Using an Iceberg Intervention Model to Understand Equivalent Fraction Learning when Students with Mathematical Learning Difficulties use Different Manipulatives

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This study examined variations in 43 fifth-grade Tier II students' learning of equivalent fractions using physical and virtual manipulatives during intervention instruction. The overarching research question focused on how different manipulatives types support learning fraction sub-concepts during mathematics intervention instruction for students with mathematical learning difficulties. Over a three-week period, students participated in ten small group sessions using different manipulatives during instruction (physical, virtual, and combination). Data were collected from pre/post testing and daily monitoring assessments. An Iceberg Intervention Model was used to document student learning for five equivalent fraction sub-concepts and nine general fraction sub-concepts. Results showed that physical manipulatives were favoured for 5 sub-concepts, virtual manipulatives for 4 sub-concepts and combined manipulatives for two sub-concepts. The results demonstrate that an understanding of relationships between manipulative type and equivalent fraction subtopics can be used to guide the use of manipulatives during intervention instruction.

1 INTRODUCTION

In the past two decades there has been an increased focus on Tier II interventions. Tier II interventions target students who have mathematical learning difficulties, but who do not receive special education services (Fuchs, 2005). However, the development of intervention practices for Tier II students has been limited by a lack of intervention research (Fuchs, Seethaler, Powell, Fuchs, Hamlett and Fletcher, 2008; Glover and DiPerna, 2007). One area of research needed is how to effectively use mathematical representations (e.g., manipulatives) during intervention instruction (Gersten, Beckmann, Clarke, Foegen, Marsh, Star and Witzel, 2009). Different types of manipulatives have unique affordances that affect learning. A craftsman, in their work, will use a variety of tools. Their understanding and use of different tools allows them to select the best tool for each job. In a similar manner, a teacher or designer of mathematics intervention must learn which tools to select when teaching different mathematical concepts. To date, much of the research on manipulatives has focused on comparing which manipulatives are most effective for teaching. This study took a different approach by examining how different manipulative types supported learning equivalent fraction sub-concepts for students participating in mathematics intervention instruction. An Iceberg Intervention Model of equivalent fraction sub-concepts was used to synthesize variations in students' learning.

2 LITERATURE REVIEW

Gersten et al. (2009) suggested that many students with mathematical learning difficulties struggle to connect abstract fraction symbols to internal visual representations and view fractions only as groups of whole numbers. Fifty-five percent of the 13-year-olds in Behr and Post's (1992) study estimated $12/13 + 7/8$ to be either 19 or 21. These students had not developed internal representations of the magnitude of the fractions. Manipulatives are external representations which are used to aid students in their development of internal visual representations (Baroody, 1989; Behr, Lesh, Post and Silver, 1983). Four meta-analyses summarising the research findings on manipulatives in mathematics instruction reported that use of manipulatives was effective for improving students' understanding of mathematics (Carbonneau, Marley and Selig, 2013; Parham, 1983; Sowell, 1989; Suydam and Higgins, 1977)

Physical and Virtual Manipulatives

Two common manipulative types used in schools are virtual and physical manipulatives. Virtual manipulatives are defined as an "interactive, web based, visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (Moyer, Bolyard and Spikell, 2002, p. 373). Moyer-Packenham and Westenskow's (2013) meta-analysis of 66 studies reported that the use of virtual manipulatives produced an overall moderate effect (Cohen $d = 0.75$) favouring virtual manipulatives when compared to traditional textbook instruction. Although most virtual manipulatives used during fraction instruction are based on the structure of the physical manipulatives, there are some important differences. Moyer-Packenham and Westenskow (2013) identified five categories of affordances of virtual manipulatives (VMs) that impact learning: simultaneous linking, focused constraint, efficient precision, creative variation and motivation. Simultaneous linking is when the VMs contain features that link symbolic and pictorial representations. Focused constraint includes VM features that focus students' attention to specific representation changes or mathematical aspects. Efficient precision is a structural feature of the VMs that enhances students' ability to use the manipulatives efficiently.
and accurately. Creative variation is when the VMs contain features that encourage creativity and a variety of solutions. The motivation affordance is when the VMs contain elements that capture students' interest and encourage persistence.

Physical manipulatives also have specific affordances. They are tangible, familiar, and easy to use. Some authors have suggested that when students initially learn concepts, they should use objects which they can handle and manipulate physically (Hunt, Nipper and Nash, 2011; Martin, Sviha and Smith, 2012; Swan and Marshall, 2010; Takahashi, 2002). Through manipulation, students explore the concrete object, and develop internal representations and conceptual understanding (Lakoff and Nunez, 2000; McNeil and Jarvin, 2007; Skemp, 1987). Martin and Schwartz's (2005) theory of Physically Distributed Learning suggests that physical manipulation of objects develops and challenges students' mathematical understanding. At an early age, students learn to manipulate physical objects and to use the objects to off-load memory and make their cognition processes easier (Manches and O'Malley, 2011). However, the use of manipulatives is only effective when learner attention is focused on the mathematics concept and not on the tool itself (Boulton-Lewis, 1998). If a manipulative is too interesting or too cumbersome, it can distract the learner from thinking about and internalizing the mathematical concepts (Belenky and Schalk, 2014; Kaminski, Sloutsky and Heckler, 2009; McNeil, Uttal, Jarvin and Stemberg, 2009; Uttal, Scudder and DeLoache, 1997). Physical manipulatives typically used for teaching fractions (e.g., fraction circles) have the affordance of being easily manipulated.

