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Components of Place Value Understanding: Targeting Mathematical Difficulties When Providing Interventions

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Place value understanding requires the same activity that students use when developing fractional and algebraic reasoning, making this understanding foundational to mathematics learning. However, many students engage successfully in mathematics classrooms without having a conceptual understanding of place value, preventing them from accessing mathematics that is more sophisticated later. The purpose of this exploratory study is to investigate how upper elementary students' unit coordination related to difficulties they experience when engaging in place value tasks. Understanding place value requires that students coordinate units recursively to construct multi-digit numbers from their single-digit number understandings through forms of unit development and strategic counting. Findings suggest that students identified as low-achieving were capable of only one or two levels of unit coordination. Furthermore, these students relied on inaccurate procedures to solve problems with millennial numbers. These findings indicate that upper elementary students identified as low-achieving are not to yet able to (de)compose numbers effectively, regroup tens and hundreds when operating on numbers, and transition between millennial numbers. Implications of this study suggest that curricula designers and statewide standards should adopt nuances in unit coordination when developing tasks that promote or assess students' place value understanding.

Place value understanding requires the same activity that students use when developing fractional and algebraic reasoning (Norton & Boyce, 2015), making this understanding foundational to mathematics learning. However, many students engage successfully in mathematics classrooms without having a conceptual understanding of place value, preventing them from accessing more sophisticated later. With that said, many of these students are still not successful on place-value related items in upper elementary. Four in seven questions on the 2015 National Assessment of Education Progress (NAEP, 2015) relate to place value understandings with only 40% of fourth-grade students performing at or above the proficient level (NAEP, various years 1992–2015). Wright, Ellemor-Collins, and Tabor (2012) have investigated upper elementary students' place value development and provided detailed means of instruction and assessment for low-achieving students. However, absent from their work are detailed stages of conceptual place value understanding illustrating degrees of underlying mental activity in low-achieving students.

Battista (2012) argues that for an educator to "remediate a learning difficulty requires a more detailed picture" (p. 7), as educators are often not aware of students' underlying mental activity. In this study, we frame this mental activity with unit coordination literature (Norton & Boyce, 2015), to describe students' construction of composite units (groups of items composed of more than one item) and their ability to coordinate these units when understanding place value. The purpose of this study was to investigate elementary low-achieving students' unit coordination when engaging in place value tasks. We define a "low-achieving student" as a student achieving below standards established by research related assessments and tasks (Ginsburg & Baroody, 2003). Our inquiry focused on the following research questions:

1. What types of unit development difficulties are prevalent in the place value understanding of these low-achieving students (Grades 3, 4, and 5)?
2. How do these difficulties appear to manifest themselves as possibly limiting low-achieving students' development of place value concepts?

In this study, the authors adopt the constructivist paradigm, meaning that students learn mathematics through their active engagement with mathematics materials (Clements & Battista, 1990). We adopt this epistemological lens as informing our conceptual framework and review the literature on children's grouping and counting actions for the construction of multi-digit number understanding.

Literature Review and Conceptual Framework

Research shows that students are capable of constructing place value understandings earlier than previously thought (Resnick, 1984). Further, researchers report that children who can engage with multiple forms of activity (i.e., counting objects, grouping, estimating) construct more robust multi-digit understandings (e.g., Chan, Au, & Tang, 2014; Fuchs et al., 2010; Resnick, 1984). However, educational researchers do not yet know which of these forms of activity can best be leveraged to support low-achieving upper elementary students. In the following sections, we consider how unit coordination and conceptual place value relate to students' place value development.

Unit Coordination as a Conceptual Framework

Place value is grounded in unit coordination. Students first construct number through a building up of empirical activity into an abstract series of coordinated actions whereby part-whole understandings are reorganized and extended to construct number as a mental object to act upon (Norton & Boyce, 2015; Steffe, 1994, 2001). The construction of mental objects forms the basis for unit construction and coordination, as students rely on concrete, pictorial, fingers, symbolic numerals, and language to evidence internalized (being able to mentally re-imagine contextual actions) or interiorized (being able to draw on de-contextualized actions) actions.

When students solve problems with multi-digit numbers, they might draw upon forms of unit construction and coordination. Students understanding how proportional relationships grow by powers of 10, suggests that they would need to coordinate as many as four units (i.e., $10^{\circ} \times 10^1 \times 10^2 \ldots$). For example, for students to conceptually understand 1,000, they would need to understand how it relates proportionally to 100 and 10,000 ($10^2 \times 10^1$ and $10^4/10^1$). This requires multiplicative operations of exponential numbers, indicative of relatively more units to coordinate than if the student were asked to simply understand 1,000 in isolation (10^3). Students unable to coordinate these exponential relationships might simply describe larger numbers (similar to what is illustrated above) as smaller numbers with several zeros and use language such as, "one with three zeros" or "one-thousand" but not be able to represent the value of this number concretely. However, once number is interiorized it becomes a mental object to act upon. Norton and Boyce (2015) delineate stages of unit coordination as attributable to students' number development (see Table 1). They explain that students capable of building a composite unit with a series of ones may be utilizing one level of unit coordination (Stage 1) to understand two levels of units. However, for students to operate at Stage 2, they would need to iterate a composite unit (i.e., three is repeated five times) to build a new composite unit (i.e., fifteen).

Table 1
Three Stages of Unit Coordination

	Students' Mental Actions
Stage 1	Students rely on one level of units (i.e., units of "one") and can build towards the coordination of two levels of units.
Stage 2	Students rely on two levels of units (i.e., units of "five" as composed with "five units of one") and can build toward the coordination of three levels of units.
Stage 3	Students rely on three levels of units (i.e., units of "fifteen" as composed with "five units of three").

