Mathematics Education in the Digital Era

Patricia S. Moyer-Packenham *Editor*

International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives

Springer

Editor
Patricia S. Moyer-Packenham
Utah State University
Logan, UT
USA

ISSN 2211-8136 ISSN 2211-8144 (electronic)
Mathematics Education in the Digital Era
ISBN 978-3-319-32716-7 ISBN 978-3-319-32718-1 (eBook)
DOI 10.1007/978-3-319-32718-1

Library of Congress Control Number: 2016937520

Printed on acid-free paper

This Springer imprint is published by Springer Nature
The registered company is Springer International Publishing AG Switzerland

Chapter 1
Revisiting the Definition of a Virtual Manipulative

Patricia S. Moyer-Packenham and Johnna J. Bolyard

Abstract In 2002, Moyer, Bolyard and Spikell defined a virtual manipulative as an "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). The purpose of this chapter is to revisit, clarify and update the definition of a virtual manipulative. After clarifying what a virtual manipulative is and what it is not, we propose an updated definition for virtual manipulative: *an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge*. The chapter describes the characteristics of five of the most common virtual manipulative environments in use in education: single-representation, multi-representation, tutorial, gaming and simulation.

Fifteen years ago, colleagues Moyer et al. (2002) proposed a definition for a virtual manipulative. They defined a virtual manipulative as an "an interactive, Web-based visual representation of a dynamic object that presents opportunities for constructing mathematical knowledge" (p. 373). The term "interactive" was used in the definition to distinguish tools that users could interact with from those that were simply static images viewed on the screen. The term "Web-based" was used in the definition to distinguish easily accessible tools on the Internet from those that were being commercially produced as computer programs. The term "visual representation" was used in the definition to highlight that a pictorial image had the potential to accurately represent some mathematical idea. The term "dynamic" was used in the definition to focus on the manipulability of the image representation that could be moved by the user. The term "object" was used to refer to the idealized mathematical object, beyond its physical inscription, that the two-dimensional

P.S. Moyer-Packenham (&)
Utah State University, Logan, USA
e-mail: Patricia.moyer-packenham@usu.edu

J.J. Bolyard
West Virginia University, Morgantown, USA
e-mail: Johnna.bolyard@mail.wvu.edu

P.S. Moyer-Packenham (ed.), *International Perspectives on Teaching and Learning Mathematics with Virtual Manipulatives*, Mathematics Education in the Digital Era 7,
DOI 10.1007/978-3-319-32718-1_1

image was used to represent (Kirby 2013). The terms "presents opportunities for constructing mathematical knowledge" were used in the definition to distinguish that virtual manipulatives are designed for the purpose of facilitating the opportunity for mathematical learning.

Since this definition was published in 2002 in *Teaching Children Mathematics*, it has been referenced and cited over 280 times (source: Google Scholar), demonstrating its usefulness to the educational and research communities. Because of the widespread use of the term *virtual manipulative* and its definition, a number of questions have arisen as new technologies have been developed that include technology tools with virtual manipulatives. What is and what is not a virtual manipulative? Are all virtual manipulatives "web-based" as described in the 2002 definition? Is a virtual manipulative simply the representation, alone, or does the virtual manipulative include some or all of the features that are designed in the environment around it? What is the relationship between games and virtual manipulatives? What is the difference between virtual manipulatives designed as Java-based apps and the newer touch-screen apps?

At the time of the release of the original definition, Moyer et al. (2002) described virtual manipulatives as "a new class of manipulatives" (p. 372). In the 2002 publication, the authors described virtual manipulatives being manipulated by a computer mouse. Today, virtual manipulatives are presented on computer screens, on touch screens of all sizes (e.g., tablets, phones, white boards), as holographs, and via a variety of different viewing and manipulation devices. The virtual manipulatives on these devices will likely be manipulated by a mouse, stylus, fingers, lasers, and a variety of other manipulation modalities in the years to come. Several collections of virtual manipulatives have been developed over the years including the National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu), National Council of Teachers of Mathematics (NCTM) Illuminations (http://illuminations.nctm.org), and Shodor Interactivate Curriculum Materials (http://shodor.com/curriculum/). There are also new libraries of virtual manipulatives being developed for the touch-screen environment, although to date, there are none as extensive as those developed for the computer.

As new technologies have developed and questions arose in the field, we believed it was time to revisit the definition of a virtual manipulative and to discuss some of the most common environments for the educational setting in which virtual manipulatives appear. The purpose of this chapter is to address questions that have arisen in the field since the publication of the original definition; revisit, clarify and update the definition of a virtual manipulative; and to describe the characteristics of five of the most common virtual manipulative environments in use in education. Describing examples of different environments in which users may find a virtual manipulative allows educators and researchers to have a common language and understanding of these important technology tools for teaching and learning mathematics.

1.1 What Is and What Is Not a Virtual Manipulative?

Moyer et al. (2002) clarified the difference between technology tools that are and are not virtual manipulatives. One of the most important distinctions made in the 2002 publication was that the virtual manipulative user needs to be able to interact with a dynamic object in such a way that these interactions provide opportunities for constructing mathematical knowledge. Therefore, as described in the 2002 article, filling in worksheets on the screen or simply answering questions in the presence of a pictorial object does not fit the definition of a virtual manipulative.

A key defining feature of a virtual manipulative is the difference between static images of the representation and dynamic images of the representation on the screen. The user needs to be able to interact with, move, or manipulate the dynamic mathematical representation in some way that accurately represents a mathematical concept, relationship, procedure, and/or students' thinking about mathematical concepts, relationships, and procedures. This movement could take place using a mouse, stylus, fingers, lasers, and a variety of other manipulation devices yet to be developed (see Fig. 1.1). This interactive feature of the visual representation of the dynamic mathematical object distinguishes a virtual manipulative from other mathematics technology tools.

