathbf{e}^a = \mathbf{d}\mathbf{e}^a - \mathbf{e}^c \wedge \alpha_c^a = 0$$

this reduces to

$$\frac{1}{2} \tilde{R}_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d = \frac{1}{2} R_{bcd}^a \mathbf{e}^c \wedge \mathbf{e}^d + 2\Delta_{db}^{ac} \mathbf{D}\phi_c \mathbf{e}^d - (2\Delta_{db}^{ce} \phi_e \mathbf{e}^d) \wedge (2\Delta_{fc}^{ag} \phi_g \mathbf{e}^f)$$

as is easily checked by expanding the Δ s. We need only simplify the final term:

$$\begin{aligned}
4\Delta_{db}^{ce}\phi_e\mathbf{e}^d \wedge \Delta_{fc}^{ag}\phi_g\mathbf{e}^f &= (2\Delta_{db}^{ce}\phi_e\mathbf{e}^d) \wedge (2\Delta_{fc}^{ag}\phi_g\mathbf{e}^f) \\
&= (\phi_b\mathbf{e}^c - \eta^{ce}\eta_{bd}\phi_e\mathbf{e}^d) \wedge (\phi_c\mathbf{e}^a - \eta^{ag}\eta_{fc}\phi_g\mathbf{e}^f) \\
&= \phi_c\phi_b\mathbf{e}^c \wedge \mathbf{e}^a - \eta^{ag}\phi_g\phi_b\eta_{fc}\mathbf{e}^c \wedge \mathbf{e}^f - \eta_{bd}\eta^{ce}\phi_c\phi_e\mathbf{e}^d \wedge \mathbf{e}^a \\
&\quad + \eta^{ag}\eta_{fc}\phi_g\eta^{ce}\eta_{bd}\phi_e\mathbf{e}^d \wedge \mathbf{e}^f \\
&= (\phi_b\phi_f\mathbf{e}^f \wedge \mathbf{e}^a - \eta_{bd}\phi^c\phi_c\mathbf{e}^d \wedge \mathbf{e}^a - \eta^{ag}\eta_{bd}\phi_g\phi_f\mathbf{e}^f \wedge \mathbf{e}^d) \\
&= ((\delta_d^a\delta_b^e\phi_e\phi_f - \eta^{ae}\eta_{bd}\phi_e\phi_f)\mathbf{e}^f \wedge \mathbf{e}^d - \eta_{bd}\phi^c\phi_c\mathbf{e}^d \wedge \mathbf{e}^a) \\
&= \left((\delta_d^a\delta_b^c - \eta^{ac}\eta_{bd})\phi_c\phi_e\mathbf{e}^e \wedge \mathbf{e}^d - \frac{1}{2}\delta_d^a\eta_{bc}\phi^f\phi_f\mathbf{e}^c \wedge \mathbf{e}^d + \frac{1}{2}\delta_c^a\eta_{bd}\phi^f\phi_f\mathbf{e}^c \wedge \mathbf{e}^d \right) \\
&= \left(2\Delta_{db}^{ac}\phi_c\phi_e - \frac{1}{2}\delta_d^a\delta_b^c\eta_{ce}\phi^f\phi_f + \frac{1}{2}\eta^{ac}\eta_{bd}\eta_{ce}\phi^f\phi_f \right) \mathbf{e}^e \wedge \mathbf{e}^d \\
&= 2\Delta_{db}^{ac} \left(\phi_c\phi_e - \frac{1}{2}\eta_{ce}\phi^f\phi_f \right) \mathbf{e}^e \wedge \mathbf{e}^d
\end{aligned}$$

Now setting $\mathbf{D}^{(\alpha)}\phi_c = \phi_{c;d}\mathbf{e}^d$ and $\mathbf{d}^{(x)}\phi = \phi_d\mathbf{e}^d$. the final result is,

$$\tilde{\mathbf{R}}_b^a = \mathbf{R}_b^a + 2\Delta_{db}^{ac} \left(\phi_{c;e} - \phi_e\phi_c + \frac{1}{2}\phi^f\phi_f\eta_{ce} \right) \mathbf{e}^e \wedge \mathbf{e}^d \quad (8)$$

In components, eq.(8) becomes

$$\tilde{R}_{bcd}^a = R_{bcd}^a + 2\Delta_{db}^{ae} \left(\phi_{e;c} - \phi_c\phi_e + \frac{1}{2}\phi^f\phi_f\eta_{ec} \right) - 2\Delta_{cb}^{ae} \left(\phi_{e;d} - \phi_d\phi_e + \frac{1}{2}\phi^f\phi_f\eta_{ed} \right)$$

The Ricci tensor and scalar are

$$\tilde{R}_{bd} = R_{bd} - (n-2)\phi_{b;d} - \eta_{bd}\phi^c{}_{;c} + (n-2)\phi_b\phi_d + (n-2)\eta_{bd}\phi^c\phi_c$$

and

$$\begin{aligned}
\tilde{R} &= \tilde{g}^{ab}\tilde{R}_{ab} \\
&= e^{-2\phi}g^{ab}(R_{bd} - (n-2)\phi_{b;d} - \eta^{ce}\phi_{e;c}\eta_{bd} + (n-2)\phi_d\phi_b - (n-2)\phi^c\phi_c\eta_{bd}) \\
&= e^{-2\phi}(R - 2(n-1)\phi^c{}_{;c} + (n-2)\phi^c\phi_c - n(n-2)\phi^c\phi_c) \\
&= e^{-2\phi}(R - 2(n-1)\phi^c{}_{;c} - (n-1)(n-2)\phi^c\phi_c)
\end{aligned}$$

2.4 Invariance of the Weyl curvature tensor

In general, we may split the Riemann curvature R_{bcd}^a into its trace, the Ricci tensor, and its traceless part, called the Weyl curvature. This decomposition is most concisely expressed if we first define the Schouten tensor,

$$\mathcal{R}_{bd} \equiv \frac{1}{(n-2)} \left(R_{bd} - \frac{1}{2(n-1)}\eta_{db}R \right) \quad (9)$$

where R_{ab} is the Ricci tensor,

$$R_{ab} \equiv R^c{}_{acb}$$

The Schouten tensor often arises as a 1-form, $\mathcal{R}_a = \mathcal{R}_{ab}\mathbf{e}^b$. Except in 2-dimensions, it is equivalent to the Ricci tensor, since we may invert eq.(9) to write

$$R_{bd} = (n-2)\mathcal{R}_{bd} + \eta_{db}\mathcal{R} \quad (10)$$

$$R = 2(n-1)\mathcal{R} \quad (11)$$

In terms of \mathcal{R}_{ab} , the Weyl curvature 2-form is defined as

$$\mathbf{C}_b^a \equiv \mathbf{R}_b^a + 2\Delta_{db}^{ae}\mathcal{R}_e \wedge \mathbf{e}^d \quad (12)$$

Expanding to find the components, C_{bcd}^a , of \mathbf{C}_b^a ,

$$\begin{aligned} C_{bcd}^a &= R_{bcd}^a + 2\Delta_{db}^{ae}\mathcal{R}_{ec} - 2\Delta_{cb}^{ae}\mathcal{R}_{ed} \\ &= R_{bcd}^a + (\delta_d^a\delta_b^e - \eta^{ae}\eta_{bd})\mathcal{R}_{ec} - (\delta_c^a\delta_b^e - \eta^{ae}\eta_{cb})\mathcal{R}_{ed} \\ &= R_{bcd}^a + \delta_d^a\mathcal{R}_{bc} - \mathcal{R}_c^a\eta_{bd} - \delta_c^a\mathcal{R}_{bd} + \mathcal{R}_d^a\eta_{bc} \\ &= R_{bcd}^a - \frac{1}{(n-2)}(\delta_c^a R_{bd} - \delta_d^a R_{bc} - R_d^a\eta_{bc} + R_c^a\eta_{bd}) + \frac{R}{(n-1)(n-2)}(\delta_c^a\eta_{bd} - \delta_d^a\eta_{bc}) \end{aligned} \quad (13)$$

we readily verify its tracelessness,

$$\begin{aligned} C_{bcd}^c &= R_{bd} - \frac{1}{n-2}(nR_{bd} - R_{bd} - R_{bd} + \eta_{bd}R) + \frac{R}{(n-1)(n-2)}(n-1)\eta_{bd} \\ &= 0 \end{aligned}$$

with all other nontrivial traces equivalent to this one. By contrast, the second term in eq.(12) is equivalent to knowing the Ricci or Schouten tensor, since the components of the Δ_{cb}^{ae} term, $D_{bcd}^a \equiv \Delta_{db}^{ae}\mathcal{R}_{ec} - \Delta_{cb}^{ae}\mathcal{R}_{ed}$ in eq.(12) give

$$\mathcal{R}_{bd} = \delta_a^c \left[-\frac{2}{n-2}D_{bcd}^a + \frac{1}{(n-1)(n-2)}(\eta^{fg}D_{fcg}^a)\eta_{bd} \right]$$

To check this we expand,

$$\begin{aligned} \mathcal{R}_{bd} &= \delta_a^c \left[-\frac{2}{n-2}D_{bcd}^a + \frac{1}{(n-1)(n-2)}(\eta^{fg}D_{fcg}^a)\eta_{bd} \right] \\ &= \delta_a^c \left[-\frac{2}{n-2}(\Delta_{db}^{ae}\mathcal{R}_{ec} - \Delta_{cb}^{ae}\mathcal{R}_{ed}) + \frac{1}{(n-1)(n-2)}(\eta^{fg}\Delta_{fg}^{ae}\mathcal{R}_{ec} - \eta^{fg}\Delta_{cf}^{ae}\mathcal{R}_{eg})\eta_{bd} \right] \\ &= -\frac{2}{n-2} \left(\frac{1}{2}(\delta_d^a\delta_b^e - \eta^{ae}\eta_{db})\mathcal{R}_{ea} - \frac{1}{2}(n-1)\mathcal{R}_{bd} \right) \\ &\quad + \frac{1}{(n-1)(n-2)} \left(\frac{1}{2}\eta^{fg}(\delta_f^a\delta_g^e - \eta^{ae}\eta_{fg})\mathcal{R}_{ea} - \frac{1}{2}(n-1)\mathcal{R} \right) \eta_{bd} \\ &= -\frac{1}{n-2}(\mathcal{R}_{bd} - \eta_{bd}\mathcal{R} - (n-1)\mathcal{R}_{bd}) + \frac{1}{2(n-1)(n-2)}(1-n-(n-1))\mathcal{R}\eta_{bd} \\ &= \frac{1}{n-2}[-\mathcal{R}_{bd} + (n-1)\mathcal{R}_{bd} + \eta_{bd}\mathcal{R} - \mathcal{R}\eta_{bd}] \\ &= \mathcal{R}_{bd} \end{aligned}$$

We now have the decomposition of the Riemann curvature into traceless and trace parts,

$$\mathbf{R}_b^a = \mathbf{C}_b^a - 2\Delta_{db}^{ae}\mathcal{R}_e \wedge \mathbf{e}^d \quad (14)$$

After a conformal transformation, the new Riemann curvature 2-form may also be decomposed in the same way,

$$\tilde{\mathbf{R}}_b^a = \tilde{\mathbf{C}}_b^a - 2\Delta_{db}^{ae}\tilde{\mathcal{R}}_e \wedge \tilde{\mathbf{e}}^d$$

Combining this with eq.(8) we have

$$\begin{aligned} \tilde{\mathbf{C}}_b^a - 2\Delta_{db}^{ae}\tilde{\mathcal{R}}_e \wedge \tilde{\mathbf{e}}^d &= \mathbf{C}_b^a - 2\Delta_{db}^{ae}\mathcal{R}_e \wedge \mathbf{e}^d + 2\Delta_{db}^{ae} \left(\phi_{e;c} - \phi_e\phi_c + \frac{1}{2}\phi^2\eta_{ce} \right) \mathbf{e}^c \wedge \mathbf{e}^d \\ &= \mathbf{C}_b^a - 2\Delta_{db}^{ae} \left(\mathcal{R}_{ec} - \phi_{e;c} + \phi_e\phi_c - \frac{1}{2}\phi^2\eta_{ce} \right) \mathbf{e}^c \wedge \mathbf{e}^d \end{aligned}$$

Equality of the traceless and trace parts shows immediately that both

$$\begin{aligned}\tilde{\mathbf{C}}_b^a &= \mathbf{C}_b^a \\ \tilde{\mathcal{R}}_e &= e^{-\phi} \left(\mathcal{R}_{ec} - \phi_{e;c} + \phi_e \phi_c - \frac{1}{2} \phi^2 \eta_{ce} \right) \tilde{\mathbf{e}}^c\end{aligned}$$

so the Weyl curvature 2-form is conformally invariant. The components of each part transform as

$$\tilde{C}_{bcd}^a = e^{-2\phi} C_{bcd}^a \quad (15)$$

$$\tilde{\mathcal{R}}_{ab} = e^{-2\phi} \left(\mathcal{R}_{ab} - \phi_{a;b} + \phi_a \phi_b - \frac{1}{2} \phi^2 \eta_{ab} \right) \quad (16)$$

$$\tilde{\mathcal{R}} = e^{-2\phi} \left(\mathcal{R} - \phi^a{}_{;a} - \frac{1}{2} (n-2) \phi^a \phi_a \right) \quad (17)$$

where the factor of $e^{-2\phi}$ comes from replacing $\tilde{\mathbf{e}}^a = e^\phi \mathbf{e}^a$. This proves that the Weyl curvature tensor is covariant with weight -2 under a conformal transformation of the metric, and yields the expression for the change in the Schouten (and therefore, Ricci) tensor under conformal transformation.