Note. Table interpreted from Norton and Boyce's (2015) Table 1, p. 212.

Finally, Norton and Boyce (2015) explain that students at Stage 3 are able to understand simultaneously how all three levels of units relate, as they have developed mental structures for number. For example, a student at a Stage 3 level of understanding can understand how a unit of "one" relates to a composite unit of a composite unit (i.e., three sets of five). Students unable to coordinate all three levels of units simultaneously may build toward these mental structures through a "counting by" strategy or may count all. When developing Stage 3 unit coordination, students in early elementary grades are required to (1) develop mental activity necessary for conceptual place value development, and (2) develop units through counting and grouping. These two themes will be discussed more fully in this article.

Mental Activity Necessary for Conceptual Place Value Development

Procedural knowledge should be an extension of conceptual knowledge (Hiebert & Lefevre, 1986). Clark and Kamii (1996) found that first- through fifth-grade students were capable of constructing tens and ones for conceptual place value development. This finding seemed to occur when tasks afforded students opportunities to operate on multi-digit addition numbers and develop their own procedures for these operations. Thus, students were much more capable when they constructed procedures and relied on their internal mental activity.

Battista (2012) delineated a set of learning progressions that explain mental activity that students' rely on when developing conceptual place value. Seven levels delineate these actions and operations that ultimately describe students as beginning with concrete manipulative engagement, then transitioning toward abstract number engagement, and eventually developing algorithmic procedures for multi-digit numbers (p. 10). The stance that Battista takes on his learning progressions is that educators need to transition from a deficit model toward a preventive model when assisting low-achieving students in their place value development. However, missing from Battista's delineation of these levels is how to serve low-achieving students who are operating in schools that use intervention models. In other words, how can educators intervene to support low-achieving students who are already many school years behind in meeting statewide standards and classroom objectives?

Ellemor-Collins and Wright (2011) studied conceptual place value development with low-achieving students in the upper elementary grades. Similar to Battista (2012), Ellemor-Collins and Wright found that students struggling with place value understanding should engage in tasks that extend from concrete to abstract material, and extend from place values of 200 and beyond. Furthermore, students capable of complex unit exchanges (i.e., transitioning between ones to tens to hundreds back to ones) were more capable of extending number understanding beyond 1,000.

Wright et al. (2012) delineate instruction for low-achieving students, and connect number "structures" to conceptual place value development. They define this *structuring of numbers* as students' understanding that number is a structure and structured with smaller structures (similar to part-whole reasoning). Three sets of mental activity are required for structuring of numbers: (a) combinations and pattern development, (b) part-whole number construction, and (c) relational thinking. These are very similar to unit coordination. However, these types of mental activity do not explain the degree of unit coordination students may be engaging with as tasks change.

In addition, Wright et al. (2012) posit that three dimensions should be used when supporting students' conceptual place value development. These dimensions include: (a) extending ranges of numbers, (b) making the increments and decrements of number more difficult, and (c) distancing the setting. Absent from this discussion are trajectories that explain specific degrees of change in mental activity when understanding place value. We believe that by adopting this unit coordination lens we can explain more detail regarding the degree of change in students' responses.

Structuring of numbers explains one form of mental activity students develop with conceptual place value. However, another form of mental activity that children develop with conceptual place value is a mental number line development (Dehaene, Izard, Spelke, & Pica, 2008; Moeller, Pixner, Kaufmann, & Nuerk, 2009; Siegler & Booth, 2004). Quite often mathematical tasks and assessments that use a physical number line measure changes in students' mental number lines. For instance, Siegler and Ramani (2008) found that when young children from low-socioeconomic status homes played games with linear number boards (similar to number lines) their estimation proficiency equaled that of their higher-achieving peers. Siegler and Booth (2004) also used a number line to measure first- and second-grade students' understanding of number. The findings indicated that when these students used a number line there were significant correlations between their estimating and mathematical achievement ($pr(15) = -.60$, $p < .01$ for first-grade students and $pr(17) = -.76$, $p < .01$ for second-grade students).

Many findings suggest that students in the early grades begin understanding magnitudes of number by relying on a mental logarithmic number line before developing a mental linear number line (Dehaene, 2011; Dehaene et al., 2008; Siegler & Booth, 2004). However, Moeller et al. (2009) found that children may be relying on two mental linear number lines (one for numbers 1–10 and one for one-digit and two-digit numbers 1–100). Moeller et al. (2009) explain why these findings conflict with Dehaene et al.'s (2008) logarithmic number line theory, as children in Moeller et al.'s study were shown Arabic numerals and children in Dehaene et al.'s study were verbally told numbers. Thus, it seems symbols and verbal number words may be processed quite differently by young children when constructing multi-digit numbers. These findings describe the comprehensive nature of children's place value development. Children developing conceptual place value need to transition from concrete to abstract experiences, transition from low numbers to high numbers, rely upon a number as a structure, and rely upon their own mental number lines. Yet, with this comprehensive perspective on place value understanding, there is still much in the literature that is unknown.

Unit Development Through Grouping and Counting

Young children develop abstract units in upper elementary grades by engaging in physical grouping actions and spatial reasoning activities (Battista, 2004; Jones, Thornton, & Putt, 1994; Long & Kamii, 2001; McGuire & Kinzie, 2013; Reynolds & Wheatley, 1996; Winer & Battista, 2015; Wright et al., 2012). For the purpose of this study, we will focus mainly on relationships between grouping and unit coordination. McGuire and Kinzie (2013) found that when young children grouped objects in units of 10, they attended to groups of tens in numbers, but could not coordinate tens and leftovers. Fuson et al. (1997) found children constructed multi-digit number understandings through a coordination of decades and ones, whereby two-digit numbers were unitized by tens (i.e., fifty) and by ones (i.e., three). Fuson et al. (1997) explain that children in their final stages of multi-digit number understanding need to understand numbers either by a sequence of tens and ones (i.e., ten, twenty, thirty, thirty-one, thirty-two) or by separate tens and ones (i.e., one ten, two tens, three tens, and one, two is 32).