A meta-analysis comparing virtual and physical manipulatives in 38 comparisons yielded only a small Cohen $d$ effect size difference of 0.15 favouring virtual manipulatives (Moyer-Packenham and Westenskow, 2013). Although a slight difference was identified, there were many variations among the individual studies with some yielding moderate to large effect sizes favouring physical manipulatives (Burns and Hamm, 2011; Kim, 1993; Lane, 2010; Nute 1997; Plett, 1991; Smith, 2006; Suh and Moyer, 2007; Terry, 1995; Yuan, Lee and Wang, 2010) and others yielding moderate to large effect sizes favoring virtual manipulatives (Berlin and White, 1986; Daghestani, Al-Nuaim and Al-Mshat, 2004; Deliyianni, Michael and Pitta-Pantazi, 2006; Kim, 1993; Lane 2010; Lin 2010; Manches, O'Malley and Benford, 2010; Martin and Lukong, 2005; Moyer-Packenham and Suh, 2012; Nute, 1997; Smith, 1995; Steen, Brooks and Lyon, 2006; Suh and Moyer-Packenham, 2007). In three of the studies, although the participants and the setting within the study remained constant, moderate to large effect sizes favouring the use of physical manipulative instruction was found for one topic and moderate to large effect sizes favouring the use of virtual manipulatives was found for a different topic (Kim, 1993; Suh and Moyer-Packenham, 2007, Lane 2010). This suggests that the advantages of using one type of manipulative over another may be specific to the relationship between manipulative affordances and the content being learned. Behr et al. (1983) suggested that because manipulatives differ in their perceptual features, they will differ in the "mathematical way they embody the concept" (p 32). If the effectiveness of manipulative affordances is specific to the relationship between manipulative features and content, then to develop optimum intervention instruction it is important to identify areas of variance. To identify areas of variance a model of intervention was developed.

2 DEVELOPMENT OF THE ICEBERG INTERVENTION MODEL FOR EQUIVALENT FRACTION

Not all children enter the classroom with the same knowledge and abilities and not all children learn concepts at the same rate. As a result of these differences or as a result of having only limited opportunities to learn, some students develop misconceptions or have "gaps" in their understandings and need mathematics intervention. To provide efficient and effective interventions, a teacher must have an understanding of students' misconceptions and areas of inadequate knowledge development (Li, 2008). One way to identify students' content weaknesses and misconceptions is to use an Iceberg Intervention Model. Iceberg models of learning were originally introduced by the Freudenthal Institute (Webb, Boswinkel and Dekker, 2008). In the iceberg model, the part of the iceberg above the water line represents the understanding typically assessed during classroom instruction. Although the tip of the iceberg is easily visible, the majority of the mass of the iceberg is below the water line. In the Iceberg model, the knowledge, understandings, and skills a student needs for mastery of a concept are the mass below the water line. The more basic the skill, the lower it is placed on the iceberg model. When adapted to intervention settings, the model provides a visual representation of the strengths and weakness of students' understanding. The Equivalent Fraction Iceberg Intervention Model developed for this study was designed through a review and synthesis of literature describing students' misconceptions and common topics of inadequate development of equivalent fraction understandings. The tip of the iceberg or Level I in the model is equivalent fraction understanding (see Figure 1); the ability to develop and use equivalent fractions in problem solving situations. Level II consists of five sub-concepts important to the understanding of equivalent fractions. Level III consists of three areas of general fraction understanding which are necessary for an understanding of the five equivalent fraction sub concepts. Each area of general fraction understanding also contains three sub concepts.

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The Equivalent Fraction Iceberg Intervention Model is not a model that shows the progression of learning, but a model used to identify the strengths and weaknesses of students' understanding at a given point in time. A table giving the descriptions of each sub concepts' understandings and common misconceptions is provided in Appendix A. The Iceberg Intervention Model is a dynamic tool which is constantly being adapted to reflect new research. The model shown here is the model used in this research study. The model has since undergone minor revisions.

When providing intervention, the model is used to identify strengths and weakness in students' equivalent fraction understanding. Concepts are interrelated and the model provides a visual representation of the specific needs of a student. For example, assessments may indicate that a student can correctly develop two equivalent fractions, but has difficulty developing groups of equivalent fractions and solving equivalent fraction equations. To understand this student’s disconnect in understanding, we would examine the student’s mastery of Level III concepts and observe that the student is using an additive rather than a multiplicative procedure for developing equivalent fractions \( \frac{a}{b} = \frac{(a + a)}{(b + b)} \). The use of an addition strategy may be limiting the student’s ability to identify and develop groups of equivalent fractions. Intervention should focus not just on developing equivalent fraction groups, but also the development and use of multiplicative thinking.

In this study, the Equivalent Fraction Intervention Model was used to identify variations in the effects of manipulative affordances and to examine connections between variances. Each sub-concept of the model requires a slightly different type of thinking or understanding and each manipulative type has unique affordances which affect student learning. Better understanding the effects of manipulative affordances on remediation of each sub-concept has the potential to enhance intervention effectiveness. The overarching research question guiding this study was: How do different manipulative types support learning fraction sub-concepts during mathematics intervention instruction for students with mathematical learning difficulties? Sub-questions focused on: a) How do different manipulative types support students’ ability to interpret and develop equivalent fraction understanding? b) How do different manipulative types support students’ ability to model, identify, group, solve and simplify fractions? and c) How do different manipulative types support students’ ability to name fractions, compare fractions, and demonstrate equivalence thinking?

2.1 METHODS

This study used an explanatory mixed methods approach of triangulating evidence from both quantitative and qualitative data in answering each of the research questions (Creswell, Plano Clark, Gutmann and Hanson, 2003). Building from constructivist epistemology, this study, through the observations of student learning, describes how students’ understanding of equivalent fraction sub concepts is affected by manipulative use. The research activities included Tier II intervention for fifth grade students who did not demonstrate mastery of equivalent fractions concepts; concepts which the Common Core State Standards (2010) suggest should be mastered in fourth grade. Data were analyzed using a transitional conceptions perspective; not only were pre and post intervention data analyzed, but the data were also analyzed at the individual lesson and student levels. This made it possible to identify the effects of the manipulatives on students’ equivalent fraction understanding at both the overall level and at the sub-concept levels (Shaughnessy, 2007). In this type of study, validity is developed through triangulation and complementarity of data from several sources. By using rigorous practices of collecting and analyzing data from multiple sources that can be used to confirm observations, and by making those practices transparent, this strengthens the validity and reliability of the study (Creswell and Plano-Clark, 2011).

2.2 PARTICIPANTS

A total of 43 fifth-grade students from four western suburban schools participated in this study. The students were predominately Caucasian and middle class. About one-
fourth of the students came from a school classified as a Title I school, indicating that they were of lower social economic status. There were 14 males and 29 female who participated in the study and completed the intervention instruction and testing. To select participants for the study, the Equivalent Fractions pre-test was administered to 182 fifth-grade students in four schools. The scores ranged from 5% to 100% correct with a mean score of 51.1 percent. All students who scored below 40% correct were invited to participate in the study. Students receiving special education services for mathematics were not included because these students typically receive Tier 3 interventions. At the teachers' request, eight additional students who struggled in learning mathematics also participated. Using a stratified selection process based on pre-test scores, participants were assigned to one of three intervention groups: physical manipulatives alone (PM group), virtual manipulatives alone (VM group) or physical and virtual manipulative combined group (CM group). Instructional groups consisted of two to four students per group. The intervention was conducted before the students studied fractions in fifth grade.