2.5 Conditions for conformal Ricci flatness

Next, we find the condition required for the metric of a Riemannian geometry to be conformally related to the metric of a Ricci-flat spacetime. This follows as a pair of integrability conditions for ϕ when we set eq.(16) equal to zero.

First, we rewrite eq.(16) as a 1-form equation,

$$\tilde{\mathcal{R}}_c = e^{-\phi} \left(\mathcal{R}_c - \mathbf{D}\phi_c + \phi_c \mathbf{d}\phi - \frac{1}{2} \eta^{ab} \phi_a \phi_b \eta_{ce} \mathbf{e}^e \right)$$

where $\mathbf{D}\phi_c = \mathbf{d}\phi_c - \phi_e \omega_c^e$. Using the vector field $\phi_c \equiv e_c{}^\mu \partial_\mu \phi$, we define the corresponding 1-form $\chi \equiv \phi_c \mathbf{e}^c = \mathbf{d}\phi$ and ask for the conditions under which $\tilde{\mathcal{R}}_c = 0$ has a solution for ϕ . This may be written as a pair of equations,

$$\mathbf{d}\phi_c = \mathcal{R}_c + \phi_e \omega_c^e + \phi_c \chi - \frac{1}{2} \eta^{ab} \phi_a \phi_b \eta_{ce} \mathbf{e}^e \quad (18)$$

$$\mathbf{d}\phi = \chi \quad (19)$$

The integrability conditions follow from the Poincaré lemma, $\mathbf{d}^2 \equiv 0$,

$$\begin{aligned}0 &\equiv \mathbf{d}^2 \phi_c \\ &= \mathbf{d}\mathcal{R}_c + \mathbf{d}\phi_e \wedge \omega_c^e + \phi_e \mathbf{d}\omega_c^e + \mathbf{d}\phi_c \wedge \chi - \eta^{ab} \phi_a \mathbf{d}\phi_b \wedge \eta_{ce} \mathbf{e}^e - \frac{1}{2} \phi^2 \eta_{ce} \mathbf{d}\mathbf{e}^e\end{aligned} \quad (20)$$

$$\begin{aligned}0 &\equiv \mathbf{d}^2 \phi \\ &= \mathbf{d}\chi\end{aligned} \quad (21)$$

The second condition, eq.(21), is identically satisfied by the definition of χ . Substituting the original equation

for $\mathbf{d}\phi_e$, eq.(18), into the first integrability condition, eq.(20),

$$\begin{aligned}
0 &= \mathbf{d}\mathcal{R}_c + \left(\mathcal{R}_e + \phi_d \boldsymbol{\omega}_e^d + \phi_e \boldsymbol{\chi} - \frac{1}{2} \phi^2 \eta_{ed} \mathbf{e}^d \right) \wedge \boldsymbol{\omega}_c^e + \phi_e \mathbf{d}\boldsymbol{\omega}_c^e \\
&\quad + \left(\mathcal{R}_c + \phi_e \boldsymbol{\omega}_c^e + \phi_c \boldsymbol{\chi} - \frac{1}{2} \phi^2 \eta_{ce} \mathbf{e}^e \right) \wedge \boldsymbol{\chi} \\
&\quad - \eta^{ab} \phi_a \left(\mathcal{R}_b + \phi_d \boldsymbol{\omega}_b^d + \phi_b \boldsymbol{\chi} - \frac{1}{2} \phi^2 \eta_{bd} \mathbf{e}^d \right) \wedge \eta_{ce} \mathbf{e}^e - \frac{1}{2} \phi^2 \eta_{ce} \mathbf{d}\mathbf{e}^e \\
&= (\mathbf{d}\mathcal{R}_c + \mathcal{R}_e \wedge \boldsymbol{\omega}_c^e) + (\mathcal{R}_c \wedge \boldsymbol{\chi} - \eta^{ab} \phi_a \mathcal{R}_b \wedge \eta_{ce} \mathbf{e}^e) + (\phi_e \mathbf{d}\boldsymbol{\omega}_c^e + \phi_d \boldsymbol{\omega}_e^d \wedge \boldsymbol{\omega}_c^e) \\
&\quad - \frac{1}{2} \phi^2 (\eta_{ce} \mathbf{d}\mathbf{e}^e + \eta_{ed} \mathbf{e}^d \wedge \boldsymbol{\omega}_c^e) - \frac{1}{2} \phi^2 \eta_{ce} \mathbf{e}^e \wedge \boldsymbol{\chi} - \eta^{ab} \phi_a \phi_d \boldsymbol{\omega}_b^d \wedge \eta_{ce} \mathbf{e}^e \\
&\quad - \phi^2 \boldsymbol{\chi} \wedge \eta_{ce} \mathbf{e}^e + \frac{1}{2} \phi^2 \phi_a \mathbf{e}^a \wedge \eta_{ce} \mathbf{e}^e \\
&= \mathbf{D}\mathcal{R}_c + \phi_a (\delta_c^b \delta_e^a - \eta^{ab} \eta_{ce}) \mathcal{R}_b \wedge \mathbf{e}^e + \phi_e \mathbf{R}_c^e - \frac{1}{2} \phi^2 \eta_{ec} (\mathbf{D}\mathbf{e}^e - \mathbf{e}^d \wedge \boldsymbol{\omega}_d^e) \\
&\quad + \phi^2 \left(\frac{1}{2} \eta_{ce} \boldsymbol{\chi} \wedge \mathbf{e}^e - \eta_{ce} \boldsymbol{\chi} \wedge \mathbf{e}^e + \frac{1}{2} \eta_{ce} \boldsymbol{\chi} \wedge \mathbf{e}^e \right) - (\phi_a \phi_d \boldsymbol{\omega}^{da}) \wedge \eta_{ce} \mathbf{e}^e \\
&= \mathbf{D}\mathcal{R}_c + \phi_a (\mathbf{R}_c^a + 2\Delta_{ec}^{ab} \mathcal{R}_b \wedge \mathbf{e}^e)
\end{aligned}$$

which we see from eq.(14) becomes

$$0 = \mathbf{D}\mathcal{R}_c + \phi_a \mathbf{C}_c^a \quad (22)$$

Though this well-known condition still depends on the gradient of the conformal factor, ϕ_a , Szekeres has shown using spinor techniques that it can be broken down into two integrability conditions depending only on the curvature [22].

2.6 Conditions for conformal Einstein equation with matter

We may apply the same approach to the Einstein equation with conformal matter. Let the matter be of definite conformal weight, $\tilde{\Psi} \rightarrow e^{k\phi} \Psi$ for a generic field Ψ . Then the covariant form of the stress-energy tensor will be of conformal weight -2 ,

$$\tilde{T}_{ab} = e^{-2\phi} T_{ab}$$

to have the correct weight for the Einstein equation. The Einstein tensor, of course, is not of definite conformal weight, but it acquires an overall factor of $e^{-2\phi}$. We assume that T_{ab} is of definite weight.

Then, writing the Einstein equation, $R_{ab} - \frac{1}{2} \eta_{ab} R = T_{ab}$ in terms of the Schouten tensor using eqs.(10) and (11), gives

$$\mathcal{R}_{ab} - \eta_{ab} \mathcal{R} = \frac{1}{n-2} T_{ab}$$

Now define, for arbitrary curvatures, not necessarily solutions,

$$E_{ab} \equiv \mathcal{R}_{ab} - \eta_{ab} \mathcal{R} - \frac{1}{n-2} T_{ab}$$

We would like to know when there exists a conformal transformation, $E_{ab} \rightarrow \tilde{E}_{ab}$ such that $\tilde{E}_{ab} = 0$. The calculation is simpler if we notice that $E_{ab} = 0$ if and only if

$$E_{ab} - \frac{1}{n-1} E \eta_{ab} = \mathcal{R}_{ab} - \frac{1}{n-2} \left(T_{ab} - \frac{1}{n-1} T \eta_{ab} \right) = 0$$

Defining

$$\mathcal{T}_{ab} \equiv \frac{1}{n-2} \left(T_{ab} - \frac{1}{n-1} T \eta_{ab} \right) \quad (23)$$

we ask for a conformal gauge in which $\tilde{E}_{ab} - \frac{1}{n-1}\tilde{E}\eta_{ab} = \mathcal{R}_{ab} - \mathcal{T}_{ab} = 0$.

We establish clearly that this is equivalent to the Einstein equation. The essential question is the condition for a conformal transformation such that $\tilde{E}_{ab} = 0$. Substituting the conformally transformed fields to find \tilde{E}_{ab} ,

$$\begin{aligned}\tilde{E}_{ab} &= \mathcal{R}_{ab} - \eta_{ab}\mathcal{R} - \frac{1}{n-2}T_{ab} \\ &= e^{-2\phi} \left(\mathcal{R}_{ab} - \phi_{a;b} + \phi_a\phi_b - \frac{1}{2}\phi^2\eta_{ab} \right) - \eta_{ab}e^{-2\phi} \left(\mathcal{R} - \phi^c{}_{;c} - \frac{1}{2}(n-2)\phi^c\phi_c \right) - \frac{1}{n-2}e^{-2\phi}T_{ab}\end{aligned}$$

so we examine integrability of

$$0 = \left(\mathcal{R}_{ab} - \phi_{a;b} + \phi_a\phi_b - \frac{1}{2}\phi^2\eta_{ab} \right) - \eta_{ab} \left(\mathcal{R} - \phi^c{}_{;c} - \frac{1}{2}(n-2)\phi^c\phi_c \right) - \frac{1}{n-2}T_{ab} \quad (24)$$

However, by solving the trace of this equation for $\phi^c{}_{;c}$,

$$\begin{aligned}0 &= \mathcal{R} - \phi^a{}_{;a} + \phi^a\phi_a - \frac{1}{2}n\phi^c\phi_c - \left(n\mathcal{R} - n\phi^c{}_{;c} - \frac{1}{2}n(n-2)\phi^c\phi_c \right) - \frac{1}{n-2}T \\ &= -(n-1)\mathcal{R} + (n-1)\phi^c{}_{;c} + \frac{1}{2}(n-1)(n-2)\phi^c\phi_c - \frac{1}{n-2}T \\ \phi^c{}_{;c} &= \mathcal{R} - \frac{1}{2}(n-2)\phi^c\phi_c + \frac{1}{(n-1)(n-2)}T\end{aligned} \quad (25)$$

and substituting eq.(25) back into eq.(24) we find the simpler form,

$$0 = \mathcal{R}_{ab} - \phi_{a;b} + \phi_a\phi_b - \frac{1}{2}\phi^2\eta_{ab} - \mathcal{T}_{ab} \quad (26)$$

and this is just $\tilde{E}_{ab} - \frac{1}{n-1}\tilde{E}\eta_{ab} = 0$. Conversely, the trace of eq.(26) reproduces the trace condition, eq.(25). Therefore, conformal vanishing of \tilde{E}_{ab} is equivalent to conformal vanishing of $\tilde{E}_{ab} - \frac{1}{n-1}\tilde{E}\eta_{ab}$.