Child using a mouse to move a virtual manipulative on a computer screen

Child using fingers to move a virtual manipulative on a touch-screen

Fig. 1.1 Users can interact with, move, or manipulate the virtual manipulative using a mouse, fingers, or other interaction modalities

1.2 What Is the History of the Term "Virtual Manipulative"?

In the late 1990s different developers proposed the creation of a new class of manipulatives, which they referred to as digital manipulatives and virtual manipulatives. For example, Resnick et al. (1998) proposed the creation of digital manipulatives. The goal of these digital manipulatives, as described by Resnick and colleagues, was to:

> …embed computational and communications capabilities in traditional children's toys. By using traditional toys as a starting point, we hope to take advantage of children's deep familiarity with (and deep passion for) these objects. At the same time, by endowing these toys with computational and communications capabilities, we hope to highlight a new set of ideas for children to think about. (Resnick et al. 1998, p. 282)

Also, in the late 1990s, colleagues Jim Dorward, Bob Heal, Larry Cannon and Joel Duffin at Utah State University proposed the creation of a library of virtual manipulatives (Dorward and Heal 1999; Heal et al. 2002). They were funded by the National Science Foundation and, in 1999, created the National Library of Virtual Manipulatives (NLVM) (http://nlvm.usu.edu/), a collection of Java-based applets for K-12 mathematics teaching and learning. The NLVM is still in use today and is available in four different languages (Chinese, English, French, and Spanish). Throughout the years, the terms *digital manipulatives* (Manches and O'Malley 2012; Resnick et al. 1998), *computer manipulatives* (Sarama and Clements 2009), and *virtual manipulatives* (Dorward and Heal 1999; Heal et al. 2002) have been used most commonly as synonyms.

1.3 Are All Virtual Manipulatives Web-Based?

Technologic innovations have exploded over the past decade. This innovation has caused virtual manipulatives to appear in a variety of forms beyond the World Wide Web. So perhaps now is the time to amend the original definition, which defined virtual manipulatives as "web-based", and revise the definition to say "technology-enabled". Currently, virtual manipulatives are available through mul- tiple technological means; thus, the term "web-based" no longer encompasses all of the forms of virtual manipulatives that are available. It is also important to recognize the shift from "based" to "enabled". In the future it is very likely that virtual manipulatives will no longer be based in any technology (e.g., they may be projected 3D objects or holographic images). Describing virtual manipulatives as technology-enabled allows for changes in future iterations of these tools.

1.4 Is a Virtual Manipulative Simply the Representation, Alone, or Does the Virtual Manipulative Include Some or All of the Features that Are Designed in the Environment Around It?

Some researchers make a subtle distinction between the visual representation (i.e., the image, the inscription) of a virtual manipulative and the features of the representation, which enable it to be acted upon as a dynamic mathematical object. Because the original definition of a virtual manipulative says "an interactive … visual representation of a dynamic object" some have interpreted this to mean that the virtual manipulative is the inscription of the representation only, while others have interpreted this to mean that the virtual manipulative is the representation including its dynamic and programmable features. In the original definition by Moyer et al. (2002), the intention of the authors was that a virtual manipulative includes the representation and its dynamic and programmable features that allow the user to come to understand it as a representation of the idealized mathematical object (Kirby 2013). The representation portion of the virtual manipulative is only "interactive" and "dynamic" when its programmable features enable capabilities for knowledge construction.

As Kirby (2013) explains, "the properties of the object derive from the relevant definition, not the inscription itself…" (p. 1). For example, in Fig. 1.2, we can see an inscription or representation of an icosahedron. From the idealized mathematical object for an icosahedron, developers created this technology representation. The representation that appears on the computer screen only represents the icosahedron. Yet the representation, because of its limitations and constraints, can never be the idealized mathematical object with all of its properties and relationships. Through an individual's mathematical development, learners begin to understand the properties and relationships of the icosahedron as an idealized mathematical object beyond the representation. This goes beyond the simple images and limited inscriptions that appear in two dimensions on the screen. Most importantly, it is the interactive and dynamic programmable features that allow the user to explore with the representation and develop the concept of the icosahedron beyond its two-dimensional screen inscription. Therefore, in a virtual manipulative, the representation cannot be separated from its interactive and dynamic programmable features.

Further, the potential of the virtual manipulative to provide opportunities for constructing mathematical knowledge is dependent upon the representation's potential to accurately provide an interaction with the mathematics and for the user to be able to perceive the mathematics through this interactivity (Simon 2013). Goldin (2003) describes representation as process and product. Representational systems are both internal (within the individual) and external (outside the individual) and it is the interaction between these two systems that is the key to learning (Goldin and Shteingold 2001).

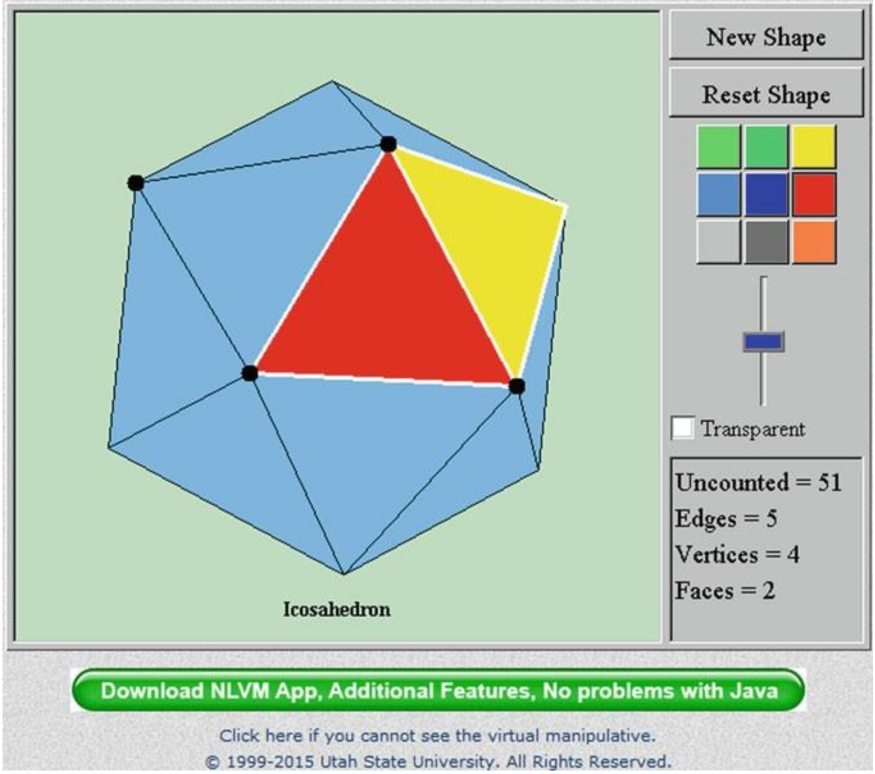


Fig. 1.2 Icosahedron virtual manipulative with marked faces, edges and vertices

Research has shown that the dynamic and interactive features of a virtual manipulative facilitate interactions between representational systems (Moyer-Packenham and Westenskow 2013). The dynamic movements of the visual representations and observation of the resulting outcomes support the structuring of the user's internal representation of the mathematics under study; likewise, the same movements and outcome observations can represent the user's current mathematical thinking, allowing the user to test and refine ideas.