2.3 PROCEDURES

Each student group participated in 10 instructional lessons over a period of three weeks. Each lesson was approximately 45 minutes in length and consisted of three phases: 1) Concept introduction and exploration (25 minutes); 2) Fraction skill practice (10 minutes); and 3) Daily assessments (10 minutes). The Rational Number Project: Initial Fraction Ideas Lessons (Cramer et al, 2009) were used during the concept introduction and exploration phase. The focus of the lessons were; Lesson 1 and 2 – naming fractions; Lesson 3 and 4 – comparing fractions; Lesson 5, 6 and 7 – identifying and developing equivalent fractions using region models; and, Lesson 8, 9 and 10 – developing equivalent fractions using set models. During the practice phase, students practiced the procedures of naming and comparing fractions and developing equivalent fractions. In the daily assessment phase, all students completed paper and pencil assessments. All three instructional groups completed approximately the same number of problems during each lesson.

Each lesson used manipulatives and pictorial representations to develop and practice fraction concepts. With the exception of the type of manipulative used, all three instructional groups received the same instruction and participated in the same activities during concept introduction and exploration phases. For one section of Lesson 7, paper strips were used by all three groups because the activity could not be duplicated with virtual manipulatives. The physical manipulatives used during the introduction and exploration phases were fraction circles and squares and two-colour counters (see Figure 2). The virtual manipulatives used were Fraction Pieces (circles and squares) and Pattern Blocks (used as counters). During the practice phase of the lessons, the virtual manipulatives used were Fractions-Naming, Fractions-Equivalence, and Fractions-Comparison, which were located in the National Library of Virtual Manipulatives website (http://nlvm.usu.edu; retrieved January 10, 2015), and Equivalent Fractions located in the NCTM Illuminations website (http://illuminations.nctm.org; retrieved January 10, 2015).

<table>
<thead>
<tr>
<th>Physical Manipulatives</th>
<th>Virtual Manipulatives</th>
</tr>
</thead>
<tbody>
<tr>
<td>Fraction Circles</td>
<td>Fraction Pieces</td>
</tr>
<tr>
<td></td>
<td>Pattern Blocks</td>
</tr>
<tr>
<td>Fraction Squares</td>
<td>Fractions-Naming</td>
</tr>
<tr>
<td></td>
<td>Fractions-Equivalence</td>
</tr>
<tr>
<td>Counter</td>
<td>Fractions-Comparison</td>
</tr>
<tr>
<td></td>
<td>Equivalent Fractions</td>
</tr>
</tbody>
</table>

Figure 2 Physical and virtual manipulatives used during instructional lessons.
During the practice phase, the physical manipulative groups used fraction circles and squares. Practice activities sheets were developed for the practice phase that were similar to the type of tasks students completed when using the VM apps. The VM groups used virtual manipulatives and the PM group used physical manipulatives during both the concept introduction/exploration and practice phases. The CM groups used physical manipulatives during the concept introduction/exploration phase of the lessons, and used virtual manipulatives during the practice phase of the lessons. There were a number of steps taken to ensure the rigor and internal validity of the procedures in this study (Kane, 2001). First, instruction was conducted by a researcher and a second trained instructor so that all lessons followed a prescribed structure. Both instructors had over 25 years of public school teaching experience. Second, to prevent teacher bias, each instructor taught the same number of PM groups, VM groups, and CM groups. This ensured that both of the instructors had the opportunity to provide instruction using each of the treatment methods. Third, at the conclusion of the tenth instructional lesson, classroom teachers administered the post-test without any knowledge of which students participated in which treatment group.

3 DATA COLLECTION AND ANALYSIS

A pre and post Equivalent Fraction Test (EFT) and a Daily Cumulative Assessment (DCA) were the two primary assessments used to answer the research questions quantitatively. The Equivalent Fraction Tests contained three types of questions: open response, short response and multiple-choice. Questions were a mixture of pictorial only, pictorial and symbolic, and symbolic only. Each test contained 20 questions, with four questions from each of the five fraction subtopics: 1) modelling equivalence, 2) identifying equivalence, 3) building an equivalent group, 4) solving equivalent sentences, and 5) simplifying fractions. Questions on the pre and post tests were similarly formatted with changes made in the values used. Researchers established content validity through a three-step process (Kane, 2001). First, questions were evaluated and refined by an expert team of three mathematics specialists. The tests were then administered in interviews to a high, medium and low achieving student to determine if the questions elicited the targeted equivalent fraction thinking and further refinements were made. In the third stage, the multiple choice questions on the pre-test were paired with similar post-test questions, aggregated evenly into three tests and administered to 81 students. An item response analysis of the questions yielded a reliability of 0.74 for the pre-test and 0.76 for the post-test (Hambleton, Swaminathan and Rogers, 1991). Pre- to post-test gains on the Equivalent Fraction Test were analysed using paired samples t tests. From the results, Cohen d effect size scores were calculated.

There were ten Daily Cumulative Assessments (DCA). A DCA was administered to students at the conclusion of each lesson. DCAs contained eight open response questions targeting the development of fraction models, comparing and ordering fractions, partitioning, simplifying and identifying and developing equivalent fractions. Questions on all ten DCAs were similarly formatted with changes made only in the values used. Using paired samples t tests, pre- to post-test gains were analyzed and Cohen d effect size scores were calculated. Assessment results were also used to develop line graphs of students' development of fraction skills.

Lesson artefacts, including activity sheets, explore papers and video tapes, were the sources of data used to answer the research questions qualitatively. These data sources were used to identify students' misconceptions and errors. The activity sheets, explore papers and video tapes supported the identification of categories of misconceptions and errors. Occurrences of misconceptions and errors were coded and tallied. Nineteen different types of errors were observed. The 19 error types were determined to be indicators of seven types of fraction misconceptions. These qualitative data were quantitized using line and scatter plot graphs to summarize the number of students exhibiting each error type.