Returning to the find the condition, we set $\mathcal{T}_a \equiv \mathcal{T}_{ab}\mathbf{e}^b$ and $\chi = \mathbf{d}\phi$, then write $\tilde{E}_{ab} - \frac{1}{n-1}\tilde{E}\eta_{ab} = 0$ as a 1-form equation,

$$0 = \left(\mathcal{R}_a - \mathbf{d}\phi_a + \phi_b\omega_a^b + \phi_a\chi - \frac{1}{2}\phi^2\eta_{ab}\mathbf{e}^b \right) - \mathcal{T}_a$$

We therefore require

$$\begin{aligned}\mathbf{d}\phi_a &= \mathcal{R}_a + \phi_b\omega_a^b + \phi_a\chi - \frac{1}{2}\phi^2\eta_{ab}\mathbf{e}^b - \mathcal{T}_a \\ \mathbf{d}\chi &= \mathbf{d}^2\phi \equiv 0\end{aligned}$$

The trace relation of eq.(25) also holds. The integrability condition is:

$$\begin{aligned}0 &\equiv \mathbf{d}^2\phi_a \\ &= \mathbf{d} \left(\mathcal{R}_a - \mathcal{T}_a + \phi_b\omega_a^b + \phi_a\chi - \frac{1}{2}\phi^2\eta_{ab}\mathbf{e}^b \right) \\ &= \mathbf{d}(\mathcal{R}_a - \mathcal{T}_a) + \phi_b\mathbf{d}\omega_a^b - \frac{1}{2}\phi^2\eta_{ab}\mathbf{d}\mathbf{e}^b \\ &\quad + \left(\mathcal{R}_b - \mathcal{T}_b + \phi_c\omega_b^c + \phi_b\chi - \frac{1}{2}\phi^2\eta_{bc}\mathbf{e}^c \right) \wedge \omega_a^b \\ &\quad + \left(\mathcal{R}_a - \mathcal{T}_a + \phi_b\omega_a^b + \phi_a\chi - \frac{1}{2}\phi^2\eta_{ab}\mathbf{e}^b \right) \wedge \chi \\ &\quad - \phi^c \left(\mathcal{R}_c - \mathcal{T}_c + \phi_b\omega_c^b + \phi_c\chi - \frac{1}{2}\phi^2\eta_{cb}\mathbf{e}^b \right) \wedge \eta_{ad}\mathbf{e}^d\end{aligned}$$

Distributing and collecting like terms,

$$\begin{aligned}
0 &= \mathbf{d}(\mathcal{R}_a - \mathcal{T}_a) + (\mathcal{R}_b - \mathcal{T}_b) \wedge \omega_a^b + (\mathcal{R}_a - \mathcal{T}_a) \wedge \mathbf{d}\phi - \phi^c (\mathcal{R}_c - \mathcal{T}_c) \wedge \eta_{ad} \mathbf{e}^d \\
&\quad + \phi_b (\mathbf{d}\omega_a^b - \omega_a^c \wedge \omega_c^b) - \frac{1}{2} \phi^2 \eta_{ab} (\mathbf{d}\mathbf{e}^b - \mathbf{e}^c \wedge \omega_c^b) \\
&\quad + (\phi_b \chi \wedge \omega_a^b + \phi_b \omega_a^b \wedge \chi) + \left(\frac{1}{2} \phi^c \phi_c - \phi^c \phi_c + \frac{1}{2} \phi^c \phi_c \right) \mathbf{d}\phi \wedge \eta_{ad} \mathbf{e}^d \\
&= \mathbf{D}(\mathcal{R}_a - \mathcal{T}_a) + \phi_b \mathbf{R}_a^b + \phi_b \delta_a^c \delta_d^b (\mathcal{R}_c - \mathcal{T}_c) \wedge \mathbf{e}^d - \phi_b \eta^{bc} \eta_{ad} (\mathcal{R}_c - \mathcal{T}_c) \wedge \mathbf{e}^d \\
&= \mathbf{D}(\mathcal{R}_a - \mathcal{T}_a) + \phi_b \mathbf{R}_a^b + 2\phi_b \Delta_{ad}^{cb} (\mathcal{R}_c - \mathcal{T}_c) \wedge \mathbf{e}^d \\
&= \mathbf{D}\mathcal{R}_a + \phi_b \mathbf{C}_a^b - \mathbf{D}\mathcal{T}_a - 2\phi_b \Delta_{ad}^{cb} \mathcal{T}_c \wedge \mathbf{e}^d
\end{aligned}$$

leaving us with

$$\mathbf{D}\mathcal{R}_a + \phi_b \mathbf{C}_a^b = \mathbf{D}\mathcal{T}_a + \phi_b 2\Delta_{da}^{bc} \mathcal{T}_c \mathbf{e}^d \quad (27)$$

This is a new result. When eq.(26) is written using the Riemann tensor instead of the Weyl tensor,

$$\mathbf{D}(\mathcal{R}_a - \mathcal{T}_a) + \phi_b \mathbf{R}_a^b + 2\phi_b \Delta_{ad}^{cb} (\mathcal{R}_c - \mathcal{T}_c) \wedge \mathbf{e}^d = 0$$

we recognize the same condition as that for Ricci flatness, but with the Schouten tensor replaced by $\mathcal{R}_a - \mathcal{T}_a$.

3 Weyl geometry

A simple extension of the Poincaré symmetry underlying Riemannian geometry leads to the Cartan structure equations for the Weyl group:

$$\mathfrak{R}_b^a = \mathbf{d}\omega_b^a - \omega_b^c \wedge \omega_c^a \quad (28)$$

$$\mathbf{T}^a = \mathbf{d}\mathbf{e}^a - \mathbf{e}^b \wedge \omega_b^a - \omega \wedge \mathbf{e}^a \quad (29)$$

$$\Omega = \mathbf{d}\omega \quad (30)$$

where the most general case includes both the torsion, $\mathbf{T}^a = \frac{1}{2} T_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c$, and the dilatational curvature, $\Omega = \frac{1}{2} \Omega_{ab} \mathbf{e}^b \wedge \mathbf{e}^c$. In our treatment of a Dirac-like theory, we will not assume vanishing torsion.

A conformal transformation of the metric, eq.(3), now transforms both the solder form and the Weyl vector, according to

$$\tilde{\mathbf{e}}^a = e^\phi \mathbf{e}^a$$

$$\tilde{\omega} = \omega + \mathbf{d}\phi$$

The final structure equation, eq.(30) then remains unchanged, since

$$\mathbf{d}\tilde{\omega} = \mathbf{d}\omega$$

The basis equation transforms as

$$\begin{aligned}
\tilde{\mathbf{T}}^a &= \mathbf{d}\tilde{\mathbf{e}}^a - \tilde{\mathbf{e}}^b \wedge \tilde{\omega}_b^a - \tilde{\omega} \wedge \tilde{\mathbf{e}}^a \\
&= (e^\phi \mathbf{d}\phi \wedge \mathbf{e}^a + e^\phi \mathbf{d}\mathbf{e}^a) - e^\phi \mathbf{e}^b \wedge \tilde{\omega}_b^a - (\omega + \mathbf{d}\phi) \wedge e^\phi \mathbf{e}^a \\
&= e^\phi (\mathbf{d}\phi \wedge \mathbf{e}^a + (\mathbf{T}^a + \mathbf{e}^c \wedge \omega_c^a + \omega \wedge \mathbf{e}^a) - \mathbf{e}^b \wedge \tilde{\omega}_b^a - \omega \wedge \mathbf{e}^a - \mathbf{d}\phi \wedge \mathbf{e}^a) \\
&= e^\phi \mathbf{T}^a + e^\phi \mathbf{e}^b \wedge (\omega_b^a - \tilde{\omega}_b^a)
\end{aligned}$$

We conclude that it is sufficient to take the spin connection to be conformally invariant, and the torsion a weight-1 conformal tensor:

$$\begin{aligned}
\tilde{\omega}_b^a &= \omega_b^a \\
\tilde{\mathbf{T}}^a &= e^\phi \mathbf{T}^a
\end{aligned}$$

These inferences are correct, as may be shown directly from the gauge transformation properties of the Cartan connection. Since the spin connection is invariant, the Lorentz curvature 2-form is also invariant, $\tilde{\mathfrak{R}}^a_b = \mathfrak{R}^a_b$.

We again use the Poincaré lemma, $\mathbf{d}^2 \equiv 0$ to find the integrability conditions:

$$\mathbf{D}\mathfrak{R}^a_b = 0 \quad (31)$$

$$\mathbf{D}\mathbf{T}^a = \mathbf{e}^b \wedge \mathfrak{R}^a_b - \Omega \wedge \mathbf{e}^a \quad (32)$$

$$\mathbf{D}\Omega = 0 \quad (33)$$

where the covariant derivatives are given by

$$\mathbf{D}\mathfrak{R}^a_b \equiv \mathbf{d}\mathfrak{R}^a_b + \mathfrak{R}^c_b \wedge \omega^a_c - \mathfrak{R}^a_c \wedge \omega^c_b$$

$$\mathbf{D}\mathbf{T}^a \equiv \mathbf{d}\mathbf{T}^a + \mathbf{T}^b \wedge \omega^a_b - \omega \wedge \mathbf{T}^a$$

$$\mathbf{D}\Omega \equiv \mathbf{d}\Omega$$

When the torsion vanishes, we have a pair of algebraic identities since the Weyl-Ricci tensor may have an antisymmetric part. From

$$\begin{aligned} \mathbf{e}^b \wedge \mathfrak{R}^a_b &= \Omega \wedge \mathbf{e}^a \\ \mathfrak{R}^a_{[bcd]} &= \delta^a_{[b} \Omega_{cd]} \end{aligned} \quad (34)$$

we find the symmetric and antisymmetric parts,

$$\begin{aligned} \mathfrak{R}^a_{bcd} + \mathfrak{R}^a_{cdb} + \mathfrak{R}^a_{dbc} &= \delta^a_b \Omega_{cd} + \delta^a_c \Omega_{db} + \delta^a_d \Omega_{bc} \\ \mathfrak{R}_{bd} - \mathfrak{R}_{db} &= -(n-2) \Omega_{bd} \end{aligned}$$

While the Lorentz curvature 2-form is conformally invariant, the components \mathfrak{R}^a_{bcd} , \mathfrak{R}_{ab} and Ω_{ab} all have conformal weight -2 .

3.1 The connection with the Weyl vector and torsion

As with Riemannian geometry, the structure equation for the solder form, eq.(29). allows us to solve for the connection.

3.1.1 Weyl connection with torsion

Look at the solder form equation,

$$\mathbf{d}\mathbf{e}^a = \mathbf{e}^b \wedge \omega^a_b + \omega \wedge \mathbf{e}^a + \mathbf{T}^a$$

Notice that when $\mathbf{T}^a = 0$ this has exactly the same form as the conformally transformed solder form structure equation of a Riemannian geometry, eq.(5), with $\mathbf{d}\phi$ replaced by $-\omega$. Thus, the solution for the Weyl spin connection is completely analogous to the effect of a dilatation on the connection of a Riemannian geometry, with the Weyl vector W_c replacing the negative of the gradient of the scale change, $-\phi_c$ in eq.(6). Taking advantage of this observation, let $\omega^a_b = \alpha^a_b + \beta^a_b + \gamma^a_b$ where α^a_b is the compatible connection and β^a_b is the required Weyl vector piece,

$$\begin{aligned} \mathbf{d}\mathbf{e}^a &= \mathbf{e}^b \wedge \alpha^a_b \\ \beta^a_b &= -2\Delta^{ac}_{db} W_c \mathbf{e}^d \end{aligned}$$

and each term has the same antisymmetry of indices as the full spin connection, i.e., $\omega_b^a = -\eta^{ac}\eta_{bd}\omega_c^d$. Then

$$\begin{aligned}
\mathbf{d}e^a &= \mathbf{e}^b \wedge (\alpha_b^a + \beta_b^a + \gamma_b^a) + \omega \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \mathbf{e}^b \wedge \alpha_b^a + \mathbf{e}^b \wedge \beta_b^a + \mathbf{e}^b \wedge \gamma_b^a + \omega \wedge \mathbf{e}^a + \mathbf{T}^a \\
0 &= (-2\Delta_{db}^{ac}W_c \mathbf{e}^b \wedge \mathbf{e}^d + \omega \wedge \mathbf{e}^a) + (\mathbf{e}^b \wedge \gamma_b^a + \mathbf{T}^a) \\
&= \mathbf{e}^b \wedge \gamma_b^a + \mathbf{T}^a \\
&= \left(\gamma_{bc}^a + \frac{1}{2}T_{bc}^a \right) \mathbf{e}^b \wedge \mathbf{e}^c
\end{aligned}$$