Strategic counting is one means in which grouping and counting have been found to be effective for developing place value understanding abstractly (Chan et al., 2014; Steffe & Cobb, 1988). Chan et al. (2014) defined strategic counting as students' grouping of objects when counting. Essentially, for students to be successful in with place value, they need to build operations of numbers by first internalizing counting actions and their associated coordinated actions before interiorizing the operations as mental structures (Chan et al., 2014; Norton & Boyce, 2015; Resnick, 1984; Steffe & Cobb, 1988).

Steffe and Cobb (1988) initially proposed that students compose and coordinate units when counting. Through counting, students understand units as sets of ones prior to the construction of tens and ones. Essentially, beginning with a number-word sequence, children are able to attribute their number-word sequence to perceptual items before re-presenting them with fingers or imagined figurative material.

Chan et al. (2014) found that first-grade students' strategic counting abilities related to their multi-digit number understanding in second grade. Additionally, young children strategically counting in different bases had a wide variety of reasoning about place value number understandings (Slovin, 2011). This suggests that children need to do more than strategic counting to understand place value (2011).

Summary

Much of the research in mathematics education provides a multi-faceted perspective on how low-achieving students conceptualize place value. However, educational researchers do not yet empirically understand relationships between low-achieving students' unit coordination and their place value development.

Much of the place value learning research within the domain of students' number understanding is in the lower elementary grades. Those who study upper elementary students' place value development describe learning progressions, where "representations of learning rely on predetermined benchmarks for student achievement across grade levels" (Weber, Walkington, & McGalliard, 2015, p. 254). The distinction here is that a learning trajectory explains "how students might engage with tasks, reflect on tasks, and develop knowledge through work on tasks" (Weber et al., 2015, p. 254). In addition, by adopting unit coordination as a theoretical model our findings can uniquely explain degrees of cognitive change with a new perspective on student learning.

Methods

This investigation was part of a larger study that used a sequential explanatory mixed-methods design to categorize student responses when understanding place value (Creswell, Plano Clark, Gutmann, & Hanson, 2003). The conceptual framework we described earlier grounded our research design, as we used student responses (qualitative data) to construct conceptual measures of student understanding (quantitative data). This article reports on the analysis of the qualitative data because nuances in student responses were analyzed with descriptive statistics. We considered the quantitative data as representing the frequency of student responses but not as an objective representation of a reality.

Participants and Setting

The participants in this study were upper elementary students referred for participation by their classroom teachers (in the school setting) or their parents (in the clinic setting) because they had experienced difficulties learning school mathematics. More specifically, teachers and parents deemed students as "low-achieving" after students failed to meet research-based standards at the same rate as their peers. For instance, students enrolled in third, fourth, or fifth grade were included in this study because research indicates that students in this age range should begin constructing rational number understandings. Researchers included three grade levels to capture the range and variance of student understandings that may occur across this age and grade span.

None of the participants were receiving special education services for mathematics. A total of 124 participants (63 male) from two western, rural school districts participated in the study. The group consisted of 41 third-grade (ages 8–9), 61 fourth-grade (ages 9–10), and 22 fifth-grade (ages 10–11) students. We did not collect socioeconomic data on individual students; however, most students (93%) attended Title I schools, suggesting a high probability that some participants were from low-socioeconomic homes.

Twenty-six classroom teachers and researchers administered diagnostic assessments with the participants in a one-to-one setting. In this study, we describe each teacher participating as a teacher-researcher, as they were fulfilling both roles as a teacher and as a researcher. The research team trained the teacher-researchers to conduct the interview assessments during a two-day training session to increase the likelihood of reliable administration of the interview procedures. During the training session, the 26 teacher-researchers learned to administer the assessment protocol and they had multiple opportunities to practice the assessment procedures with feedback from the researchers. Teacher-researchers practiced, reflected, and discussed the logistics and decisions that would need to make when administering the assessment and categorizing students' responses. The two-day training session increased the likelihood that all members of the data collection team were consistent in the administration of the assessment.

Assessment Procedures and Analysis

The data collected for this study is a subset of the pre-intervention data collected from the administration of the Place Value Iceberg Intervention Diagnostic Assessment (IIDA) to students scheduled to participate in intervention sessions (Westenskow, Moyer-Packenham, & Child, 2017). In contrast to other place value assessments, the IIDA was designed to be used by teachers when providing Tier II place value interventions (Assessments are available at: http://targetingmathinterventions.weebly.com). The IIDA tool explicitly illustrates which underlying understandings students successfully and unsuccessfully rely on during Tier I instruction. To develop IIDA assessment questions, the place value literature was analyzed and essential components of place value learning and the types of student misconceptions and errors were identified. During the pilot assessments, interviews were video recorded and responses were analyzed to identify categorizations of responses and to ensure that the questions adequately assessed understanding. Assessments were also piloted with teacher administrators to ensure that the categories of responses were easily understood and reliably identified by teachers. Three different grade levels of assessments were developed to reflect the progressions typically used in school instruction. Although all three levels assess the same place value components, the grade-level questions varied in complexity. For example, third-grade questions were limited to numbers one thousand and less, fourth grade were limited to one million and less, and fifth grade to one billion and less. All students were tested only on their grade level and the results of further assessment to determine if the students had mastered low levels was not reported.

This study used 13 IIDA questions which integrated the foundational activity (as described by Sarama & Clements, 2009) and different dimensions (as described by Wright et al., 2012) so we could clearly see how students used previous knowledge when structuring numbers. Unique to the mathematics education field was the theoretical lens we adopted when analyzing student responses.