4 RESULTS

The results presented in the following sections are organized in relation to the three levels of the Equivalent Fraction Iceberg Intervention Model.

Level I – Equivalent Fraction Understanding

The first research sub-question was: How do different manipulative types support students’ ability to interpret and develop equivalent fraction understanding? This is Level I in the Iceberg Intervention Model. Data for this level were collected from the Equivalent Fraction Tests (EFT) and are summarised in Table 1 to answer the first research sub-question. Although there were numerical differences among the groups (PM, VM, and CM), a one-way ANOVA indicated that pre-test differences were not significant (F (2,43) = 0.01, p = 0.99). Paired samples t tests of gains scores from the pre to post testing indicated that all three types of interventions were significant at the 95% level and calculations of Cohen d effect size scores resulted in large positive effect size scores of 2.79 for PM intervention, 2.03 for VM intervention and 1.98 for CM intervention. Although, differences among groups were not statistically significant at the 95% level, effect size comparisons indicated a small moderate positive effect when PM intervention was compared to VM (d = 0.27) and CM intervention (d = 0.24). In summary, the findings of the EFT suggest that all three of the interventions were effective in increasing students' equivalent fraction achievement scores and although the differences among the intervention groups were not significant, there was a small effect favouring the use of physical manipulative intervention.
Table 1 Comparison of Pre- to Post-Test Gains of EFT  

<table>
<thead>
<tr>
<th>Sub-concept</th>
<th>Pre-test</th>
<th>Post Test</th>
<th>Cohen d Effect Sizes</th>
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</thead>
<tbody>
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<td></td>
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<tr>
<td>PM</td>
<td>7.00</td>
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<td>5.21</td>
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<tr>
<td>CM</td>
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<tr>
<td>PM</td>
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<td>12.00</td>
<td>0.13</td>
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<tr>
<td>VM</td>
<td>6.79</td>
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<tr>
<td>CM</td>
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<td>13.21</td>
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<tr>
<td>VM</td>
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<td>0.14</td>
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<tr>
<td>CM</td>
<td>6.79</td>
<td>13.36</td>
<td>0.15</td>
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<td>17.00</td>
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<tr>
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<tr>
<td>CM</td>
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<td>VM</td>
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<tr>
<td>CM</td>
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<td>14.64</td>
<td>0.15</td>
</tr>
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</table>

Table 2 Pre- and Post-Test Results for Five Equivalent Fraction Sub-Concepts.

The second research sub-question was: How do different manipulative types support students' ability to model, identify, group, solve and simplify fractions? The five equivalent fraction sub-concepts of Level II in the Iceberg Intervention Model are: modelling, identifying, grouping, solving and simplifying. To answer this research sub-question, pre- to post-test gains for the EFT fraction sub-concepts were aggregated. This resulted in large Cohen $d$ effect sizes and significant differences at the 95% level for all sub-concept treatments except identifying (see Table 2). For the identifying sub-concept, only the PM intervention gains were significant at the 95% level, and effect size scores for the VM and CM interventions were moderate (0.77 and 0.71 respectively).

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two fractions, whereas the sub-concept of grouping relates to three or more fractions. We would expect that students with a stronger understanding of identifying and solving (indicated by the large effect for PM) would be able to extend this knowledge to grouping; but this was not the case.

**Level III – General Fraction Understanding and Sub-concepts**

The third research sub-question was: How do different manipulative types support students' ability to name fractions, compare fractions, and demonstrate equivalence thinking? The three general fraction concepts essential to the development of equivalent fraction understanding in Level III in the Iceberg Intervention Model are: naming fractions, comparing fractions and equivalence thinking. Within the three groups are nine sub-concepts. To answer this sub-question, data for comparisons of each fraction sub-concept were collected from the Daily Cumulative Assessment (DCA) questions and the observation of errors students made during daily work and assessments.

**Naming fractions.** Naming fractions is the ability to name the symbolic part-whole relationships of concrete and pictorial representations and to develop models of the part-whole relationship. Naming Fractions includes three sub-skills: labelling, partitioning and building models of fractions.

![Image of Line Graph and Scatter Plot trajectories of Labelling Misconceptions](image)

**Figure 3** Line Graph and Scatter Plot trajectories of Labelling Misconceptions.

**Building models:** Data comparing student learning for the building models concept were collected from a DCA question in which students were asked to draw a model of a given fraction using a rectangular region. Responses were evaluated using a six point rubric ranging from not drawing a model to correct shading and equal-sized partitioning. Paired samples $t$ tests indicated that the pre- to post-test gains for the VM and CM interventions were significant at the 95% level, but not for the PM intervention (see Table 3). The Cohen $d$ effect size comparison among intervention groups yielded large effect sizes favouring CM groups when compared with PM groups ($d = 1.36$) and VM groups ($d = 0.84$) and a moderate effect ($d = 0.39$) favouring VM groups when compared to PM groups. This suggests that for the building models intervention CM intervention was favoured.

<table>
<thead>
<tr>
<th>Intervention</th>
<th>DCA Pre Test M</th>
<th>SD</th>
<th>DCA Post Test M</th>
<th>SD</th>
<th>$df$</th>
<th>$t$</th>
<th>$p$</th>
<th>Cohen's $d$</th>
<th>Within</th>
<th>Among</th>
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<tbody>
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<td>0.01</td>
<td>1.05</td>
<td>VM to PM 0.39</td>
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Table 3. Summary of Analysis of DCA-Q1 Building Models Note. N=43
Partitioning. Partitioning fractions is the ability to correctly partition representations and sets. The DCA partitioning question asked students to divide a given number of pizzas with a given number of friends. Responses were evaluated using a six point rubric ranging from not drawing the correct number of pizzas to partitioning the models correctly and identifying the fractional amount each friend would receive. Paired samples t tests indicated that the pre-to post-test gain was significant for the VM and the CM intervention groups, but not the PM intervention group (see Table 4). Comparisons among intervention groups yielded a large effect size favouring VM intervention when compared with PM intervention (1.85) and a small effect when compared with CM intervention (0.20). CM intervention compared to PM intervention resulted in a large effect size (1.31). However, the PM groups' pre-test scores (4.27 out of 6 possible points) were considerably higher than those of the VM (2.21) and CM (2.43). Although the PM groups made significantly less gains, the post-test achievement levels of the three groups were similar. Results of the data analysis suggest that VM intervention had a higher rate of gain for the development of partitioning skills.