The final equation must involve antisymmetric $\eta_{ae}\gamma_{bc}^e = \gamma_{abc} = -\gamma_{bac}$. Lowering indices in the remaining condition and cycling,

$$\begin{aligned}
0 &= \gamma_{abc} - \gamma_{acb} + T_{abc} \\
0 &= \gamma_{bca} - \gamma_{bac} + T_{bca} \\
0 &= \gamma_{cab} - \gamma_{cba} + T_{cab}
\end{aligned}$$

we combine with the usual sum-sum-difference and solve.

$$\begin{aligned}
0 &= \gamma_{abc} - \gamma_{acb} + \gamma_{bca} - \gamma_{bac} - \gamma_{cab} + \gamma_{cba} + (T_{abc} - T_{cab} + T_{bca}) \\
&= (\gamma_{abc} - \gamma_{bac}) - (\gamma_{acb} + \gamma_{cab}) + (\gamma_{bca} + \gamma_{cba}) + (T_{abc} - T_{cab} + T_{bca}) \\
&= 2\gamma_{abc} + (T_{abc} - T_{cab} + T_{bca}) \\
\gamma_{abc} &= -\frac{1}{2}(T_{abc} - T_{cab} + T_{bca})
\end{aligned}$$

Therefore,

$$\gamma_{bc}^a = -\frac{1}{2}(T_{bc}^a + T_{cb}^a + T_{bc}^a)$$

and the spin connection is given by

$$\omega_b^a = \alpha_b^a - 2\Delta_{db}^{ac}W_c \mathbf{e}^d - C_{bc}^a \mathbf{e}^c \quad (35)$$

where we define the contorsion tensor to be

$$C_{bc}^a \equiv \frac{1}{2}(T_{bc}^a + T_{cb}^a + T_{bc}^a) \quad (36)$$

Now check,

$$\begin{aligned}
\mathbf{d}e^a &= \mathbf{e}^b \wedge \omega_b^a + \omega \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \mathbf{e}^b \wedge (\alpha_b^a - 2\Delta_{db}^{ac}W_c \mathbf{e}^d - C_{bc}^a \mathbf{e}^c) + \omega \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \mathbf{e}^b \wedge \alpha_b^a - (\delta_d^a \delta_b^c - \eta^{ac}\eta_{bd}) W_c \mathbf{e}^b \wedge \mathbf{e}^d - C_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c + \omega \wedge \mathbf{e}^a + \mathbf{T}^a \\
&= \mathbf{e}^b \wedge \alpha_b^a - W_b \mathbf{e}^b \wedge \mathbf{e}^a + \omega \wedge \mathbf{e}^a - \frac{1}{2}(T_{bc}^a + T_{cb}^a + T_{bc}^a) \mathbf{e}^b \wedge \mathbf{e}^c + \frac{1}{2}T_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c \\
&= \mathbf{e}^b \wedge \alpha_b^a - \frac{1}{2}T_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c + \frac{1}{2}T_{bc}^a \mathbf{e}^b \wedge \mathbf{e}^c \\
&= \mathbf{e}^b \wedge \alpha_b^a
\end{aligned}$$

3.1.2 The covariant derivative of Weyl geometry in a coordinate basis

When a tensor transforms linearly and homogeneously with a power λ of the conformal transformation that applies to the solder form $\tilde{\mathbf{e}}^a = e^\phi \mathbf{e}^a$,

$$\tilde{T}^A = e^{\lambda\phi} T^A$$

it is a *conformal tensor of weight* λ . When differentiating a conformal tensor of weight λ the covariant derivative in Weyl geometry is not just the partial derivative, but includes the weight of the field times the field, times the Weyl vector. For example, for a scalar field we have

$$D_\mu \varphi = \partial_\mu \varphi - \lambda W_\mu \varphi$$

This means that metric compatibility gives a different expression for the connection.

$$\begin{aligned} 0 &= D_\mu g_{\alpha\beta} \\ &= \partial_\mu g_{\alpha\beta} - g_{\nu\beta} \tilde{\Gamma}^\nu_{\alpha\mu} - g_{\alpha\nu} \hat{\Gamma}^\nu_{\beta\mu} - 2W_\mu g_{\alpha\beta} \\ &= \partial_\mu g_{\alpha\beta} - \hat{\Gamma}_{\beta\alpha\mu} - \hat{\Gamma}_{\alpha\beta\mu} - 2W_\mu g_{\alpha\beta} \end{aligned}$$

The derivative of a contravariant vector of weight λ is given by

$$D_\mu v^\alpha = \partial_\mu v^\alpha + v^\beta \hat{\Gamma}^\alpha_{\beta\mu} - \lambda v^\alpha W_\mu \quad (37)$$

and, checking the transformed derivative,

$$\begin{aligned} \tilde{D}_\mu \tilde{v}^\alpha &= \tilde{D}_\mu (e^{\lambda\phi} v^\alpha) \\ &= \partial_\mu (e^{\lambda\phi} v^\alpha) + e^{\lambda\phi} v^\beta \tilde{\Gamma}^\alpha_{\beta\mu} - \lambda e^{\lambda\phi} v^\alpha (W_\mu + \partial_\mu \phi) \\ &= e^{\lambda\phi} \left(\partial_\mu v^\alpha + v^\beta \tilde{\Gamma}^\alpha_{\beta\mu} - \lambda v^\alpha W_\mu \right) \\ &= e^{\lambda\phi} D_\mu v^\alpha \end{aligned}$$

and is therefore properly covariant. Notice that the Weyl connection is invariant under a conformal transformation, $\tilde{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu}$.

3.1.3 Weyl connection with torsion in a coordinate basis

The corresponding expression in a coordinate basis starts with the definition of torsion as the antisymmetric part of the connection,

$$T_{\alpha\mu\beta} \equiv \hat{\Gamma}_{\alpha\mu\beta} - \hat{\Gamma}_{\alpha\beta\mu} \quad (38)$$

Then, starting from metric compatibility,

$$\begin{aligned} 0 &= D_\mu g_{\alpha\beta} \\ &= \partial_\mu g_{\alpha\beta} - g_{\nu\beta} \tilde{\Gamma}^\nu_{\alpha\mu} - g_{\alpha\nu} \hat{\Gamma}^\nu_{\beta\mu} - 2W_\mu g_{\alpha\beta} \\ &= \partial_\mu g_{\alpha\beta} - \hat{\Gamma}_{\beta\alpha\mu} - \hat{\Gamma}_{\alpha\beta\mu} - 2W_\mu g_{\alpha\beta} \end{aligned}$$

we cycle the expression in the usual way

$$\begin{aligned} \hat{\Gamma}_{\beta\alpha\mu} + \hat{\Gamma}_{\alpha\beta\mu} &= \partial_\mu g_{\alpha\beta} - 2W_\mu g_{\alpha\beta} \\ \hat{\Gamma}_{\alpha\mu\beta} + \hat{\Gamma}_{\mu\alpha\beta} &= \partial_\beta g_{\mu\alpha} - 2W_\beta g_{\mu\alpha} \\ \hat{\Gamma}_{\mu\beta\alpha} + \hat{\Gamma}_{\beta\mu\alpha} &= \partial_\alpha g_{\beta\mu} - 2W_\alpha g_{\beta\mu} \end{aligned}$$

Each of these three expressions is a conformal tensor since

$$\begin{aligned} \partial_\mu \tilde{g}_{\alpha\beta} - 2\tilde{W}_\mu \tilde{g}_{\alpha\beta} &= \partial_\mu (e^{2\phi} g_{\alpha\beta}) - 2(W_\mu + \partial_\mu \phi) e^{2\phi} g_{\alpha\beta} \\ &= e^{2\phi} (\partial_\mu g_{\alpha\beta} - 2W_\mu g_{\alpha\beta}) \end{aligned}$$

Adding the first two and subtracting the third we no longer assume the connection is symmetric,

$$\begin{aligned}
0 &= \hat{\Gamma}_{\beta\alpha\mu} + \hat{\Gamma}_{\alpha\beta\mu} + \hat{\Gamma}_{\alpha\mu\beta} + \hat{\Gamma}_{\mu\alpha\beta} - \hat{\Gamma}_{\mu\beta\alpha} - \hat{\Gamma}_{\beta\mu\alpha} \\
&\quad - \partial_\mu g_{\alpha\beta} + 2W_\mu g_{\alpha\beta} - \partial_\beta g_{\mu\alpha} + 2W_\beta g_{\mu\alpha} + \partial_\alpha g_{\beta\mu} - 2W_\alpha g_{\beta\mu} \\
&= 2\hat{\Gamma}_{\alpha\beta\mu} + \left(\hat{\Gamma}_{\alpha\mu\beta} - \hat{\Gamma}_{\alpha\beta\mu}\right) + \left(\hat{\Gamma}_{\beta\alpha\mu} - \hat{\Gamma}_{\beta\mu\alpha}\right) + \left(\hat{\Gamma}_{\mu\alpha\beta} - \hat{\Gamma}_{\mu\beta\alpha}\right) \\
&\quad - (\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\mu\alpha} - \partial_\alpha g_{\beta\mu}) + 2(W_\mu g_{\alpha\beta} + W_\beta g_{\mu\alpha} - W_\alpha g_{\beta\mu}) \\
&= 2\hat{\Gamma}_{\alpha\beta\mu} + (T_{\alpha\mu\beta} + T_{\beta\alpha\mu} + T_{\mu\alpha\beta}) \\
&\quad - (\partial_\mu g_{\alpha\beta} + \partial_\beta g_{\mu\alpha} - \partial_\alpha g_{\beta\mu}) + 2(W_\mu g_{\alpha\beta} + W_\beta g_{\mu\alpha} - W_\alpha g_{\beta\mu})
\end{aligned}$$

we find

$$\hat{\Gamma}_{\alpha\beta\mu} = \Gamma_{\alpha\beta\mu} - (W_\mu g_{\alpha\beta} + W_\beta g_{\mu\alpha} - W_\alpha g_{\beta\mu}) - \frac{1}{2}(T_{\alpha\mu\beta} + T_{\beta\alpha\mu} + T_{\mu\alpha\beta})$$

Now, if we raise the first index,

$$\hat{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} - (\delta^\alpha_\beta W_\mu + \delta^\alpha_\mu W_\beta - W^\alpha g_{\beta\mu}) - \frac{1}{2}(T^\alpha_{\mu\beta} + T_\beta{}^\alpha{}_\mu + T_\mu{}^\alpha{}_\beta) \quad (39)$$

we arrive at the coordinate form of the connection.

3.2 The Weyl-Schouten tensor

The invariance of the full curvature, $\tilde{\mathfrak{R}}^a_b = \mathfrak{R}^a_b$, means that not only is the Weyl curvature of a Weyl geometry conformally covariant, but so is the Weyl-Schouten tensor, \mathfrak{S}_a . By the Weyl-Schouten tensor, we mean the conformally covariant Schouten tensor of a Weyl geometry. To compute it for a torsion-free Weyl geometry, we must expand the curvature, eq.(28) using the Weyl connection,

$$\omega^a_b = \alpha^a_b - 2\Delta^{ac}_{db} W_c e^d \quad (40)$$

with α^a_b still the metric-compatible spin connection. The difference is that now *all* of \mathfrak{R}^a_b will be conformally covariant because the connection is scale invariant,

$$\begin{aligned}
d\tilde{e}^a &= \tilde{e}^b \wedge \tilde{\omega}^a_b + \tilde{\omega} \wedge \tilde{e}^a \\
d(e^\phi e^a) &= e^\phi e^b \wedge \tilde{\omega}^a_b + (\omega + d\phi) \wedge e^\phi e^a \\
e^\phi d\phi \wedge e^a + e^\phi de^a &= e^\phi e^b \wedge \tilde{\omega}^a_b + (\omega + d\phi) \wedge e^\phi e^a \\
e^b \wedge \omega^a_b &= e^b \wedge \tilde{\omega}^a_b
\end{aligned}$$

and therefore $\tilde{\omega}^a_b = \omega^a_b$. It follows from eq.(28) that the full Weyl curvature tensor is scale invariant, $\tilde{\mathfrak{R}}^a_b = \mathfrak{R}^a_b$.