Opportunities for constructing mathematical knowledge consist of more than the visual representation. The use of a virtual manipulative has maximum potential to support learning by behaving in a way that represents the idealized mathematical object when manipulated by the user and by accurately representing the user's mathematical thinking. Consequently, the manipulative representation alone, is not the virtual manipulative. It is the interactive and dynamic capabilities of the manipulative representation that makes it a virtual manipulative. Therefore, the programmable features of the application that support its interactivity are part of the virtual manipulative. The features that allow the representation to be manipulated, to be interactive, and to be dynamic are an inherent part of the virtual manipulative. Without these features, it is simply a static inscription.

To clarify the original definition, it could be amended to say a “representation of a dynamic object, including all of the programmable features that allow it to be manipulated; or that allow it to be dynamic; or that allow it to be interactive”. For example, in Fig. 1.2 which shows the three-dimensional representation of an icosahedron, the features of the app that allow the user to change the color of the faces, mark the vertices with black dots, mark the edges with white lines, move the slider to change the object’s size and change the solid to a transparent view are all part of the interactivity and manipulability of the virtual manipulative that can be acted upon by the user to draw attention to or highlight the relevant properties of the solid. In addition, using a mouse to click on and drag the icosahedron or using fingers to swipe the icosahedron allows the user to move and rotate it.

All of these actions take the user beyond the simple representation of the object to a greater understanding of the properties and relationships imposed by the definitions and theorems of the idealized icosahedron. Therefore, the virtual manipulative is not simply the visual representation of the icosahedron, the virtual manipulative is the visual representation of the icosahedron and all of the programmable features surrounding it that allow it to be dynamic, interactive and manipulated by the user to explore and observe its properties. These programmable features allow it to be manipulated and are an inherent part of it being classified as a virtual manipulative. Without these programmable features, the icosahedron is simply a visual/pictorial representation on a computer screen. With these programmable features, it is a virtual manipulative because it is an interactive and dynamic representation that can be manipulated.

1.5 What Is the Relationship Between Games and Virtual Manipulatives?

There are some virtual manipulatives that are embedded within gaming environments. When virtual manipulatives are embedded within a gaming environment, the environment is designed to host the virtual manipulative with its dynamic features. Some gaming environments are very basic, while other gaming environments can be highly developed and multi-layered. The game may have increasing levels, points, goals, timers, and other elements of game design (Deterding et al. 2011). Therefore, the entire gaming environment and everything in it is not a virtual manipulative, but there are often virtual manipulatives embedded in gaming environments. This could be the result of a designer taking a virtual manipulative and gamifying it to make it more appealing to learners.

Deterding et al. (2011) define *gamification* as “the use of game design elements in non-game contexts” (p. 10). For example, in the Motion Math Zoom app, a virtual manipulative is housed in a gaming environment (see Fig. 1.3, Zoom app).

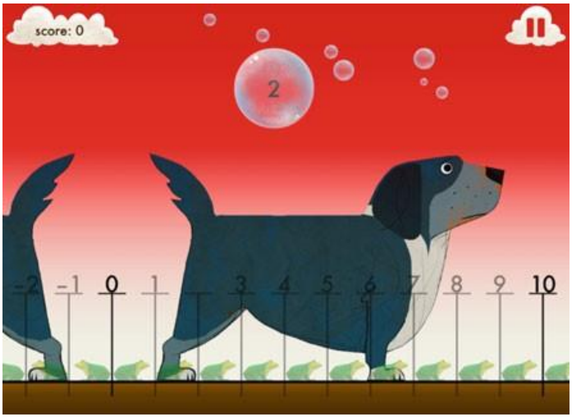


Fig. 1.3 Motion math zoom game app

The virtual manipulative is the dynamic number line that can be expanded, contracted and swiped by the user. This dynamic number line is placed inside a gaming environment where there are levels for the user to achieve using the virtual manipulative number line.

The gaming environment in which the virtual manipulative number line is housed could be changed; however, the dynamic number line remains the virtual manipulative for the learner to manipulate. For example, the virtual manipulative number line that is used in the Motion Math Zoom app could be placed in a different environment where the user is not playing a game. The environment could have number line tasks for the user to complete. Therefore, the relationship between games and virtual manipulatives is that virtual manipulatives are sometimes embedded in gaming environments.

1.6 What Is the Difference Between Virtual Manipulatives Designed as Java-Based Apps and the Newer Touch-Screen Apps?

Virtual manipulatives have been developed over the years in a variety of different formats from Java- and Flash-based applications, largely for Windows computers and Android devices to Swift-based applications for Apple iOS products (e.g., iPads). Whether these dynamic objects are Java-based, Swift-based, or developed using a host of available programming languages and tools, they are still virtual manipulatives. The programming language or tool used to develop the virtual manipulative or the platform through which it is delivered does not change the essence of the virtual manipulative. As long as the product that is created is a dynamic representation of a mathematical object, having the characteristics of interactivity and manipulability that presents opportunities for constructing mathematical knowledge, it is a virtual manipulative. New programming languages may allow new and different capabilities, but these capabilities simply allow the virtual manipulative to have different kinds of interactivity and manipulability.

1.7 How Is the Term "Virtual Manipulative" Confused with Other Technology Terminology?

Over the years, there have been subtle, yet important, distinctions made in the literature among the terminology used to describe technologies for mathematics teaching and learning. Some of the terminology related to *virtual manipulatives* includes: *cognitive technology tools* (Pea 1985), *learning objects* (Kay 2012), *virtual math objects* (Bos 2009b), and *computer-based mathematical cognitive tools* (Sedig and Liang 2006). This similar terminology has led to confusion about virtual manipulatives. Some publications have used terminology other than the term *virtual manipulative* to refer to technologies that actually fit the definition of a virtual manipulative; conversely, the term virtual manipulative has been used to refer to technologies that do not fit the definition of a virtual manipulative.

Using a term other than virtual manipulative to refer to a virtual manipulative in a research study makes it challenging for researchers to determine what mathematics technologies were actually used in the study, to identify if the tools investigated meet the definition of a virtual manipulative, and to conduct rigorous evaluations and meta-analyses (Moyer-Packenham and Westenskow 2013) that summarize the effects of virtual manipulatives on student achievement and learning. When a term other than virtual manipulative is used in a research publication, it is unclear if the authors are simply using another term when they actually mean virtual manipulative, or if the authors are actually referring to something different than a virtual manipulative. These distinctions among terminology warrant some clarification.