The research team administered the Place Value assessment to 112 of the participants in their school of residence and to 12 participants in a university clinical setting. Assessment completion typically required 30–45 minutes per student. All assessments were conducted in a one-to-one setting. For each question, the person administering the test presented a card to the student and then read aloud the question for the card. Although students were encouraged to use mental math strategies, the interviewers provided the students with paper and pencil to use if needed. Students were asked to explain their thinking. When categorization of a response was not obvious, test administrators probed the students by asking clarifying questions. If the meaning of students' verbal responses were unclear, the students were encouraged to write down their responses for further clarification. During the assessment the test administrator either recorded the category of the response or wrote a description of the response. Students' written work was collected. Assessments conducted in the clinical sessions were videotaped, but assessments conducted in the school were not. Teacher-researchers administered the assessments in the morning and met in the afternoon with the researchers to discuss student responses. For the quantitative portion of the analysis, which investigated the prevalence of unit development difficulties, we calculated the percentage of students' responses for each category. We then used these descriptive data to consider conceptually how students' responses within each component (i.e., regrouping, counting) might relate to their ability to coordinate units when understanding place value. Using qualitative methods, we employed a thematic analysis (Patton, 1990) and open and axial coding (Strauss & Corbin, 1990) to categorize students' responses for each of the question types.

Results

We identified assessment questions within six constructs of place value, which related to students' use of unit coordination with place value: Counting, Groups of Ten, Decomposing, Regrouping, Position Relationships, and Comparing. We discuss these results in the sections below.

Counting

The Counting questions assessed students' ability to count sequences and to count across transitions (see Figure 1).

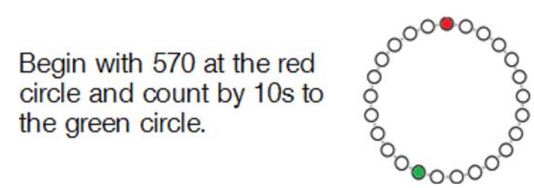

Figure 1. Example of counting questions. Adapted from Shumway (2011) number sense tasks. [Color figure can be viewed at wileyonlinelibrary.com]

Table 2
Percent of Rubric Responses to Counting Questions

		Incorrect		Correct	
Grade	Concept	Sequence	Transition	Pauses	Fluent
3	Counting backward from 106 to 92	4.9	24.4	9.8	61.0
4	Counting backward from 106 to 92	10.0	23.3	1.7	65.0
5	Counting backward from 106 to 92	0	19.0	4.8	76.2
3	Counting by 10's from 570 to 710	9.8	19.5	9.8	61.0
4	Counting by 10's from 970 to 1,110	15.0	61.7	1.7	21.7
5	Counting by 10's from 570 to 710	9.5	9.5	23.8	57.2
3	Counting by 2's from 888 to 916	17.1	39.0	12.2	31.7
5	Counting by 2's from 888 to 916	28.6	23.8	14.3	33.3

Note. N = 3–41, 4–61, 5–22.

We first examined students' ability to sequence numbers. The results indicate that most student (more than 85%) correctly sequenced numbers when counting backwards by ones and forwards by tens (see Table 2). However, a higher number of students (17.1–28.6%) struggled when counting by twos. An explanation for this difference may be that both counting by tens and counting backwards are done in increments of one in the tens and ones positions, while counting by twos requires students to think in increments of two.

For students who correctly counted in sequence, we next examined their counting across century transitions. Counting across century transitions requires the coordination of at least two units because the number to count on from (e.g., 99, 199) needs to be reorganized with the new century (e.g., 100, 200). We identified three categories of transition responses; Incorrect; Transitions with Long Pauses (three seconds or more), and Fluent transitions. Incorrect century transition errors occurred when students stopped counting at the century, substituted a different century, or incorrectly skipped to an incorrect millennial position.

As shown in Table 2, the analysis suggested two trends: First, students tended to make more transition errors when counting by twos than when counting backward or by tens. This may be due to the need to coordinate both groups of two and the regrouping of the tens unit when making a century transition. Second, when asked to count across the one thousands position, 61.5% of fourth-grade students made a millennial transition error (see Table 2). This action required the coordination of three units as students had to count by 1 and reorganize the ten's and hundred's places. Thus, students' coordination of units may develop slowly from grade level to grade level, but the tasks also develop requiring more sophisticated coordination with higher composite units (millennial transitions) and decomposing (backward counting).

Groups of Ten

We used two questions to assess students' use of groups of ten when interpreting numbers (see Figure 1). We identified four categories of responses for question 1; No Grouping, Partial Grouping, Immediate Grouping, and Unitizing (see Figure 2 and Table 3).

Students in the No Grouping category (18.3–24.4%) counted the hearts one-by-one, suggesting that students were at Stage 1 unit coordination (i.e., not acting upon groups of ten as an object) Students in the Partial Grouping category, (0–21.7%) counted the first two columns by ones and then used groups of ten to finish counting. Students in the Immediate Grouping category (23.3–47.6%) counted one group of ten and then counted the remaining groups by tens. These two types of responses suggest that students were at Stage 1 unit coordination, but using the material effectively to build toward coordinating two sets of units. Students in the Unitizing category (28.6–36.7%) answered 42 immediately. Their explanations suggested that they anticipated and then subitized the groups of ten. This response suggests that students were at Stage 2 in their use of unit coordination.