Substituting into the curvature, the algebra is identical to that leading up to eq.(8), with ϕ_a replaced by $-W_a$. This results in

$$\mathfrak{R}^a_b = \mathbf{R}^a_b - 2\Delta^{ac}_{db} \left(W_{c;e} + W_e W_c - \frac{1}{2} W^2 \eta_{ce} \right) e^e \wedge e^d$$

which decomposes into three parts when we separate the symmetric and antisymmetric parts of the trace term. With

$$\begin{aligned}
\Omega &= d\omega \\
\Omega_{ab} &= W_{[b;a]}
\end{aligned}$$

we have

$$\mathfrak{R}_b^a = \mathbf{C}_b^a - 2\Delta_{db}^{ac} \left(\mathcal{R}_{ce} + W_{(c;e)} + W_e W_c - \frac{1}{2} W^2 \eta_{ce} \right) \mathbf{e}^e \wedge \mathbf{e}^d - 2\Delta_{db}^{ac} \Omega_{ec} \mathbf{e}^e \wedge \mathbf{e}^d \quad (41)$$

where $\mathbf{R}_b^a = \mathbf{d}\alpha_b^a - \alpha_b^c \wedge \alpha_c^a$ is the Riemannian part of the curvature. Carrying out the decomposition of the curvature into trace and trace-free parts, we find the relationship between the Weyl and Schouten tensors of the Weyl and Riemannian geometries. In addition, the asymmetry of the Ricci tensor gives rise to a third independent component, the dilatational curvature:

$$\begin{aligned} \mathfrak{C}_b^a &= \mathbf{C}_b^a \\ \mathcal{R}_a &= \mathbf{R}_a + \left(W_{(a;c)} - W_a W_c + \frac{1}{2} W^2 \eta_{ac} \right) \wedge \mathbf{e}^c \\ \Omega &= W_{[b;a]} \mathbf{e}^a \wedge \mathbf{e}^b \end{aligned}$$

or in components,

$$\mathcal{R}_{ab} = \mathcal{R}_{ab} + W_{(a;b)} - W_a W_b + \frac{1}{2} W^2 \eta_{ab} \quad (42)$$

We define the Weyl-Schouten tensor \mathcal{R}_{ab} to be this symmetric part only.

In an integrable Weyl geometry, defined as one in which the dilatational curvature, Ω , vanishes, there exists a conformal transformation which makes the Weyl vector vanish, $W_a = 0$. In this gauge, the Weyl-Schouten tensor reduces to the Schouten tensor. The gravitational field, \mathbf{C}_b^a , is the same in all gauges.

The modification of eq.(41) in the presence of torsion follows immediately since it only changes the Weyl connection of eq.(40) by the contorsion tensor,

$$\hat{\omega}_b^a = \alpha_b^a - 2\Delta_{db}^{ac} W_c \mathbf{e}^d - C_{bc}^a \mathbf{e}^c = \omega_b^a - \mathbf{C}_b^a$$

This changes the curvature to

$$\hat{\mathfrak{R}}_b^a = \mathfrak{R}_b^a - \mathbf{D}\mathbf{C}_b^a + \mathbf{C}_b^c \wedge \mathbf{C}_c^a$$

4 Scale invariant gravity

We now turn to the formulation of a scale invariant gravity theory, based in a Weyl geometry. For this we must construct a Lorentz- and dilatation-invariant action functional from the curvature and any other available tensors. As we have noted, the conformal weight of the curvature components in an orthonormal basis is -2 . Since $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$, the Minkowski metric has conformal weight zero, making the conformal weight of the Weyl-Ricci scalar equal to -2 as expected. This introduces a difficulty in writing a scale invariant action in dimensions greater than two, since the volume element has weight $+n$ in n -dimensions.

4.1 Actions nonlinear in the curvature

In $2n$ -dimensions, we may use n -products of the curvature:

$$S = \int \mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \wedge \dots \wedge \mathfrak{R}^{ef} Q_{abcd\dots efg}$$

where $Q_{abcd\dots efg}$ is a rank- n invariant tensor. In 4-dim the most general such action is curvature-quadratic action, and takes the form

$$S = \int (\alpha \mathfrak{R}^{abcd} \mathfrak{R}_{abcd} + \beta \mathfrak{R}^{ab} \mathfrak{R}_{ab} + \gamma \mathfrak{R}^2) \sqrt{-g} d^4x$$

However, this may be simplified using the invariance of the Euler character. Variation of the Gauss-Bonnet combination for the Euler character $\chi = -\frac{1}{32\pi^2} \int \mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd}$, gives

$$\begin{aligned}
\delta\chi &= \delta \int \mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd} \\
&= 2 \int (\mathbf{d}(\delta\omega^{ab}) - (\delta\omega^{eb}) \wedge \omega_e^a - (\delta\omega^{ae}) \wedge \omega_e^b) \mathfrak{R}^{cd} \varepsilon_{abcd} \\
&= 2 \int \mathbf{D}(\delta\omega^{ab}) \wedge \mathfrak{R}^{cd} \varepsilon_{abcd} \\
&= 2 \int (\mathbf{D}(\delta\omega^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd}) + \delta\omega^{ab} \wedge \mathbf{D}\mathfrak{R}^{cd} \varepsilon_{abcd})
\end{aligned}$$

and this vanishes identically when we use the second Bianchi identity, $\mathbf{D}\mathfrak{R}^{cd} \equiv 0$, and let the variation vanish on the boundary,

$$\begin{aligned}
\delta\chi &= 2 \int_V \mathbf{D}(\delta\omega^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd}) \\
&= 2 \int_V \mathbf{d}(\delta\omega^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd}) \\
&= 2 (\delta\omega^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd}) \Big|_{\delta V} \\
&= 0
\end{aligned}$$

The addition of any multiple of the Euler character to the action therefore makes no contribution to the field equations.

We expand the Euler character as follows. Define a convenient volume element as the dual of unity, $\Phi \equiv *1$. Then:

$$\begin{aligned}
\Phi &\equiv *1 \\
&= \frac{1}{4!} \varepsilon_{abcd} \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d \\
*\Phi &= * \left(\frac{1}{4!} \varepsilon_{abcd} \mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d \right) \\
&= \frac{1}{4!} \varepsilon_{abcd} \varepsilon^{abcd} \\
&= -1
\end{aligned}$$

In a coordinate basis, $\Phi = \frac{1}{4!} \sqrt{-g} \varepsilon_{\mu\nu\alpha\beta} \mathbf{d}x^\mu \wedge \mathbf{d}x^\nu \wedge \mathbf{d}x^\alpha \wedge \mathbf{d}x^\beta$, and if we ignore orientation this is simply $\sqrt{-g} d^4x$. From the definition of Φ it follows that

$$\begin{aligned}
\varepsilon^{abcd} \Phi &= \frac{1}{4!} \varepsilon^{abcd} \varepsilon_{efgh} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^g \wedge \mathbf{e}^h \\
&= \frac{1}{4!} (-4! \delta_{efgh}^{abcd}) \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^g \wedge \mathbf{e}^h \\
&= -\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d
\end{aligned}$$

where² $\varepsilon^{abcd}\varepsilon_{efgh} = -4!\delta_{efgh}^{abcd}$. Therefore, substituting $\mathbf{e}^a \wedge \mathbf{e}^b \wedge \mathbf{e}^c \wedge \mathbf{e}^d = -\varepsilon^{abcd}\Phi$ into the expression for the Euler character,

$$\begin{aligned}\chi &= -\frac{1}{32\pi^2} \int \mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd} \\ &= -\frac{1}{128\pi^2} \int \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^g \wedge \mathbf{e}^h \varepsilon_{abcd} \\ &= \frac{1}{128\pi^2} \int \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \varepsilon^{efgh} \varepsilon_{abcd} \Phi \\ &= \frac{1}{128\pi^2} \int \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} 4!\delta_{abcd}^{efgh} \Phi\end{aligned}$$

Expanding the antisymmetric δ_{abcd}^{efgh} ,

$$\begin{aligned}4!\delta_{abcd}^{efgh} &\equiv \delta_a^e \left(\delta_b^f (\delta_c^g \delta_d^h - \delta_c^h \delta_d^g) + \delta_b^g (\delta_c^h \delta_d^f - \delta_c^f \delta_d^h) + \delta_b^h (\delta_c^f \delta_d^g - \delta_c^g \delta_d^f) \right) \\ &\quad - \delta_a^f (\delta_b^e (\delta_c^g \delta_d^h - \delta_c^h \delta_d^g) + \delta_b^g (\delta_c^h \delta_d^e - \delta_c^e \delta_d^h) + \delta_b^h (\delta_c^e \delta_d^g - \delta_c^g \delta_d^e)) \\ &\quad - \delta_a^g (\delta_b^f (\delta_c^e \delta_d^h - \delta_c^h \delta_d^e) + \delta_b^e (\delta_c^h \delta_d^f - \delta_c^f \delta_d^h) + \delta_b^h (\delta_c^f \delta_d^e - \delta_c^e \delta_d^f)) \\ &\quad - \delta_a^h (\delta_b^f (\delta_c^g \delta_d^e - \delta_c^e \delta_d^g) + \delta_b^g (\delta_c^e \delta_d^f - \delta_c^f \delta_d^e) + \delta_b^e (\delta_c^f \delta_d^g - \delta_c^g \delta_d^f))\end{aligned}$$

we explicitly write out the integrand of the Euler character in the Gauss-Bonnet form³,

$$\begin{aligned}\mathfrak{R}^{ab} \wedge \mathfrak{R}^{cd} \varepsilon_{abcd} &= -\frac{1}{4} \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} 4!\delta_{abcd}^{efgh} \Phi \\ &= -(\mathfrak{R}^2 - 4\mathfrak{R}^b{}_d \mathfrak{R}^d{}_b + \mathfrak{R}^{abcd} \mathfrak{R}_{abcd})\end{aligned}$$

²We check the normalization by contracting all pairs of indices, $(ae), (bf), (cg), (dh)$:

$$\begin{aligned}4! &= \delta_a^a \left(\delta_b^b (\delta_c^c \delta_d^d - \delta_c^d \delta_d^c) + \delta_b^c (\delta_c^d \delta_d^b - \delta_c^b \delta_d^d) + \delta_b^d (\delta_c^b \delta_d^c - \delta_c^c \delta_d^b) \right) \\ &\quad - \delta_a^b \left(\delta_b^a (\delta_c^c \delta_d^d - \delta_c^d \delta_d^c) + \delta_b^c (\delta_c^d \delta_d^a - \delta_c^a \delta_d^d) + \delta_b^d (\delta_c^a \delta_d^c - \delta_c^c \delta_d^a) \right) \\ &\quad - \delta_a^c \left(\delta_b^a (\delta_c^c \delta_d^d - \delta_c^d \delta_d^c) + \delta_b^c (\delta_c^d \delta_d^a - \delta_c^a \delta_d^d) + \delta_b^d (\delta_c^a \delta_d^c - \delta_c^c \delta_d^a) \right) \\ &\quad - \delta_a^d \left(\delta_b^a (\delta_c^c \delta_d^d - \delta_c^d \delta_d^c) + \delta_b^c (\delta_c^d \delta_d^b - \delta_c^b \delta_d^d) + \delta_b^d (\delta_c^b \delta_d^a - \delta_c^a \delta_d^b) \right) \\ &\quad - \delta_a^e \left(\delta_b^e (\delta_c^c \delta_d^d - \delta_c^d \delta_d^c) + \delta_b^c (\delta_c^d \delta_d^e - \delta_c^e \delta_d^d) + \delta_b^d (\delta_c^e \delta_d^c - \delta_c^c \delta_d^e) \right) \\ &= 4(4(16-4) + 4-16+4-16) - (64-16+4-16+4-16) \\ &\quad - (64-16+4-16+4-16) - (64-16+4-16+4-16) \\ &= 24\end{aligned}$$