<table>
<thead>
<tr>
<th>Intervention Type</th>
<th>Pre Test</th>
<th>Post Test</th>
<th>df</th>
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<th>p</th>
<th>Pre to Post Cohen's d</th>
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<td>SD</td>
<td>M</td>
<td>SD</td>
<td>p</td>
<td>Among</td>
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<td>CM</td>
<td>2.43</td>
<td>1.95</td>
<td>4.71</td>
<td>1.68</td>
<td>13</td>
<td>4.02 0.00 1.25</td>
</tr>
</tbody>
</table>

Table 4 Summary of Analyses of DCA-Partitioning Question Note. N=43

In summary, for the three subcategories of naming fractions, although the differences were not great, they suggest that PM intervention may be favoured for the labelling of fractions, CM intervention for building models and VM intervention for partitioning.

Evaluating fraction magnitudes. To conceptually determine if two fractions are equivalent, students need an understanding of the magnitude of fractions and must know the strategies needed to compare fractions. From the literature, the three skills of comparing, ordering, and developing were identified and three corresponding DCA questions were developed. Table 5 contains a summary of the paired samples t tests for the gain scores. For the comparing question and the developing question the interventions were not significant, indicating that students made only limited gains. This was a surprising result because we anticipated that instruction using multiple representations (e.g., virtual manipulatives and physical manipulatives) would have positive effects on students' ability to judge magnitude. However, as the table shows, not only were there limited significant gains, there were also instances where students' post-test scores were lower than their pre-test scores. This result could call into question the use of the part-whole model for instruction with these students. Therefore, these two questions were not analysed further.

<table>
<thead>
<tr>
<th>Question/ DCA</th>
<th>Pre-test</th>
<th>DCA Post Test</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intervention Type</td>
<td>M</td>
<td>SD</td>
</tr>
<tr>
<td>Comparing PM</td>
<td>4.93</td>
<td>1.58</td>
</tr>
<tr>
<td>Comparing VM</td>
<td>4.21</td>
<td>1.58</td>
</tr>
<tr>
<td>Comparing CM</td>
<td>3.86</td>
<td>1.61</td>
</tr>
<tr>
<td>Ordering PM</td>
<td>3.47</td>
<td>1.61</td>
</tr>
<tr>
<td>Ordering VM</td>
<td>4.00</td>
<td>1.52</td>
</tr>
<tr>
<td>Ordering CM</td>
<td>3.07</td>
<td>1.59</td>
</tr>
<tr>
<td>Developing PM</td>
<td>0.40</td>
<td>0.51</td>
</tr>
<tr>
<td>Developing VM</td>
<td>0.64</td>
<td>0.50</td>
</tr>
<tr>
<td>Developing CM</td>
<td>0.36</td>
<td>0.50</td>
</tr>
</tbody>
</table>

Table 5 Summary of Data Analysis for the Comparing, Ordering and Developing Questions

For the ordering question the difference between students' placement of the fraction on a number line and the correct location were measured and evaluated on a six point rubric ranging from a total difference of greater than 20 centimetres to 0 centimetres. Paired samples t tests indicated that the gains were significant at the 95% level for the PM and CM interventions, but not the VM intervention. The effect size analysis among intervention groups yielded a large effect size favouring PM groups when compared to VM groups (d = 0.84) and a small moderate effect size when compared to CM groups (d = 0.25). Comparison of CM groups to VM groups yielded a moderate effect size of 0.60. Analysis results suggest that PM intervention may be favored for the learning of ordering fractions.

Developing equivalence thinking. Each of the five sub-concepts of equivalent fraction understanding requires an understanding of equivalence. Students must have a working
Using an Iceberg Intervention Model to Understand Equivalent Fraction Learning

understanding of three concepts: 1) the meaning of equivalence, 2) the conservation of the part-whole relationship, and 3) the ability to think multiplicatively. Since the importance of this concept emerged through the study analysis of student errors there were no assessment questions targeting equivalence thinking, and the sources of data were limited to error analysis.

Meaning of equivalence (moe). Error analysis of the data revealed that many of the students focused on incorrect fraction relationships to determine equivalence. Four types of incorrect relationships were observed: 1) Identifying equivalent fractions as being two fractions naming the relationship of the parts making up a whole (e.g., \(1/3 = 2/3\)) (51.1% of the 247 moe error cases); 2) Identifying equivalent fractions as the original fraction and a second fraction whose value is equal to one and contains numerals that were either in the original fraction or factors or multiples of the numerals in the original fraction (e.g., \(6/8 = 6/6\) and \(8/8 - numerals from original fraction, 6/8 = 2/2 - factor or 2/3 = 4/4 or 6/6 - multiples) (20.6% of the moe cases); 3) Identifying equivalent fractions as being a fraction and its reciprocal (e.g., \(1/3 = 3/1\)) (15.4% of the moe cases); and, 4) Identifying equivalent fractions as being a fraction and a second fraction which is derived by determining the number of times a number will divide into either the numerator or the denominator of the original fraction (e.g., \(5/10 = 2/5\) because five goes into 10 twice) (8.9% of the moe cases).

Figure 4 shows the meaning of equivalency line graph and scatter plot trend line for the error cases. All three interventions were effective in reducing the number of moe errors. Over the first four lessons, the number of error cases for the VM intervention group increased, while the PM and CM intervention groups generally decreased. After the fourth lesson all the trajectories follow approximately the same path until the last lesson when the VM cases increased again. The greater slope of the VM trend suggests that the rate of resolution was greater for this group. The VM group had the greatest decrease in errors with an average difference of 9.66 errors from the first three lessons to the last three lessons. The decrease for the PM groups was 8.33 and for the CM group it was 5.66. Although these results suggest that VM intervention had the greatest rate of reduction of errors, this may in part be the result of the VM group also having the greatest number of errors originally.

Conservation of the part-whole relationship. A second equivalency thinking concept which many students struggle with is understanding that partitioning creates proportional changes in both the numerator and denominator of the fraction. Symbolically, to conserve the relationship of the fraction you multiply or divide the numerator and the denominator by the same number. Error analysis of the lesson artifacts identified 147 part/whole conservation errors in which students incorrectly varied the numbers and denominators of fractions independently of each other. Two types of errors were observed: 1) multiplying the numerator and denominator by different numbers (42.9% of the 147 part/whole conservation cases); and, 2) increasing or decreasing only the denominator or only the numerator (58.7% of the 147 part/whole conservation errors).