Pea (1985) defined *cognitive technology tools* as "any medium that helps transcend the limitations of the mind, such as memory, in activities of thinking, learning, and problem solving" (p. 168). Because cognitive technology tools include the broad class of "any medium," we consider virtual manipulatives as a sub-category of the term cognitive technology tools because there are also many other types of medium that can be considered cognitive technology tools. Therefore, cognitive technology tools and virtual manipulatives are not synonymous.

Kay (2012) defines *learning objects* as "interactive Web-based tools that support the learning of specific concepts by enhancing, amplifying, and/or guiding cognitive processes of learners" (p. 351). Kay (2012) gives two examples of learning objects in his study: "adding integers with virtual colored tiles" and "three-dimensional objects transform to two-dimensional nets in order to examine surface area" (p. 351). Based on Kay's definition of a learning object, virtual manipulatives would be considered learning objects because the examples of the learning objects he describes in his study fit the definition of a virtual manipulative. However, if learning objects include other tools, beyond those described in the study that do not fit the definition of a virtual manipulative, then learning objects and virtual manipulatives are not synonymous.

Bos (2009b) writes about *virtual math objects*: "A math object enhanced with technology offers manipulations, multiple representations, multiple entry points, and provides opportunity to test, revisit, revise, and apply mathematical patterns" (p. 522). "The *math object* uses multiple representations that are interactive and change with the given input" (Bos 2009a, p. 110). Given this description, virtual manipulatives may be the same as *virtual math objects* or one type of *math object* because virtual manipulatives contain "multiple representations that are interactive and change with the given input." Although Bos (2009b) wrote, "Virtual manipulatives…are often mistaken as math objects…" (p. 522), the description of virtual math objects in these publications implies that virtual math objects and virtual manipulatives may be synonymous.

Sedig and Liang (2006) describe *computer-based mathematical cognitive tools* (MCTs) as "a category of external aids intended to support and enhance learning and cognitive processes of learners. MCTs often contain interactive visual mathematical representations…" (p. 179). Sedig and Liang (2006) go on to describe these visual mathematical representations as "graphical representations that encode causal, functional, structural, logical, and semantic properties and relationships of mathematical structures, objects, concepts, problems, patterns, and ideas" (p. 180). Based on these definitions, virtual manipulatives are a subcategory of computer-based mathematical cognitive tools because there are some tools that would be considered computer-based mathematical cognitive tools but that would not fit the definition of a virtual manipulative.

An additional source of confusion comes from the science literature in which virtual science materials are sometimes referred to as virtual manipulatives. In some studies, the research uses the terms *physical and virtual manipulatives* and *physical and virtual material* interchangeably. For example, Triona and Klahr (2003) compared the effectiveness of two instructional conditions, which they called the "physical, manipulable materials" condition and the "virtual, computer-based materials" condition (p. 152). Olympiou and Zacharia (2012) compared the effectiveness of three instructional conditions which they called experimenting with physical manipulatives (PM), with virtual manipulatives (VM), and with a blended combination of PM and VM, to determine students' understanding of concepts in the domain of Light and Color. Zacharia and deJong (2014) compared the effectiveness of five instructional conditions that included "virtual material" and a "Virtual Labs Electricity environment" in which students manipulated "virtual objects and virtual instruments" to develop an understanding of electric circuits (p. 112). In another comparison study, Lazonder and Ehrenhard (2013) compared the effectiveness of physical and virtual manipulatives in an inquiry task about falling objects. Just like the mathematics literature, it is unclear how closely aligned the "virtual manipulatives" being used in these science studies are with the 2002 definition of virtual manipulatives for mathematics. It may be important for the science education community to define virtual manipulatives and virtual materials in the context of science.

1.8 An Updated Definition for Virtual Manipulatives

As these questions posed over the past decade show, there is a need for greater clarification of the definition of a virtual manipulative. Based on the discussion in the preceding sections, which included proposed revisions, here we suggest an updated definition of a virtual manipulative: *an interactive, technology-enabled visual representation of a dynamic mathematical object, including all of the programmable features that allow it to be manipulated, that presents opportunities for constructing mathematical knowledge*. This updated definition preserves the term "interactive" in the definition because this is a defining characteristic of a virtual manipulative. The updated definition takes into account that all virtual manipulatives do not have to be "web-based", and replaces this terminology with the term "technology-enabled". The updated definition also preserves the terms "visual representation of a dynamic object" and adds the term "mathematical" to clarify that we are referring to a representation of a mathematical object.

The updated definition clarifies that the visual representation of a dynamic object is accompanied by all of its programmable features, because without these features it would not be interactive and dynamic. Implied in this updated definition is that a virtual manipulative may: (a) appear in many different technology-enabled environments; (b) be created in any programming language; and (c) be delivered via any technology-enabled device.

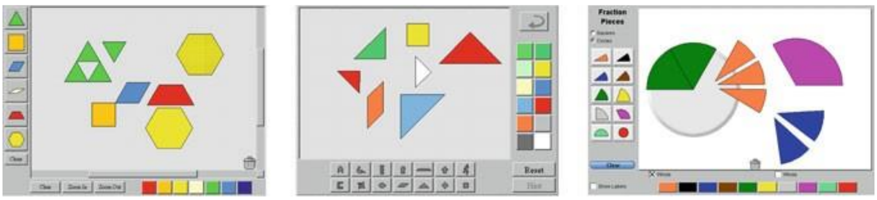
1.9 Common Virtual Manipulative Environments

One source of confusion about what is and what is not a virtual manipulative has been that virtual manipulatives have been designed to be housed in various technological environments. Other authors have outlined categories of computer-based learning technologies for mathematics education. For example, Handal and Herrington (2003) reported that there are six categories of computer-based learning in mathematics and these include: drills, tutorials, games, simulations, hypermedia, and tools. Kurz et al. (2005) reported that there are five categories of tool-based mathematics software and these include: review and practice, general, specific, environment, and communication. Although there are some commonalities between these categories and virtual manipulative environments, the categories are not specific to virtual manipulatives. In an NCTM conference presentation, Bolyard and Moyer (2007) discussed four virtual manipulative environments. However, there has been no publication that has described these environments.

This section of the chapter seeks to put that discussion into print by describing the common environments in which virtual manipulatives frequently appear. Currently, there are five common virtual manipulative environments that have been used by developers. These environments include: single-representation, multi-representation, tutorial, gaming and simulation. While other environments may exist and new environments may be developed, these five environments have stood the test of time and can be found most commonly among the virtual manipulatives currently available to users.