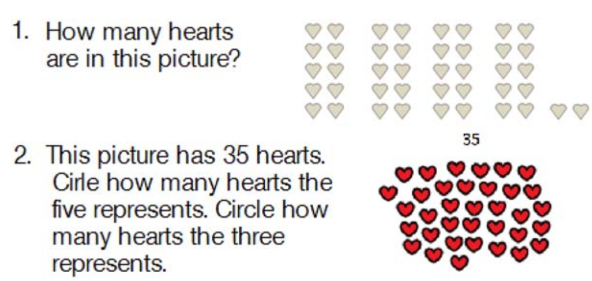

Figure 2. Groups of ten questions. [Color figure can be viewed at wileyonlinelibrary.com]

Table 3
Percent of Students in Each Group of Ten Category

		Rubric Percent			
Grade	Concept	No Grouping	Partial Grouping	Immediate Grouping	Unitizing
3	42 hearts grouped in 10's	24.4	12.2	31.7	31.7
4	42 hearts grouped in 10's	18.3	21.7	23.3	36.7
5	42 hearts grouped in 10's	23.8	0	47.6	28.6

Note. N = 3–41, 4–61, 5–22.

Table 4
Percent of Rubric Responses to the 35 Hearts Question

		Percent Incorrect	Percent Correct	
Grade	Concept	No Place Value	Counts 30	Unitizing
3	Model 3 and 5 in 35	39.0	17.1	43.9
4	Model 3 and 5 in 35	32.8	26.3	41.0
5	Model 3 and 5 in 35	61.9	0	38.1

Note. N = 3–41, 4–61, 5–22.

The second Groups of Ten question assessed students' symbolic understanding that two digit numbers consist of groups of ten and single units. Three categories of responses were identified; No Place Value, Counts 30, and Unitizing) (see Table 4). Student responses in the No Place Value category circled five hearts and then three more hearts. (32.8–61.9%). This response did not reflect an understanding of the difference between the place value positions. This response also suggests students were not coordinating place value units, indicative of Stage 1 unit coordination. In contrast, student responses in the other two categories did reflect the place value positions as students circled five hearts and then circled 30 hearts. The difference between the two categories was that students in the Counts 30 category (0–26.3%) counted and then circled 30 hearts while students in the Unitizing category circled the remaining hearts without counting (see Table 4), The Counts 30 category is indicative of Stage 1 coordination whereas the Unitizing category is indicative of Stage 2 unit coordination.

Decomposing

To assess decomposing, we used an addition and a multiplication question (see Figure 3). The questions assessed students' use of Stage 2 unit coordination. Through these tasks students were required to act upon units of ten by partitioning composite units and then disembedding these composite units with their complements of ten. Students need to coordinate at least two units to be successful in the addition task and possibly three units to be successful in the multiplication task.

These two questions revealed the degree of difficulty many students had when decomposing numbers. For the third-, fourth-, and fifth-grade students 12.2, 18.0, and 18.2%, respectively, percent of the students did not write a correct addition combination (see Table 5). The percent of third-, fourth-, and fifth-grade students who identified three combinations were 56.1, 42.7, and 45.5%, respectively. The most common combination written was 100 + 20.

For decomposing using multiplication, the percent of third-, fourth-, and fifth-grade students who did not write a correct combination (120 × 1 or 60 × 1 was accepted) were 22.0, 14.8, and 36.4%, respectively, and the number of students who wrote four combinations was 4.9, 9.8, and 0%, respectively. An important difference between students' accuracy when decomposing additively versus multiplicatively was students often additively compensated with a "+ 1" or a "+ 2" strategy rather than using groups of 10. The need to use multiplicative decomposing prevented students from using this compensation strategy when decomposing multiplicatively. Further, students responding in this way were at a Stage 1 or 2 unit coordination because they could coordinate one or two

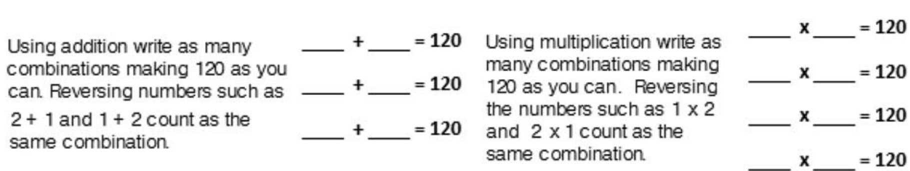

Figure 3. Examples of the addition and multiplication decomposing questions.

Table 5
Percent of the Number of Responses for Each Composing/Decomposing Question

Grade	Question	Number of Combinations			
		0	1	2	3
3	Addition combinations to 120	12.2	29.3	2.4	56.1
4	Addition combinations to 120	18.0	24.6	14.8	42.7
5	Addition combinations to 120	18.2	22.7	13.6	45.5

		Number of Combinations				
		0	1	2	3	4
3	Multiplication combinations to 60	22.0	24.4	41.5	7.3	4.9
4	Multiplication combinations to 120	14.8	26.2	26.2	23.0	9.8
5	Multiplication combinations to 120	36.4	50.0	13.6	0	0

Note. N = 3–41, 4–61, 5–22.

units, but struggled to multiplicatively understand the operative structure of unit needed for multiplication.

Regrouping

Regrouping requires students to act upon their counting and place value construction to regroup numbers. The three skills relevant to unit coordination assessed in this component were addition regrouping, subtraction regrouping, and adding/subtracting one at the transitions to tens and thousands (see Figure 4).

For the regrouping questions we identified three types of regrouping responses: No Regrouping, Incorrect Procedure, and Correct Regrouping (see Table 6). In the first level, No Regrouping, students did not attempt to regroup. For example, to name 2 hundreds, 13 tens, and 1 ones, some students wrote 2,131 and when subtracting, some students subtracted the digit of the least value from the digit of the greatest value. These responses suggest that students were at Stage 1 unit coordination, as they did not appear to understand the conceptual need for a reorganization of units when operating on multi-digit numbers.