Figure 5 shows the line graph and scatter plot trend lines for the part/whole conservation resolution of errors. The learning trajectory shows multiple increases and decreases in errors, with only the PM group having an averaged decrease of 3.33 errors from the first three lessons to the last three lessons. There was no decrease of observed errors for the VM group and 0.3 decreases of errors for the CM group. This suggests that only the PM intervention was effective for the resolution of part/whole conservation errors.
**Multiplicative thinking.** Initially, many students mistakenly use additive instead of multiplicative thinking when developing equivalent fractions. Figure 6 shows the line graph and scatter plot trend lines and for the resolution of additive thinking errors. A high number of errors occurred in lesson 2 because developing multiplicative thinking was the topic of the lesson and students had many more opportunities to make additive errors. The decrease in the average number of errors observed in lessons one, three and four to the average number of errors observed in the last three lessons was 2.0 less for the VM group, and 0.4 less for the CM and 0.4 more for the PM intervention. The line graph indicates that except in lesson two, few errors were made by the PM and CM interventions. In contrast, the VM intervention results suggest a steady decrease in the number of additive errors. This suggests that the intervention was effective for the VM intervention. Within this data set, it is not possible to determine the effectiveness of the PM and CM interventions or to make comparisons among interventions.

Figure 6 Trajectories of Additive Thinking Error Cases.

In summary, three types of student misconceptions were identified which related to developing equivalent thinking. A comparison of the resolution of equivalence meanings suggested that the VM group had the greatest rate of error resolution. Analysis of part/whole conservation indicated that the intervention was effective only for the PM intervention. For multiplicative thinking the intervention was effective for the VM group, but comparisons among intervention groups could not be made.

**Iceberg Synthesis**

The overarching research question in the study was: How do different manipulative types support learning fraction sub-concepts during mathematics intervention instruction for students with mathematical learning difficulties? To answer this overarching question, the final step in the analysis was the synthesis of results into the Equivalent Fraction Iceberg Intervention Model (see Figure 7). Although the PM intervention showed the greatest gain in Equivalent Fraction Test scores (Level I of the Iceberg Intervention Model), when broken into equivalent fraction sub-concepts (Level II of the Iceberg Intervention Model), the results suggest that PM and VM intervention were each favoured for two concepts and CM intervention for one concept. At Level III of the Iceberg Intervention Model, the analysis of the general fraction concepts needed for the development of the equivalent fraction sub-concepts indicated that PM interventions were favoured for three of the general concepts, VM interventions for three general concepts and CM intervention for one general concept.

![Figure 7 Favoured Manipulative for Iceberg Intervention Model Concepts.](www.technologyinmatheducation.com)
5 DISCUSSION

This study examined variations in gain scores and patterns of error resolution of students with mathematical learning difficulties when they used different manipulative types to learn equivalent fraction sub-concepts during intervention instruction. The results of this study indicate that the three types of intervention were effective for teaching equivalent fraction concepts and sub-concepts. Overall, pre to post gains favoured the PM intervention, but there were important sub-concept variations related to the type of manipulative intervention used.

The five concepts for which PM intervention was favoured were concepts which involved the development and comparison of the part-whole relationships of one or two fractions. Both labelling and ordering fraction values require an understanding of the part-whole relationship. When labelling fractions, students must be able to identify the fraction parts and the whole. In ordering students must also be able to determine the magnitude of a fraction by evaluating the relationship between the part and the whole. An affordance of physical manipulatives is the opportunity students have to physically touch and manipulate the fraction pieces. When Martin and Schwartz (2005) compared student learning between groups who manipulated objects and those who made marks on pictorial representations, the group who physically moved the objects performed better than those who used pictorial representations. The authors suggested that physically moving the pieces helped the children to transfer from the use of whole number understanding to fraction understanding. Likewise Arnon, Nesher and Nirenburg (2001) found that students physically lying two pieces on one piece of an equivalent fraction helps students understand the size relationships of the fractions. It may be that the tangible nature of physical manipulations is an important element in the development of part-whole understandings for labelling and ordering.

The remaining three sub-concepts favoured by PM intervention extend part-whole understanding to evaluating the equivalency of two fractions. Identifying and solving equivalent fractions and part-whole conservation requires an understanding that partitioning a fraction results in proportional changes in the numerator and denominator. In the practice activities, the PM intervention groups used pipe cleaners to partition fractions pieces into halves and thirds to obtain equivalent fractions which were double and triple the original fractions (e.g., ½, 2/4, 3/6). In this way the PM intervention constrained students' attention to the equivalence of fractions which are doubled and tripled. The students observed that halving a fraction halved both the parts and the whole and they quickly became proficient at doubling and tripling fractions. However, it may be that this procedure also limited students' development of multiplicative thinking. When doubling and tripling fractions, it is possible to use additive thinking by adding numerators and denominators two and three times to obtain correct answers. From the data collected, it was not possible to distinguish if students used multiplicative or additive thinking.

VM intervention was favoured for the sub-concepts of modelling and grouping and the sub skills of partitioning. Results suggest that the VM intervention was most effective for developing multiplicative thinking. The VM intervention also had the greatest reduction of meaning of equivalence errors. Partitioning fractions is a sub skill of modelling equivalent fractions; meaning of equivalence and multiplicative thinking are sub skills of grouping. Since the PM intervention made the greatest gains in identifying and solving two equivalent fractions, it would be expected that the PM intervention students would also make the greatest gains in the similar, but more difficult, skill of developing groups of equivalent fractions. However, this was not the case. There are two virtual manipulative affordances which may, in part, explain this disconnect: creative variation and simultaneous linking (Moyer-Packenham and Westenskow, 2013). Three of the virtual manipulatives used for instruction included features that enabled students to partition fraction pieces into as many as 99 partitions. Because of this affordance, students often chose to work with fractions with higher numbers of partitions. Most students experienced making equivalent fractions that were more than double or triple the original fraction. Unlike the physical manipulatives, which are difficult to partition multiple times, the virtual manipulatives were easily partitioned multiple times. The larger number of partitions may have encouraged students to abandon additive methods and encouraged their use of multiplicative thinking.