The single-representation virtual manipulative environment. The single-representation virtual manipulative environment contains an interactive pictorial/visual representation (i.e., image) of the dynamic mathematical object and is not accompanied by any numerical or text information. Bolyard and Moyer (2007) referred to this as "pictorial-only" in their NCTM presentation. The single-representation environment typically relies on only one type of representation of the mathematics and, most commonly, that single representation is a pictorial image. In some cases, the pictorial image is based on a physical manipulative, and in some cases the virtual manipulative image has no physical counterpart. Some publications mistake this notion, which implies that all virtual manipulatives are patterned after physical manipulatives: "Virtual manipulatives are screen-based instantiations of physical manipulatives…" (Manches and O'Malley 2012, p. 406).

Three examples of the single-representation environment are the Pattern Blocks, the Tangrams, and the Fraction Pieces found at the National Library of Virtual Manipulatives (NLVM; nlvm.usu.edu) website (see Fig. 1.4). The virtual manipulative pattern blocks contain six different geometric shapes that users can move and alter (e.g., change color, change location, and change the orientation). The tangrams also contain several different geometric shapes that users can move and alter (e.g., change color, change location, and change the orientation). The fraction pieces contain different fractional portions of a circle region that users can move and compare with the whole. In the single-representation environment, the pictorial image is the predominant representation, with limited information provided in numerical or text form. As can be seen in Fig. 1.4, this environment simply includes the pictorial representation of the objects for the user to manipulate along with all of the accompanying programmable features.



Pattern Blocks Tangrams Fraction Pieces

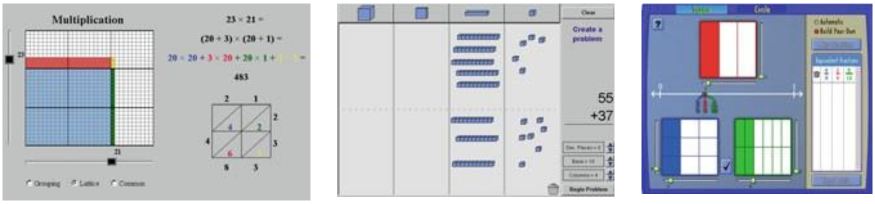
Fig. 1.4 Examples of the single-representation virtual manipulatives environment found at the nlvm.usu.edu

The single-representation environment requires the teacher to design specific tasks for learners that will help draw their attention to the mathematical ideas under study. However, this environment also allows the teacher more flexibility with the tools to design specific tasks that meet the needs and goals of the curriculum. Because of its open-ended nature, the single-representation environment can easily be used as the basis for independent practice activities (Wight and Kitchenham 2015). Anderson-Pence (2014) reported that, because the single-representation environment relies only on pictorial images, this environment is more versatile for use in teaching because the pictorial images can be used for many different types of mathematical explorations.

The single-representation environment also places responsibility on the student for attending to and making sense of connections between the pictorial representations and numeric representations of the mathematics, because the numeric representations do not appear simultaneously with the pictorial images, as is the case in other virtual manipulative environments. Anderson-Pence (2014) reported that, when student pairs worked with the single-representation environment (which she called "pictorial"), they had the largest amount of discussion and the highest use of gestures (both physical gestures and computer-based gestures). However, these discussions were not at a high level that would lead to mathematical generalizations.

Other reports on the single-representation environment have noted that this environment leads to more creative variation during problem solving (Moyer-Packenham and Westenskow 2013). For example, Moyer et al. (2005) reported that children using the virtual manipulative pattern blocks (a single-representation environment) exhibited more creative behaviors with the blocks. Because this environment contains only visual images, students working in pairs must put forth more effort in communicating how to manipulate the objects, how to solve problems, and what mathematics these activities represent.

The multi-representation virtual manipulative environment. The multi-representation virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and, sometimes, text information. Therefore, the multi-representation environment typically relies on two or more forms of representations, and these are often pictorial and numeric representations. Bolyard and Moyer (2007) referred to this as "combined pictorial and numeric" in their NCTM presentation. Three examples of the multi-representation environment are the Rectangle Multiplication of Fractions and Base Blocks Addition found at the NLVM and Equivalent Fractions found at the NCTM Illuminations website (nctm.org; see Fig. 1.5). The Rectangle Multiplication of Fractions app shows a pictorial image of a grid with numerical information to accompany the visual changes in the amounts in the grid. The Base Blocks Addition app shows a pictorial image of base-10 blocks with numerical information that represents the changing amounts displayed by the blocks. The Equivalent Fractions app shows a pictorial image of three rectangular regions that can be divided and shaded to show fraction amounts that are displayed on a number line and recorded in a table for the user.



Rectangle Multiplication of Fractions

Base Blocks Addition

Equivalent Fractions

Fig. 1.5 Examples of the multi-representation virtual manipulatives environment found at the nlvm.usu.edu and nctm.org

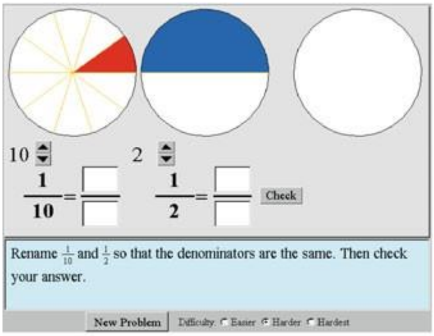
In each of these applications, the environment contains multiple representations and the pictorial images are commonly linked simultaneously with the numeric information. As the user interacts with the pictorial images, the numeric information provides an abstract model that accompanies the images. The presentation of two or more different representations (e.g., pictorial, numeric, text) simultaneously enables the user to link images with abstractions in numeric mathematical form. As can be seen in Fig. 1.5, the multi-representation environment often contains primarily pictorial representations and numerical representations in a linked form along with all of the accompanying programmable features.

For many years, researchers have recognized the importance of linking features in computational media to promote *representational fluency* and learners' ability to see relationships among representations (Kaput 1986). Sarama and Clements (2009) describe this as "linking the concrete and the symbolic with feedback" (p. 147). A meta-analysis of the research on virtual manipulatives shows that simultaneous linking of representations has positive impacts on students' mathematics achievement (Moyer-Packenham and Westenskow 2013). For example, Suh and Moyer (2007) reported that their students observed the links between the algebra symbols and the movement of a balance scale. Haistings (2009) reported that her students preferred the linked pictorial/symbolic apps because the mathematical information appeared for them on the screen and they did not have to remember or recount the blocks during problem solving. Additionally, the numbers changed as they performed actions with the blocks allowing them to see the result of their actions.