3

$$\begin{aligned}4!\mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \delta_{abcd}^{efgh} &= \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \left(\delta_a^e \left(\delta_b^f (\delta_c^g \delta_d^h - \delta_c^h \delta_d^g) + \delta_b^g (\delta_c^h \delta_d^f - \delta_c^f \delta_d^h) + \delta_b^h (\delta_c^f \delta_d^g - \delta_c^g \delta_d^f) \right) \right) \\ &\quad - \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \delta_a^f \left(\delta_b^e (\delta_c^g \delta_d^h - \delta_c^h \delta_d^g) + \delta_b^g (\delta_c^h \delta_d^e - \delta_c^e \delta_d^h) + \delta_b^h (\delta_c^e \delta_d^g - \delta_c^g \delta_d^e) \right) \\ &\quad - \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \delta_a^g \left(\delta_b^f (\delta_c^e \delta_d^h - \delta_c^h \delta_d^e) + \delta_b^e (\delta_c^h \delta_d^f - \delta_c^f \delta_d^h) + \delta_b^h (\delta_c^f \delta_d^e - \delta_c^e \delta_d^f) \right) \\ &\quad - \mathfrak{R}^{ab}{}_{ef} \mathfrak{R}^{cd}{}_{gh} \delta_a^h \left(\delta_b^f (\delta_c^g \delta_d^e - \delta_c^e \delta_d^g) + \delta_b^g (\delta_c^e \delta_d^f - \delta_c^f \delta_d^e) + \delta_b^e (\delta_c^f \delta_d^g - \delta_c^g \delta_d^f) \right) \\ &= 2 \left(\mathfrak{R}^{ab}{}_{ab} \mathfrak{R}^{cd}{}_{cd} + \mathfrak{R}^{ab}{}_{ad} \mathfrak{R}^{cd}{}_{bc} + \mathfrak{R}^{ab}{}_{ac} \mathfrak{R}^{cd}{}_{bd} \right) - 2 \left(\mathfrak{R}^{ab}{}_{ba} \mathfrak{R}^{cd}{}_{cd} + \mathfrak{R}^{ab}{}_{da} \mathfrak{R}^{cd}{}_{bc} + \mathfrak{R}^{ab}{}_{ca} \mathfrak{R}^{cd}{}_{db} \right) \\ &\quad - 2 \left(\mathfrak{R}^{ab}{}_{cb} \mathfrak{R}^{cd}{}_{ad} + \mathfrak{R}^{ab}{}_{bd} \mathfrak{R}^{cd}{}_{ac} + \mathfrak{R}^{ab}{}_{dc} \mathfrak{R}^{cd}{}_{ab} \right) - 2 \left(\mathfrak{R}^{ab}{}_{db} \mathfrak{R}^{cd}{}_{ca} + \mathfrak{R}^{ab}{}_{bc} \mathfrak{R}^{cd}{}_{da} + \mathfrak{R}^{ab}{}_{cd} \mathfrak{R}^{cd}{}_{ba} \right) \\ &= 4 \left(\mathfrak{R}^2 - 4\mathfrak{R}^b{}_d \mathfrak{R}^d{}_b + \mathfrak{R}^{abcd} \mathfrak{R}_{abcd} \right)\end{aligned}$$

so that

$$\chi = \frac{1}{32\pi^2} \int (\mathfrak{R}^2 - 4\mathfrak{R}_d^b \mathfrak{R}_b^d + \mathfrak{R}^{abcd} \mathfrak{R}_{abcd}) \Phi$$

The invariance of χ allows us to replace

$$\int \mathfrak{R}^{abcd} \mathfrak{R}_{abcd} \Phi = 32\pi^2 \chi - \int (\mathfrak{R}^2 - 4\mathfrak{R}_d^b \mathfrak{R}_b^d) \Phi$$

Dropping the invariant first term leaves the most general curvature-quadratic action in the form

$$S = \int (a\mathfrak{R}^2 + b\mathfrak{R}^{ab}\mathfrak{R}_{ab}) \sqrt{-g} d^4x \quad (43)$$

for constants a, b . Quadratic gravity theories, especially the $b = 0$ case, have often been studied because the scale invariance allows the theory to be renormalizable. However, fourth order field equations such as those resulting from eq.(43) are sometimes found to introduce ghosts in the quantum theory. The Einstein-Hilbert term may be included as well, but while this has desirable effects, it breaks the scale invariance we examine here. For further discussion and references on quadratic gravity, see [24].

The particular case of Weyl (conformal) gravity deserves mention. As first shown by Bach [9], the *fully* conformally action for Weyl gravity may be written as

$$\begin{aligned} S_W &= \alpha \int C^{abcd} C_{abcd} \sqrt{-g} d^4x \\ &= \alpha \int \left(R^{abcd} R_{abcd} - 2R^{bd} R_{bd} + \frac{1}{3} R^2 \right) \sqrt{-g} d^4x \\ &= \left(32\pi^2 \alpha \chi - \alpha \int (R^2 - 4R_d^b R_b^d) \sqrt{-g} d^4x \right) + \alpha \int \left(-2R^{bd} R_{bd} + \frac{1}{3} R^2 \right) \sqrt{-g} d^4x \\ &= 32\pi^2 \alpha \chi + 2\alpha \int \left(R_d^b R_b^d - \frac{1}{3} R^2 \right) \sqrt{-g} d^4x \end{aligned} \quad (44)$$

Naturally, fourth order field equations result from *metric* variation of eq.(44). However, it is shown in [25] that varying the full conformal connection in the action 44 gives the additional integrability condition to reduce the fourth order equations to the Einstein equation.

Quadratic gravity applies only in four dimensions, with dimension $2n$ theories having correspondingly more complicated field equations. Instead, we consider Weyl invariant theories linear in the curvature.

4.2 Actions linear in the curvature

There are two classes of gravitational theories with Lorentz and dilatational symmetry with actions linear in the curvatures. Both may be written in any dimension.

4.2.1 Biconformal gravity

Based in geometries first developed by Ivanov and Niederle [16, 17] and Wheeler [18], what is now called biconformal gravity takes place in a $2n$ -dimensional symplectic manifold. The canonical conjugacy of this space makes the volume element dimensionless, allowing for a curvature-linear action in any dimension [19] of the form

$$S = \int (\alpha \Omega_{b_1}^{a_1} + \beta \delta_{b_1}^{a_1} \Omega + \gamma e^{a_1} \wedge \mathbf{f}_{b_1}) \wedge e^{a_2} \wedge \cdots \wedge e^{a_n} \wedge \mathbf{f}_{b_1} \wedge \cdots \wedge \mathbf{f}_{b_1} \varepsilon^{b_1 \dots b_n} \varepsilon_{a_1 \dots a_n} \quad (45)$$

The resulting theory describes n -dimensional gravity on an n -dimensional Lagrangian submanifold. Arising as the quotient of the conformal group by the Weyl group, the theory leads to scale invariant general relativity. Because the underlying structure is the full conformal group instead of the Weyl group, resulting in Kähler geometry instead of Weyl geometry, we will not go into further detail here.

4.2.2 The Dirac theory

An alternative approach to scale invariant gravity was developed by Dirac in an attempt to give rigor to his Large Numbers Hypothesis [26], the idea that extremely large dimensionless numbers in the description of nature should be related to one another. In [27], Dirac presents a scale invariant gravity theory in which the gravitational constant varies with time in such a way that the large dimensionless magnitude constructed from the fundamental charge e and G is related to the age of the universe. The result follows from a single, simple solution to the scale invariant theory. Here we examine the scale invariant theory without further discussion of the Large Numbers Hypothesis.

In the Dirac theory, scale invariance is achieved by including a gravitationally coupled scalar field in addition to the curvature. There is a wide literature on scalar fields coupled to gravity. Fierz [29] and Jordan [28] showed that the energy-momentum tensor of scalar theories may sometimes be unphysical. This flaw is corrected by the Brans-Dicke scalar-tensor theory [30], and discussion continues. In more recent work, [31], Romero, Fonseca-Neto, and Pucheu study the relationship between Brans-Dicke theory and integrable Weyl geometry is studied. See also the history by Brans [32].

Dirac begins with an action which in our notation takes the form

$$S_D = \int \left(\frac{1}{4} \Omega^{\mu\nu} \Omega_{\mu\nu} - \varphi^2 R + 6g^{\mu\nu} D_\mu \varphi D_\nu \varphi + c\varphi^4 \right) \sqrt{-g} d^n x \quad (46)$$

where $\Omega_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu}$ is the dilatational curvature. Dropping the dilatation, and allowing an arbitrary constant multiplying the kinetic term gives the Brans-Dicke scalar-tensor theory, usually written as

$$S_D = \int \left(\varphi R + \frac{\omega_0}{\varphi} g^{\mu\nu} D_\mu \varphi D_\nu \varphi + \mathcal{L}_m \right) \sqrt{-g} d^n x$$

where \mathcal{L}_m is a matter Lagrangian.

In the next Section, we take a similar but slightly different approach, allowing torsion and a mass term but no quadratic dilatational term. Our treatment also differs by our use of a Palatini variation, so the metric and connection are regarded as independent variables. Finally, we consider arbitrary spacetime dimension. We find a locally scale invariant theory that exactly reproduces general relativity as soon as a suitable definition of the unit of length is made.

5 Curvature linear, scale invariant gravity in any dimension

Beginning with the action for a Klein-Gordon scalar field, φ , in curved, n -dimensional Weyl geometry, we add a non-quadratic term analogous to Dirac's φ^4 potential,

$$S_\varphi = \int \left(g^{\mu\nu} D_\mu \varphi D_\nu \varphi + \frac{m^2 c^2}{\hbar^2} \varphi^2 + \beta \varphi^s \right) \sqrt{-g} d^n x \quad (47)$$

where the covariant derivative of φ is $D_\mu \varphi = \partial_\mu \varphi - \lambda W_\mu$, we include a gravitational term of the form where $\Phi \equiv *1 = \frac{1}{n!} \varepsilon_{a\dots b} \mathbf{e}^a \wedge \dots \wedge \mathbf{e}^b$ in n -dim and the power k will be chosen to make the full action scale invariant. There is no scalar we can form which is linear in the dilatational curvature. We choose units $\hbar = c = 1$.

To include gravity, we multiply a power of the scalar field times the scalar curvature. Thus, we arrive at

$$S = \int (\alpha \varphi^k \mathfrak{R} + g^{\mu\nu} D_\mu \varphi D_\nu \varphi + m^2 \varphi^2 + \beta \varphi^s) \sqrt{-g} d^n x \quad (48)$$

If the scalar field, metric, curvature and geometric mass scale as

$$\begin{aligned}
\varphi &\rightarrow e^{\lambda\phi}\varphi \\
g_{\mu\nu} &\rightarrow e^{2\phi}g_{\mu\nu} \\
g &\rightarrow e^{2n\phi}g \\
\mathfrak{R} &\rightarrow e^{-2\phi}\mathfrak{R} \\
m^2 = \frac{m^2c^2}{\hbar^2} &\rightarrow e^{-2\phi}m^2 \\
\alpha, \beta &\text{ dimensionless}
\end{aligned}$$

then S scales as

$$\begin{aligned}
\tilde{S} &= \int \left(\alpha\tilde{\varphi}^k\tilde{\mathfrak{R}} + \tilde{g}^{\mu\nu}D_\mu\tilde{\varphi}D_\nu\tilde{\varphi} + m^2\tilde{\varphi}^2 + \beta\tilde{\varphi}^s \right) \sqrt{-\tilde{g}}d^n x \\
&= \int \left(\alpha e^{(\lambda k - 2)\phi}\varphi^k\mathfrak{R} + e^{-2\phi}e^{2\lambda\phi}g^{\mu\nu}D_\mu\varphi D_\nu\varphi + e^{-2\phi}e^{2\lambda\phi}m^2\varphi^2 + e^{s\lambda\phi}\beta\varphi^s \right) e^{n\phi}\sqrt{-g}d^n x \\
&= \int \left[e^{(\lambda k - 2 + n)\phi}(\alpha\varphi^k\mathfrak{R}) + e^{(n - 2 + 2\lambda)\phi}(g^{\mu\nu}D_\mu\varphi D_\nu\varphi + m^2\varphi^2) + e^{(n + s\lambda)\phi}\beta\varphi^s \right] \sqrt{-g}d^n x
\end{aligned}$$

and is therefore locally scale invariant if

$$\begin{aligned}
\lambda &= -\frac{n-2}{2} \\
k &= 2 \\
s &= \frac{2n}{n-2}
\end{aligned}$$

We may make these assignments in any dimension greater than two, resulting in

$$S = \int \left(\alpha\varphi^2\mathfrak{R} + g^{\mu\nu}D_\mu\varphi D_\nu\varphi + m^2\varphi^2 + \beta\varphi^{\frac{2n}{n-2}} \right) \sqrt{-g}d^n x \quad (49)$$

5.1 Variation of the action

We consider the Palatini variation of S , varying the solder form, spin connection, Weyl vector and scalar field, $(e^a, \omega^a_b, W_a, \varphi)$, independently. The variation of the solder form is most easily accomplished by varying the metric. These are equivalent, since

$$\delta g_{\alpha\beta} = 2\eta_{ab}e_\alpha^a \delta e_\beta^b$$

and conversely

$$\delta e_\alpha^a = 2\eta^{ab}e_b^\beta \delta g_{\alpha\beta}$$

The connection variation is easiest using differential forms and varying ω^a_b directly.