In the Incorrect Procedure category, students attempted to regroup but made procedural errors, such as adding 10 instead of 1 to the next position when adding or not exchanging ten rods for one hundred (fifth grade) or ten blocks for one ten rod (third, fourth, and fifth grades). These responses suggest that students were at Stage 1 unit coordination because students had procedures not grounded in conceptual understandings of number. In the Correct category, 85.4% of third graders correctly regrouped ones in a two-digit number, and 39.3% of fourth graders and 54.5% of fifth graders correctly regrouped tens in a three-digit number when naming the modeled numbers (see Table 6). Less than one-half of the students correctly modeled regrouping when subtracting. These responses suggest Stage 2 or 3 unit coordination, as students were either building three levels of units or utilizing three levels of units to coordinate trading and (de)composition of number.

The Before and After regrouping questions assessed students' ability to regroup when crossing transitions (see Figure 5). These types of questions required students to engage in Stage 2 or 3 unit coordination.

Analysis of the Before and After Question results yielded four categories of responses; Last Digit, Both Incorrect, One Correct, and Both Correct (see Table 7). For the Last Digit category, students added and subtracted one to and one from the last non-zero digit in the number. For example, students would respond that 102,000 and 104,000 were one less and one more than 103,000. This suggests that students were not coordinating units of one thousand, but were focused on counting in sequence without consideration of the place value unit of the non-zero digit.

In the second category, students incorrectly attempted regrouping. For example, some student responses to one

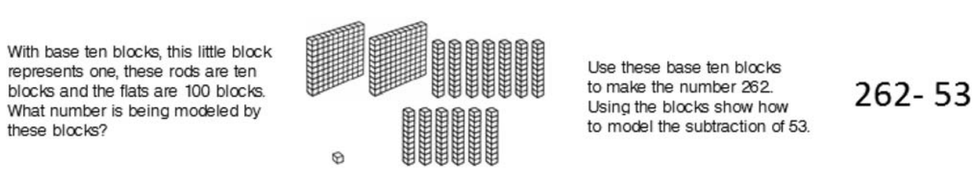

Figure 4. Example of an addition and a subtraction regrouping question.

Table 6
Percent of the Number of Responses for Each No Regrouping/Incorrect Procedure/Correct Procedure Question

Grade	Concept	No Regrouping	Incorrect Procedure	Correct
3	Regroup 5 tens 12 ones	7.3	7.3	85.4
4	Regroup 2 hundreds 13 tens 1 one	29.5	31.1	39.3
5	Regroup 2 hundreds 13 tens 1 one	18.2	27.3	54.5
3	81–3	4.9	46.3	48.8
4	262–53	3.3	52.5	44.3
5	632–253	9.1	68.1	27.7

Note. N = 3–41, 4–61, 5–22.

less than 103,000 were 103,999; 103,009; or 193,999. In the third category, One Correct, students answered one question correctly which typically was the "one after question," suggesting that students could operate at Stage 1 or 2 unit coordination. Students would need to have an operative structure for addition and subtraction relationships to be successful with the "one less" question, indicative of level three unit coordination. In the Both Correct level, there was a large contrast in the number of third-grade students (41.5%) and the number of fourth (12.3%) and fifth (14.3%) grade students who answered both before and after correctly (see Table 7). These findings suggest that students may have operative procedures for lower digit numbers, but struggle with recursive operative relationships when extending these procedures to higher digit numbers.

Relationships

Relationship questions assessed students' conceptual understanding that each place value position is ten times the position to the right and 1/10 of the position to the left (a Stage 3 unit coordination requirement). Three types of questions were used to evaluate students' understanding; (a) How Many Times Larger; (b) Multiplying by Multiples of Ten; and (c) Dividing by Multiples of Ten. At the beginning of the relationship component of the assessment, test administrators asked students how many times larger 200 is than 20. In response, 34.1% of third graders, 37.7% of fourth graders and 22.7% of fifth graders responded with 10 times larger. The other two groups of Relationship questions asked students' to mentally calculate the solution to four multiple of ten multiplication and four multiple of ten division expressions of increasing complexity (see Figure 6).

If you were counting what number would come one before 203,000. What number would come one after 203,000

203,000

Figure 5. Example of before and after question.

Table 7
Percent of Rubric Responses to Before and After Regrouping Question

Grade	Question	Last Digit	Both Incorrect	One Correct	Both Correct
3	500	24.4	17.1	17.1	41.5
3	770	24.4	7.3	7.3	61.0
4	203,000	49.2	13.1	16.4	21.3
5	103,000	28.6	9.5	61.9	0
5	1,030,000	19.0	33.3	33.3	14.3

Note. N = 3–41, 4–61, 5–22.

Most students (90% or higher) correctly multiplied at least one of the four multiplying questions (see Table 8). Typically students correctly answered 3×10, a fact which is often memorized as part of multiplication fact practice. However, less than half of the students were able to correctly solve all four multiplication/division expressions. When asked to explain their strategies, many students did not refer to place value relationships but only to adding or subtracting zeros. It is probable that students were following procedural rules and were not engaging in unit coordination. This lack of unit coordination is further supported by the low number of students who responded that 200 was 10 times greater than 20. Students did not appear to understand or use the concept that a place value position is 1/10 of the position value to the left and 10 times the position value to the right.

Most students tended to correctly solve more multiplication than division expressions. This suggests that some students struggled with the inverse relationship between multiplication and division and is indicative of students operating at Stage 2 unit coordination.

Comparing Numbers

The Comparing Numbers question relevant to unit coordination assessed students' understanding of number density by asking students to identify a number between two given numbers (see Figure 7). Correctly naming numbers between two or three digit numbers required students to coordinate two units. Correctly naming numbers between numbers in the thousands and millions required students to coordinate three units. Students needed to use the relationships established when coordinating two units to understand the abstract relationships when coordinating three units.