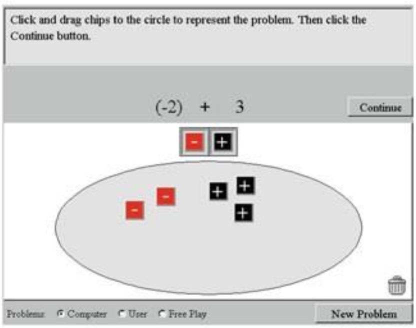
Viewing numeric and pictorial information that changes simultaneously allows the user to adapt and reinterpret the representations (Martin and Schwartz 2005). Anderson-Pence (2014) reported that, when students worked in pairs using the multi-representation environment (which she referred to as "combined"), students' discussions reflected higher levels of mathematical generalization, justification, and collaboration. The multiple representations encouraged students to make connections, make comparisons among the representations, and see patterns more easily. A similar finding was also reported by Ares et al. (2008), who noted that interacting with multiple representations promoted mathematical discourse among students.

The tutorial virtual manipulative environment. The tutorial virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object and is accompanied by numerical and text information in a format that guides the user through a tutorial of the mathematical procedures and processes being presented. Therefore, the tutorial environment provides a guiding and tutoring support structure for the user and relies on multiple forms of representation—pictorial, numeric, and text. The guiding and tutoring features are what make the tutorial environment different from the multi-representation environment.

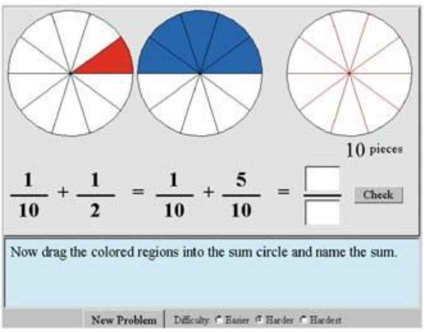
Two examples of the tutorial environment are Fractions Adding and Color Chips Addition found at the National Library of Virtual Manipulatives (see Fig. 1.6). The Fractions Adding app presents the user with two fractions that have unlike denominators. The prompt in the tutorial guides the user to rename the two fractions so that they have a denominator that is common to both fractions. As students use the arrow button to change the number of pieces of each fraction, they can see how the total number of pieces changes on each fraction region until they find divisions of the regions that are common. Once the common denominator is found, students are prompted to rename the two fractions and check to see if their answer is correct.



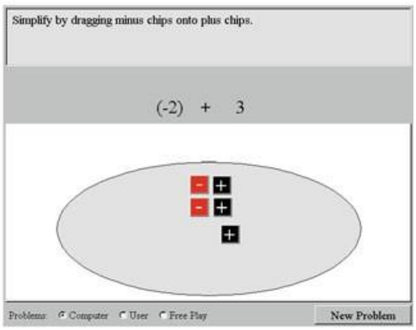
Fractions Adding – screen 1



Color Chips Addition – screen 1



Fractions Adding – screen 2



Color Chips Addition – screen 2

Fig. 1.6 Examples of the tutorial virtual manipulatives environment found at the nlvm.usu.edu

When they have created correct common denominators, they continue to the next screen and are guided to add the renamed fractions by dragging the fraction pieces into a sum region. When students type the answer in symbolic form that represents the pictorial image they have created, they receive feedback that tells them if their response is correct or that guides them to make an adjustment to their answer if it is incorrect.

The Color Chips Addition app presents the user with a numeric expression and prompts the user to use the positive and negative chips to build the expression. Students continue to the next screen where they are prompted to simplify the expression and type in a solution. The tutorial environment generally follows this format of guiding and tutoring students to understand a process in a step-by-step manner. As can be seen in Fig. 1.6, this environment can include multiple steps that guide students through a process or procedure using a variety of representations.

Anderson-Pence (2014) reported that the tutorial environment is better suited to students working individually because the tutorial essentially serves as an individual tutor that walks students through the steps of solving a problem or learning a mathematical procedure. This environment discourages communication among student pairs because of the step-by-step format that allows little exploration or deviation from the tutoring process.

While this environment is not as useful for students working in pairs, the tutorial environment has been shown to have significant positive effects in classroom studies where students were working individually at their own computers (Reimer and Moyer 2005; Steen et al. 2006; Suh and Moyer 2007). For example, in one study with low, average and high achievement groups, researchers reported that the low achievers benefited from the treatment because of the step-by-step presentation format in the tutorial environment. Researchers stated: "The low achieving group used a step-by-step methodical process to find multiples and common denominators…" (Moyer-Packenham and Suh 2012, p. 53). The step-by-step tutorial environment led the low achieving group through this process to successfully complete the mathematical procedures.

The gaming virtual manipulative environment. The gaming virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object that is embedded in a format that allows the user to play a game with the objective to reach goals that are reflected in the game play. Therefore, the gaming environment relies on multiple forms of representation embedded in an environment with a variety of gaming features that might include levels, badges, time constraints, clear goals, challenge and play-centric design (Deterding et al. 2011).

Three examples of the gaming environment are Motion Math Zoom, Dragon Box Algebra, and Hungry Guppy found on the Apple iTunes store (see Fig. 1.7). The Motion Math Zoom app is an interactive number line that users can swipe left and right to view higher numbers and lower numbers on the number line, respectively. To quickly move from ones to tens to hundreds to thousands, users employ a

Motion Math Zoom Dragon Box Algebra Hungry Guppy

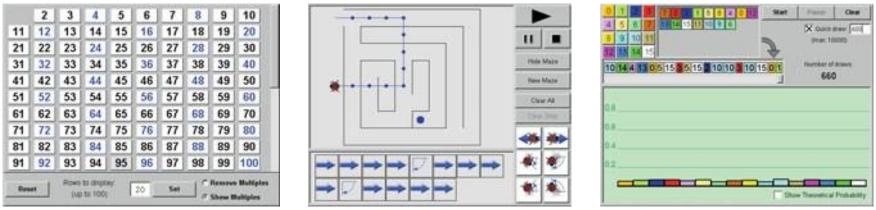
Fig. 1.7 Examples of the gaming virtual manipulatives environment

two-finger pinching and stretching motion to "zoom in and out" on the number line. In the game, numbers appear in bubbles above the number line. The user must move the number line to the correct location so that it is below the number in the bubble and then pop the bubble so that the number lands at the correct placement on the number line. The game has 24 levels, with multiple tasks in each level, that increase in difficulty. There is a needle that can be turned on or off that acts as a timer to encourage the user to become increasingly more efficient at identifying where the numbers go on the interactive number line.