5.1.1 Metric variation

Writing $\mathfrak{R} = g^{\mu\nu}\mathfrak{R}_{\mu\nu}$ where $\mathfrak{R}_{\mu\nu}$ depends only on the spin connection, and noting that $\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}g_{\mu\nu}\delta g^{\mu\nu}$, the metric variation is

$$0 = \int \delta g^{\mu\nu} \left[\alpha\varphi^2\mathfrak{R}_{\mu\nu} + D_\mu\varphi D_\nu\varphi - \frac{1}{2}g_{\mu\nu} \left(\alpha\varphi^2\mathfrak{R} + g^{\mu\nu}D_\mu\varphi D_\nu\varphi + m^2\varphi^2 + \beta\varphi^{\frac{2n}{n-2}} \right) \right] \sqrt{-g}d^n x$$

so we immediately get the field equation,

$$\alpha\varphi^2 \left(\mathfrak{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathfrak{R} \right) = -D_\mu\varphi D_\nu\varphi + \frac{1}{2}g_{\mu\nu} \left(D^\alpha\varphi D_\alpha\varphi + m^2\varphi^2 + \beta\varphi^{\frac{2n}{n-2}} \right)$$

This takes the form of the scale covariant Einstein equation

$$\mathfrak{R}_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\mathfrak{R} = \frac{1}{\alpha} \left[-\frac{1}{\varphi^2}D_\mu\varphi D_\nu\varphi + \frac{1}{2}g_{\mu\nu} \left(\frac{1}{\varphi^2}D^\alpha\varphi D_\alpha\varphi + m^2 + \beta\varphi^{\frac{4}{n-2}} \right) \right] \quad (50)$$

5.1.2 Scalar field variation

The scalar field variation is

$$\begin{aligned} 0 &= \int \left(2\alpha\delta\varphi\varphi\mathfrak{R} + 2g^{\mu\nu}D_\mu\delta\varphi D_\nu\varphi + 2m^2\varphi\delta\varphi + \frac{2n}{n-2}\beta\delta\varphi\varphi^{\frac{n+2}{n-2}} \right) \sqrt{-g}d^n x \\ &= \int \delta\varphi \left(2\alpha\varphi\mathfrak{R} - 2D_\mu(g^{\mu\nu}D_\nu\varphi) + 2m^2\varphi + \frac{2n}{n-2}\beta\varphi^{\frac{n+2}{n-2}} \right) \sqrt{-g}d^n x \end{aligned}$$

and therefore we find a nonlinear wave equation coupled to the scalar curvature,

$$D^a D_a\varphi - m^2\varphi - \alpha\mathfrak{R}\varphi - \frac{n}{n-2}\beta\varphi^{\frac{n+2}{n-2}} = 0 \quad (51)$$

5.1.3 Weyl vector variation

The Weyl vector only appears in the kinetic term for the scalar field, where $D_\mu\varphi = \partial_\mu\varphi - \lambda W_\mu\varphi$. Varying W_μ ,

$$0 = \int (2g^{\mu\nu}\lambda\delta W_\mu\varphi D_\nu\varphi) \sqrt{-g}d^n x$$

and therefore,

$$D_\alpha\varphi = 0 \quad (52)$$

This may immediately be solved for the Weyl vector

$$\begin{aligned} \mathbf{d}\varphi - \lambda\omega\varphi &= 0 \\ \omega &= \mathbf{d} \left(\frac{1}{\lambda} \ln \varphi \right) \\ &= -\frac{2}{n-2} \frac{1}{\varphi} \mathbf{d}\varphi \end{aligned}$$

which implies an integrable Weyl geometry and the existence of a gauge in which the Weyl vector vanishes. We easily find the gauge transformation ϕ required to remove the Weyl vector.

$$\begin{aligned} \tilde{\omega} &= \omega + \mathbf{d}\phi \\ 0 &= \mathbf{d} \left(\frac{1}{\lambda} \ln \varphi \right) + \mathbf{d}\phi \\ \phi &= -\frac{1}{\lambda} \ln \frac{\varphi}{\varphi_0} \end{aligned}$$

With this transformation, we have $\tilde{\omega} = 0$. This now remains the case for arbitrary *global* scale transformations.

The same transformation changes the scalar field according to

$$\begin{aligned}\tilde{\varphi} &= \varphi e^{\lambda\phi} \\ &= \varphi e^{\lambda\left(-\frac{1}{\lambda}\ln\frac{\varphi}{\varphi_0}\right)} \\ &= \varphi_0\end{aligned}$$

so the transformation that removes the Weyl vector simultaneously makes the scalar field constant. In an arbitrary gauge ϕ , the Weyl vector and scalar field become

$$\begin{aligned}W_\mu &= \partial_\mu\phi \\ \varphi &= \varphi_0 e^{\lambda\phi}\end{aligned}$$

but the physical content of the theory remains the same.

If we were to allow curvature squared terms in the action, it would involve $\Omega^*\Omega$ where $\Omega_{\mu\nu} = W_{\mu,\nu} - W_{\nu,\mu}$. Such a dilatational curvature would lead, in general, to a nonintegrable Weyl geometry. However, the physical constraints against such a geometry are extremely strong – we do not experience changes of relative physical size.

A quadratic term, $\Omega^*\Omega$, is a kinetic term for a nonintegrable Weyl vector. Except in dimension $n = 4$, scale invariance of such a kinetic term would also require a factor of the scalar field,

$$\frac{1}{4} \int \varphi^{\frac{2(n-4)}{n-2}} g^{\mu\alpha} g^{\nu\beta} \Omega_{\mu\nu} \Omega_{\alpha\beta} \sqrt{-g} d^n x$$

This would lead to a field equation of the form

$$\left(\varphi^{\frac{2(n-4)}{n-2}} \Omega^{\mu\nu}\right)_{;\nu} = 2\lambda D^\mu \varphi$$

which allows nontrivial φ but also a nontrivial dilatation, $\Omega_{\mu\nu}$. The dilatational curvature would also act as a source to the wave equation for φ ,

$$D^\mu D_\mu \varphi + m^2 \varphi + \frac{n}{n-2} \beta \varphi^{\frac{n+2}{n-2}} + \alpha \varphi \mathfrak{R} + \frac{n-4}{n-2} \varphi^{\frac{n-6}{n-2}} \Omega^{\alpha\beta} \Omega_{\alpha\beta} = 0$$

and as an energy source for the Einstein equation in the form

$$T_{\mu\nu} = \varphi^{\frac{2(n-4)}{n-2}} \left(\Omega_\mu^\beta \Omega_{\nu\beta} - \frac{1}{4} g_{\mu\nu} \Omega^{\alpha\beta} \Omega_{\alpha\beta} \right)$$

We continue without the kinetic term, because of the unphysical nature of dilatations.

5.1.4 Connection variation

The variation of the spin connection is much easier to work with than the variation of the Christoffel connection. Since only the curvature depends on the connection, we need only the first term in the action and the equivalence

$$\begin{aligned}\frac{\alpha}{(n-2)!} \varphi^2 \mathfrak{R}^{ab} \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} &= \frac{\alpha}{(n-2)!} \varphi^2 \frac{1}{2} \mathfrak{R}^{ab}{}_{ef} \mathbf{e}^e \wedge \mathbf{e}^f \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} \\ &= -\frac{\alpha}{(n-2)!} \varphi^2 \frac{1}{2} \mathfrak{R}^{ab}{}_{ef} \varepsilon^{efc\dots d} \varepsilon_{abc\dots d} \Phi \\ &= \frac{\alpha}{(n-2)!} \varphi^2 \frac{1}{2} \mathfrak{R}^{ab}{}_{ef} (n-2)! \left(\delta_a^e \delta_b^f - \delta_a^f \delta_b^e \right) \Phi \\ &= \alpha \varphi^2 \mathfrak{R} \Phi\end{aligned}$$

Therefore, replacing $\int \alpha \varphi^k \mathfrak{R} \sqrt{-g} d^n x$ with $\int \frac{\alpha}{(n-2)!} \varphi^2 \delta_{\omega_b^a} \mathfrak{R}^{ab} \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d}$, we vary ω^a_b ,

$$\begin{aligned}
\delta_{\omega^{ab}} S &= \int \frac{\alpha}{(n-2)!} \varphi^2 \delta_{\omega_b^a} \mathfrak{R}^{ab} \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} \\
&= \int \frac{\alpha}{(n-2)!} \varphi^2 \delta_{\omega^{ab}} (\mathbf{d}\omega^{ab} - \eta_{ef} \omega^{eb} \wedge \omega^{af}) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} \\
&= \int \frac{\alpha}{(n-2)!} \varphi^2 (\mathbf{d}\omega^{ab} - \delta\omega^{eb} \wedge \omega^a_e - \delta\omega^{ae} \wedge \omega^b_e) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} \\
&= \int \frac{\alpha}{(n-2)!} \varphi^2 \mathbf{D}(\delta\omega^{ab}) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} \\
&= \int \frac{\alpha}{(n-2)!} \mathbf{D}(\varphi^2 (\delta\omega^{ab}) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d}) - \int \frac{\alpha}{(n-2)!} (2\varphi (\mathbf{D}\varphi) \delta\omega^a_b \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d}) \\
&\quad + \int \frac{\alpha}{(n-2)!} ((n-2) \varphi^2 \delta\omega^{ab} \wedge \mathbf{D}\mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d})
\end{aligned}$$

The first term is a total divergence which we discard. Then, writing $\delta\omega^{ab} = \delta\omega^{ab}_k \mathbf{e}^k$ and noticing that $\mathbf{D}\mathbf{e}^c$ is the torsion 2-form, \mathbf{T}^c

$$0 = \int \frac{\alpha}{(n-2)!} \delta\omega^{ab}_k \mathbf{e}^k \wedge (2\varphi (\mathbf{D}\varphi) \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} + (n-2) \varphi^2 \mathbf{T}^c \wedge \mathbf{e}^d \wedge \cdots \wedge \mathbf{e}^e \varepsilon_{abcd\dots e})$$

and therefore

$$\begin{aligned}
0 &= -\frac{2}{n-2} \frac{1}{\varphi} \mathbf{D}\varphi \wedge \mathbf{e}^k \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} + \mathbf{T}^c \wedge \mathbf{e}^k \wedge \mathbf{e}^d \wedge \cdots \wedge \mathbf{e}^e \varepsilon_{abcd\dots e} \\
&= -\frac{2}{n-2} \frac{1}{\varphi} D_m \varphi \mathbf{e}^m \wedge \mathbf{e}^k \wedge \mathbf{e}^c \wedge \cdots \wedge \mathbf{e}^d \varepsilon_{abc\dots d} + \frac{1}{2} T^c_{mn} \mathbf{e}^m \wedge \mathbf{e}^n \wedge \mathbf{e}^k \wedge \mathbf{e}^d \wedge \cdots \wedge \mathbf{e}^e \varepsilon_{abcd\dots e} \\
&= \left(\frac{2}{n-2} \frac{1}{\varphi} D_m \varphi \varepsilon^{mkc\dots d} \varepsilon_{abc\dots d} - \frac{1}{2} T^c_{mn} \varepsilon^{mnkd\dots e} \varepsilon_{abcd\dots e} \right) \Phi
\end{aligned}$$