This analysis revealed four categories of responses: Correct; No number; Last Non-zero Digit; and Others. For smaller numbers such as between 30 and 40, and between 600 and 700, 73.2 and 70.6%, respectively, of the third-

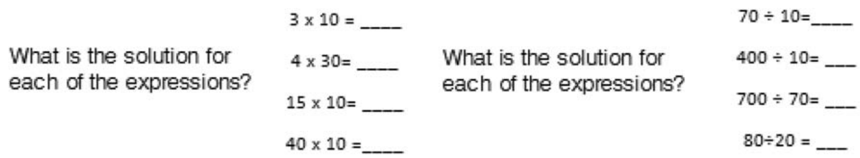

Figure 6. Examples of multiplying by multiples of ten and dividing by multiples of ten questions.

grade students correctly identified a number (see Table 9). However, for larger numbers in the thousands and millions periods, less students correctly identified numbers between the two given numbers. This change in accuracy may be due to students' inability to operate at Stage 3 unit coordination. Operating with larger and less familiar numbers requires more abstract, recursive, unit structure understandings.

Students who gave No Number category responses typically first hesitated and then stated there were no numbers between the two given numbers. Students who gave Last Non-zero Digit category responses added or subtracted one to the last non-zero digit of one of the numbers (e.g., 412,000 is between 410,000 and 411,000). These two types of responses suggest that students were relying on sequence counting and were not coordinating the place value units. Students who gave Other category responses named numbers that were outside the range of the two numbers given and were not of the Last Non-zero Digit category (e.g., 411,001 or 499,000 for numbers between 410,000 and 411,000).

Discussion

In this paper, we examined elementary students' unit coordination when engaging in tasks to explain Place Value understandings. In this section, we discuss the difficulties students exhibited with place value concepts and the educational implications of our findings.

Difficulties Children Experienced When Understanding Place Value Concepts

The third-, fourth-, and fifth-grade students in this study struggled with foundational aspects of place value relationships within multi-digit numbers. This was associated with their inability to flexibly operate on numbers and use their knowledge of place value between millennial numbers. Wright et al. (2012) posit that students need to engage with hundreds materials when extending their understanding to 1000. However, in our tasks we only gave the students in this particular study symbolic material; this prevented them from be able to make connections between concrete and symbolic material when engaging with millennials. Students were also unable to treat digits in their respective place value positions as anything more than the value of the digits themselves when regrouping and unitizing (i.e., the "3" in 34 is treated as a 3, not a 30). These findings are not new as they are very similar to the findings from Fuson et al. (1997) wherein students struggled when transitioning from single-unitary digit numbers to multi-digit numbers. These findings suggest that students in this study may have been low-achieving because they were not yet able to interiorize concepts of ten (Fuson et al. 1997). In addition, we posit that students were unable to coordinate three levels of units and may have benefitted from more opportunities to use concrete material to build toward three levels. The Fuson et al. (1997) task may explain when students are (un)able to coordinate two levels of units because they are not given concrete material to build toward two levels of unit coordination. For instance, because of the lack of concrete experiences, students may not be able to develop and use composite units as mathematical objects when operating on smaller and larger numbers, relatively. This aligns with Norton and Boyce's (2015) discussion on unit coordination, which

Table 8
Percent of Students Who Correctly Answered None, One, Two, Three, and Four of the Multiplying/Dividing Questions

		Rubric Percent				
Grade	Concept	None	One	Two	Three	Four
3	310, 4 × 100, 12 × 100 and 60 × 100	7.3	2.4	34.1	41.5	14.6
3	30/3, 40/10, 420/10, 400/100	17.5	17.5	32.5	20.0	12.5
4	3 × 10, 4 × 30, 15 × 10, and 40 × 10	1.6	4.9	31.1	18.0	44.3
4	70/10, 400/10, 700/70, 80/20	0	23.1	10.3	25.6	41.0
5	3 × 10, 4 × 1,000, 23 × 10 and 40 × 100	4.5	0.0	22.7	31.8	40.9
5	100/10, 6,000/10, 1200/40, 25/100	36.4	13.6	36.4	13.6	0

Note. N = 3–41, 4–61, 5–22.

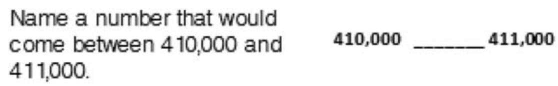

Figure 7. Example of a density question.

suggests that students at the Stage 1 unit coordination understanding are not capable of developing multi-digit or composite units absent from concrete material and that this development only allows them access to tasks that require two levels of unit coordination, not three levels of unit coordination.

Our findings also suggest that students must understand the relationships between column values (density). What this means for students is that they should understand how one place value column is one multiple of ten difference from a subsequent place value column. For students to understand and operate on multi-digit numbers, their understanding of the operational structures for multi-digit numbers must be recursive. Essentially, students should be able to extend their number understanding to larger numbers by incorporating operational structures of numbers developed through their prior unit coordination (Norton & Boyce, 2015; Wright et al., 2012). Students in this study simply manipulated the place values by adding or taking away zeros without understanding the changes in value that these actions produced. Wright et al. (2012) posit that this type of activity is what students and teachers expect when engaging with conventional place value instead of conceptual place value. We further argue that it is far more effective to understand how this conventional activity illustrates stages in students' unit coordination. For instance, when engaging in this task students were typically operating at a Stage 1 unit coordination, preventing them from accessing Stage 3. If students were able to operate at a Stage 2 they did not always have strategies (i.e., counting by multiples of ten) to build toward Stage 3. Students' inability to have an interiorized structure of the proportional relationships may explain why they had difficulty transitioning from two- and three-digit number understanding and operations to six- or seven-digit number understanding and operations.