The Dragon Box Algebra app engages the user with operations, additive and multiplicative thinking, solving expressions and equations, and fractions. The game has ten 20-level chapters where the user moves game pieces to solve expressions or equations to complete the game levels. The Hungry Guppy app requires the user to combine bubbles of different numbered dots to create a target number and feed the hungry fish. When the correct number of dots is fed to the fish, the fish gets larger and the user completes the level. As can be seen in Fig. 1.7, the gaming environment typically has multiple representations and a more developed background design and visual images that enhance the appearance of the app when compared with the other virtual manipulative environments.

Tucker (2015) reported that a user's mathematical and technological distance (with *distance* defined as the "degree of difficulty in understanding how to act upon [something] and interpret its responses" (Sedig and Liang 2006, p. 184)) changed as they interacted with the Zoom app. Other studies have reported that virtual manipulatives in gaming environments can have positive effects on the development of mathematics learning (Carr 2012). For example, Barendregt et al. (2012) reported that, when five- and six-year-old children played the Fingu game during a three-week period, it supported the development of their subitizing and arithmetic skills. Riconscente's (2013) research using the Motion Math Fractions game for the iPad with 122 fourth-grade students showed that when the students played the game for 20 min daily for a 5-day period, there was a 15 % improvement in students' fraction test scores.

The simulation virtual manipulative environment. The simulation virtual manipulative environment contains the interactive visual representation (i.e., image) of the dynamic mathematical object along with other representations (e.g., numeric,



Sieve of Eratosthenes   Lady Bug Maze   Box Model

Fig. 1.8 Examples of the simulation virtual manipulatives environment

text) that are embedded in a format that allows the user to run a simulation intended to represent or draw attention to embedded mathematics concepts. Therefore, the simulation environment may rely on one or multiple forms of representation that can be used to run the simulation. Three examples of the simulation environment are the Sieve of Eratosthenes, Lady Bug Maze, and the Box Model found at the National Library of Virtual Manipulatives (see Fig. 1.8).

The Sieve of Eratosthenes app allows users to run a simulation showing the multiples of the numbers on a number board. Running the simulation of each successive number on the board (e.g., the multiples of 2, 3, 4, 5, etc.) reveals patterns in the multiples and helps users to identify the prime numbers on the number board. The Lady Bug Maze allows the user to create a program for the path of a lady bug in order to help the lady bug reach a point within the maze. Each time the user creates and modifies the program, there is a “play” button that allows the user to run the simulation to see if the programing commands that they have created allow the lady bug to successfully navigate the maze. By repeatedly running the simulation, the user can make adjustments to their programing commands until the lady bug is successful.

The Box Model app simulates multiple random draws of numbers from a box and plots the numbers on a chart comparing actual probability to theoretical probability. The simulation environment allows the user to efficiently perform and model multiple trials over and over again. Clements et al. (2001) research with a virtual manipulative in the simulation environment used Logo Geometry (which has a similar design to the Lady Bug Maze pictured in Fig. 1.8) to simulate geometric shapes, paths and motions. In a study of 1624 Kindergarten through 6th grade students, those who used the Logo Geometry curriculum made significant gains, which were almost double the gains of those students who participated in traditional geometry instruction. This study of the simulation virtual manipulative environment showed that Logo Geometry helped students link symbolic and visual representations, demanded greater precision in geometric thinking from students, and encouraged students to make and test geometric conjectures.

1.10 Concluding Remarks

This chapter provided an update to the definition of a virtual manipulative. This new definition reflects attention to technology developments and clarification about what is and is not included in the technology for it to be defined as a virtual manipulative. The chapter also described five different environments in which virtual manipulatives are commonly embedded and provided examples of each to show the structure of the most common designs of virtual manipulative environments. As these examples demonstrate, there are a variety of virtual manipulative environments currently in use today. This updated definition and the descriptions of the five environments provide guidance for educators and researchers on a common language and understanding of the meaning of a virtual manipulative for teaching and learning mathematics.

The potential of virtual manipulatives to support students' developing mathematical ideas relies on judicious, appropriate, and effective use. Learners must experience the virtual manipulative and interact with its characteristics and features in ways that represent the relevant mathematics. Virtual manipulatives are technologies, and like any technology, virtual manipulatives do not create learning; rather, it is the quality of the engagement with the technology that presents opportunities for learning mathematics.

References

Anderson-Pence, K. L. (2014). *Examining the impact of different virtual manipulative types on the nature of students' small-group discussions: An exploratory mixed-methods case study of techno-mathematical discourse* (Doctoral dissertation). ProQuest Dissertations and Theses database (UMI No. 3683422).

Ares, N., Stroup, W. M., & Schademan, A. R. (2008). The power of mediating artifacts in group-level development of mathematical discourses. *Cognition and Instruction, 27*(1), 1–24. doi:10.1080/07370000802584497

Barendregt, W., Lindström, B., Rietz-Leppänen, E., Holgersson, I., & Ottosson, T. (2012). Development and evaluation of Fingu: A mathematics iPad game using multi-touch interaction. In H. Schelhowe (Ed.), *Proceedings of the 11th international conference on interaction design and children* (pp. 204–207). ACM, New York, NY. http://doi.org/10.1145/2307096.2307126

Bolyard, J. J., & Moyer, P. S. (2007, March). *Selecting dynamic technology representations for mathematics teaching*. Research presentation, 85th annual meeting of the national council of teachers of mathematics (NCTM), Atlanta, GA.

Bos, B. (2009a). Technology with cognitive and mathematical fidelity: What it means for the Math classroom. *Computers in the Schools, 26*(2), 107–114. doi:10.1080/07380560902906088

Bos, B. (2009b). Virtual math objects with pedagogical, mathematical, and cognitive fidelity. *Computers in Human Behavior, 25*, 521–528.

Carr, J. M. (2012). Does math achievement happen when iPads and game-based learning are incorporated into fifth-grade mathematics instruction? *Journal of Information Technology Education: Research, 11*, 269–286.

Clements, D. H., Battista, M. T., & Sarama, J. (2001). Logo and geometry. *Journal for Research in Mathematics Education. Monograph, 10*, 1–177.