Another virtual manipulative affordance that may have enhanced learning is simultaneous linking (Moyer-Packenham and Westenskow, 2013). As students manipulated the virtual fraction pieces, they could observe the corresponding change in the symbolic representations of the fractions. This simultaneous linking has been found to enhance users' cognitive abilities (e.g., Baturo, Cooper and Thompson, 2003; Clements, Battista and Sarama, 2001; Moreno and Mayer, 1999; Suh and Moyer-Packenham, 2007; Takahashi, 2002). It has been suggested that simultaneous linking of pictorial to symbolic representations encourages the development of students' internal representations. When students solved more complex grouping questions, their internalized representations for the symbols may have made the questions more meaningful.

CM intervention was favored for simplifying and for building models. CM interventions used the physical manipulatives for the introduction and exploration phases of each lesson and virtual manipulatives for the practice phases. One advantage to the combined use of manipulatives, and a possible explanation for why CM intervention was favoured for model building instruction is that the CM group was exposed to more types of representations (i.e., models) than the other groups. The CM group used fraction circles and squares, counters, paper strips, and three different virtual manipulative applets. As students transferred learning from one model to another, the unique features of the new models may have challenged students' understandings and supported the development of new concepts (Kiezek, Maher and Speiser, 2001). Friedlander and Tabach (2001) found that presenting problems using different models tended to increase students' flexibility. Other studies report that as
students interact with manipulatives, they experience visual proof of their solutions and build understanding of mathematical concepts (Durmus and Karakirik, 2006; Moyer and Bolyard, 2002; Moyer-Packenham, 2008; Moyer-Packenham, Salkind and Bolyard, 2012). Exposure to multiple types of models may have helped to deepen the CM groups' ability to think about and to develop fraction models. An increase in flexibility could also, in part, explain the CM intervention being favoured for the equivalent fraction concept of simplifying. Simplifying requires not only an understanding of partitioning to develop equivalency, but also the ability to reverse partitioning and an understanding of unit fractions. It requires a deeper and more flexible conceptual understanding of equivalent fractions. Further testing is needed to determine if the thinking of the CM intervention students' tends to be more flexible and show a deeper conceptual understanding. Additionally, for the CM interventions in this study, concepts were first introduced using physical manipulatives and were then practiced using virtual manipulatives. Further research is needed to determine if the order of manipulative presentation affects student learning.

There are some limitations to note with these findings. First, this study was designed to be an exploratory study that identified areas of possible differences in manipulative affordances. The results can now be used to design future large-scale studies that are experimental. Second, the study was limited in the number and the diversity of the participating students. Students were all of the same age and had similar demographic characteristics. The schools from which the students were drawn all used the same classroom textbooks and these texts tended to focus primarily on the part-whole interpretation of fractions. Further research is needed to determine if similar results are found for more diverse populations. Third, the study did not examine the differences of manipulative effects in relation to types of mathematical learning difficulties. For example, it may be that students with different types of mathematical learning difficulties (e.g., learning disabled or attention deficit disorder) or students who experience instruction that is ineffective to meet their particular mathematics needs, may respond differently to the variations in manipulative affordances. As variations in effects of specific manipulative affordances are identified, more structured research comparing the effects of manipulative presentation, order, and features will help teachers to optimize the effective use of both physical and virtual manipulatives.

6 CONCLUSION

As the Iceberg Intervention Model shows, developing an understanding of fraction equivalence is a complex multi-level accomplishment for students that is built on foundational concepts and an integration of multiple understandings. The results of this study suggest that learning variations favouring the use of one manipulative type over another are specific to sub-concepts of the overarching mathematical concept that students are studying. If in further research, similar trends are found, an understanding of the manipulative affordances specific to each sub-concept will be useful in the design and implementation of effective intervention fraction curriculum. Results of this study indicated that there may be advantages to the constraint affordance of the physical manipulatives when students are first developing the procedures of partitioning objects into equivalent fractions, but that students benefit from the variation affordance of virtual manipulatives as they expand their understanding to include objects which are partitioned multiple times into equivalent fractions. A deeper understanding of how to balance manipulative affordances is vital to improving both the effectiveness and efficiency of intervention instruction.

In this process the Iceberg Intervention Model served as an effective tool for comparing, organizing and synthesizing study results into findings which can easily be interpreted for guiding curriculum development and creating intervention instruction.

REFERENCES


Using an Iceberg Intervention Model to Understand Equivalent Fraction Learning


Aria Westenskow and Patricia Moyer-Packenham


Torbeyns, J., Schneider, M., Xin, Z. and Siegler, R. S. (2014) Bridging the gap: Fraction understanding is central to mathematics achievement in students from three different continents, Learning and Instruction. http://dx.doi.org/10.1016/j.learninstruc.2014.03.022


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BIOGRAPHICAL NOTES

Patricia Moyer-Packenham is Director and Professor of Mathematics Education and Leadership at Utah State University. She received her PhD from the University of North Carolina at Chapel Hill in 1997 with an emphasis in mathematics education. Moyer-Packenham's research focuses on uses of mathematics representations (including virtual, physical, pictorial, and symbolic) for technology devices (mouse driven and touch screen). She is often referenced for her definition of virtual manipulatives (appearing in Teaching Children Mathematics, 2002), and her expertise in teaching and research using physical and virtual manipulatives. Her publications include two books, Teaching K-8 Mathematics with Virtual Manipulatives and What Principals Need to Know about Teaching Mathematics, and over 80 scholarly contributions including numerous journal articles, book chapters, refereed proceedings, and contributions to mathematics methods textbooks. Moyer-Packenham has served as a PI on numerous grants totaling over $16 million dollars in funding for mathematics teacher development.