Counting, Grouping, and Unit Coordination

We analyzed the data through the theoretical lens of students' unit coordination abilities (Norton & Boyce, 2015). Students were successful when engaging in tasks when characteristics of the tasks only required students to coordinate units at Stage 1 or 2. This was evident when students used groups to build toward three tens or "3" in 35, but they were not able to immediately draw on different groups of tens when (de)composing number. Also, students' ability to find numbers in between two multi-digit numbers seemed to require two or three levels of interiorized units.

This may explain why students in third grade were able to utilize a strategy when finding numbers in between two and three-digit numbers, but almost half of the students in fourth and fifth grade struggled to name a number in between six and nine digits. These findings suggest that students may be internalizing units to coordinate each time, but are not interiorizing recursive relationships of multi-digit number understandings as a mental structure. Wright et al. (2012) suggest that when students are developing place value they need to engage in the making of complex sets of increments and decrements with multiples of ones, tens, and hundreds. These actions further press students' development of larger numbers later. In this study, we could argue that in engaging in these increments and decrements, students are constructing operations on and around numbers, which promotes more sophisticated unit coordination.

It was also evident that, for students to provide multiple combinations for number and to develop operations for those combinations, students would need to coordinate more groups and develop more flexible, sophisticated part-whole understandings. For instance, when students were asked to describe as many as four multiplicative combinations for a number, they were required to coordinate units at stages 2 and 3, but when students were asked to simply solve multiplication problems, many of

Table 9
Percent of Rubric Responses to Density Questions

		Incorrect			
Grade	Question	No Number	Last Non-Zero Digit	Other	Correct
3	Between 30 and 40	7.3	17.1	2.4	73.2
3	Between 600 and 700	12.2	14.6	2.4	70.6
4	Between 410,000 and 411,000	48.3	1.6	9.8	44.3
5	Between 610,000,000 and 611,000,000	40.9	13.6	0	68.2

Note. N = 3–41, 4–61, 5–22.

their responses required them to coordinate units only at Stage 2. This finding aligns with that of Wright et al. (2012), who reported that the more number structures children create, the more sophisticated their understanding for that number.

Finally, findings from the regrouping and grouping while counting tasks suggest that the degree of the number of units needed to coordinate and the material provided to the students may (dis)allow them opportunities to build from one unit coordination stage to the next. For instance, in the regrouping tasks more than 85% of the third-grade students were successful when using a picture and two numbers to regroup. However, when the fourth- and fifth-grade students were given the same task, but with three numbers to regroup, only about half of the students were correct. Also, when third and fifth graders were asked to count by “two’s” their accuracy rates were much lower than their fourth-grade student counterparts who were asked to count by “10’s.” These findings suggest that unit coordination became quite difficult for three reasons: (a) more numbers to coordinate required more abstract unit coordination, (b) novel units to coordinate required more flexible unit coordination, and (c) images of blocks disallowed active engagement for some students when building from one stage of unit coordination to another stage.

These findings suggest that when students in third grade were successful it was because they were given tasks that only required them to engage in Stage 2 unit coordination. Whereas fourth- and fifth-grade students were less successful when given relatively more sophisticated numbers, more units, and fewer pictorial items. These changes in assessment questions were fundamental in how students were able to access their counting, grouping, and unit coordination and suggest that students at different grade levels require different stages of unit coordination when engaging in place value tasks.

Educational Implications

These results have implications regarding the design of assessments and interventions for students’ understanding of place value. For instance, it is still not clear how educators can vary their instruction to address specific underlying mental actions that low-achieving students may require who are already many school years behind in meeting statewide standards and classroom objectives. Findings from this study do not directly address these concerns, but suggest that curricula and statewide content standards do not utilize important mental activity (e.g., Common Core State Standards for Mathematical Practices, 2009; Principles to Action, 2014) necessary for low-achieving students when designing interventions. The Common Core Mathematical practices that illustrate effective student engagement is not nuanced enough to describe how student engagement may appear when students are performing well below grade level and, more specifically, how this activity should differ for conceptual place value instruction. Thus, addressing these specific concerns would benefit these students with this particular form of number development. Assessment and intervention should focus on inferred mental actions that students rely on when solving problems and not just accurate procedures. Quite often statewide standards guide students’ place value assessments rather than engaging students in the necessary mental activity needed for success for these problems. Curricula designers and those designing statewide standards should consider these necessary mental actions when designing assessments and interventions. Many U.S. students without conceptual place value understandings are unable to reach Stage 3 unit coordination to successfully meet place value standards. This suggests that common standards and assessments need to change to reflect mental activity necessary for students’ conceptual place value understandings.

More specifically, Wright et al. (2012) address the “structuring” of numbers students need to engage in when developing early number and conceptual place value understandings. Many of the interventions Wright et al. posit for place value development address activity with materials (e.g., incrementing and decrementing) and the proximity students have with materials (e.g., concrete, visualized concrete, verbal). Findings from this study echo these intervention models and explain the stage of unit coordination, which may be used by students when relying on activity and materials. Thus, we argue that curriculum should take on these aspects of of activity and materials while addressing the stages of unit coordination.

A final educational implication is that curricula designers should design intervention procedures that provide more nuanced options that closely align with student’s place value understandings and abilities. Low-achieving students, are not yet able to master particular conceptual place value knowledge in lower grade levels, which then further distances them from achieving at the same rate as their peers in number and operations in upper grade levels. Thus, future research should use unit coordination to more fully develop learning trajectories that can support this curricula design. This new theoretical perspective can provide opportunities for educational stakeholders to consider the degrees of learning necessary for such a multi-faceted concept in mathematics.

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Authors' Note

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