Deterding, S., Dixon, D., Khaled, R., & Nacke, L. (2011). From game design elements to gamefulness: Defining "Gamification". *Communications of the ACM, 978*, 1–15.

Dorward, J. & Heal, R. (1999). National library of virtual manipulatives for elementary and middle level mathematics. In *Proceedings of WebNet world conference on the WWW and internet 1999* (pp. 1510–1511). Association for the Advancement of Computing in Education (AACE), Chesapeake, VA. Retrieved August 12, 2015 from http://www.editlib.org/p/7655

Goldin, G. A. (2003). Representation in school mathematics: A unifying research perspective. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to Principles and Standards for School Mathematics* (pp. 275–285). Reston, VA: NCTM.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics NCTM yearbook 2001* (pp. 1–23). Reston, VA: NCTM.

Haistings, J. L. (2009). *Using virtual manipulatives with and without symbolic representation to teach first grade multi-digit addition* (Doctoral dissertation). Available from ProQuest Dissertations and Theses database (UMI No. 3366234).

Handal, B., & Herrington, A. (2003). Re-examining categories of computer-based learning in mathematics education. *Contemporary Issues in Technology and Teacher Education, 3*(3), 275–287.

Heal, R., Dorward, J., & Cannon, L. (2002). Virtual manipulatives in mathematics: Addressing Conceptual dilemmas. In D. Willis, J. Price & N. Davis (Eds.), *Proceedings of society for information technology and teacher education international conference 2002* (pp. 1056–1060). Association for the Advancement of Computing in Education (AACE), Chesapeake, VA.

Kaput, J. J. (1986). Information technology and mathematics: Opening new representational windows. *The Journal of Mathematical Behavior, 5*(2), 187–207.

Kay, R. H. (2012). Examining factors that influence the effectiveness of learning objects in mathematics classrooms. *Canadian Journal of Science, Mathematics, and Technology Education, 12*(4), 350–366.

Kirby, K. D. (2013, April). *The development of an idealized number line: Differentiating physical inscription from mathematical object.* Paper presented at the American Educational Research Association, San Francisco, CA.

Kurz, T. L., Middleton, J. A., & Yanik, H. B. (2005). A taxonomy of software for mathematics instruction. *Contemporary Issues in Technology and Mathematics Teacher Education, 5*(2), 1–13.

Lazonder, A. W., & Ehrenhard, S. (2013). Relative effectiveness of physical and virtual manipulatives for conceptual change in science: How falling objects fall. *Journal of Computer Assisted learning, 30*(2), 110–120.

Manches, A., & O'Malley, C. (2012). Tangibles for learning: A representational analysis of physical manipulation. *Personal and Ubiquitous Computing, 16*(4), 405–419.

Martin, T., & Schwartz, D. L. (2005). Physically distributed learning: Restructuring and reinterpreting physical environments in the development of fraction concepts. *Cognitive Science, 29*, 587–625.

Moyer, P. S., Bolyard, J. J., & Spikell, M. A. (2002). What are virtual manipulatives? *Teaching Children Mathematics, 8*(6), 372–377.

Moyer, P. S., Niezgoda, D., & Stanley, J. (2005). Young children's use of virtual manipulatives and other forms of mathematical representations. In W. J. Masalski & P. C. Elliott (Eds.), *Technology-supported mathematics learning environments: Sixty-seventh yearbook* (pp. 17–34). Reston, VA: NCTM.

Moyer-Packenham, P. S., & Suh, J. M. (2012). Learning mathematics with technology: The influence of virtual manipulatives on different achievement groups. *Journal of Computers in Mathematics and Science Teaching, 31*(1), 39–59.

Moyer-Packenham, P. S., & Westenskow, A. (2013). Effects of virtual manipulatives on student achievement and mathematics learning. *International Journal of Virtual and Personal Learning Environments, 4*(3), 35–50.

Olympiou, G., & Zacharia, Z. C. (2012). Blending physical and virtual manipulatives: An effort to improve students' conceptual understanding through science laboratory experimentation. *Science Education, 96*(1), 21–47.

Pea, R. D. (1985). Beyond amplification: Using the computer to reorganize mental functioning. *Educational Psychologist, 20*, 167–182.

Reimer, K., & Moyer, P. S. (2005). Third graders learn about fractions using virtual manipulatives: A classroom study. *Journal of Computers in Mathematics and Science Teaching, 24*(1), 5–25.

Resnick, M., Martin, F., Berg, R., Borovoy, R., Colella, V., Kramer, K., et al. (1998). *Digital manipulatives: New toys to think with*. Proceedings of the SIGCHI conference on human factors in computing systems (pp. 18–23). MIT Media Laboratory, Boston, MA.

Riconscente, M. M. (2013). Results from a controlled study of the iPad fractions game motion math. *Games and Culture, 8*(4), 186–214. http://doi.org/10.1177/1555412013496894

Sarama, J., & Clements, D. H. (2009). "Concrete" computer manipulatives in mathematics education. *Child Development Perspectives, 3*(3), 145–150.

Sedig, K., & Liang, H.-N. (2006). Interactivity of visual mathematical representations: Factors affecting learning and cognitive processes. *Journal of Interactive Learning Research, 17*(2), 179–212.

Simon, M. A. (2013). The need for theories of conceptual learning and teaching of mathematics. In K. R. Leatham (Ed.), *Vital directions for mathematics education research* (pp. 95–118). New York: Springer.

Steen, K., Brooks, D., & Lyon, T. (2006). The impact of virtual manipulatives on first grade geometry instruction and learning. *Journal of Computers in Mathematics and Science Teaching, 25*(4), 373–391.

Suh, J., & Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching, 26*(2), 155–173.

Triona, L. M., & Klahr, D. (2003). Point and click or grab and heft: Comparing the influence of physical and virtual instructional materials on elementary school students' ability to design experiments. *Cognition and Instruction, 21*(2), 149–173.

Tucker, S. I. (2015). *An exploratory study of attributes, affordances, abilities, and distance in children's use of mathematics virtual manipulative iPad apps* (Doctoral dissertation). ProQuest Dissertations and Theses database.

Wight, H., & Kitchenham, A. (2015). Virtual 10 frames and mobile technology in a Canadian primary classroom. In *Mobile learning and mathematics* (pp. 135–149). New York, NY: Routledge.

Zacharia, Z. C., & deJong, T. (2014). The effects on students' conceptual understanding of electric circuits of introducing virtual manipulatives within a physical manipulatives-oriented curriculum. *Cognition and Instruction, 32*(2), 101–158.