Taking the dual eliminates the volume form. Then, resolving the pairs of Levi-Civita tensors,

$$\begin{aligned}
0 &= -\frac{2}{n-2} \frac{1}{\varphi} D_m \varphi (n-2)! (\delta_a^m \delta_b^k - \delta_a^k \delta_b^m) \\
&\quad + \frac{1}{2} (n-3)! T^c_{mn} (\delta_a^m (\delta_b^k \delta_c^n - \delta_b^n \delta_c^k) + \delta_a^n (\delta_b^k \delta_c^m - \delta_b^m \delta_c^k) + \delta_a^k (\delta_b^m \delta_c^n - \delta_b^n \delta_c^m)) \\
0 &= \frac{1}{2} (T^k_{ab} - \delta_b^k T^m_{an} + T^m_{ma} \delta_b^k - T^k_{ba} + 2\delta_a^k T^c_{bc}) - \frac{2}{\varphi} (\delta_b^k D_a \varphi - \delta_a^k D_b \varphi) \\
&= T^k_{ab} + \delta_b^k T^m_{ma} - \delta_a^k T^c_{cb} - \frac{2}{\varphi} (\delta_b^k D_a \varphi - \delta_a^k D_b \varphi)
\end{aligned}$$

The trace gives

$$\begin{aligned}
0 &= (n-2) T^m_{ma} - \frac{2}{\varphi} (n-1) D_a \varphi \\
T^m_{ma} &= \frac{2}{\varphi} \frac{n-1}{n-2} D_a \varphi
\end{aligned}$$

so that

$$\begin{aligned}
T^k_{ab} &= \delta_a^k T^c_{cb} - \delta_b^k T^m_{ma} + \frac{2}{\varphi} (\delta_b^k D_a \varphi - \delta_a^k D_b \varphi) \\
&= \frac{2}{\varphi} \left(\frac{n-1}{n-2} \delta_a^k D_b \varphi - \frac{n-1}{n-2} \delta_b^k D_a \varphi + \delta_b^k D_a \varphi - \delta_a^k D_b \varphi \right) \\
&= \frac{2}{\varphi} \left(\left(\frac{n-1}{n-2} - 1 \right) \delta_a^k D_b \varphi - \left(\frac{n-1}{n-2} - 1 \right) \delta_b^k D_a \varphi \right)
\end{aligned}$$

and we arrive at a solution for the torsion,

$$T^c{}_{ab} = \frac{1}{n-2} \frac{2}{\varphi} (\delta_a^c D_b \varphi - \delta_b^c D_a \varphi) \quad (53)$$

5.2 Collected field equations

Collecting the variational field equations, eqs.(50-53), and restoring the orthonormal basis,

$$\mathfrak{R}_{ab} - \frac{1}{2} \eta_{ab} \mathfrak{R} = \frac{1}{\alpha} \left[-\frac{1}{\varphi^2} D_a \varphi D_b \varphi + \frac{1}{2} \eta_{ab} \left(\frac{1}{\varphi^2} D^c \varphi D_c \varphi + m^2 + \beta \varphi^{\frac{4}{n-2}} \right) \right] \quad (54)$$

$$T^c{}_{ab} = \frac{2}{n-2} \frac{1}{\varphi} (\delta_a^c D_b \varphi - \delta_b^c D_a \varphi) \quad (55)$$

$$D_a \varphi = 0 \quad (56)$$

$$D^a D_a \varphi - m^2 \varphi - \alpha \mathfrak{R} \varphi - \frac{n}{n-2} \beta \varphi^{\frac{n+2}{n-2}} = 0 \quad (57)$$

It is interesting to note that the form of the torsion in eq.(55) is that of Einstein's lambda transformation, which is from the most general transformation of the connection leaving the curvature unchanged. However, substituting eq.(56) into eq.(55), we see that the torsion vanishes,

$$T^c{}_{ab} = 0$$

Again using eq.(56) in eq.(57) get a relationship between the mass, the scalar field and the scalar curvature,

$$\mathfrak{R} = -\frac{m^2}{\alpha} - \frac{n}{n-2} \frac{\beta}{\alpha} \varphi^{\frac{4}{n-2}}$$

A similar but different relation arises when we look at the trace of eq.(54) with $\varphi_a = 0$,

$$\begin{aligned} \mathfrak{R}_{ab} - \frac{1}{2} \mathfrak{R} \eta_{ab} &= \frac{1}{2\alpha\varphi^2} \eta_{ab} \left(m^2 \varphi^2 + \beta \varphi^{\frac{2n}{n-2}} \right) \\ \mathfrak{R} &= -\frac{1}{\alpha} \frac{n}{n-2} \left(m^2 + \beta \varphi^{\frac{4}{n-2}} \right) \end{aligned}$$

Comparing the two,

$$\begin{aligned} -\frac{m^2}{\alpha} - \frac{n}{n-2} \frac{\beta}{\alpha} \varphi^{\frac{4}{n-2}} &= -\frac{1}{\alpha} \frac{n}{n-2} m^2 - \frac{n}{n-2} \frac{\beta}{\alpha} \varphi^{\frac{4}{n-2}} \\ \frac{2}{n-2} m^2 &= 0 \end{aligned}$$

so the mass of φ must vanish in any dimension. However, there is no constraint on $\frac{\beta}{\alpha} \varphi^{\frac{4}{n-2}}$, which in the Riemannian gauge leads to the usual Einstein equation with cosmological constant.

As note following eq.(52), $\mathbf{D}\varphi = 0$ requires that there exist a local gauge in which the Weyl vector vanishes. In this gauge, the Weyl-Riemann tensor reduces to the usual Riemann curvature,

$$\mathfrak{R}^a{}_b|_{W_a=0, T^a{}_{bc}=0} = \mathbf{R}^a{}_b$$

and the Einstein tensor takes the usual Riemannian form. The system has reduced to exactly the vacuum Einstein equation with cosmological constant Λ in a Riemannian geometry,

$$\begin{aligned} R_{ab} - \frac{1}{2} \eta_{ab} R - \eta_{ab} \Lambda &= 0 \\ T^c{}_{ab} &= 0 \\ \varphi &= \varphi_0 \\ W_a &= 0 \end{aligned}$$

where

$$\Lambda \equiv \frac{\beta}{2\alpha} \varphi_0^{\frac{4}{n-2}}$$

the only difference being that in this formulation we retain local scale invariance with an integrable Weyl vector. In a general gauge ϕ , the solution takes the form:

$$\begin{aligned} \mathfrak{R}_{ab} - \frac{1}{2} \mathfrak{R} \eta_{ab} &= 0 \\ T^a_{bc} &= 0 \\ W_a &= \partial_a \phi \\ \varphi &= \varphi_0 e^{-\frac{n-2}{2} \phi} \end{aligned}$$

where this form of the scalar field φ insures that $D_a \varphi = 0$. The form in a general gauge describes exactly the same physical situation, but in a set of units which may vary from place to place.

6 Conclusion

We developed the properties of Weyl geometry using the Cartan formalism for gauge theories, including enough details of the calculations to illustrate the techniques and show the advantages of the Cartan approach. Beginning with a review of the conformal properties of Riemannian spacetimes, we present an efficient form of the decomposition of the Riemann curvature into trace and traceless parts. This allows an easy proof that the Weyl curvature tensor is the conformally invariant part of the Riemann curvature, and shows the explicit change in the Ricci and Schouten tensors produced by a conformal transformation.

By writing the change in the Schouten tensor as a system of differential equations for the conformal factor, we reproduce the well-known condition for the existence of a conformal transformation to a Ricci-flat spacetime as a pair of integrability conditions. Continuing with the streamlined approach, we generalize this condition to a derivation of the condition for the existence of a conformal transformation to a spacetime satisfying the Einstein equation with matter sources. The inclusion of matter sources is a new result.

Next, enlarging the symmetry from Poincaré to Weyl, we develop the Cartan structure equations of Cartan-Weyl geometry without assuming vanishing torsion. We find the form of the curvature tensor and its relationship to the Riemann curvature of the corresponding Riemannian geometry, and show that the spin connection reproduces the expected coordinate form of Weyl connection plus contorsion tensor. We then look at possible gravity theories. We use the Gauss-Bonnet form of the Euler character to write the general form of quadratic-curvature action in terms of the Ricci tensor and scalar, then turn to a detailed description of a modified form of the Dirac scalar-tensor action. Our approach differs from that of either Dirac or Brans-Dicke in three ways: we allow nonvanishing torsion, we vary the solder form and spin connection independently, and we work in arbitrary dimension. We find that the torsion and gradient of the scalar field must both vanish, exactly reducing the system to locally scale-covariant general relativity with cosmological constant.

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Appendix: Bianchi identities

The Cartan structure equations,

$$\begin{aligned} d\omega_b^a &= \omega_b^c \wedge \omega_c^a + \mathfrak{R}_b^a \\ de^a &= e^b \wedge \omega_b^a + \omega \wedge e^a + \mathbf{T}^a \\ d\omega &= \Omega \\ W_{\mu,\nu} dx^\nu \wedge dx^\mu &= \frac{1}{2} \Omega_{\mu\nu} dx^\mu \wedge dx^\nu \\ W_{\mu,\nu} - W_{\nu,\mu} &= \Omega_{\mu\nu} \end{aligned}$$

have integrability conditions, which for gravity theories are called Bianchi identities. These follow from the Poincaré lemma, $d^2 \equiv 0$:

$$\begin{aligned} d\mathfrak{R}_b^a &= d^2\omega_b^a - d\omega_b^c \wedge \omega_c^a + \omega_b^c \wedge d\omega_c^a \\ 0 &= d\mathfrak{R}_b^a + (\mathfrak{R}_b^c + \omega_b^e \wedge \omega_e^c) \wedge \omega_c^a - \omega_b^c \wedge (\mathfrak{R}_c^a + \omega_c^e \wedge \omega_e^a) \\ &= d\mathfrak{R}_b^a + \mathfrak{R}_b^c \wedge \omega_c^a - \mathfrak{R}_c^a \wedge \omega_b^c \\ D\mathfrak{R}_b^a &= 0 \\ d^2e^a &= de^b \wedge \omega_b^a - e^b \wedge d\omega_b^a + d\omega \wedge e^a - \omega \wedge de^a + d\mathbf{T}^a \\ 0 &= (e^c \wedge \omega_c^b + \omega \wedge e^b + \mathbf{T}^b) \wedge \omega_b^a - e^b \wedge (\omega_b^c \wedge \omega_c^a + \mathfrak{R}_b^a) + \Omega \wedge e^a - \omega \wedge (e^b \wedge \omega_b^a + \omega \wedge e^a + \mathbf{T}^a) + d\mathbf{T}^a \\ 0 &= -e^b \wedge \mathfrak{R}_b^a + \Omega \wedge e^a + d\mathbf{T}^a + \mathbf{T}^b \wedge \omega_b^a - \omega \wedge \mathbf{T}^a \\ D\mathbf{T}^a &= e^b \wedge \mathfrak{R}_b^a - \Omega \wedge e^a \\ d\Omega &= 0 \end{aligned}$$

Summary:

$$\begin{aligned} D\mathfrak{R}_b^a &\equiv d\mathfrak{R}_b^a + \mathfrak{R}_b^c \wedge \omega_c^a - \mathfrak{R}_c^a \wedge \omega_b^c \equiv 0 \\ D\mathbf{T}^a &= e^b \wedge \mathfrak{R}_b^a - \Omega \wedge e^a \\ D\Omega &= 0 \end{aligned}$$

where

$$\begin{aligned} D\mathfrak{R}_b^a &\equiv d\mathfrak{R}_b^a + \mathfrak{R}_b^c \wedge \omega_c^a - \mathfrak{R}_c^a \wedge \omega_b^c \\ D\mathbf{T}^a &\equiv d\mathbf{T}^a + \mathbf{T}^b \wedge \omega_b^a - \omega \wedge \mathbf{T}^a \\ D\Omega &\equiv d\Omega \end{aligned}$$

When the torsion vanishes, we have an algebraic identity,

$$\begin{aligned} e^b \mathfrak{R}_b^a &= \Omega \wedge e^a \\ \mathfrak{R}_{[bcd]}^a &= \delta_{[b}^a \Omega_{cd]} \\ \mathfrak{R}_{bcd}^a + \mathfrak{R}_{cdb}^a + \mathfrak{R}_{dbc}^a &= \delta_b^a \Omega_{cd} + \delta_c^a \Omega_{db} + \delta_d^a \Omega_{bc} \\ \mathfrak{R}_{bd} - \mathfrak{R}_{db} &= -(n-2) \Omega_{bd} \end{aligned}$$