Arla Westenskow is Director of the Tutoring Intervention and Mathematics Enrichment (TIME) Clinic in the Early Childhood Education Research Center at Utah State University. She received her PhD from Utah State University in 2012 with an emphasis in mathematics education. Westenskow's research focuses on Response to Intervention methods for children with mathematical difficulties and mathematics representations including virtual manipulatives. She has developed a unique "iceberg" model for assessing and remediating children's mathematical learning difficulties. Her scholarly activity includes numerous presentations, publications, and conference papers.
Appendix A. Equivalent Fraction Intervention Model Descriptions

<table>
<thead>
<tr>
<th>Level 1 – Equivalent Fraction Understanding</th>
<th>Ability or Understanding</th>
<th>Common Difficulties</th>
</tr>
</thead>
</table>
| -ability to flexibly use equivalent fractions in operations and in problem solving situations | -incorrectly develops equivalent fractions
- inability to use equivalent fractions in new problem solving contexts
(e.g., Arnon, Nesher and Nierenburg, 2001; Bright, Behr, Post and Wachsmuth, 1988; Kamii and Clark, 1995; Smith 2002) |

<table>
<thead>
<tr>
<th>Level 2- Equivalent Fraction Sub Concepts</th>
<th>Ability or Understanding</th>
<th>Common Difficulties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub Concepts</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Modeling                                  | -creating EF using visual representations
-identifying EF within visual representations | - partitioning models to show equivalence
- visualizing EF nested inside models used in instruction
(e.g., Kamii and Clark, 1995; Westenskow, Moyer-Packenham, Anderson, Shumway and Jordan, 2014) |
| Identifying                                | -determining if two fractions are equivalent | -misconceptions of the meaning of equivalency
- conserving the N/D relationship of EF
(e.g., Behr, Wachsmith, Behr and Lesh, 1984) |
| Grouping                                   | -understanding that each fractional quantity can be identified with an infinite number of EF names
-developing groups of equivalent fractions | -identifying the common N/D relationship of fractions in an EF group
- use of only additive procedures when developing EF groups
- understanding that multiple partitioning results in EF naming the same quantity
(e.g., Chan and Leu, 2007; Ni, 2001; Smith, 2002) |
| Solving                                    | -determining the value of a missing N or D in an EF sentence
-use of additive instead of multiplicative procedures
\[
a/b = (a + c)/(b + c)
\] | -use of inverse operation of reducing fractions
- identification of smallest possible units
(e.g., Chan and Leu, 2007; Spangler, 2011) |
| Simplifying                                | -finding the fraction with the smallest possible units in a set of EF |                      |

<table>
<thead>
<tr>
<th>Level 3-Naming Fractions Sub Concepts</th>
<th>Ability or Understanding</th>
<th>Common Difficulties</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sub Concepts</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
| Labelling                                  | -identifying and writing the part/whole relationship of a concrete or pictorial representation | -identification of the parts and wholes (labelling \(a/b\) as \(b/a\); \(a/(b-a)\); \((b-a)/a\); \((b-a)/b\); or, \(b/(b-a)\)
- interpreting fractions as a pair of numbers instead of a relationship
(Smith, 2002; Steffe, 2004; Steffe and Olive, 1991) |
| Building Models                           | -developing concrete models or pictorial representations of fractions | -Unequal partitions
- Incorrect part/whole relationships
- Inconsistency in sizes
- Incorrect model type selection
(e.g., Jigyel and Afamasaga-Fuata, 2007; Ng and Lee, 2009; Reave and Pattison, 1996; Steinle and Price, 2008) |
| Partitioning                              | -determining equal or “fair” shares of a quantity
- partitions models into equally sized partitions | -interpreting all partitions as halves
- partitioning into unequal portions
- incorrect number of partitions
- labelling partitions
(e.g., Ball, 1993; Behr, & Post, 1992; Empson, 2001; Lamon, 1996; 2005). |
<table>
<thead>
<tr>
<th>Level 3 - Evaluating Fraction Magnitude</th>
<th>Common Difficulties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Comparing</strong></td>
<td></td>
</tr>
<tr>
<td>- determining which of two fractions is greater</td>
<td>- use of whole number bias to interpret that the fraction with the digit of the greatest magnitude is the greater fraction</td>
</tr>
<tr>
<td>- comparing fractions to benchmarks</td>
<td>- focuses only on the magnitude of either the numerator or denominator (e.g., Armstrong and Larson, 1995; Arnon, Nesher and Nirenburg, 2001; Behr and Post, 1992; Hecht, Vagi and Torgesen, 2007; Mack, 1990; Torbeyns, Schneider, Zin and Siegler, 2014)</td>
</tr>
<tr>
<td><strong>Ordering</strong></td>
<td></td>
</tr>
<tr>
<td>- placing fractions in order by magnitude</td>
<td>- lack of efficient comparing strategies when comparing three or more fractions</td>
</tr>
<tr>
<td>- placing fractions on a number line</td>
<td>- estimating fraction placement on number lines (e.g., Behr, Wachsmith, Behr and Lesh, 1984; Bills, 2003; Schneider and Siegler, 2010; Smith, 2002)</td>
</tr>
<tr>
<td><strong>Developing</strong></td>
<td></td>
</tr>
<tr>
<td>- develops fractions greater, less than or between other fractions</td>
<td>- Sees fractions as finite, that there are no fractions between ( \frac{a}{b} ) and ( \frac{a+1}{b} ). (e.g., Arnon, Nesher and Nirenburg, 2001; Behr and Post, 1992; Bills, 2003; Vosniadou and Vamvakoussi, 2006)</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Level 3 – Equivalence Thinking</th>
<th>Common Difficulties</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Sub Concepts</strong></td>
<td></td>
</tr>
<tr>
<td><strong>Meaning of Equivalence</strong></td>
<td>- incorrect interpretation of equivalence; ( \frac{a}{b} = \frac{(b-a)}{b} ) or, ( \frac{a}{b} = \frac{b}{a} ) (e.g., Chan and Leu, 2007; Chval, Lannin and Jones, 2013; Cramer, Behr, Post and Lesh, 2009; Empson, 1995)</td>
</tr>
<tr>
<td><strong>Conservation of part-whole relationship</strong></td>
<td>- misconception that changing shape will change fraction value (e.g., Ball, 1993; Behr and Post, 1992; Charalambos and Pitta-Pantazi, 2007; Grobecker, 2000; Kamii and Clark, 1995).</td>
</tr>
<tr>
<td><strong>Multiplicative Thinking</strong></td>
<td>- focus on counting and adding strategies instead of multiplicatively determining amounts (e.g., Ball, 1993; Kamii and Clark, 1995; Kent, Arnosky and McMonagle, 2002; Lamon, 2005; Moss, 2005).</td>
</tr>
</tbody>
</table>

D- Denominator; EF – Equivalent Fractions; N-Numerator